

IMPROVED INTEGRATION METHODS FOR P-ADAPTIVE BOUNDARY ELEMENTS TECHNIQUES

A.M.Cutillas (*) M. Doblaré (**) J.Sanz-Serna (&)
E.Alarcón (*)

(*) Departamento de Mecánica Estructural y Construcciones Industriales.
Universidad Politécnica de Madrid.

(**) Departamento de Ingeniería Mecánica. Universidad de Zaragoza.

(&) Departamento de Matemática Aplicada. Universidad de Valladolid

Abstract

Non linear transformations are a good alternative for the numerical evaluation of singular and quasisingular integrals appearing in Boundary Element Methods specially in the p-adaptive version. Some aspects of its numerical implementation in 2-D Potential codes is discussed and some examples are shown.

1 Introduction

One of the most important computational problems in Boundary Element Methods (BEM) is the accurate evaluation of the integrals appearing in the influence coefficients matrices. Integral kernels in Potential and Elasticity problems produce singular and quasisingular integrals depending on the collocation point position.

Special techniques have been developed in order to evaluate strongly singular integrals and some rules-of-thumb have been established to avoid inaccurate results in the quasisingular ones. Current computer programs take three courses of action to solve the above mentioned integrals:

- Analytical solution, specially in constant or linear elements.
- Different numerical quadratures depending whether the collocation point is on or outside the elements.

- Special rules for the computations at internal points.

However there are some typical problems in which these techniques produce inaccurate results: elements round a corner, specially in non homogeneous media, internal points near the boundary and slender bodies. The main source of inaccuracy is the short distance between collocation point and the element under integration.

The new development of p-adaptive version of BEM needs accurate computation of integrals with high order approximation functions and corresponding collocation points inside the elements that tends to the ends as the order of approximation grows.

It is then desirable to have an integration rule that can be used for every situation of the collocation point.

In what follows we shall show some improvements in the evaluation of singular and quasisingular integrals based on non linear transformation techniques as well as the results of the implementation of these techniques in a 2D BEM Potential code.

2 Non linear transformations for the computation of 2-D singular and quasisingular integrals

Several techniques have been suggested to evaluate strongly singular or quasisingular integrals.

Lachat and Watson [1] and Watson [2] proposed, to solve different singularities appearing in 2D and 3D Elasticity problems, by dividing the element into subelements when the collocation point is on it. Quasisingular integrals are evaluated with different gaussian formulation depending on the collocation point distance to the element by controlling element characteristic size and number of gaussian points accordingly with an error bound.

Gomez-Lera et al [3] proposed a non linear transformation to accurately compute the integrals appearing in axisymmetric elastodynamic problems, although no justification was presented about the observed improvements.

Telles [4] proposed a non linear coordinate transformation to solve these integrals. Second and third degree polynomials were suggested for singular integrals with the first and second derivatives null at the singularity. while exponential and polynomial transformations were proposed to treat quasisingular integral evaluation.

Guiggiani and Casalini [5],[6] evaluated numerically the Cauchy Principal Value of integrals (CPV) in 2D Elastostatic problems. CPV can be obtained

with standard Gauss-Legendre quadratures and a logarithmic term because of the coordinate transformation at both sides of the singularity are not equal in different curved elements.

Guiggiani [7] showed the error if gaussian quadratures are employed over curved elements and logarithmic terms are not considered. This logarithmic term vanishes if the collocation point is on the element or linear elements with linear mappings are employed.

Cerrolaza et al [8] used a numerical procedure to evaluate the Cauchy Principal Value integrals. Two cubic coordinate transformations are suggested at each side of the singularity. This procedure mimic the limit process involved in the CPV analysis. These transformations map each subinterval in the two intervals in which the singularity divide the element. The transformations must have null jacobian at the singularity and the last (or first) gaussian point is mapped into the singularity minus (or plus) the parameter ϵ which is a small free real value. Numerical good results are obtained specially with high order gaussian quadratures.

Sanz Serna et al [9] showed a hidden error in the numerical evaluation of singular integrals by non linear coordinate transformations. The source of the error is a logarithmic relation between the main coefficients of the transformation not considered in a numerical quadrature. This term vanishes if the two main coefficients are equal and tends rapidly to zero if high order gaussian quadratures are employed.

Alarcon et al [10],[11] explained the reasons of the good behaviour of non linear coordinate transformations in the numerical evaluation of singular and quasisingular 2D integrals. Intrinsic errors in some of these methods are detected. New transformations, a bicubic a single 4th degree transformation and bitangent (Alcantud [12]), are suggested with a good behaviour in both singular and quasisingular integrals.

Non linear transformations produce a mapping from the integration interval η into the gaussian integration interval ξ

This mapping has three objectives:

- Concentration of integration points near the singularity η_s to evaluate the function where its contribution is more important.
- Zero jacobian at the singularity η_s
- Smooth behaviour of the jacobian near the singularity η_s

The integration interval can be divided into two subintervals at each side of the singular point $(-1, \eta_s)$ and $(\eta_s, 1)$ and each of them needs a new mapping into the integration interval $(-1, 1)$ as is shown in Fig.1.

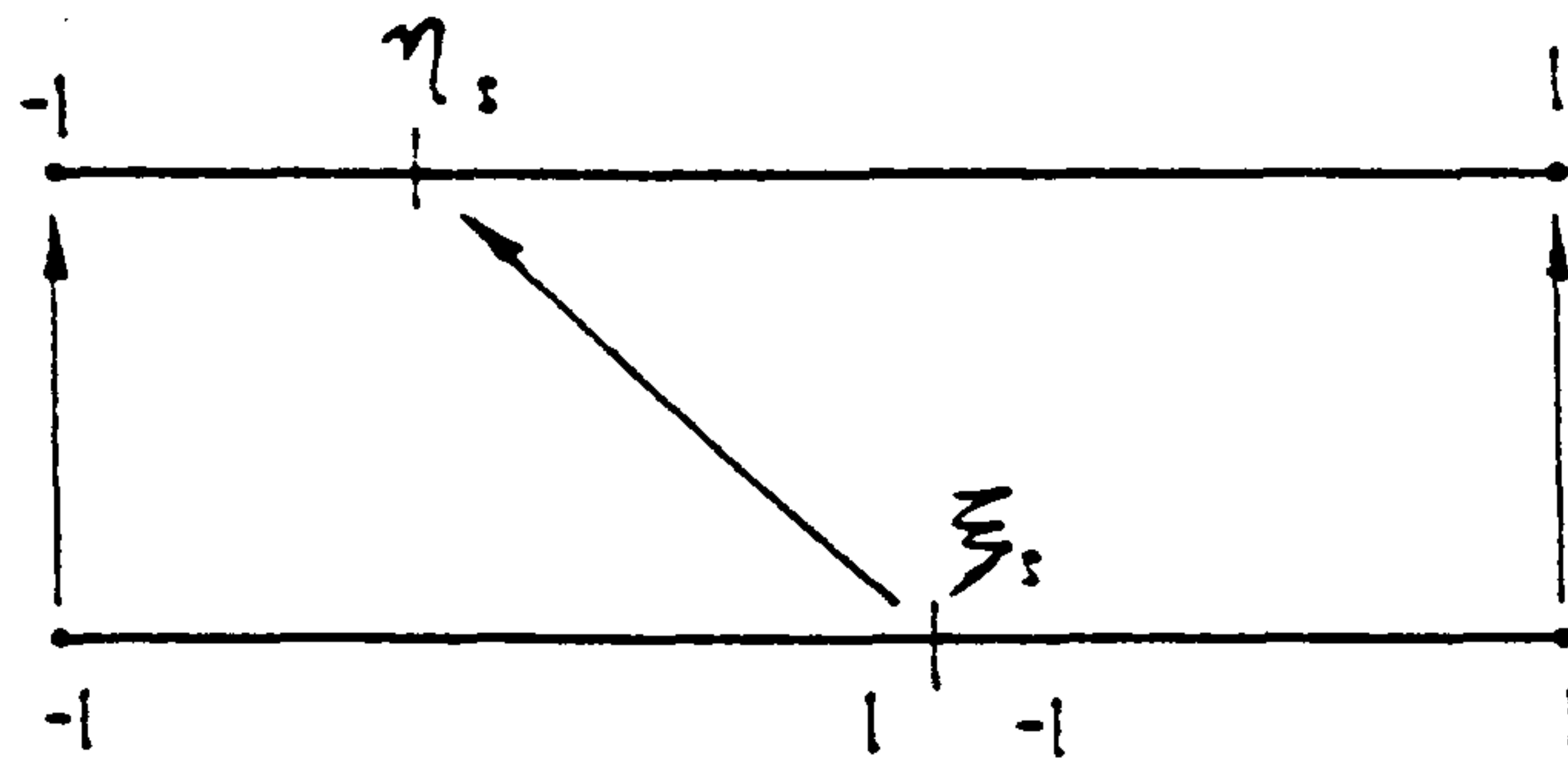


Figure 1:

Let us consider the transformations

$$\begin{aligned}\eta_1 &= \eta_1(\xi) = \eta_s + a_1(\xi - 1) + b_1(\xi - 1)^2 + c_1(\xi - 1)^3 \\ \eta_2 &= \eta_2(\xi) = \eta_s + a_2(\xi + 1) + b_2(\xi + 1)^2 + c_2(\xi + 1)^3\end{aligned}\quad (1)$$

These transformations at the left and the right side of the singularity verify:

$$\begin{aligned}\eta_1(\xi = -1) &= -1 & \eta_2(\xi = -1) &= \eta_s \\ \eta_2(\xi = 1) &= \eta_s & \eta_2(\xi = 1) &= 1\end{aligned}\quad (2)$$

$$\frac{d\eta_1}{d\xi}(\xi = 1) = 0 \quad \frac{d\eta_2}{d\xi}(\xi = -1) = 0$$

In the case of a logarithmic singularity the integrand may be always expressed as

$$I = aLnr + I_r \quad (3)$$

I_r is a regular function.

A cubic transformation with a null jacobian at the singularity produce

$$\begin{aligned}\eta(\xi) &= \eta_s + A_1(\xi - \xi_s) + B_1(\xi - \xi_s)^2 + C_1(\xi - \xi_s)^3 \\ I(\xi) &= aLnr(\xi)2A_1(\xi - \xi_s) + \dots + I_r(\xi)2A_1(\xi - \xi_s) + \dots\end{aligned}\quad (4)$$

This integrand is a regular function what explains the good performance of all these transformations.

In the case of a Cauchy Principal value integral in 2D

$$I = \frac{1}{r} + I_r \quad (5)$$

I_r is a regular function.

Let us investigate the performance of an integral of the type

$$I = \int_{-1}^1 \frac{d\eta}{\eta - \eta_s} \quad \eta_s \in (-1, 1) \quad (6)$$

With the definition of Cauchy Principal Value and the two transformations considered in (1)

$$I = \int_{-1}^{\eta_s - \epsilon} \frac{d\eta}{\eta - \eta_s} + \int_{\eta_s + \epsilon}^1 \frac{d\eta}{\eta - \eta_s} = \int_{-1}^{\eta_1^{-1}(\eta_s - \epsilon)} \frac{\eta_1'(\xi) d\xi}{\eta_1(\xi) - \eta_s} + \int_{\eta_2^{-1}(\eta_s + \epsilon)}^1 \frac{\eta_2'(\xi) d\xi}{\eta_2(\xi) - \eta_s}$$

$$I = \int_{-1}^{\eta_1^{-1}(\eta_s - \epsilon)} \frac{[a_1 + 2b_1(\xi - 1) + \dots] d\xi}{a_1(\xi - 1) + b_1(\xi - 1)^2 + \dots} + \int_{\eta_2^{-1}(\eta_s + \epsilon)}^1 \frac{[a_2 + 2b_2(\xi + 1) + \dots] d\xi}{a_2(\xi + 1) + b_2(\xi + 1)^2 + \dots} \quad (7)$$

This integral can be expressed as

$$\int_{-1}^{\eta_1^{-1}(\eta_s - \epsilon)} \frac{d\xi}{\xi - 1} + \int_{\eta_2^{-1}(\eta_s + \epsilon)}^1 \frac{d\xi}{\xi + 1} + I_r = I_r + I_s \quad (8)$$

Where I_r is a new regular integral

A numerical evaluation of I_s is zero by symmetry if the same number of gaussian points are employed.

Analytical evaluation of I_s is trivial

$$I_s = \ln \left| \frac{\eta_1^{-1}(\eta_s - \epsilon) - 1}{\eta_2^{-1}(\eta_s + \epsilon) + 1} \right| \quad (9)$$

The limits of I_s when ϵ tends to zero

$$\lim_{\epsilon \rightarrow 0} I_s = \ln \left| \frac{a_2}{a_1} \right| \quad (10)$$

If both $a_i = 0$ the limits of I_s becomes the logarithmic of the ratio between b_i etc.

It is clear that the analytical value of I_s will be the same as the numerical one when the dominant coefficient of both transformations are the same.

Then, the transformations considered in (1) must be of the form:

$$\begin{aligned}\eta_1(\xi) &= \eta_s + B(\xi - 1)^2 + C(\xi - 1)^3 & \xi \in [-1, 1] \\ \eta_2(\xi) &= \eta_s + B(\xi + 1)^2 + D(\xi + 1)^3\end{aligned}\quad (11)$$

If boundary conditions in (2) are included, the transformations become

$$\begin{aligned}\eta_1(\xi) &= \eta_s + \frac{1 + \eta_s}{8}(\xi - 1)^3 - \frac{B}{2}(\xi^2 - 1)(\xi - 1) \\ \eta_2(\xi) &= \eta_s + \frac{1 - \eta_s}{8}(\xi + 1)^3 + \frac{B}{2}(1 - \xi^2)(\xi + 1)\end{aligned}\quad (12)$$

B is a free parameter. The transformation is monotonic if

$$B > 0 \quad \text{and} \quad B \leq \frac{3}{4}(1 - |\eta_s|) \quad (13)$$

Good results have been obtained with $B = \frac{3}{4}(1 - |\eta_s|)$

3 BEM in potential theory

Potential problems may be in general expressed as:

$$-\Delta\Phi = f \quad \text{in} \quad \Omega \subset \mathbb{R}^3 \quad \Gamma = \partial\Omega \quad (14)$$

where Δ is the Laplace's operator, $\Phi \in \mathcal{C}^2(\Omega)$ is the unknown function.

Boundary conditions may be expressed in terms of the function, normal derivatives around the boundary or both defining Dirichlet, Neumann or Mixed problems.

3.1 Green's Theorems

Let us consider functions $u, v \in \mathcal{C}^2(\Omega)$ If second Green's Theorem is considered:

$$\int_{\Omega} (u\Delta v - v\Delta u) d\Omega = \int_{\Gamma} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) d\Gamma \quad (15)$$

Let us consider $u = \Phi$ and $v = \Phi^*$

$$\begin{aligned}-\Delta\Phi &= f \\ -\Delta\Phi^* &= 4\pi\delta(P)\end{aligned}$$

where

$\delta(P)$ is the Dirac's function at point P

Φ^* is called the fundamental solution in potential problems

$$\Phi^*(P, Q) = \frac{1}{r(P) - r(Q)} = \frac{1}{r}$$

If the second Green's theorem is applied avoiding the singularity building a sphere centered in P with radius ϵ .

Taking limits when ϵ tends to zero, Green's equation becomes:

$$\begin{aligned} -C(P)\Phi(P) &= \int_{\Gamma} (\Phi(Q) \frac{\partial \Phi^*(P, Q)}{\partial n} - \Phi^*(P, Q) \frac{\partial \Phi(Q)}{\partial n}) d\Gamma - \int_{\Omega} f \Phi^* d\Omega \\ -C(P)\Phi(P) &= \int_{\Gamma} (\Phi(P) q^*(P, Q) - \Phi^*(P, Q) q(Q)) d\Gamma - \int_{\Omega} f \Phi^* d\Omega \quad (16) \end{aligned}$$

where $q = \frac{\partial \Phi}{\partial n}$ and $q^* = \frac{\partial \Phi^*}{\partial n}$

In 3D:

$$C(P) = \begin{cases} 4\pi & \text{if } P \in \Omega \\ 2\pi & \text{if } P \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi^*(P, Q) = \frac{1}{r}$$

$$q^*(P, Q) = \frac{d(1/r)}{dn}$$

In 2D:

$$C(P) = \begin{cases} 2\pi & \text{if } P \in \Omega \\ \pi & \text{if } P \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi^*(P, Q) = Ln \frac{1}{r}$$

$$q^*(P, Q) = \frac{dLn(1/r)}{dn}$$

3.2 Numerical treatment

The boundary is discretized in n elements $\Gamma = \bigcup_{e=1}^n \Gamma_e$. Using the procedures of projective methods, looking for the best approximation in the subspace generated by the finite base functions $\{N_j\}_{j \in I_n}$.

In a local system of coordinates in element e :

$$\begin{aligned}\varphi^e(\xi) &= \sum_{j=1}^m N_j \varphi_j^e \\ q^e(\xi) &= \sum_{j=1}^m N_j q_j^e\end{aligned}$$

Substituting in eq ():

$$c_i \varphi_i + \sum_{e=1}^n \int_{\Gamma_e} \left[\sum_{j=1}^m N_j \varphi_j q^*(P, Q) \right] = \sum_{e=1}^n \int_{\Gamma_e} \left[\sum_{j=1}^m N_j q_j \Phi^*(P, Q) \right] + \int_{\Omega} f \Phi^*(P, Q)$$

Let us consider

$$\underline{a}_{i,j} = \sum_e^{e_j} \int_{\Gamma_e} N_j q^*(P, Q) = \int_{\Gamma_e} N_j q^*(\mathbf{x}_i, \xi) \quad (17)$$

$$b_{i,j} = \sum_e^{e_j} \int_{\Gamma_e} N_j \Phi^*(P, Q) = \int_{\Gamma_e} N_j \Phi^*(\mathbf{x}_i, \xi) \quad (18)$$

$$f_i = \int_{\Gamma_e} f \Phi^*(P, Q) = \int_{\Gamma_e} f \Phi^*(\mathbf{x}_i, \xi) \quad (19)$$

Where e_j are the elements to which node j belongs to. Let

$$\underline{\underline{a}}_{i,j} = c_i \delta_{i,j}$$

$$a_{i,j} = \underline{a}_{i,j} + \underline{\underline{a}}_{i,j}$$

A linear system of equations for Poisson's problems is obtained:

$$a_{i,j} \varphi_j = b_{i,j} q_j + f_i$$

In the matrix form:

$$\mathbf{A} \Phi = \mathbf{B} \mathbf{Q} + \mathbf{f} \quad (20)$$

for Laplace's equation $\mathbf{f} = 0$:

$$\mathbf{A}\Phi = \mathbf{B}Q \quad (21)$$

If all the unknowns are included in vector δ

$$\mathbf{K}\delta = \mathbf{p} \quad (22)$$

is the linear system of equations to be solved.

4 P-adaptive Boundary Elements

Once the basic theory and the numerical implementation difficulties were understood in Finite Element Methods (FEM), researchers tried to establish "a posteriori" errors of the solution by refining the order of the approach in a selfadaptive fashion.

Two ways were proposed the h-version, where the accuracy is obtained reducing the size of the elements, and the p-version, increasing the order of the approximation functions in the areas where it is necessary.[13],[14].

Those ideas have been recently introduced in Boundary Elements field, both in h and p versions.[15]-[17].

The basic ideas of p-adaptive boundary elements can be obtained comparing the fundamental differences between FEM and BEM:

- In FEM the basis of interpolating and weighting functions is the same. They are functions locally defined over the elements. Sparse and symmetric global matrices are then obtained.
- In BEM weighting functions are the fundamental solutions of the problem under consideration. They are globally based functions while the interpolating ones are generally locally based. Full and non-symmetric global matrices are then obtained.

As the weighting functions are globally based it seems logical to consider as well the possibility of globally based interpolating functions, at least over macro-elements, and to analyse the consequences that such decision can have in the whole computational scheme. In some macroelements the order of interpolating functions must be increased depending on local indicators of error.

A general scheme of a p-adaptive boundary element code is shown in Table-1.

Representation of the geometry and boundary conditions

Geometry representation																	
Boundary conditions representation																	
Linear interpolation solver																	
Global error estimation $ e $																	
DO WHILE																	
<table> <tr> <td colspan="2">FOR I=1</td></tr> <tr> <td colspan="2">Indicator IN computation</td></tr> <tr> <td colspan="2">If $IN - \text{Tolerance}$</td></tr> <tr> <td>≤ 0</td><td>> 0</td></tr> <tr> <td>Continue</td><td>Add a new polynomial</td></tr> <tr> <td colspan="2">TO Number of elements</td></tr> <tr> <td colspan="2">Global error computation e</td></tr> <tr> <td colspan="2">$e - \text{Tolerance} > 0$</td></tr> </table>		FOR I=1		Indicator IN computation		If $ IN - \text{Tolerance}$		≤ 0	> 0	Continue	Add a new polynomial	TO Number of elements		Global error computation $ e $		$ e - \text{Tolerance} > 0$	
FOR I=1																	
Indicator IN computation																	
If $ IN - \text{Tolerance}$																	
≤ 0	> 0																
Continue	Add a new polynomial																
TO Number of elements																	
Global error computation $ e $																	
$ e - \text{Tolerance} > 0$																	

Table 1: p-adaptive BEM

Macro-elements must be defined at this step. Geometrical criteria would be the first to be employed: Changes of geometry from straight to curved lines, changes of alignments created by corners, define natural macro-elements.

Boundary conditions may define them. Different boundary conditions define different macro-elements.

Interpolating hierarchy functions may be used in this representation.

Interpolating hierarchy

In two dimensional cases, the family is monodimensional. It is possible to work with the Peano approach:

$$\begin{aligned} N_0 &= \frac{1}{2}(1 - \xi) \\ N_1 &= \frac{1}{2}(1 + \xi) \\ N_p &= \frac{1}{p!}(\xi^p - b) \end{aligned} \tag{23}$$

where $p > 2$ and $\xi \in [-1, 1]$

$$b = \begin{cases} 1 & \text{if } p \text{ is even} \\ \xi & \text{if } p \text{ is odd} \end{cases}$$

Influence matrices analysis

During the adaptive process the degree of the interpolating functions is increased what induces a 'nesting' influence matrices so that all the coefficients obtained at each step of the adaptivity process may be used in the next one.

In order to see the nesting of influence matrices let us imagine that the equations corresponding to a certain discretization are as follows:

$$\mathbf{A}\Phi = \mathbf{B}Q \tag{24}$$

Each row corresponds to a collocation point, while each column reflects the integration of a shape function over an element.

If it is decided to introduce new interpolating functions, it is necessary to compute three blocks of integrals, producing the new system:

$$\mathbf{A}^*\Phi^* = \mathbf{B}^*Q^* \tag{25}$$

$$\Phi^* = \begin{Bmatrix} \Phi \\ \Phi' \end{Bmatrix} \quad Q^* = \begin{Bmatrix} Q \\ Q' \end{Bmatrix}$$

Φ, Q boundary conditions in the previous step
 Φ', Q' boundary conditions in the new step

$$A^* = \begin{bmatrix} A & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

A, B Influence coefficients in the previous step.
 A_{12}, B_{12} Influence coefficients of new interpolating functions viewed from the old collocation points.
 A_{21}, B_{21} Influence coefficients of old interpolating functions viewed from the new collocation points.
 A_{22}, B_{22} Influence coefficients of new interpolating functions viewed from the new collocation points.

This nesting is then transmitted to the global stiffness matrix:

$$K\delta = p \quad K^*\delta^* = p^* \quad (26)$$

Error indicator and estimators

Error indicators show in which elements the order of interpolating functions must be increased.

Two indicators were used in "Queimada" program:

Peano's criterion:

$$Q_j = \frac{\sum_{j \neq i} K_{ji} \delta_i - p_j}{\sqrt{K_{jj}}} \quad (27)$$

ETSIIM criterion:

$$\epsilon_j^2 = \delta_h^2 \frac{\int_{\Gamma_j} N_h^2}{\int_{\Gamma_j} \left| \sum_1^{h-1} \delta_i N_i \right|^2} \quad (28)$$

where

$$\delta_h \sim \frac{p_h}{K_{hh}}$$

These terms are obtained nesting the stiffness matrix, loads matrix and unknowns matrix:

$$K_{hi} \delta_i + K_{hh} \delta_h = p_h \quad (29)$$

Error estimators are the stop criteria. In FEM analysis 'a posteriori error estimators' are used. In BEM techniques the mathematical support is still lacking. In potential problems equilibrium conditions are used:

$$\int_{\Gamma} q = 0 \quad (30)$$

for Dirichlet and Mixed problems.

5 Numerical implementation of non-linear transformations in a 2D p-adaptive BEM code

5.1 Standard influence coefficients computation

Let us study the standard techniques used in the evaluation of influence coefficients in a 2D p-adaptive BEM potential code.

- Matrix A influence coefficients computation

-Collocation point is outside the element: standard Gauss-Legendre quadratures are used.

Numerical problems may appear if the collocation point is close to the element whether the collocation point belongs to the boundary or it is an internal point.

-Collocation point is on the element: the integral has a singularity in the form $O(1/r)$ and it only exists in the Cauchy Principal Value sense. Two standard Gauss-Legendre quadratures are needed.

If the collocation point is on the edge of the element the influence coefficient belongs to matrix A main diagonal and may indirectly be computed with the sum of the rows of A. (Constant potential hypothesis).

- Matrix B influence coefficients computation

-Collocation point is outside the element: standard Gauss-Legendre quadratures are used.

As coefficient A computations, numerical problems may appear if the collocation point is close to the element.

-Collocation point is on the element: the integrand has a singularity in the form $O(\ln r)$, although the integral exists.

If the collocation point is on an edge of the element the integrand can be divided into two functions: A regular one in which standard Gauss-Legendre may be used and a function of $\ln 1/x f(x)$ type where a Stroud-Secrest quadrature may be used.

If the collocation point is on the element, the integral can be divided into two integrals each of them with the collocation point on a subelement edge. The numerical quadrature is then obtained after two standard Gauss-Legendre and two Stroud-Secrest quadratures.

5.2 Influence coefficient computation with non-linear transformations

As it has been shown, standard BEM codes compute influence coefficients, with a great casuistic. Singular integrals are computed depending whether the collocation point is on the element. Quasisingular integrals are computed increasing the number of gaussian points in a standard Gauss-Legendre quadrature when the collocation point is outside the element, etc.

Non-linear transformations allow to treat singular and quasisingular integrals under the same algorithm.

Let us consider the bicubic transformation (12). Once the B parameter has been established, η_s singularity position must be established in each case.

In the bicubic transformation, gaussian points are concentrated where the contribution of the function to the integral value is more important and points which tend to infinity are well regulated with first and second derivatives transformation properties above mentioned.

These characteristics are valid for both singular and quasisingular integrals

Accordingly with that approach the two type of singularities $O(1/r)$ and $O(\ln r)$ may be treated in the same way and then influence coefficients A and B can be computed with the same formulae.

- Collocation point on the element

If the collocation point is on the element the abscissa η_s is the collocation point one.

If the collocation point is in an edge of the element there are two possibilities:

- Influence coefficient A is computed in an indirect way by the sum of the rows of the matrix. Influence coefficient B may be computed taking the abscissa 1 or -1 as η_s . Bicubic transformation degenerates as a single cubic one.

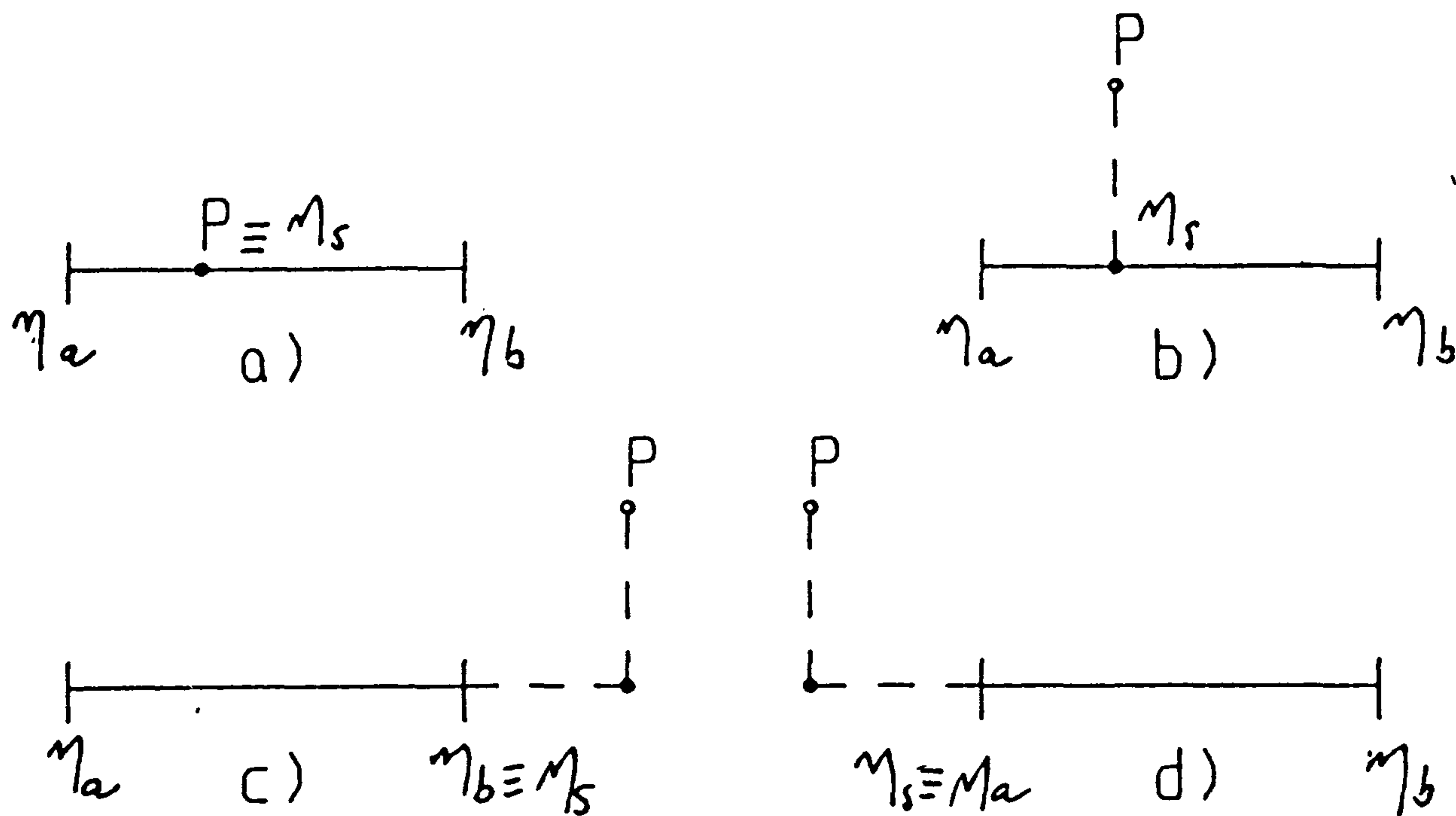


Figure 2:

- The transformation would be defined over the element we are treated and the next one. Influence coefficients A and B would be the contribution of this element. Because of each element has different coordinate mapping a logarithmic term must be added [5],[6]. This possibility would make the code more tedious, so it has not been considered

- Collocation point outside the element.

In these cases there are no singularities in the integrands terms, but depending on collocation point distance to the element quasisingular integrals can be obtained. Non-linear transformations may be used in order to concentrate gaussian points where the function contribution to the integral is more important as it was mentioned.

Maxima values of integrands are close to the collocation point projection over the element as it is shown in Figs.5-7.

Integrands (17) and (18) are represented for different values of N_j shape functions and X_0/L and Y_0/L values accordingly to Fig. 3 scheme.

A general algorithm may be proposed depending on the collocation point projection.(Fig.2).

- *Collocation point projection lies on the element.* Abscissa projection will be the singularity abscissa η_s .
- *Collocation point projection lies outside the element.* No transformations could be used in this case. However the nearest edge of the element could be used as singularity abscissa η_s in order to take into account the close position of the collocation point.

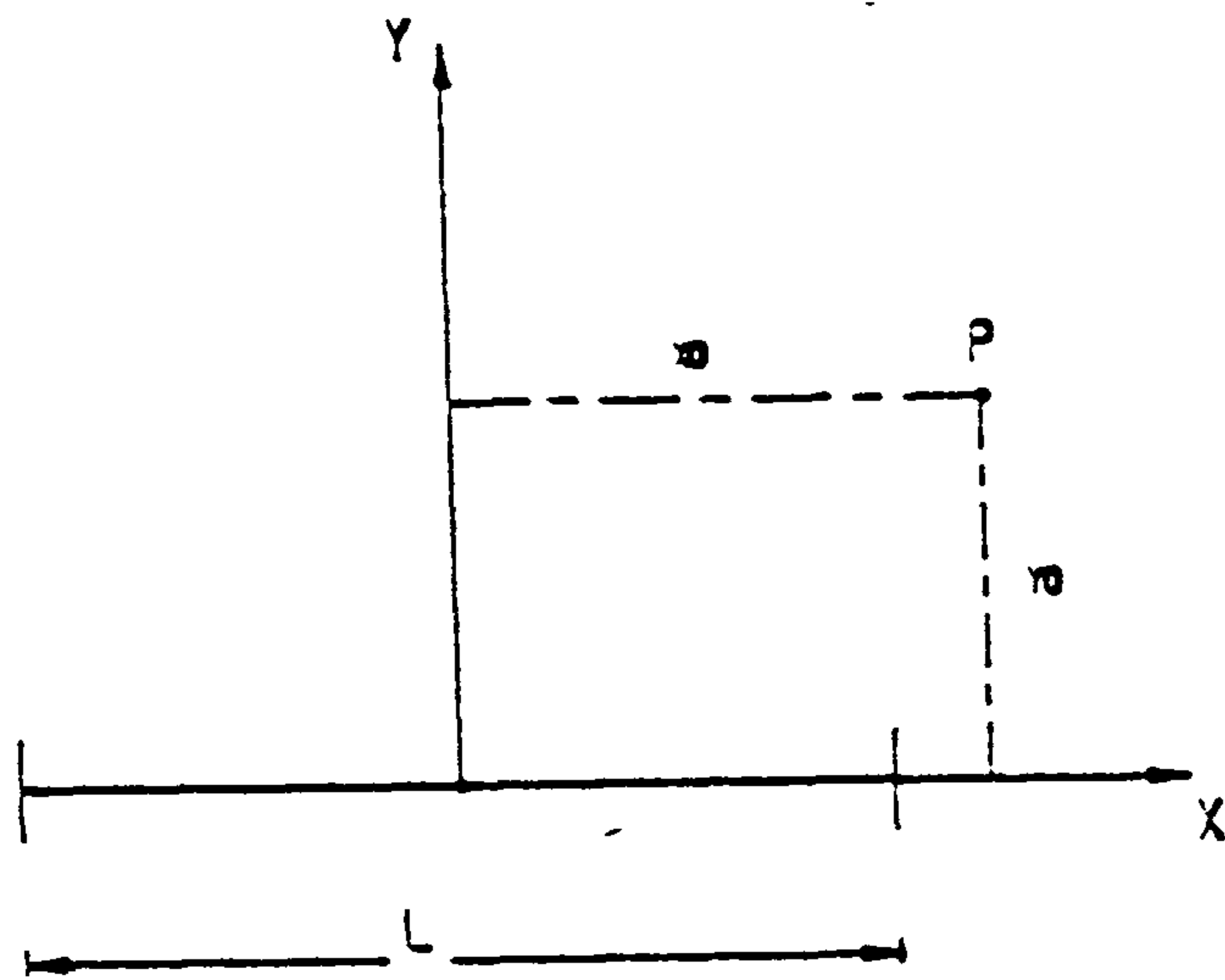


Figure 3:

5.3 Parametric studies

In order to check the criteria above mentioned, parametric studies have been developed in linear elements to compare analytical with numerical results.

In all these studies, relative errors is represented depending on relative position of collocation point and a standard linear element as it is shown in Fig.3.

All these integrals have been computed with polynomial N_1 .

Two numerical solutions have been compared. Influence coefficients computed with bicubic transformations accordingly the criteria established above, each transformation with 4 Gaussian points and influence coefficients computed with 8 Gaussian points.

In general, when the integrals are not singular or quasisingular, an eight-point quadrature is better than two four-point quadratures. However non-linear transformations give better results in singular and quasisingular cases.

As it is shown in Figs 8 and 9 with Gaussian quadratures very good results are obtained in A and B/L influence coefficients up to collocation point adimensional distances y_0/L equal to 0.50 or 0.25 . For lesser values of y_0/L , numerical instabilities have been observed and high relative errors have been obtained.

With bicubic transformations higher relative errors are obtained when the collocation point distance to the element is big, specially if its projection is near the element edge so a four-points quadrature is obtained against an eight-points quadrature in the Gaussian case. However very well bounded relative errors are obtained for small values of y_0/L .

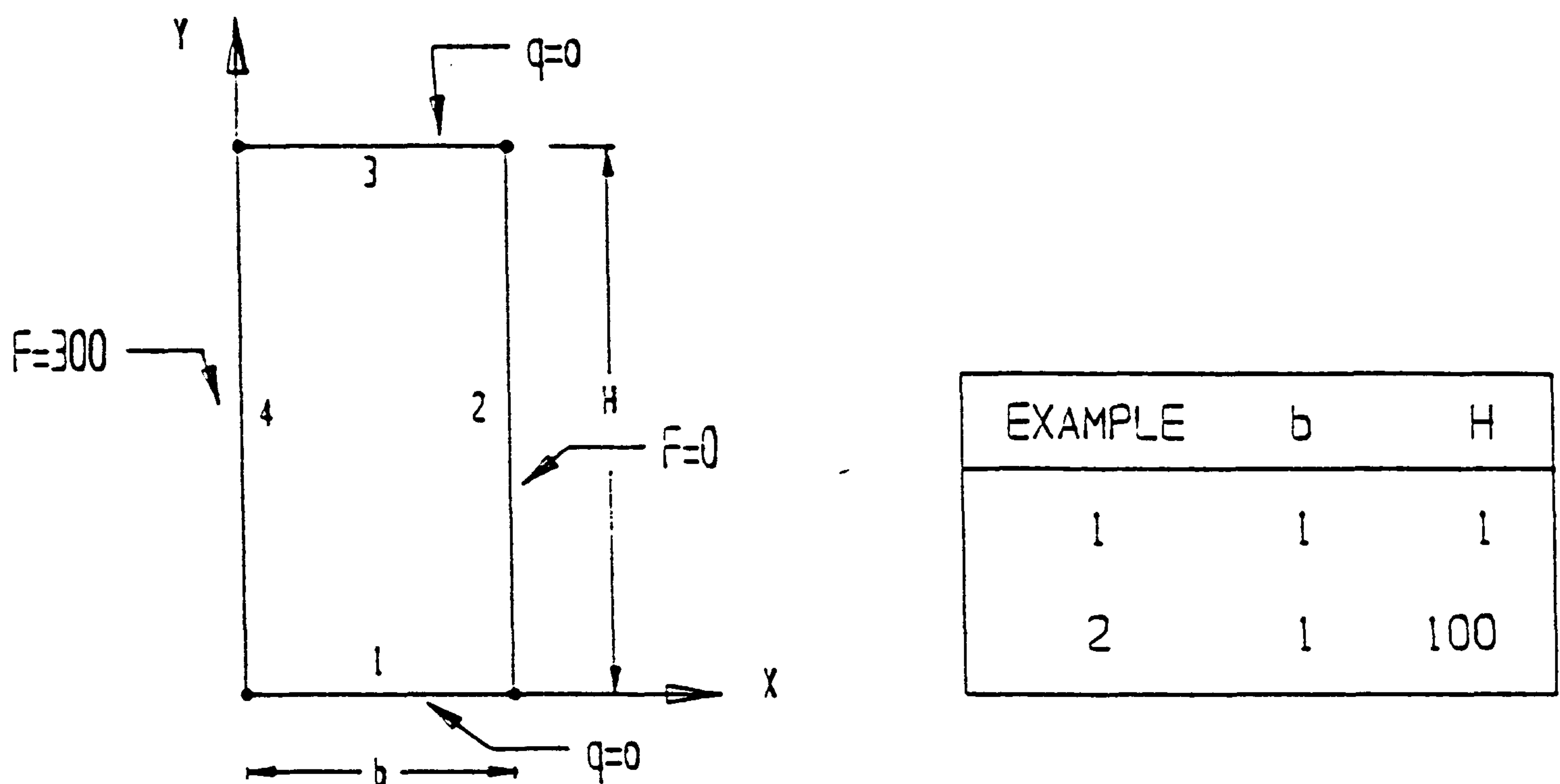


Figure 4:

5.4 Examples

Two examples are shown in order to analyse non linear transformations behaviour in 2D BEM Potential problems:

- *Example 1:*(Fig.4). Square domain. Null flux in sides 1 and 3 and potential 0. and 300. in the others two opposite sides. Analytical solution is linear with one variable dependent.
- *Example 2:*(Fig.4). Same problem as example 1 with a rectangular domain in which one side is 100 times larger than the other one (slender domain).

With examples 1 and 2 influence of singular and quasisingular integrals in both boundary and internal points solution is expected

- Example 1:

Boundary solutions are the exact ones in both cases bicubic and gaussian quadratures.

Both quadratures have four-Gaussian points because in this case bicubic transformation always degenerates in a cubic one.

Because of boundary solutions are equal, internal points solutions could be well compared.

In Fig.10 comparisons along $x=0.25$ and $x=0.01$ at different y coordinates of potential function F is shown.

Good behaviour of bicubic transformations is observed.

- Example 2:

Boundary solution has important relative errors with Gaussian quadratures (up to 88%.)

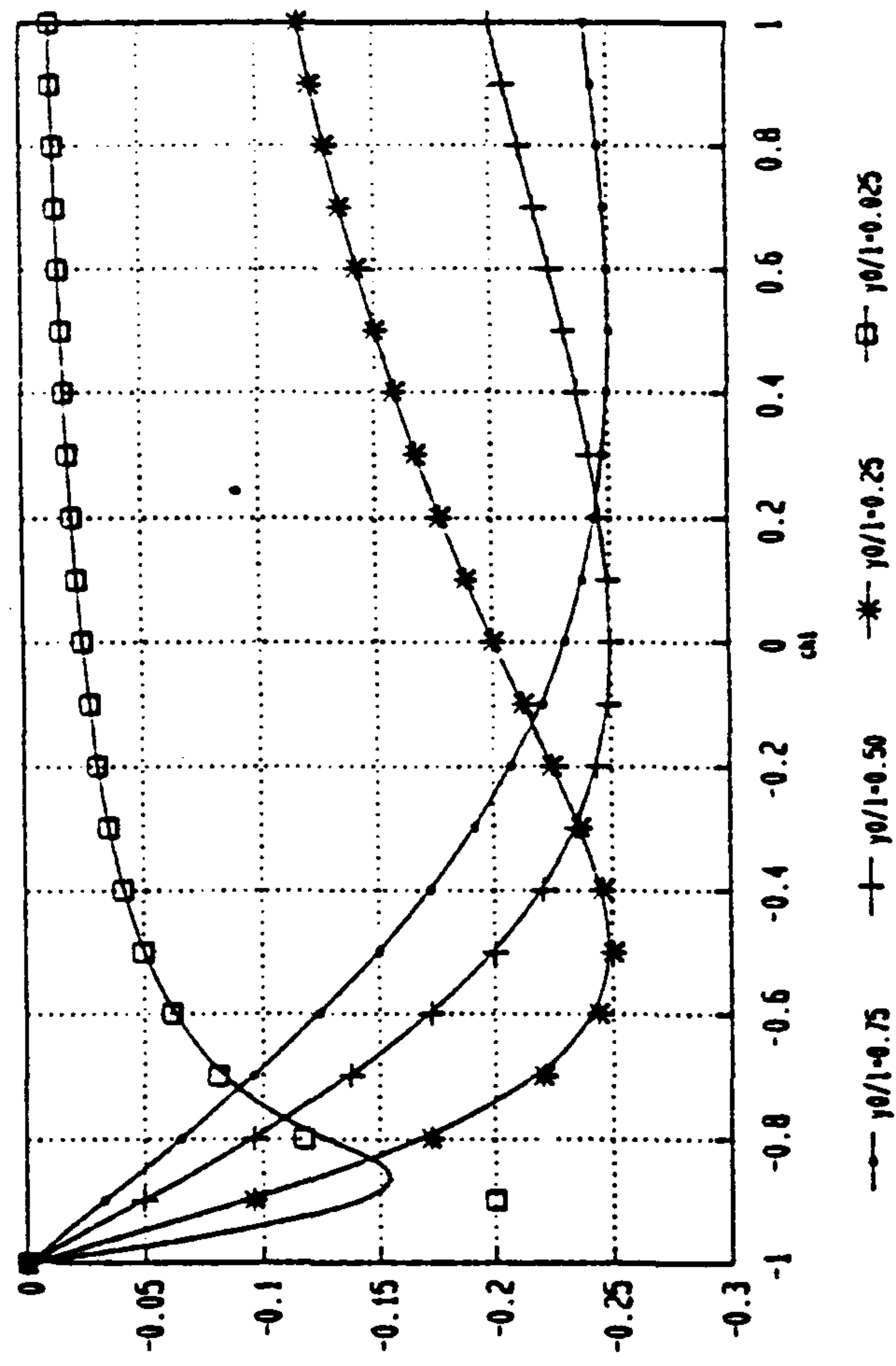
Solution at internal points depends on boundary values so very bad results are obtained with gaussian quadratures and better ones are obtained with bicubic transformations as is shown in Fig.11.

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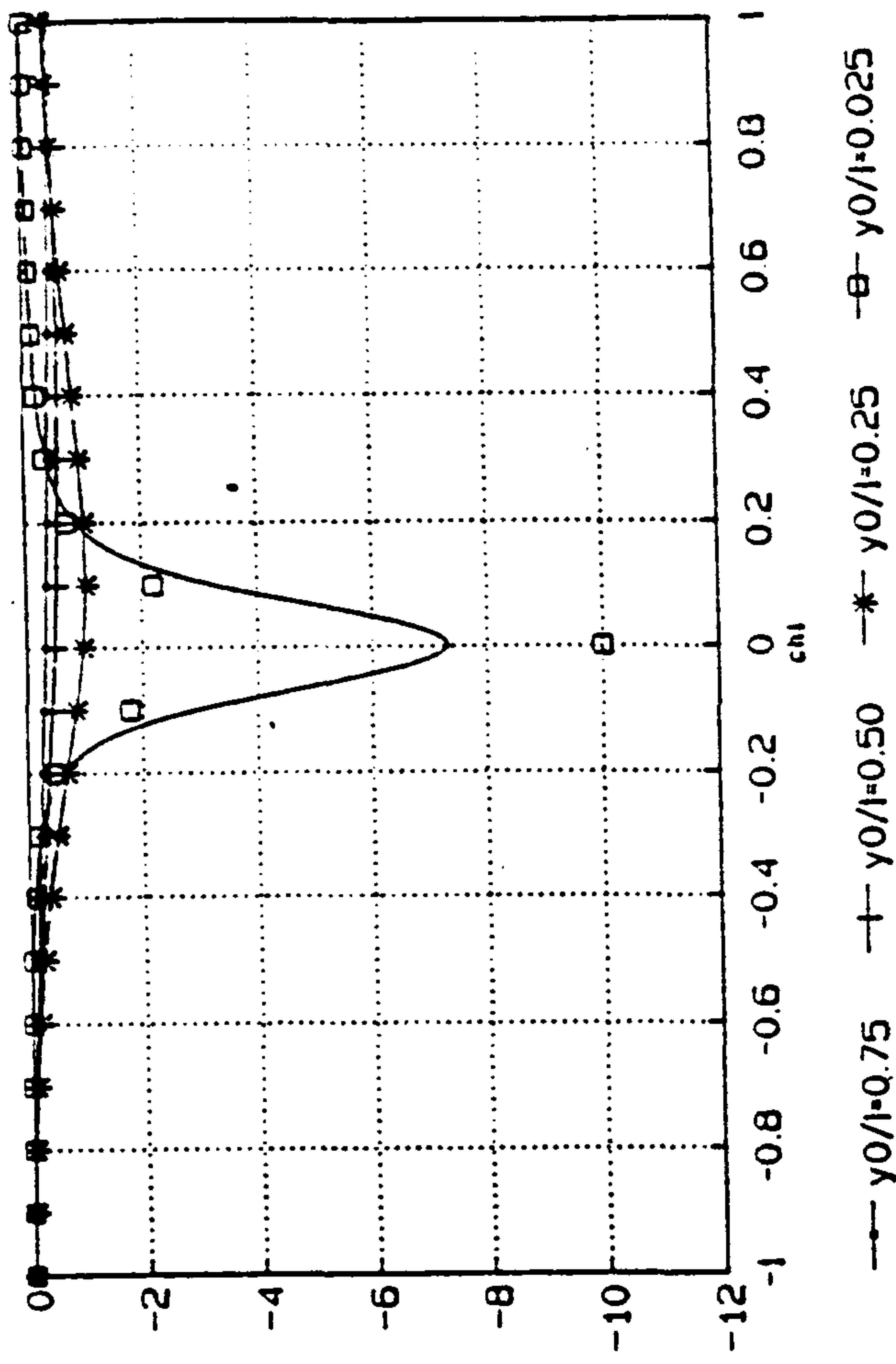
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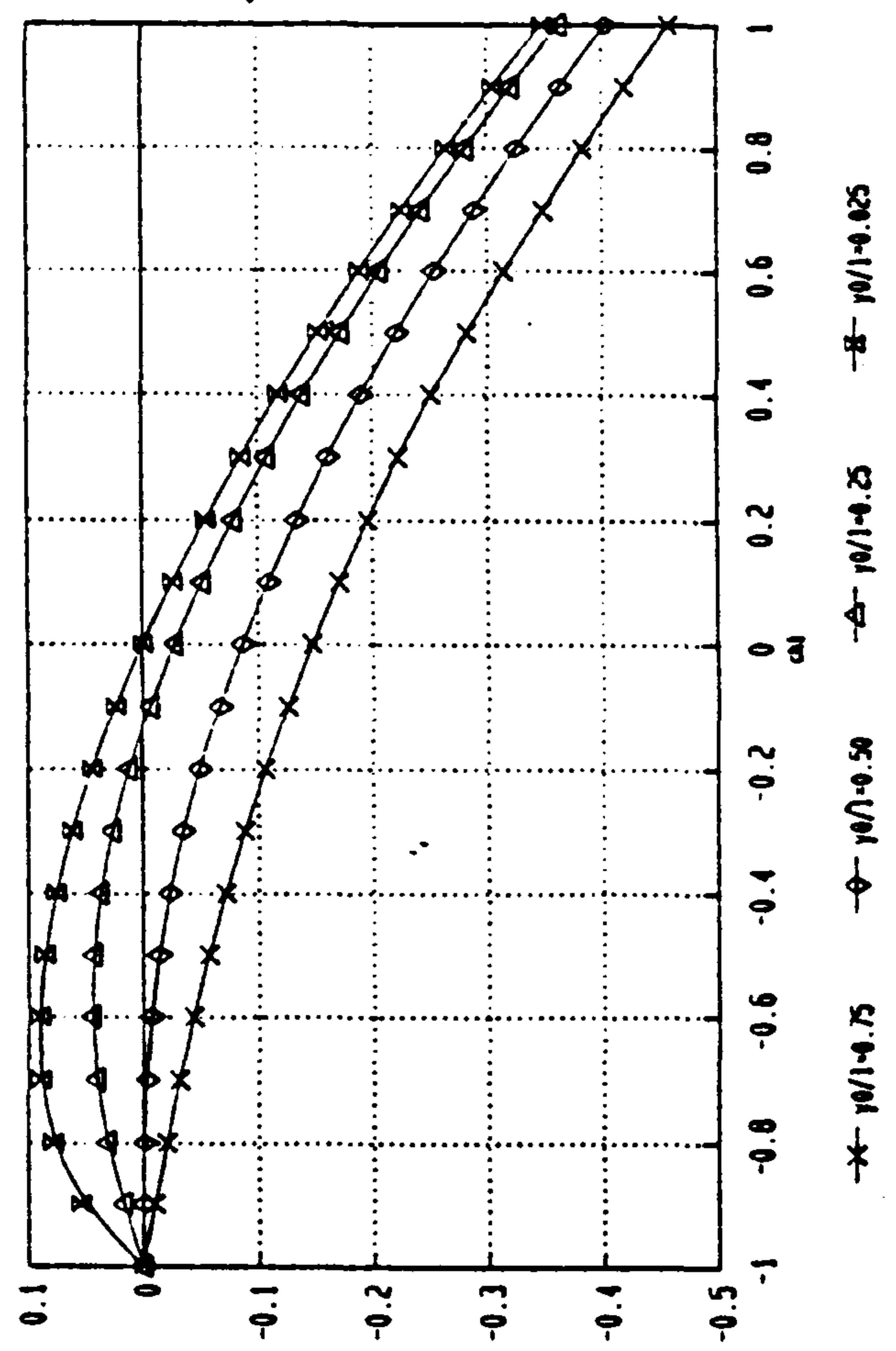
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$N0(\chi) \times Q(P, Q)$
 $x0/l=0$



$N0(\chi) \times F1(P, Q)$
 $x0/l=0.5$



$N0(\chi) \times F1(P, Q)$
 $x0/l=0$

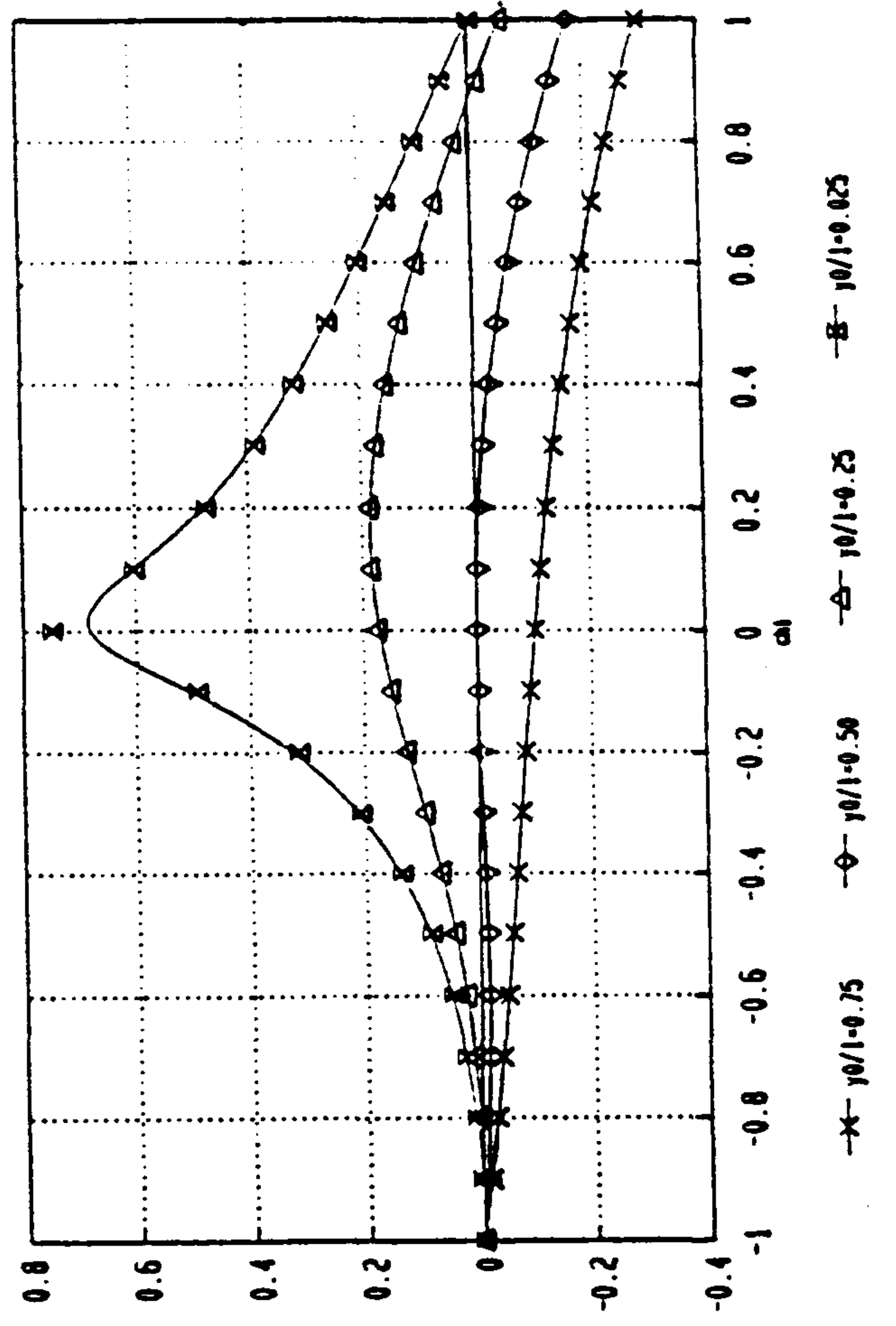


Figure 6:

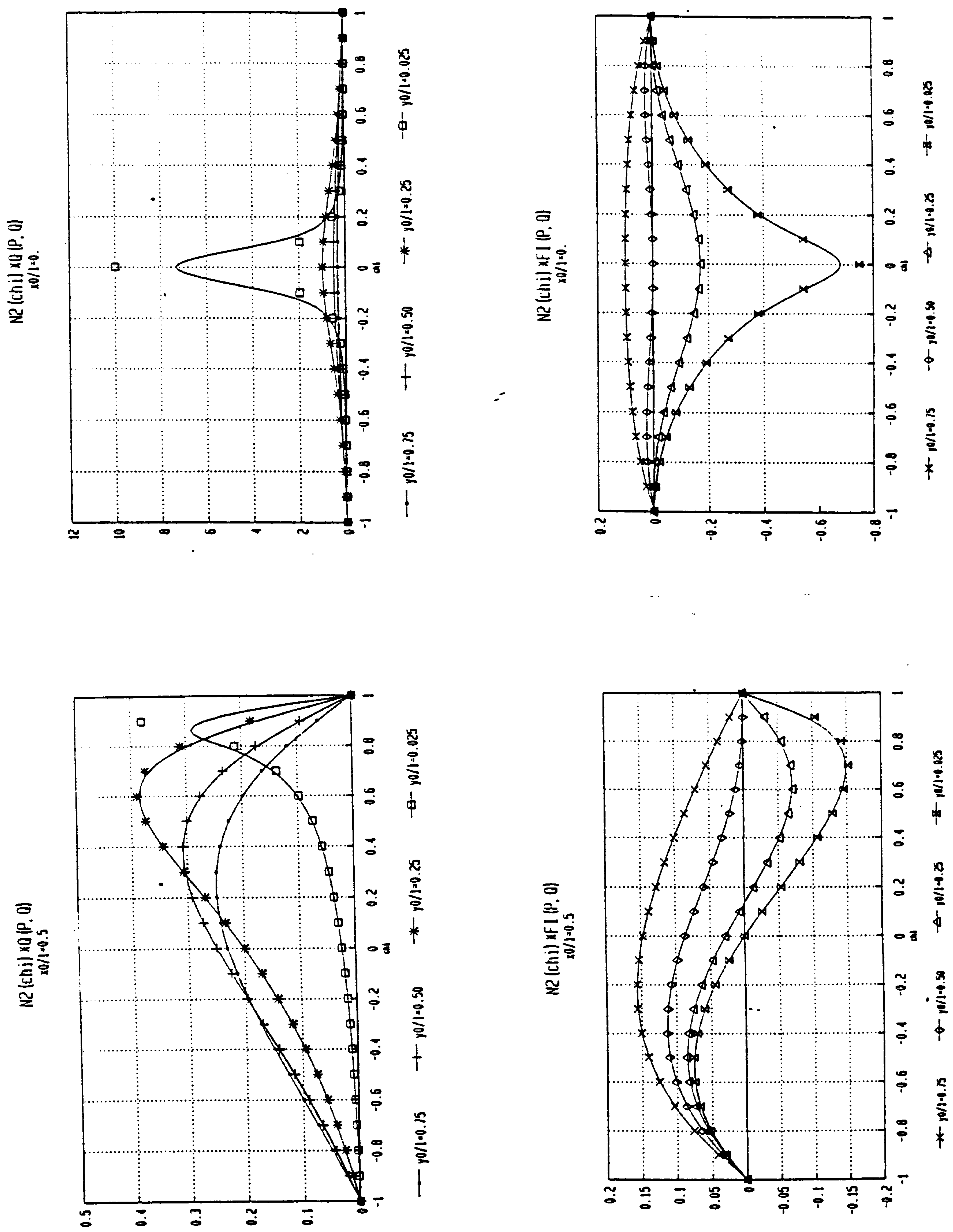
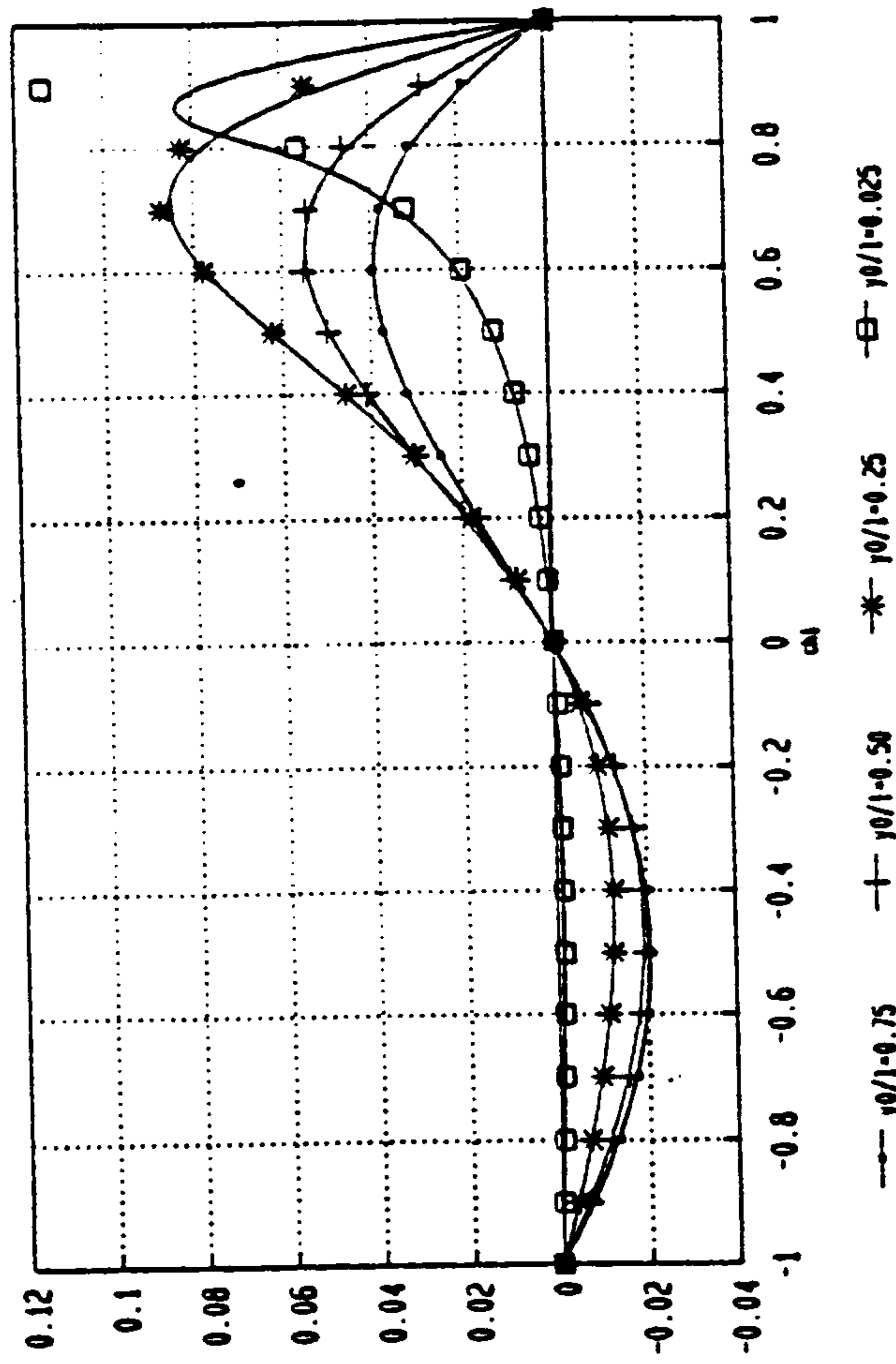
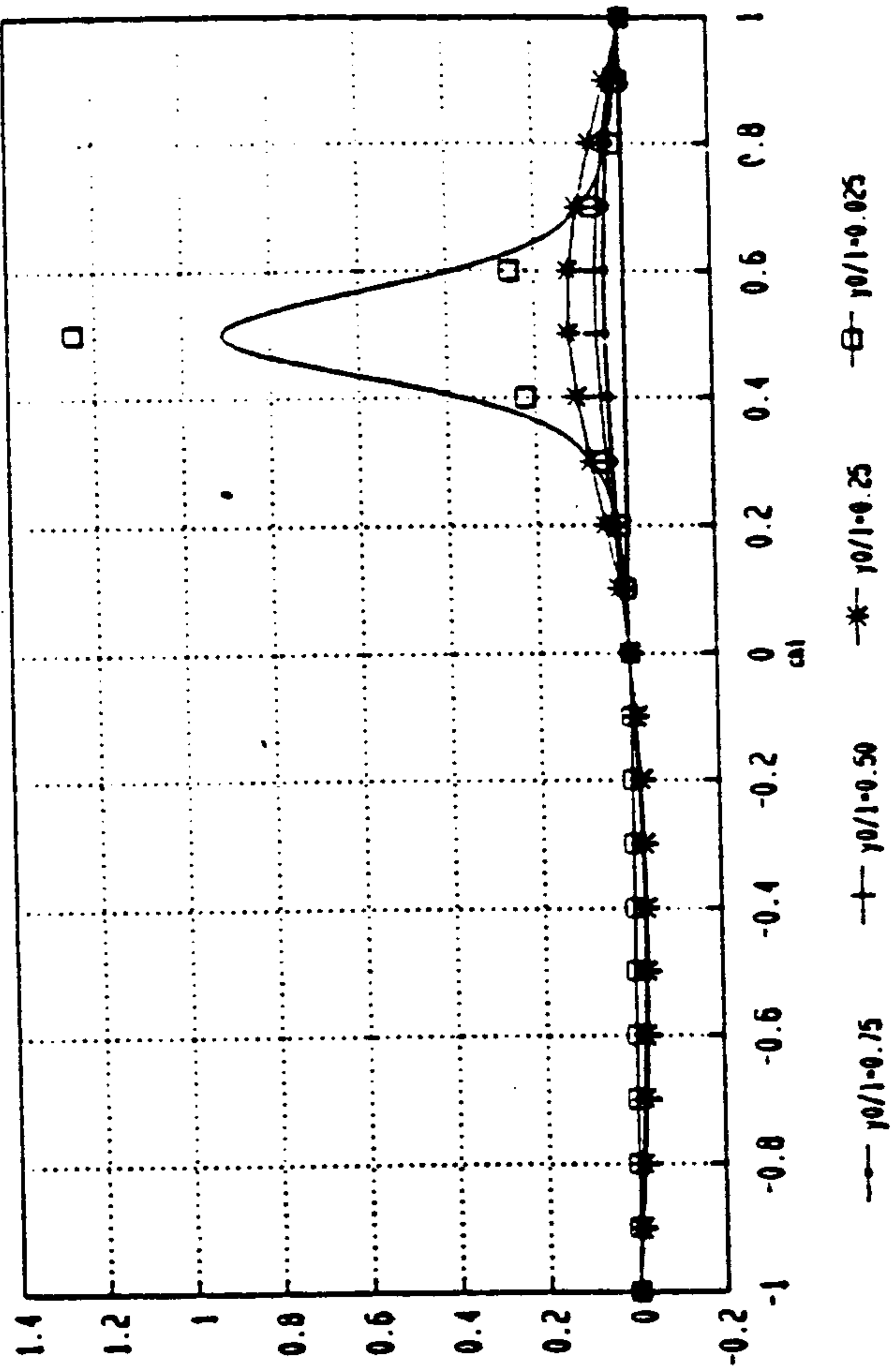


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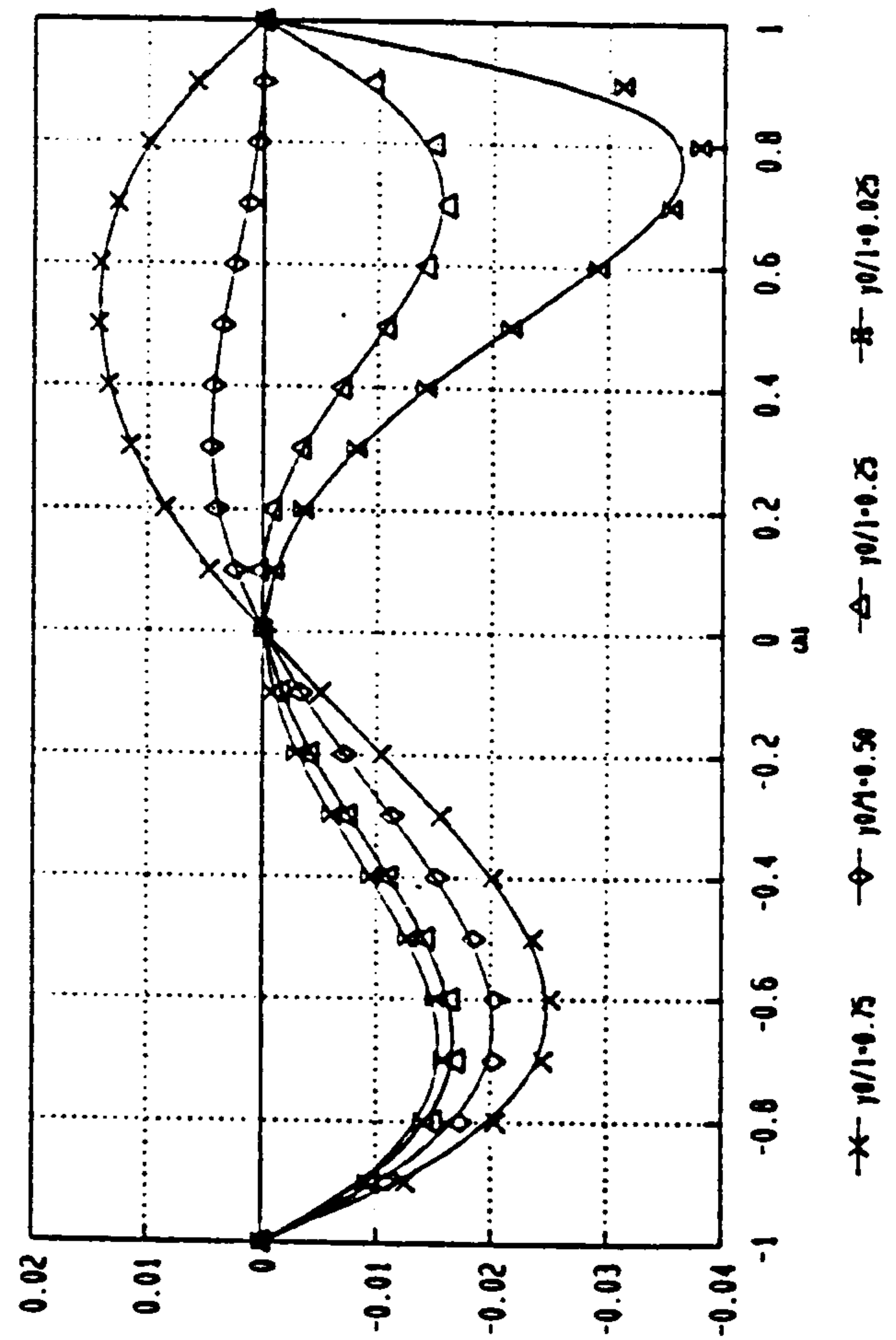
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 $\times 0/1=0.5$



$N3(\chi) \times 0(P, 0)$
 $\times 0/1=0.25$



$N3(\chi) \times FI(P, 0)$
 $\times 0/1=0.5$



$N3(\chi) \times FI(P, 0)$
 $\times 0/1=0.25$

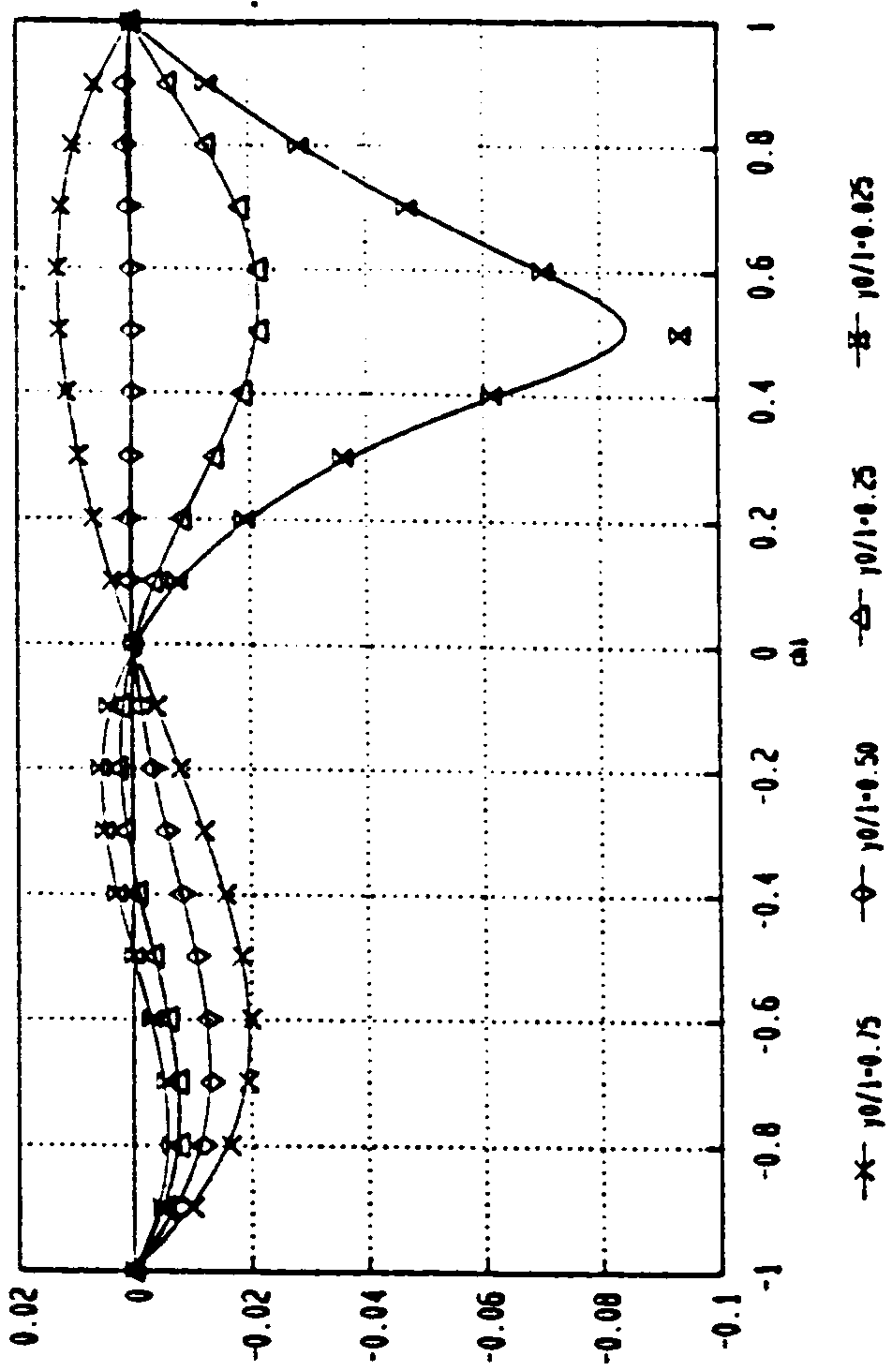
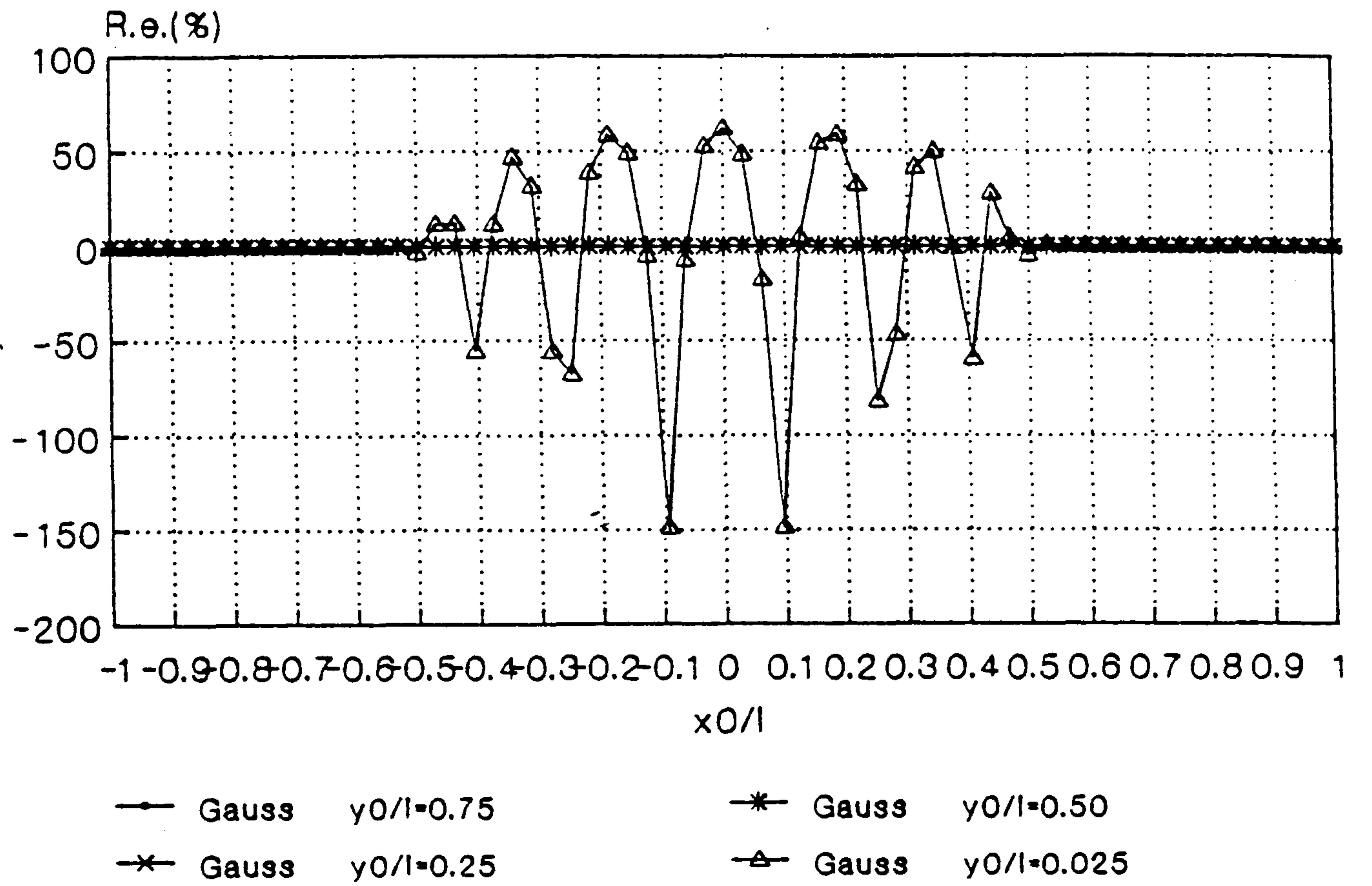


Figure 8:

COEFFICIENT A

Relative errors



COEFFICIENT A

Relative errors

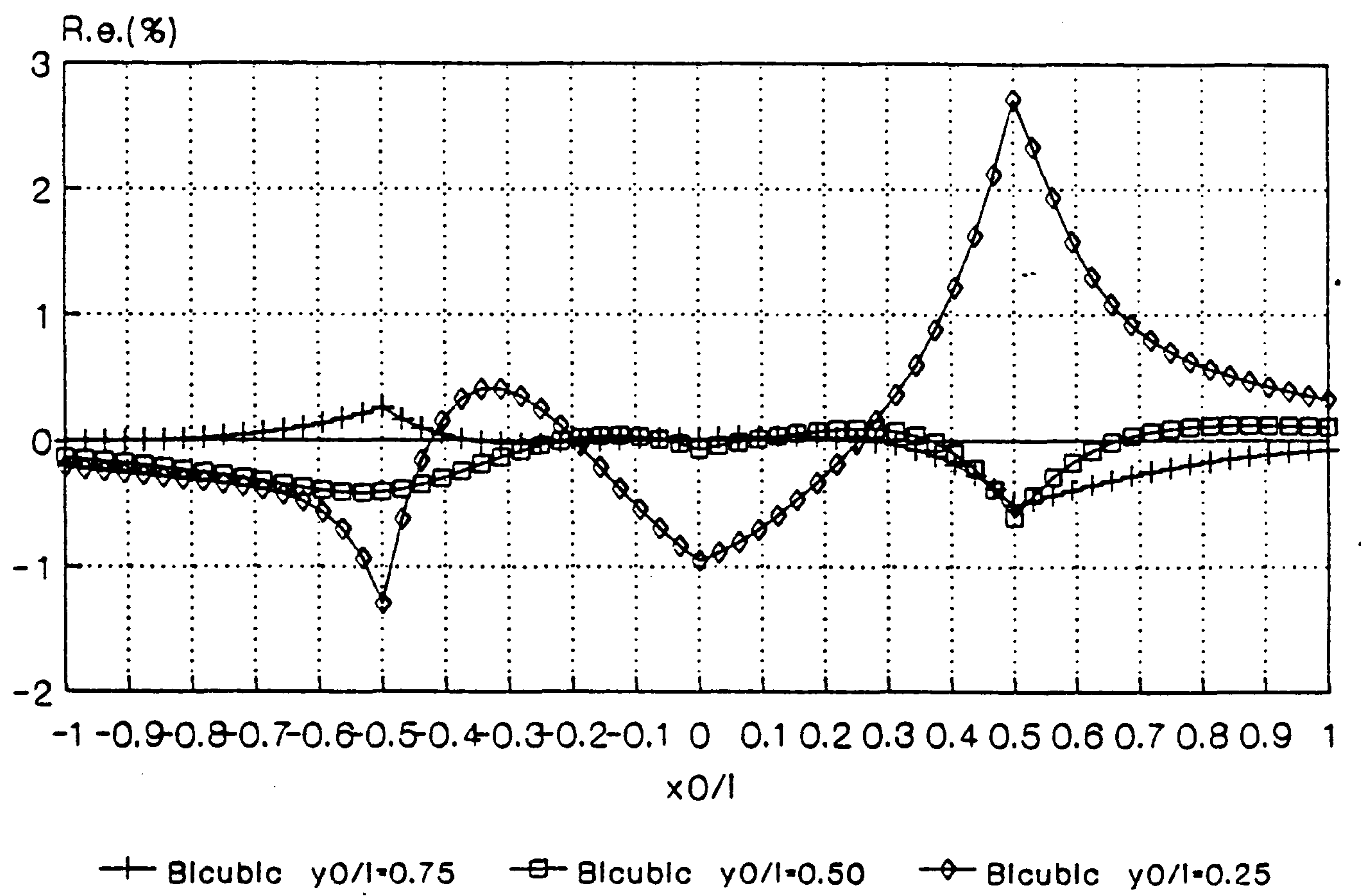
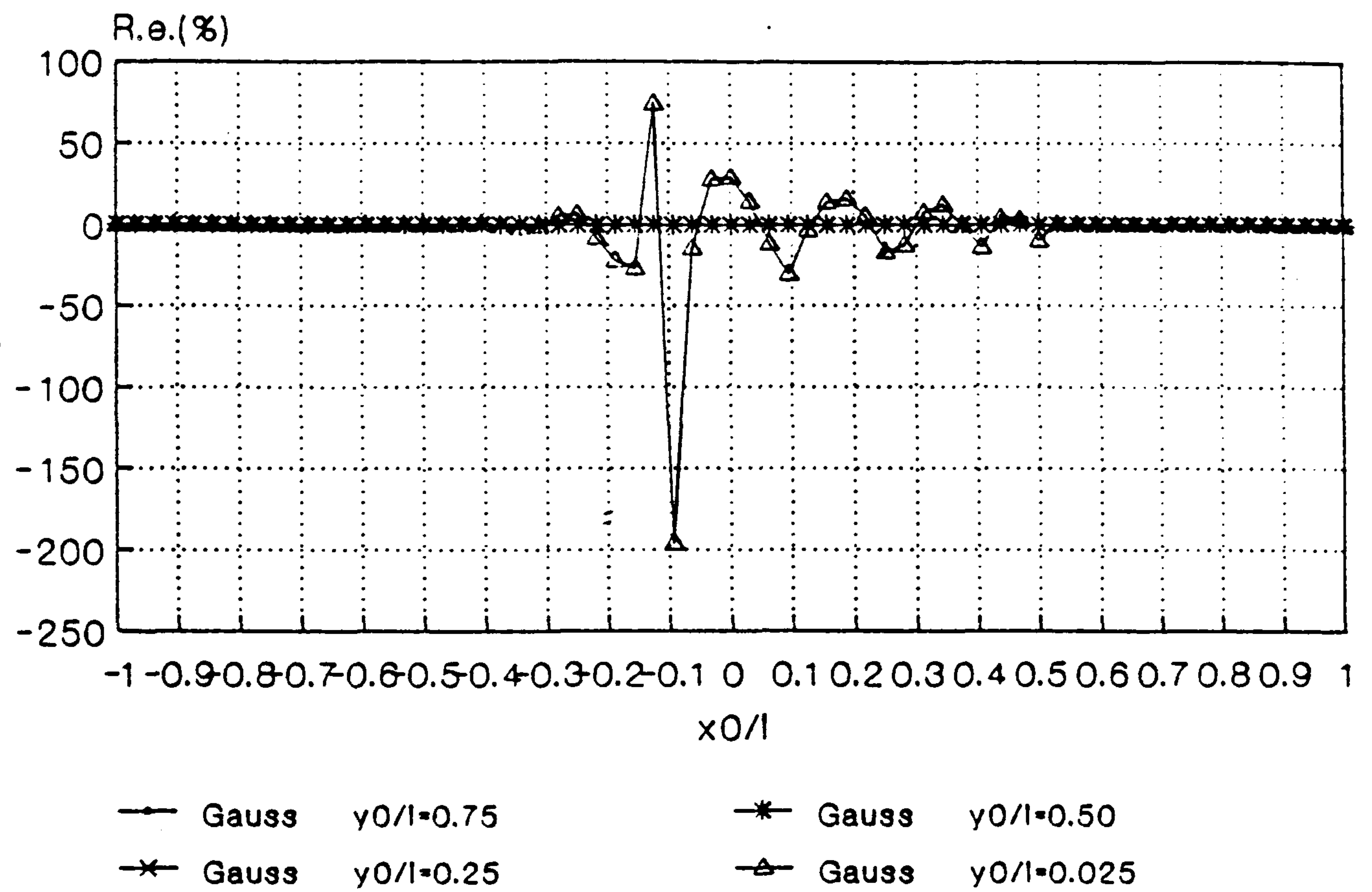


Figure 8:

COEFFICIENT B/L

Relative errors



COEFFICIENT B/L

Relative Errors

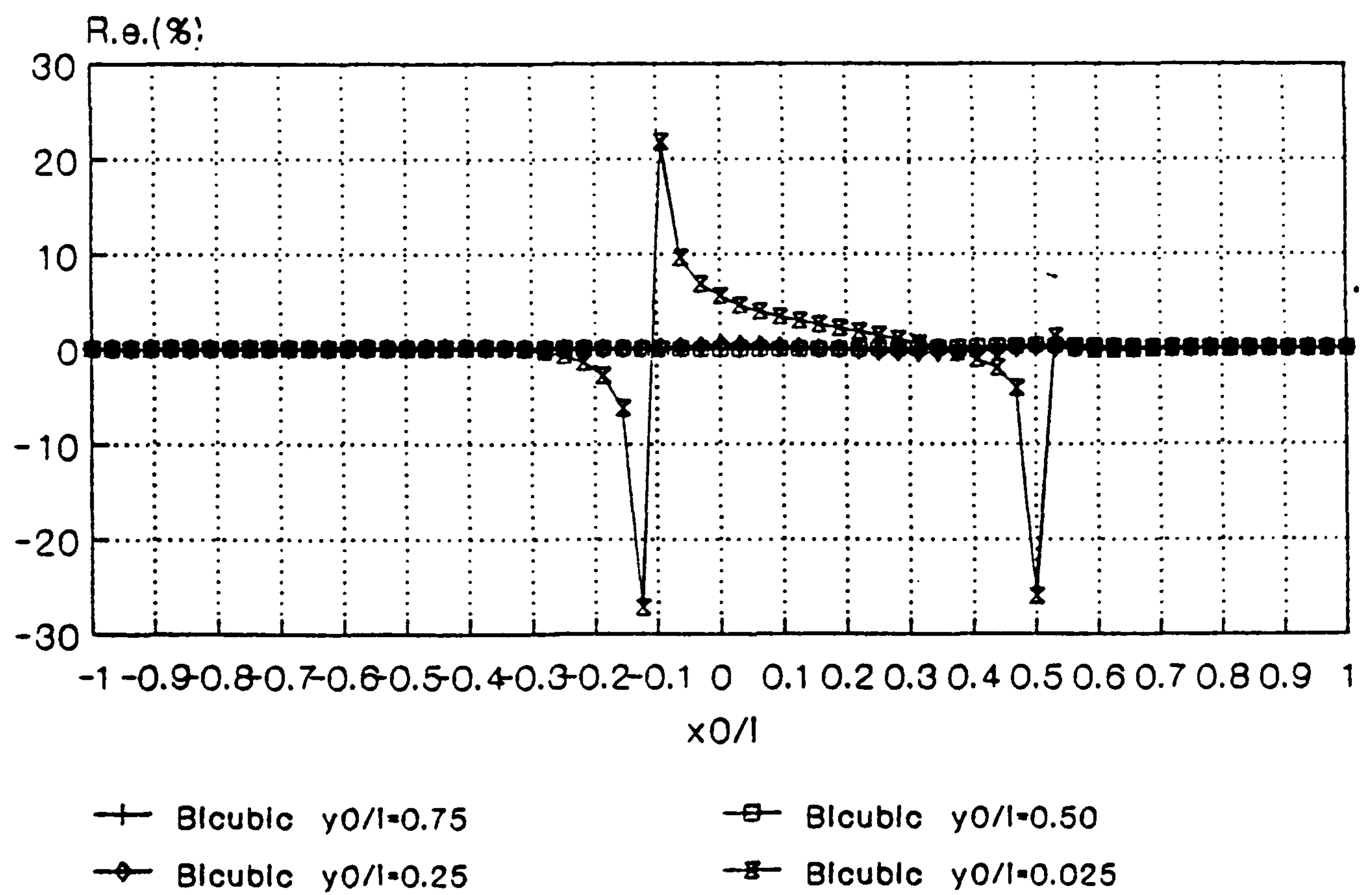
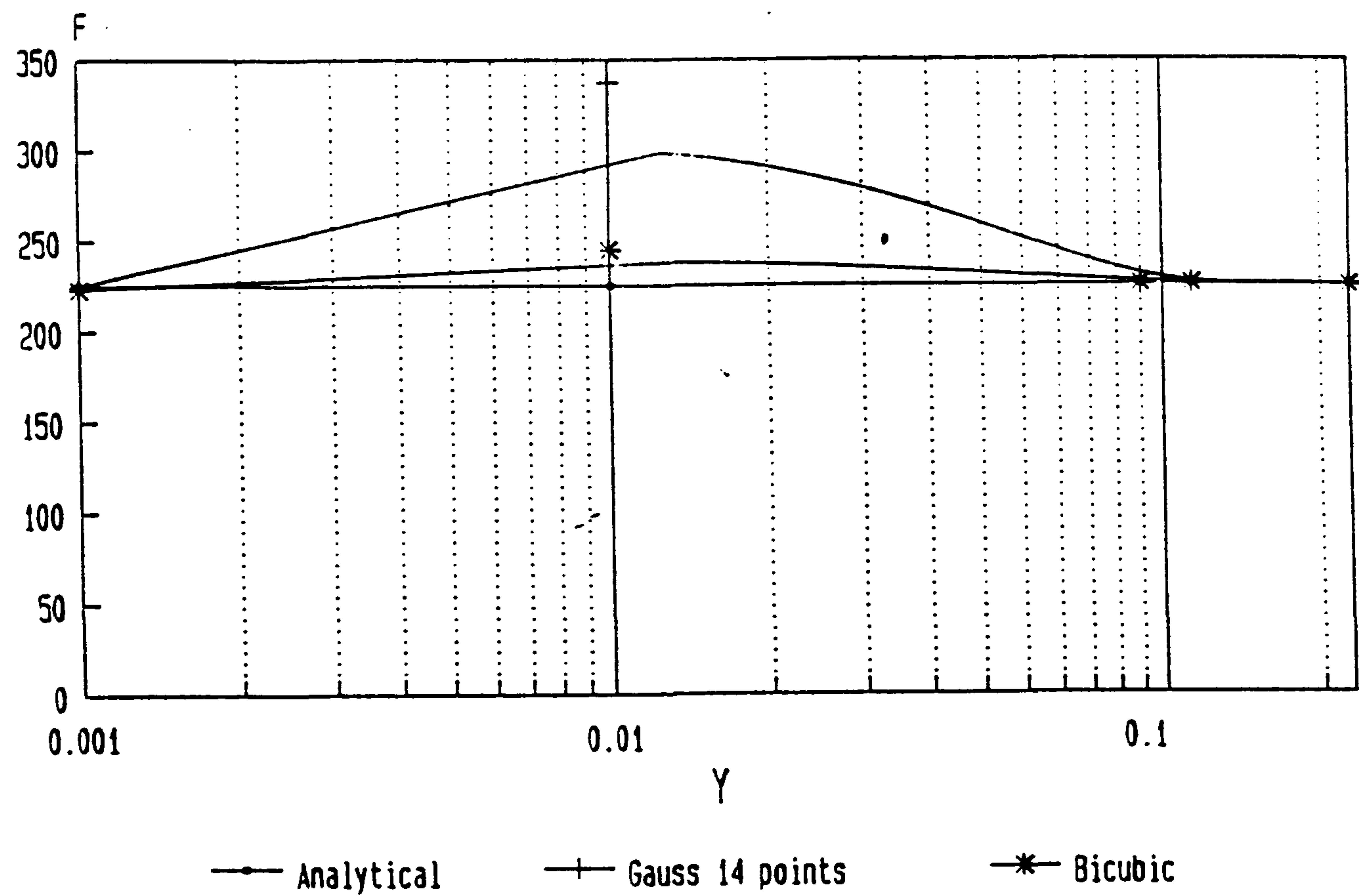


Figure 10:

EXAMPLE 1
X=0.25



EXAMPLE 1
X=0.01

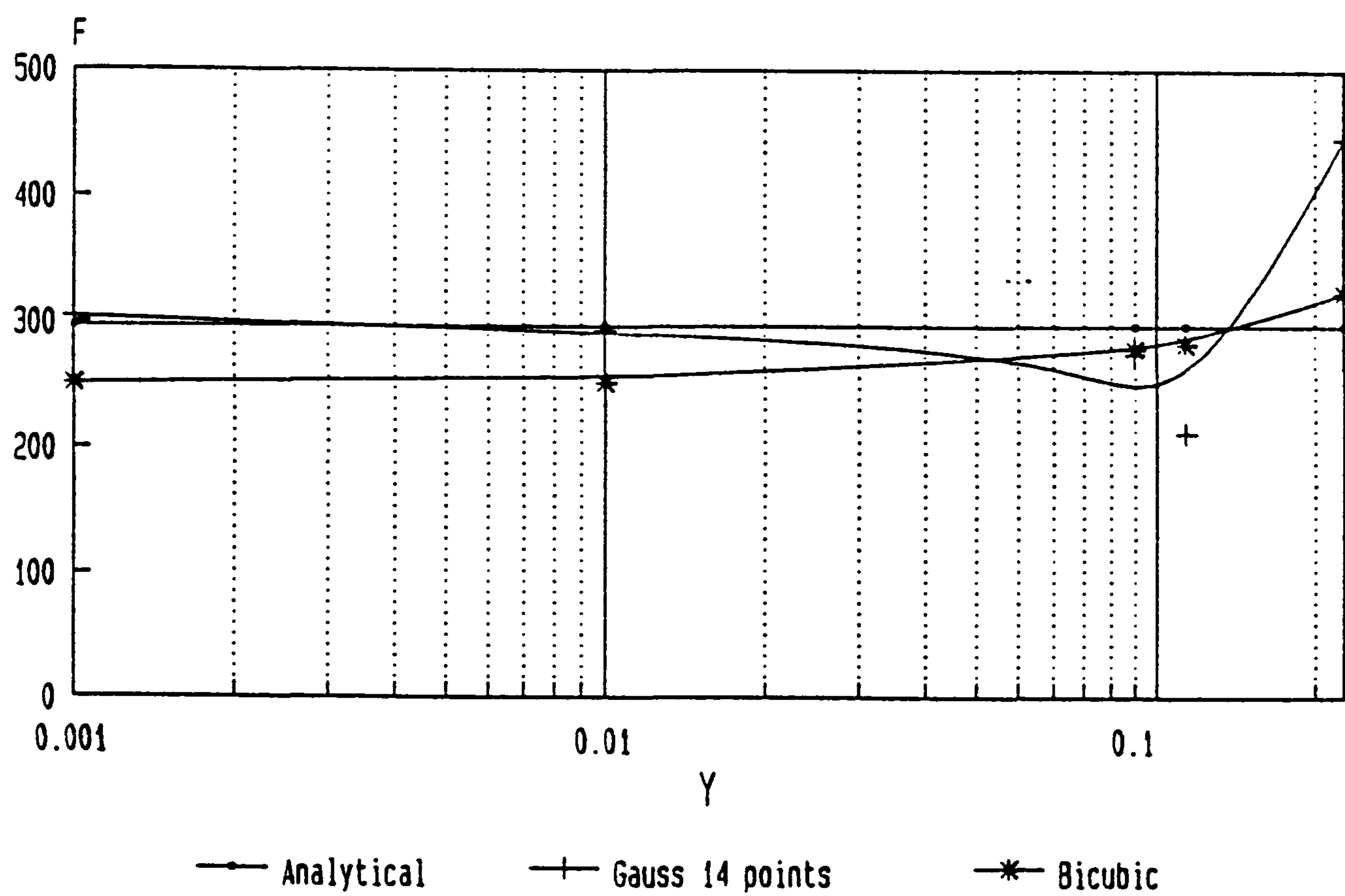
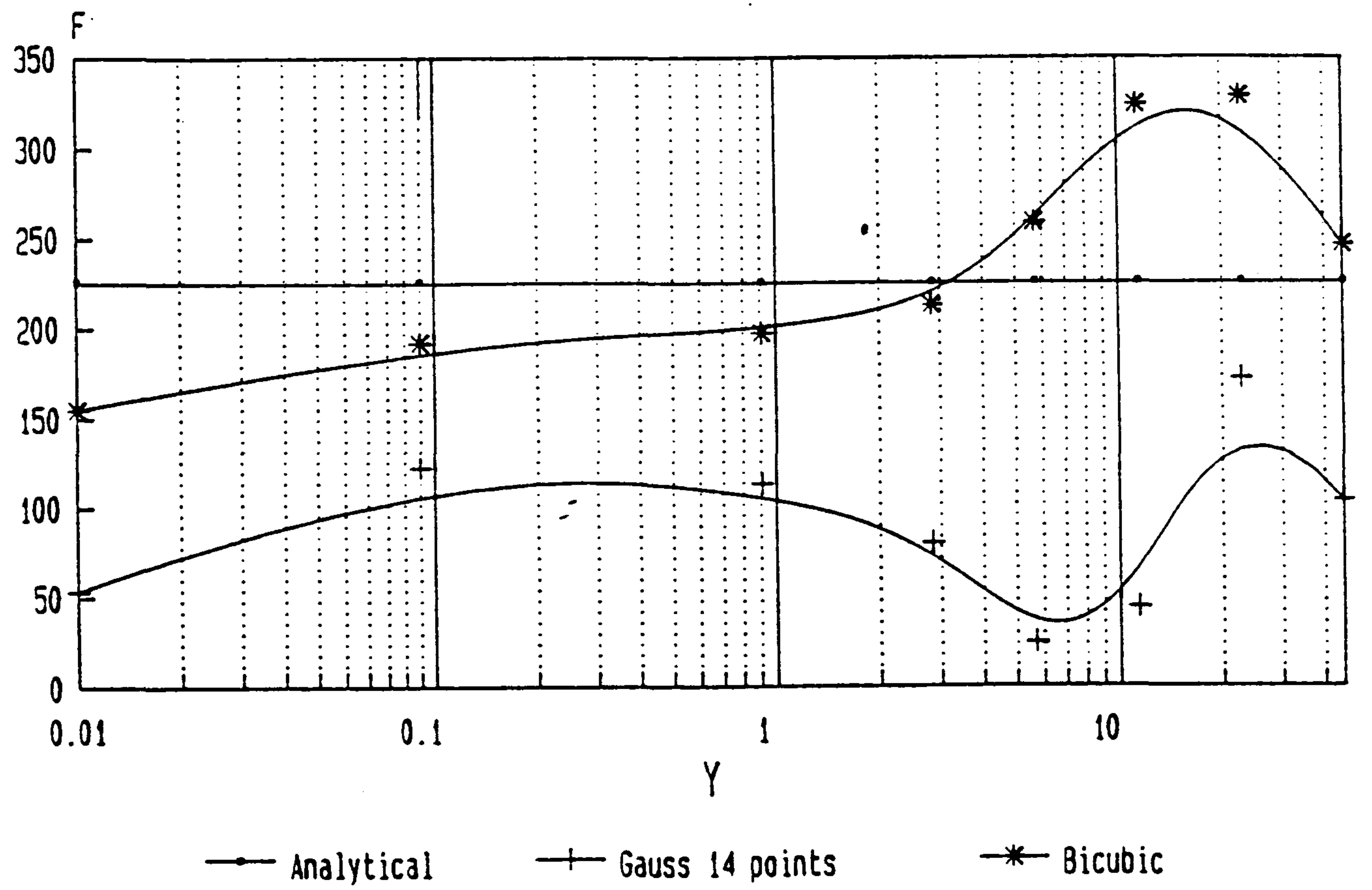


Figure 10:

EXAMPLE 2
X=0.25



EXAMPLE 2
X=0.01

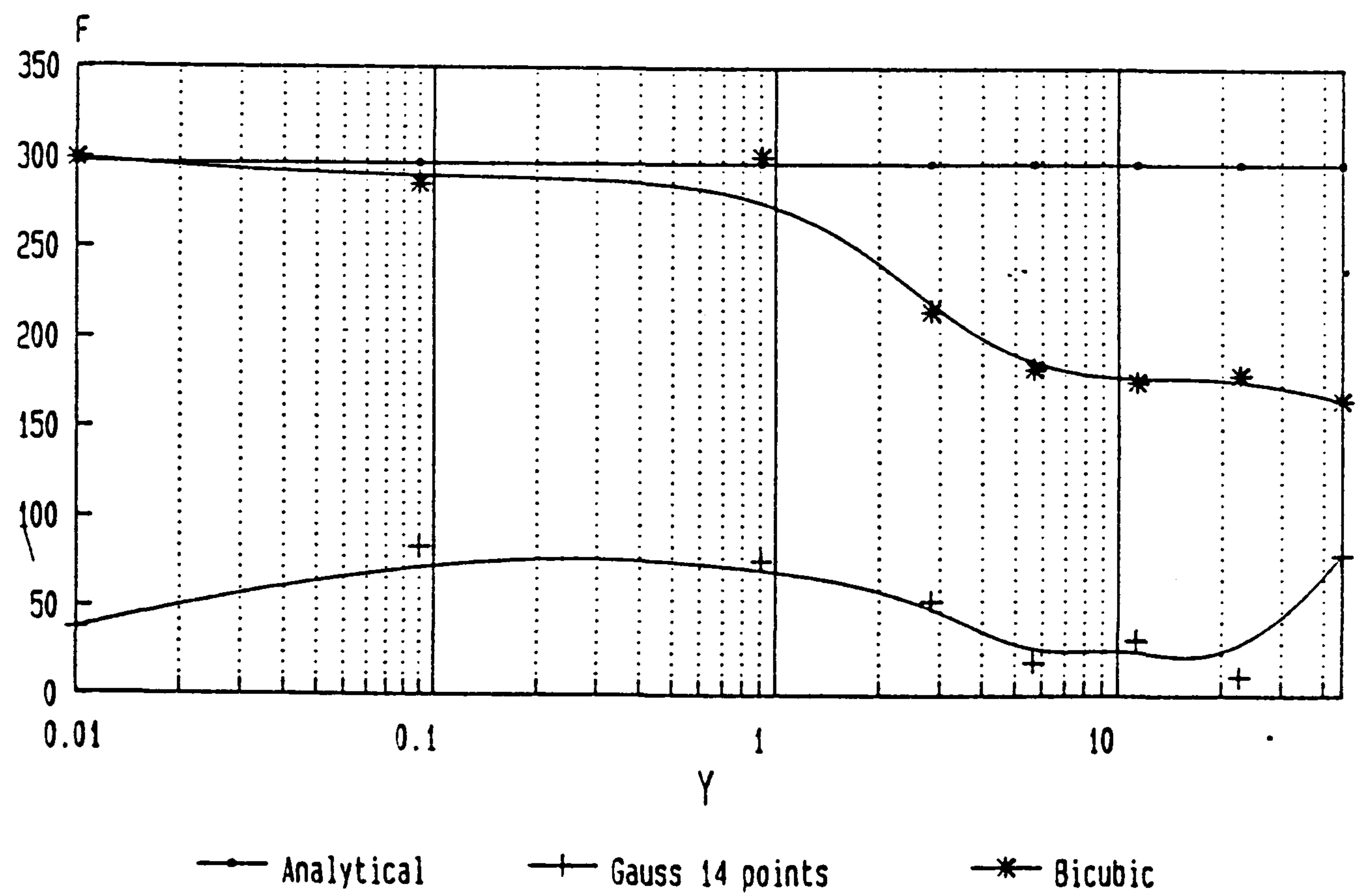


Figure 4: