EXTRAPOLATION WITH WEIGHTS, REARRANGEMENT INVARIANT FUNCTION SPACES, MODULAR INEQUALITIES AND APPLICATIONS TO SINGULAR INTEGRALS

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ABSTRACT. We present an extrapolation theory that allows us to obtain, from weighted L^p inequalities on pairs of functions for p fixed and all A_∞ weights, estimates for the same pairs on very general rearrangement invariant quasi-Banach function spaces with A_∞ weights and also modular inequalities with A_∞ weights. Vector-valued inequalities are obtained automatically, without the need of a Banach-valued theory. This provides a method to prove very fine estimates for a variety of operators which include singular and fractional integrals and their commutators. In particular, we obtain weighted, and vector-valued, extensions of the classical theorems of Boyd and Lorentz-Shimogaki. The key is to develop appropriate versions of Rubio de Francia's algorithm.

CONTENTS

1. Introduction

There is a number of important inequalities in both Harmonic Analysis and P.D.E. which are of the form

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \le C \int_{\mathbb{R}^n} |Sf(x)|^p \, w(x) \, dx, \qquad (1.1)$$

where typically T is an operator with some degree of singularity (e.g., some singular integral operator) and S is an operator which is easier to handle (e.g., a maximal operator), and w is in some class of weights. As it is well known, the usual technique for proving such results is to establish a good-λ inequality between T and S. This method was introduced by Burkholder and Gundy [BG]. These inequalities compare the measure of the level sets of S and T: for every $\lambda > 0$ and small $\varepsilon > 0$,

$$w\{y \in \mathbb{R}^n : |Tf(y)| > 2\,\lambda, |Sf(y)| \le \lambda\, C_\varepsilon\} \le C\,\varepsilon\, w\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\}. \quad (1.2)$$

Here, the weight w is usually assumed to be in the Muckenhoupt class A_∞. Given inequality (1.2), it is straightforward to prove the strong-type inequality (1.1) for any p, $0 < p < \infty$.

Inspired by the extrapolation theory for A_p weights discovered by J.L. Rubio de Francia in [Rub] (see also [Ga1] or [GR]; [Duo] for a new and short proof, and [Ga2] for extrapolation results on Banach lattices), another approach is presented in [CMP] to derive inequalities like (1.1) without using the good-λ technique. Namely, assume that it is known that

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} \, w(x) \, dx \le C \int_{\mathbb{R}^n} |Sf(x)|^{p_0} \, w(x) \, dx \qquad (1.3)$$

holds for some fixed exponent $0 < p_0 < \infty$, for all $w \in A_\infty$ and for all (reasonable) functions f for which the left-hand side is finite. Then, the authors show that there is a very general extrapolation principle that allows one to get the full range of exponents $0 < p < \infty$. This means that all the information contained in (1.1) is indeed encoded in the corresponding estimates where the exponent is fixed, say $p = 1$ or $p = 2$. This extrapolation method has been extensively applied in [CMP] to deal with different examples. Also, in [MPT] this result is used to show that the classical Hörmander condition for a singular integral operator is not sufficient to guarantee Coifman inequality, see (1.5) below and the original sources [Coi] and [CF].

As a consequence of this general extrapolation principle, in [CMP] vector-valued inequalities are obtained in a very easy way. Namely, it is proved that (1.3) also implies

$$\left\| \left(\sum_j |Tf_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \le C \left\| \left(\sum_j |Sf_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \qquad (1.4)$$

for all $0 < p, r < \infty$ and all $w \in A_\infty$.

Estimates (1.3) and (1.4) are very useful in applications. For some operators T, a natural choice of S is the Hardy-Littlewood maximal function M. If this is the case, by using the well-known weighted and vector-valued inequalities for M, one can get that T is bounded on $L^p(w)$ for $1 < p < \infty$ and for $w \in A_p$ and that T satisfies the corresponding ℓ^r-valued weighted norm inequalities provided $1 < r < \infty$. This is indeed part of the motivation of this kind of extrapolation results: the study of

a singular operator T can be done through an appropriate maximal function which turns out to be easier.

One of the most interesting examples is provided by Coifman's estimate [Coi], see also [CF]: Let T be any Calderón-Zygmund operator with standard kernel (see [Duo, p. 100] for the precise definition) and let M be the Hardy-Littlewood maximal operator, then for any p, $0 < p < \infty$, and $w \in A_\infty$, there is a constant C depending on p and w such that,

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p \, w(x) \, dx, \tag{1.5}$$

for any function f such that the left hand side is finite. As mentioned before, and just by using well-known estimates for M, one can get that T is bounded on $L^p(w)$ for all $1 < p < \infty$ and every $w \in A_p$ as well as the corresponding vector-valued extensions. We would like to point out that this way to get the vector-valued inequalities does not use the Banach-valued Calderón-Zygmund theory developed in [BCP], [RRT].

We also want to call attention to the example given by the geometrical maximal operator M_0, which will be considered in Section 6.5. Its behavior is unusual in Harmonic Analysis since it is bounded on $L^p(w)$ for all $0 < p < \infty$ and all $w \in A_\infty$. The interesting observation is that it is enough to prove the $L^1(w)$ case since, for every $0 < p < \infty$, by definition of the operator we have $M_0 f(x)^p = M_0(|f|^p)(x)$. This illustrates the fact that it is not important for which power one has the starting estimate. This special feature occurs repeatedly in the extrapolation results that will be proved in the present paper. Another "exotic" example is the minimal operator which will be also considered in Section 6.5.

With the extrapolation results that we have mentioned so far, no useful estimate is obtained at the endpoint $p = 1$. For example, when T is a Calderón-Zygmund operator, knowing that T is controlled by M in $L^1(w)$ seems to be pointless since M is not bounded in this space. Rather than the weighted Lebesgue spaces $L^p(w)$, one should seek estimates on $L^{1,\infty}(w)$. As a matter of fact, in [CMP] we can also find a way to extrapolate from (1.3) to get inequalities in the scale of Lorentz spaces. Namely,

$$\|Tf\|_{L^{p,q}(w)} \leq C \, \|Sf\|_{L^{p,q}(w)} \tag{1.6}$$

for all $0 < p < \infty$, $0 < q \leq \infty$ and all $w \in A_\infty$. As before, vector-valued extensions of this estimate are also obtained. When T is a Calderón-Zygmund operator as before, focusing on the case $p = 1$ and $q = \infty$, one gets the control of T by M in $L^{1,\infty}(w)$ which leads us to derive that T maps $L^1(w)$ into $L^{1,\infty}(w)$ for $w \in A_1$ since M does. Again, one can also get an ℓ^r-valued version of this inequality with no use of the Banach-valued Calderón-Zygmund theory.

Still, there are operators for which the natural endpoint space is not $L^{1,\infty}$. For instance, this is true for the commutators

$$[b, T]f(x) = b(x) \, Tf(x) - T(b \, f)(x),$$

where T is a Calderón-Zygmund operator as before and $b \in \mathrm{BMO}$. In this case, the appropriate maximal operator is $M^2 = M \circ M$. Indeed, it is proved in [Pe3] that we have:

$$\int_{\mathbb{R}^n} \left| [b, T]f(x) \right|^p w(x) \, dx \leq C \, \|b\|_{\mathrm{BMO}}^p \int_{\mathbb{R}^n} M^2 f(x)^p \, w(x) dx,$$

for any $0 < p < \infty$ and any $w \in A_\infty$. Using the extrapolation results in [CMP] we can get a collection of estimates on $L^p(w)$ and $L^{p,q}(w)$ for all $0 < p < \infty$, $0 < q \leq \infty$ and $w \in A_\infty$. This implies, among other things, that $[b, T]$ is a bounded operator on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$ and also satisfies the corresponding vector-valued estimates. Regarding the endpoint $p = 1$, the estimate in $L^{1,\infty}$

$$\left\| [b, T] f \right\|_{L^{1,\infty}(w)} \leq C \left\| M^2 f \right\|_{L^{1,\infty}(w)}$$

seems to be useless. This can be seen, for instance, by taking T the Hilbert transform in $\mathbb{R}$, $b(x) = \log x \in \mathrm{BMO}$ and $f(x) = \chi_{(0,1)}(x)$. In this case, $\| M^2 f \|_{L^{1,\infty}(w)} = \| [b, T] f \|_{L^{1,\infty}(w)} = \infty$. Indeed, what is behind this fact is that $L^{1,\infty}$ is not the suitable endpoint space for M^2 (as occurs with M and L^1).

The goal of the present paper is to provide a more general framework in which this kind of examples can be treated. The extrapolation results in [CMP] on Lebesgue or Lorentz spaces seem to be insufficient to deal with some operators that appear naturally on Harmonic Analysis. Our aim is to use extrapolation to derive more general estimates that allow us to handle a wider class of operators and their corresponding maximal functions. This will be done by using two different approaches, which are independent although philosophically close. Our first idea is to change $L^p(w)$ or $L^{p,q}(w)$ by more general spaces of functions, and the natural class seems to be the rearrangement invariant quasi-Banach function spaces which will be extensively considered below. The second approach is based on obtaining modular inequalities, since they are natural for operators like M^2.

The technique we present allows us, among other things, to get vector-valued and weighted extensions of the classical results of Boyd and Lorentz-Shimogaki (see Theorem 1.1 below) for the Hilbert transform, the Hardy-Littlewood maximal function and also for Calderón-Zygmund singular integrals. As mentioned above, our method has its roots on the extrapolation theory of weights discovered by J.L. Rubio de Francia in [Rub] and is flexible enough to yield results for other operators such as commutators with BMO functions, multilinear Calderón-Zygmund operators, as well as their vector-valued extensions. We would like to emphasize that no interpolation result is used. Roughly speaking, we only need a good knowledge of the boundedness of the Hardy-Littlewood maximal function and some weighted version of it.

Let us start by briefly outlining the second approach, namely, the one dealing with modular inequalities. As we observed before, the maximal operator to be used for the commutator $[b, T]$ is M^2. For the Hardy-Littlewood maximal function, the estimate for the measure of the set of level λ has a decay of order λ^{-1} which gives that M maps L^1 into $L^{1,\infty}$. However, this is not the case for M^2 and the appropriate estimate is

$$\left| \{ x \in \mathbb{R}^n : M^2 f(x) > \lambda \} \right| \leq C \int_{\mathbb{R}^n} \phi \left(\frac{|f(x)|}{\lambda} \right) dx,$$

where $\phi(t) = t \, (1 + \log^+ t)$. It seems that this result does not fit well within the context of reasonable function spaces. These kind of estimates are called *modular inequalities* (see [KK]), they provide a good endpoint result for M^2 and have good interpolation properties. As expected, $[b, T]$ satisfies the same estimate. Indeed, in [Pe1] it was proved that

$$\sup_{\lambda > 0} \varphi(\lambda) \, w\{ y \in \mathbb{R}^n : \left| [b, T] f(y) \right| > \lambda \} \leq C \sup_{\lambda > 0} \varphi(\lambda) \, w\{ y \in \mathbb{R} : M^2 f(y) > \lambda \}, \quad (1.7)$$

where $\varphi(\lambda) = \frac{\lambda}{1+\log^+ \frac{1}{\lambda}}$, $w \in A_\infty$ and f is any nice function such that the left hand side is finite. As a consequence, one can obtain

$$\left|\{y \in \mathbb{R}^n : \left|[b,T]f(y)\right| > \lambda\}\right| \leq C_{\|b\|_{\mathrm{BMO}}} \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Estimates like (1.7), which are some sort of weak modular estimates, can not be developed within the framework considered in [CMP]. Note that on both sides of the inequality we have a functional that is not homogeneous and so it is not a norm or quasi-norm. In the present paper we show that this estimate holds in a very general way. Indeed, in Theorem 3.1 we prove that (1.3) implies the modular inequality (of strong type)

$$\int_{\mathbb{R}^n} \phi\big(|Tf(x)|\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\big(|Sf(x)|\big)\, w(x)\, dx,$$

and its corresponding weak version, of which (1.7) is a special case, namely

$$\sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : |Tf(y)| > \lambda\} \leq C \sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : |Sf(y)| > \lambda\},$$

where $\phi \geq 0$ is an increasing function satisfying some very mild condition. We will apply this result to the commutators introduced above in Section 6.1.

As mentioned, we will also follow another approach seeking for estimates on function spaces. Note that the previous inequalities are not associated to linear spaces in general. The estimates on function spaces that we will be looking for are of the following type

$$\|Tf\|_{\mathbb{X}(w)} \leq C \, \|Sf\|_{\mathbb{X}(w)}, \tag{1.8}$$

where $w \in A_\infty$ and $\mathbb{X}$ is any rearrangement invariant quasi-Banach function space satisfying certain mild geometric condition. Our model examples are given by estimates (1.6). We would like to remark that in either the Lebesgue spaces $L^p(w)$, $0 < p < \infty$; or in the scale of the Lorentz spaces $L^{p,q}(w)$, $0 < p < \infty$, $0 < q \leq \infty$, we have both Banach and quasi-Banach spaces. Nevertheless, in both scales for p big enough the spaces turn out to be Banach. This observation is crucial since the method originating in [CMP] is based upon some duality argument. To extend the extrapolation principle to more general spaces one should take into account these two facts: they can be quasi-Banach but there are Banach spaces in the same scale. This last fact can be translated into some convexity assumption on the space. See Theorem 2.1 for the precise statements. We would like to emphasize that an analog of this property will also appear in the context of modular inequalities, see Theorem 3.1.

Another motivation for the present paper is the extension of the classical results of Boyd and Lorentz-Shimogaki to a wider class of operators and also to weighted, and vector-valued, estimates. These two theorems are basic in the theory of rearrangement invariant Banach function spaces (RIBFS in the sequel). They characterize those RIBFS on which the Hilbert transform, in the case of Boyd, or the Hardy-Littlewood maximal function, in the case of Lorentz-Shimogaki, are bounded operators.

Theorem 1.1. *Let $\mathbb{X}$ be a Rearrangement Invariant Banach Function Space associated to $(\mathbb{R}, dx)$, let H be the Hilbert transform and let M be the Hardy-Littlewood maximal function. Then,*

- **[Boyd, 1967]** *H is bounded on $\mathbb{X}$ if and only if $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$.*

• [**Lorentz, 1955; Shimogaki, 1965**] *M is bounded on $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$.*

Here $p_{\mathbb{X}}$ and $q_{\mathbb{X}}$ denote the Boyd indices of $\mathbb{X}$ (see Section 2.1 below). The proofs of these results (see [Bo], [Lor], [Shi] or [BS, p. 154]) are based on the pointwise estimates:

$$(Hf)^*(t) \leq C \left(\frac{1}{t} \int_0^t f^*(s)\,ds + \int_t^\infty f^*(s) \frac{ds}{s} \right)$$

and

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s)\,ds,$$

for $0 < t < \infty$, where f^* is the decreasing rearrangement of f. Observe that the right hand side of the second estimate is just the classical Hardy operator acting on f^* and that in the inequality for H we have the sum of the Hardy operator and its adjoint. Then, by using the fact that the Hardy operator is bounded on $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$, we get the restrictions on the Boyd indices that guarantee the boundedness of H and M. Both results were originally proved for Banach spaces, but they have been extended to the quasi-Banach case in [Mon] with the same restriction on the Boyd indices.

We recall the definition of the Muckenhoupt classes of weights. A weight w belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if for every cube $Q \in \mathbb{R}^n$,

$$\left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\,dx \right)^{p-1} \leq C_p.$$

When $p = 1$, w belongs to A_1 if $Mw(x) \leq C_1\, w(x)$ for almost every $x \in \mathbb{R}^n$. We also consider $A_\infty = \bigcup_{p \geq 1} A_p$. The smallest constant C_p for which the A_p condition holds will be denoted by $[w]_{A_p}$. The A_∞-constant of $w \in A_\infty$ is the limit (or infimum) of $[w]_{A_p}$ as $p \to \infty$. The reader is referred to [GR] for a complete account of the theory of Muckenhoupt weights.

We finish this introduction by stating the main theorem from [CMP], which will be used all throughout this paper. First we explain the notation. Although inequality (1.1) is written in terms of two operators T and S, the operators do not need to appear explicitly. All that is used is that there are pairs of functions $(|Tf|, |Sf|)$ such that (1.1) holds. Therefore, as was already done in [CMP], we are going to eliminate the superfluous operators and work with couples of functions. In what follows, $\mathcal{F}$ will be a family of ordered pairs of non-negative, measurable functions (f, g). If we say that for some p, $0 < p < \infty$, and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^p\, w(x)\,dx \leq C \int_{\mathbb{R}^n} g(x)^p\, w(x)\,dx, \qquad (f, g) \in \mathcal{F}, \qquad (1.9)$$

we always mean that (1.9) holds for any $(f, g) \in \mathcal{F}$ such that the left hand side is finite, and that the constant C depends only upon p and the A_∞ constant of w. We will make similar abbreviated statements involving other function norms or quasi-norms, or even modular type estimates; they will be always interpreted in the same way.

As promised above, here is the main result in [CMP].

Theorem 1.2 ([CMP]). *Let $0 < p_0 < \infty$ and $\mathcal{F}$ be a family of couples of non-negative functions such that*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\,dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\,dx, \qquad (f, g) \in \mathcal{F}, \qquad (1.10)$$

for all $w \in A_\infty$. Then, for all $0 < p, q < \infty$ and for all $w \in A_\infty$ we have

$$\int_{\mathbb{R}^n} f(x)^p \, w(x) \, dx \ \leq \ C \int_{\mathbb{R}^n} g(x)^p \, w(x) \, dx, \qquad (f, g) \in \mathcal{F}, \qquad (1.11)$$

$$\left\| \Big(\sum_j (f_j)^q \Big)^{\frac{1}{q}} \right\|_{L^p(w)} \ \leq \ C \left\| \Big(\sum_j (g_j)^q \Big)^{\frac{1}{q}} \right\|_{L^p(w)}, \qquad \{(f_j, g_j)\}_j \subset \mathcal{F}, \quad (1.12)$$

where these estimates hold whenever the left-hand sides are finite.

Remark 1.3. As it is shown in [CMP], the initial assumption (1.10) can be replaced by the following: there is $0 < p_0 < \infty$ such that

$$\int_{\mathbb{R}^n} f(x)^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} g(x)^p \, w(x) \, dx, \qquad (f, g) \in \mathcal{F},$$

for every $0 < p < p_0$ and every $w \in A_1$. This implies the same conclusion and has the advantage that the weights to be considered are in A_1, which is a class with much better properties. Observe that the same applies to the extrapolation results obtained in the present paper.

The plan of the paper is as follows: in Section 2 we state our main extrapolation result on quasi-norm estimates, which is Theorem 2.1. Starting from Coifman's estimate, we apply Theorem 2.1 to Calderón-Zygmund singular integrals. As a consequence, in Theorem 2.3 we obtain weighted and vector-valued extensions of the classical results of Boyd and Lorentz-Shimogaki. Next, we introduce the basic theory of rearrangement invariant quasi-Banach function spaces and we discuss some examples of scales of spaces to which the main results can be applied. The proofs of the results of Section 2 are given in Section 4. In Section 3 we give the main results we obtain on modular inequalities, postponing their proofs until Section 5. As in the quasi-norm case, in Theorem 3.1 we present an extrapolation result in the context of weighted modular inequalities. Via Coifman's estimate, in Theorem 3.7 we obtain weighted and vector-valued modular inequalities for the Hardy-Littlewood maximal function and, consequently, for Calderón-Zygmund operators. The last section of the paper, Section 6, is devoted to applications. The first one deals with commutators of Calderón-Zygmund operators with BMO functions, for which we find endpoint estimates, both in function spaces and also in the form of modular inequalities. Next, we carry out the same program for multilinear commutators and for fractional integrals and their corresponding commutators. In Section 6.4 we take up the theory of multilinear Calderón-Zygmund operators developed in [GT1],[GT2] and obtain that these operators are bounded on different examples of rearrangement invariant function spaces. Finally in Section 6.5 we discuss a couple of "exotic" maximal functions which exhibit a rather peculiar behavior.

2. Main results on quasi-norm estimates

Now we can state one of our main results that allows us to extrapolate from (1.9) to rearrangement invariant quasi-Banach function spaces (RIQBFS from now on). By means of this extrapolation technique we will be able to get the aforementioned extension of the theorems of Boyd and Lorentz-Shimogaki. The precise definitions and the needed background are presented below the statements of these results.

Theorem 2.1. *Let $0 < p_0 < \infty$ and $\mathcal{F}$ be a family of couples of non-negative functions such that*

$$\int_{\mathbb{R}^n} f(x)^{p_0} \, w(x) \, dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} \, w(x) \, dx, \qquad (f,g) \in \mathcal{F}, \qquad (2.1)$$

for all $w \in A_\infty$. Let $\mathbb{X}$ be a RIQBFS such that

(i) *$\mathbb{X}$ is p-convex for some $0 < p \leq 1$ or, equivalently, $\mathbb{X}^r$ is Banach for some $r \geq 1$.*

(ii) *$\mathbb{X}$ has upper Boyd index $q_{\mathbb{X}} < \infty$.*

Then for all $w \in A_\infty$ we have

$$\|f\|_{\mathbb{X}(w)} \leq C \, \|g\|_{\mathbb{X}(w)}, \qquad (f,g) \in \mathcal{F}. \qquad (2.2)$$

Furthermore, the following vector-valued inequalities also hold:

$$\left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \leq C \left\| \left(\sum_j (g_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)}, \qquad \{(f_j, g_j)\}_j \subset \mathcal{F}, \qquad (2.3)$$

for all $0 < q < \infty$ and $w \in A_\infty$.

Recall that throughout the paper inequalities like (2.1), (2.2) and (2.3) are understood in the sense that they hold whenever the left hand side is finite. Furthermore, the constant C depends upon the A_∞ constant of w. We postpone the proof of this result until Section 4.

Applying Theorem 2.1 to estimate (1.5), we can extend it to weighted rearrangement invariant function spaces in the following way:

Theorem 2.2. *Let T be a Calderón-Zygmund operator with standard kernel and let M be the Hardy-Littlewood maximal function. Let $\mathbb{X}$ be a RIQBFS satisfying (i) and (ii) in Theorem 2.1. Then, for all $w \in A_\infty$, we have*

$$\|Tf\|_{\mathbb{X}(w)} \leq C \, \|Mf\|_{\mathbb{X}(w)}. \qquad (2.4)$$

Furthermore, the following vector-valued inequality holds for any $0 < q < \infty$

$$\left\| \left(\sum_j |Tf_j|^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \leq C \left\| \left(\sum_j (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)}.$$

Next we are going to state the desired extension of the results of Boyd and Lorentz-Shimogaki.

Theorem 2.3. *Let T be a Calderón-Zygmund operator with standard kernel and let M be the Hardy-Littlewood maximal function. Let $\mathbb{X}$ be a RIQBFS which is p-convex for some $p > 0$.*

(i) *If $1 < p_{\mathbb{X}} \leq \infty$, then M is bounded on $\mathbb{X}(w)$ for all $w \in A_{p_{\mathbb{X}}}$.*

(ii) *If $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$, for all $w \in A_{p_{\mathbb{X}}}$, T satisfies the following weighted inequality*

$$\|Tf\|_{\mathbb{X}(w)} \leq C \, \|f\|_{\mathbb{X}(w)}.$$

In particular, T is bounded on $\mathbb{X}$.

(iii) If $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$ we have that for all $1 < q < \infty$ and for all $w \in A_{p_{\mathbb{X}}}$, M satisfies the following weighted vector-valued inequality

$$\left\|\left(\sum_j (Mf_j)^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}(w)} \le C \left\|\left(\sum_j |f_j|^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}(w)}. \tag{2.5}$$

In particular,

$$\left\|\left(\sum_j (Mf_j)^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}} \le C \left\|\left(\sum_j |f_j|^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}}.$$

Analogously, T satisfies the same estimates.

The first conclusion extends Lorentz-Shimogaki's result to the case of weighted RIQBFS. Its proof does not use the extrapolation procedure. Part (ii) generalizes Boyd's theorem both including more general operators and also Muckenhoupt weights. Its proof uses (i) and Theorem 2.2. The weighted (and unweighted) vector-valued extensions of both Lorentz-Shimogaki's and Boyd's classical results contained in (iii) follow by extrapolation, as we will see later, and also by Theorem 2.2. The proof will be presented in Section 4.

2.1. **Basics on RIQBFS.** We collect several basic facts about rearrangement invariant quasi-Banach function spaces (RIQBFS). We start with the Banach case. For a complete account the reader is referred to [BS]. Let (Ω, Σ, μ) be a σ-finite non-atomic measure space. We write $\mathcal{M}$ for the set of measurable functions and $\mathcal{M}^+$ for the non-negative ones. A Banach function norm ρ is a mapping $\rho : \mathcal{M}^+ \longrightarrow [0, \infty]$ such that the following properties hold:

- $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(f+g) \le \rho(f) + \rho(g)$; $\rho(a\,f) = a\,\rho(f)$, for $a \ge 0$.

- If $0 \le f \le g$ μ-a.e., then $\rho(f) \le \rho(g)$.

- If $f_n \nearrow f$ μ-a.e., then $\rho(f_n) \nearrow \rho(f)$.

- If E is a measurable set such that $\mu(E) < \infty$, then $\rho(\chi_E) < \infty$ and $\int_E f\,d\mu \le C_E\,\rho(f)$ for some constant $0 < C_E < \infty$, depending on E and ρ, but independent of f.

By means of ρ, a function space $\mathbb{X} = \mathbb{X}(\rho)$ can be defined:

$$\mathbb{X} = \big\{f \in \mathcal{M} : \rho(|f|) < \infty\big\}.$$

If for each $f \in \mathbb{X}$ we define $\|f\|_{\mathbb{X}} = \rho(|f|)$, then $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ becomes a Banach space. The associate space of $\mathbb{X}$ is the space $\mathbb{X}'$ given by the Banach function norm ρ' defined by

$$\rho'(f) = \sup\Big\{\int_\Omega f\,g\,d\mu : g \in \mathcal{M}^+,\ \rho(g) \le 1\Big\}.$$

Note that, by definition, it follows that for all $f \in \mathbb{X}$, $g \in \mathbb{X}'$ the following generalized Hölder's inequality holds:

$$\int_\Omega |f\,g|\,d\mu \le \|f\|_{\mathbb{X}}\,\|g\|_{\mathbb{X}'}.$$

Moreover, it is an important fact that for every $f \in \mathbb{X}$

$$\|f\|_{\mathbb{X}} = \sup\left\{\left|\int_{\Omega} f\, g\, d\mu\right| : g \in \mathbb{X}', \ \|g\|_{\mathbb{X}'} \leq 1\right\}, \qquad (2.6)$$

see [BS, p. 10]). The distribution function μ_f of a measurable function f is

$$\mu_f(\lambda) = \mu\{x \in \Omega : |f(x)| > \lambda\}, \qquad \lambda \geq 0.$$

A Banach function norm is rearrangement invariant if $\rho(f) = \rho(g)$ for every pair of functions f, g which are equimeasurable, that is, $\mu_f = \mu_g$. This means that the norm of a function f in $\mathbb{X}$ depends only on its distribution function. In this case, we say that the Banach function space $\mathbb{X} = \mathbb{X}(\rho)$ is rearrangement invariant. It follows that $\mathbb{X}'$ is also rearrangement invariant. The decreasing rearrangement of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \qquad t \geq 0.$$

The main property of f^* is that it is equimeasurable with f, that is,

$$\mu\{x \in \Omega : |f(x)| > \lambda\} = \left|\{t \in \mathbb{R}^+ : f^*(t) > \lambda\}\right|.$$

This allows one to obtain a representation of $\mathbb{X}$ on the measure space $(\mathbb{R}^+, dt)$. That is, there exists a RIBFS $\overline{\mathbb{X}}$ over $(\mathbb{R}^+, dt)$ such that $f \in \mathbb{X}$ if and only if $f^* \in \overline{\mathbb{X}}$, and in this case $\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}}$ (Luxemburg's representation theorem, see [BS, p. 62]). Furthermore, the associate space $\mathbb{X}'$ of $\mathbb{X}$ is represented in the same way by the associate space $\overline{\mathbb{X}}'$ of $\overline{\mathbb{X}}$, and so $\|f\|_{\mathbb{X}'} = \|f^*\|_{\overline{\mathbb{X}}'}$.

A useful tool in the study of a RIBFS $\mathbb{X}$ is the fundamental function defined by $\varphi_{\mathbb{X}}(t) = \|\chi_{[0,t]}\|_{\overline{\mathbb{X}}}$, $t \geq 0$. This function is increasing with $\varphi_{\mathbb{X}}(0) = 0$ and quasi-concave, that is, $\varphi_{\mathbb{X}}(t)/t$ is decreasing. By renorming, if necessary, we can always assume that $\varphi_{\mathbb{X}}$ is concave.

We will restrict ourselves to rearrangement invariant Banach function spaces where the underlying measure space is $(\mathbb{R}^n, dx)$ and we will define weighted versions of them. Let $\mathbb{X}$ be a RIBFS in $(\mathbb{R}^n, dx)$ and consider $\overline{\mathbb{X}}$ which is its corresponding space in $(\mathbb{R}^+, dt)$. Take w an A_∞-weight on $\mathbb{R}^n$. We use the standard notation $w(E) = \int_E w(x)\, dx$. We consider as underlying measure space $(\mathbb{R}^n, w(x)\, dx)$. Note that, since w is a measurable function and $0 < w < \infty$ a.e. (because $w \in A_\infty$) then $\mathcal{M}(\mathbb{R}^n, dx) = \mathcal{M}(\mathbb{R}^n, w(x)\, dx)$. The distribution function and the decreasing rearrangement with respect to w are given by

$$w_f(\lambda) = w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}; \qquad f_w^*(t) = \inf\{\lambda \geq 0 : w_f(\lambda) \leq t\}.$$

We define the weighted version of the space $\mathbb{X}$:

$$\mathbb{X}(w) = \{f \in \mathcal{M} : \|f_w^*\|_{\overline{\mathbb{X}}} < \infty\},$$

and the norm associated to it $\|f\|_{\mathbb{X}(w)} = \|f_w^*\|_{\overline{\mathbb{X}}}$. By construction $\mathbb{X}(w)$ is a Banach function space built over $\mathcal{M}(\mathbb{R}^n, w(x)\, dx)$. By doing the same procedure with the associate spaces we can see that the associate space $\mathbb{X}(w)'$ coincides with the weighted space $\mathbb{X}'(w)$.

Next, we define the Boyd indices of a RIBFS, which are closely related to some interpolation properties, see [BS, Ch. 3] for a complete account. First we introduce the dilation operator

$$D_t f(s) = f(s/t), \qquad 0 < t < \infty, \ f \in \overline{\mathbb{X}},$$

and its norm
$$h_{\mathbb{X}}(t) = \|D_t\|_{\mathcal{B}(\overline{\mathbb{X}})}, \qquad 0 < t < \infty,$$
where $\mathcal{B}(\overline{\mathbb{X}})$ denotes the space of bounded linear operators on $\overline{\mathbb{X}}$. Then, the lower and upper Boyd indices are defined respectively by
$$p_{\mathbb{X}} = \lim_{t \to \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \sup_{1 < t < \infty} \frac{\log t}{\log h_{\mathbb{X}}(t)}, \qquad q_{\mathbb{X}} = \lim_{t \to 0^+} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \inf_{0 < t < 1} \frac{\log t}{\log h_{\mathbb{X}}(t)}.$$
We have that $1 \leq p_{\mathbb{X}} \leq q_{\mathbb{X}} \leq \infty$. For instance, if $\mathbb{X} = L^{p,q}$ then it is very easy to see that $h_{\mathbb{X}}(t) = t^{\frac{1}{p}}$ and thus $p_{\mathbb{X}} = q_{\mathbb{X}} = p$. See more examples below.

The relationship between the Boyd indices of $\mathbb{X}$ and $\mathbb{X}'$ is the following: $p_{\mathbb{X}'} = (q_{\mathbb{X}})'$ and $q_{\mathbb{X}'} = (p_{\mathbb{X}})'$, where, as usual, p and p' are conjugate exponents.

Given a Banach function space $\mathbb{X}$, for each $0 < r < \infty$, as in [JS], we define
$$\mathbb{X}^r = \{ f \in \mathcal{M} : |f|^r \in \mathbb{X} \}$$
and the norm (or r-norm)
$$\|f\|_{\mathbb{X}^r} = \big\| |f|^r \big\|_{\mathbb{X}}^{\frac{1}{r}}.$$
Let us note that this notation differs from the one used in [LT], since there, $\mathbb{X}^r$ consists of the r-powers of elements on $\mathbb{X}$. This notation is more natural for the Lebesgue spaces: for example, with the present definition, L^r coincides with $(L^1)^r$, that is, the space of measurable functions f with $|f|^r \in L^1$. If $\mathbb{X}$ is a RIBFS and $r \geq 1$ then, $\mathbb{X}^r$ still is a RIBFS but, in general, for $0 < r < 1$, the space $\mathbb{X}^r$ is not necessarily Banach. This leads us naturally to consider the quasi-Banach case. Actually, we will impose a convexity condition on our quasi-Banach space $\mathbb{X}$ in order to guarantee that $\mathbb{X}^r$ is indeed a Banach space for some large r.

As we have just mentioned, in this context it is natural to consider rearrangement invariant quasi-Banach function spaces, RIQBFS; see [GK] or [Mon] for further details. We define a quasi-Banach function norm as in the Banach case, with the difference that the triangular inequality holds now with some constant, that is,
$$\rho(f + g) \leq C\left(\rho(f) + \rho(g)\right),$$
and that the very last condition in the definition of RIBFS is not required. The constant in the triangular inequality forces several changes in the properties of the space. For example, the Boyd indices satisfy now $0 < p_{\mathbb{X}} \leq q_{\mathbb{X}} \leq \infty$. As mentioned, our RIQBFS will have the property that some large power is indeed a Banach space. This condition can be written in terms of some convexity of the space: a quasi-Banach function space $\mathbb{X}$ is said to be p-convex for some $0 < p \leq 1$, if there exists C such that for all $f_1, \ldots, f_N \in \mathbb{X}$ we have
$$\left\| \left(\sum_{j=1}^{N} |f_j|^p \right)^{\frac{1}{p}} \right\|_{\mathbb{X}} \leq C \left(\sum_{j=1}^{N} \|f_j\|_{\mathbb{X}}^p \right)^{\frac{1}{p}}. \tag{2.7}$$
We may assume that $C = 1$, by renorming $\mathbb{X}$ if necessary. In this case, (2.7) is equivalent to the fact that $\mathbb{X}^{\frac{1}{p}}$ is a RIBFS. So we can use what we know about RIBFS as a tool to understand the RIQBFS. Namely, due to (2.6), the norm in $\mathbb{X}$ can be equivalently represented in the following way
$$\|f\|_{\mathbb{X}} \approx \sup \left\{ \left(\int_{\mathbb{R}^n} |f(x)|^p \, g(x) \, dx \right)^{\frac{1}{p}} : g \geq 0, \, \|g\|_{\mathbb{Y}'} \leq 1 \right\} \tag{2.8}$$

where $\mathbb{Y}'$ is the associate space of the RIBFS $\mathbb{Y} = \mathbb{X}^{\frac{1}{p}}$. Also, since powers commute with f^*, for a RIBFS $\mathbb{X}$ and $w \in A_\infty$, we can define $\mathbb{X}(w)^r$ for every r with $0 < r < \infty$, and we have $\mathbb{X}(w)^r = \mathbb{X}^r(w)$. It follows also that $p_{\mathbb{X}^r} = p_{\mathbb{X}} \cdot r$ and equivalently for $q_{\mathbb{X}}$. These facts will be used throughout this paper.

There are examples of rearrangement invariant quasi-Banach function spaces which are not p-convex for any $p > 0$ (see [JS]) but they are very rare, so that we can say with Grafakos and Kalton [GK] that *"all practical spaces are p-convex for some $p > 0$"*. For a discussion of convexity in Banach function spaces see [LT] and for quasi-Banach function spaces see [Kal].

Regarding the statement of Theorem 2.1 we have to make several remarks.

Remark 2.4. Note that in Theorem 2.1 we have restricted ourselves to the case of $\mathbb{X}$ p-convex with $q_{\mathbb{X}} < \infty$. As we have just mentioned, this means that $\mathbb{X}^r$ is a Banach space (with $r = 1/p$). Thus $q_{\mathbb{X}} < \infty$ is equivalent to the boundedness of the Hardy-Littlewood maximal function on $(\mathbb{X}^r)'$, see Theorem 1.1.

Remark 2.5. Theorem 2.1 can be equivalently formulated in terms of RIBFS rather than quasi-Banach spaces. The conclusion would be as follows:

> *Then, for all RIBFS $\mathbb{X}$ such that $q_{\mathbb{X}} < \infty$ —or equivalently, that the Hardy-Littlewood maximal function is bounded on $\mathbb{X}'$—, all p such that $0 < p < \infty$, and all $w \in A_\infty$, we have*
>
> $$\|f\|_{\mathbb{X}^p(w)} \le C \|g\|_{\mathbb{X}^p(w)}, \qquad (f,g) \in \mathcal{F},$$
>
> *and the corresponding vector-valued inequalities also hold.*

The equivalence is based on the fact that if $\mathbb{Y} = \mathbb{X}^r$ then $q_{\mathbb{Y}} = r \cdot q_{\mathbb{X}}$.

Remark 2.6. The formulation given in Theorem 2.1 and the equivalent one presented in the previous remark reflect that there are two different points of view: suppose that we want to get estimates in $L^{\frac{1}{2}}$ (note that these estimates are indeed proved in [CMP], we just want to illustrate the two different approaches). The first formulation consists of looking at the RIQBFS $\mathbb{X} = L^{\frac{1}{2}}$ which has the property that $\mathbb{X}^2 = L^1$ is a Banach space. This convexity allows us to apply Theorem 2.1 to $\mathbb{X}$. Alternatively we can start from $\mathbb{X} = L^1$ which is a RIBFS and by the second formulation get estimates in $\mathbb{X}^p$ for all $0 < p < \infty$, and in particular in $\mathbb{X}^{\frac{1}{2}} = L^{\frac{1}{2}}$.

2.2. **Examples.** Next we give examples where we can apply Theorem 2.1.

- **Orlicz spaces.** Let ψ be an increasing continuous function defined on $[0, \infty)$ such that $\psi(0) = 0$. The Orlicz space L^ψ is generated by the functional (Luxemburg functional):

$$\|f\|_{L^\psi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \psi \left(\frac{|f(x)|}{\lambda} \right) \, dx \le 1 \right\}. \tag{2.9}$$

  This functional is homogeneous but in general is not a quasi-norm. If we assume that ψ is convex (i.e. a Young function), then $\| \cdot \|_{L^\psi}$ is a norm and L^ψ a Banach space. The fundamental function of L^ψ is given by

$$\varphi(t) = \frac{1}{\psi^{-1}(\frac{1}{t})}. \tag{2.10}$$

A particular case of interest is $\psi(t) = t^p(1 + \log^+ t)^\alpha$, $0 < p < \infty$, $\alpha \in \mathbb{R}$ defining the Zygmund spaces $L^p(\log L)^\alpha$ see [BR]. Regarding the Boyd indices, if $\mathbb{X} = L^p(\log L)^\alpha$ we have $p_{\mathbb{X}} = q_{\mathbb{X}} = p$. In this case $\mathbb{X}^r = L^{pr}(\log L)^\alpha$ for any $0 < r < \infty$. Note that this is a Banach space provided r is big enough, indeed $r > 1/p$.

- **Classical Lorentz spaces.** The spaces $L^{p,q}$ are defined by the function quasi-norm

$$\|f\|_{L^{p,q}} = \Big(\int_0^\infty f^*(s)^q \, s^{q/p} \, \frac{ds}{s} \Big)^{\frac{1}{q}},$$

  when $0 < p, q < \infty$, and

$$\|f\|_{L^{p,\infty}} = \sup_{0 < s < \infty} f^*(s) \, s^{1/p}.$$

  If $\mathbb{X} = L^{p,q}$, $p_{\mathbb{X}} = q_{\mathbb{X}} = p$ and $\mathbb{X}^r = L^{pr,qr}$ which is a Banach space for r large enough, $r > \max\{1/p, 1/q\}$.

- **Lorentz Λ-spaces.** These spaces are extensively studied in [CRS]. The Lorentz spaces $\Lambda^q(v)$ are defined by the functional

$$\|f\|_{\Lambda^q(v)} = \Big(\int_0^\infty f^*(s)^q \, v(s) ds \Big)^{\frac{1}{q}},$$

  where $0 < q < \infty$ and v is a weight on $(0, \infty)$. By choosing $v(s) = s^{q/p-1}$ one obtains $\Lambda^q(v) = L^{p,q}$. If we take $v(s) = s^{q/p-1}(1 + \log^+ \frac{1}{s})^\alpha$ then $\Lambda^q(v) = L^{p,q}(\log L)^\alpha$ are the Lorentz-Zygmund spaces, see [BR]; or if we take $v(s) = s^{q/p-1}(1 + \log^+ \frac{1}{s})^\alpha(1 + \log^+ \log^+ \frac{1}{s})^\beta$ then $\Lambda^q(v) = L^{p,q}(\log L)^\alpha(\log\log L)^\beta$ are the generalized Lorentz-Zygmund spaces, see [EOP].

  We claim that Theorem 2.1 can be applied to the space $\mathbb{X} = \Lambda^p(v)$ whenever the weight v satisfies the following two conditions: There exists large enough $r > 1$ such that

$$\int_t^\infty v(s) \, \frac{ds}{s^r} \le \frac{C}{t^r} \int_0^t v(s) \, ds, \qquad t > 0, \tag{2.11}$$

  and

$$\frac{1}{t} \int_0^t v(s) \, ds \le C \, v(t) \qquad t > 0. \tag{2.12}$$

Indeed, it follows from [Saw] that condition (2.11) for $1 < r < \infty$, usually called the B_r condition, is equivalent to the fact that $\Lambda^r(v)$ is a Banach space. This, combined with the observation that $\Lambda^p(v) = (\Lambda^1(v))^p$, implies that $\mathbb{X}^r$ is a Banach space for large r. We now check that (2.12) implies that $q_{\mathbb{X}} < \infty$. Taking $\mathbb{Y} = \Lambda^1(v)$, it suffices to show that $q_{\mathbb{Y}} < \infty$ (since $\mathbb{X} = \mathbb{Y}^p$ and so $q_{\mathbb{X}} = p \cdot q_{\mathbb{Y}}$). Observe that $q_{\mathbb{Y}} < \infty$ if and only if the adjoint of the Hardy operator

$$Qf(t) = \int_t^\infty f^*(s) \, \frac{ds}{s}$$

is bounded on $\overline{\mathbb{Y}}$, see [BS, p.150] and [Mon]. Therefore, by (2.12), for $0 \le f \in \mathcal{M}(\mathbb{R}^+, dt)$ we obtain, as desired,

$$\|Qf\|_{\overline{\mathbb{Y}}} = \int_0^\infty (Qf)^*(t) \, v(t) \, dt = \int_0^\infty \int_t^\infty f^*(s) \, \frac{ds}{s} \, v(t) \, dt$$

$$= \int_0^\infty \int_0^s v(t)\, dt\, f^*(s)\, \frac{ds}{s} \leq C \int_0^\infty f^*(s)\, v(s)\, ds$$
$$= C\, \|f\|_{\overline{\mathbb{Y}}}.$$

Particular cases of Lorentz Λ-spaces are Λ_φ defined as $\Lambda_\varphi = \Lambda^1(\varphi')$, where φ is an increasing concave function on $[0, \infty)$ with $\varphi(0^+) = 0$. This space is a RIBFS with fundamental function φ and appears naturally when one looks for the smallest function space with a given fundamental function. Indeed, for a RIBFS $\mathbb{X}$ with fundamental function $\varphi_{\mathbb{X}}$, assumed to be concave, we have $\Lambda \varphi_{\mathbb{X}} \hookrightarrow \mathbb{X}$. For example, if $\mathbb{X} = L^p$, then $\Lambda \varphi_{\mathbb{X}} = L^{p,1}$.

- **Marcinkiewicz spaces.** Let φ be an increasing quasi-concave function —that is, $\varphi(t)/t$ decreasing— on $[0, \infty)$ with $\varphi(0^+) = 0$. The Marcinkiewicz space $\mathbb{M}_\varphi$ is defined by the function norm

$$\|f\|_{\mathbb{M}_\varphi} = \sup_{t>0} \frac{\varphi(t)}{t} \int_0^t f^*(s)\, ds.$$

The space $\mathbb{M}_\varphi$ is a RIBFS with fundamental function φ. If $\overline{\varphi}(t) = t/\varphi(t)$, then $(\mathbb{M}_\varphi)' = \Lambda_{\overline{\varphi}}$ and $(\Lambda_\varphi)' = \mathbb{M}_{\overline{\varphi}}$. The Marcinkiewicz space is the largest space with a given fundamental function, since for a RIBFS $\mathbb{X}$ with fundamental function $\varphi_{\mathbb{X}}$ we always have $\mathbb{X} \hookrightarrow \mathbb{M}_{\varphi_{\mathbb{X}}}$.

We also consider another type of Marcinkiewicz space $\widetilde{\mathbb{M}}_\varphi$ given by the functional

$$\|f\|_{\widetilde{\mathbb{M}}_\varphi} = \sup_{t>0} \varphi(t)\, f^*(t),$$

which is a RIQBFS. Note that $\mathbb{M}_\varphi \subset \widetilde{\mathbb{M}}_\varphi$, since $f^*(t) \leq \frac{1}{t} \int_0^t f^*(s)\, ds$. The condition

$$\frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)}\, ds \leq C, \tag{2.13}$$

gives the equivalence of both function norms and hence in this case $\widetilde{\mathbb{M}}_\varphi$ coincides with $\mathbb{M}_\varphi$, which is a RIBFS. This estimate can be also written in the following way that will appear later

$$\overline{\varphi}(t) \sim \int_0^t \frac{\overline{\varphi}(s)}{s}\, ds, \qquad \text{where, as above,} \quad \overline{\varphi}(t) = \frac{t}{\varphi(t)}. \tag{2.14}$$

Observe that if φ satisfies that $\varphi(t)/t^\varepsilon$ is decreasing for some $0 < \varepsilon < 1$, then (2.13) holds and, consequently, $\mathbb{M}_\varphi = \widetilde{\mathbb{M}}_\varphi$.

We claim that we can apply Theorem 2.1 to the spaces $\mathbb{X} = \widetilde{\mathbb{M}}_\varphi$. Indeed, although in general $\mathbb{X}$ is just a RIQBFS, $\mathbb{X}^r$ is always a Banach space for any $r > 1$. To see this, consider the quasi-concave function $\varphi_r(t) = \varphi(t)^{\frac{1}{r}}$. Note that $\varphi_r(t)/t^{\frac{1}{r}}$ is decreasing since $\varphi(t)/t$ also is. Hence, (2.13) implies $\widetilde{\mathbb{M}}_{\varphi_r} = \mathbb{M}_{\varphi_r}$ and so $\widetilde{\mathbb{M}}_{\varphi_r}$ is a RIBFS. Direct computation shows that $(\widetilde{\mathbb{M}}_\varphi)^r = \widetilde{\mathbb{M}}_{\varphi_r}$ and therefore $(\widetilde{\mathbb{M}}_\varphi)^r$ is a RIBFS. This allows us to apply Theorem 2.1 to the spaces $\widetilde{\mathbb{M}}_\varphi$, see Section 6.1.

To illustrate the relationship between $\mathbb{M}_\varphi$ and $\widetilde{\mathbb{M}}_\varphi$, let us consider the case of the Lebesgue spaces. For L^p, $1 \leq p < \infty$, the fundamental function is $\varphi(t) = t^{\frac{1}{p}}$. If $p = 1$, then $\widetilde{\mathbb{M}}_\varphi = L^{1,\infty}$ whereas $\mathbb{M}_\varphi = L^1$. Note that in this case $\mathbb{M}_\varphi \subsetneq \widetilde{\mathbb{M}}_\varphi$ and that $L^{1,\infty}$ is not normable. However, $(L^{1,\infty})^r$ is a Banach space for any $r > 1$. For

$p > 1$ it is clear that (2.13) is satisfied since $\varphi(t)/t^{\frac{1}{p}} = 1$, which is decreasing. In this case, both $\widetilde{\mathbb{M}}_\varphi$ and $\mathbb{M}_\varphi$ coincide with $L^{p,\infty}$, which is a Banach space for $p > 1$.

The Boyd indices of the spaces $\mathbb{M}_\varphi$ and $\widetilde{\mathbb{M}}_\varphi$ can be computed from φ. For this we need to recall the lower and upper dilation indices of a positive increasing function ϕ on $[0, \infty)$ which are defined respectively by

$$i_\phi = \lim_{t \to 0^+} \frac{\log h_\phi(t)}{\log t} = \sup_{0 < t < 1} \frac{\log h_\phi(t)}{\log t}, \qquad I_\phi = \lim_{t \to \infty} \frac{\log h_\phi(t)}{\log t} = \inf_{1 < t < \infty} \frac{\log h_\phi(t)}{\log t},$$

where

$$h_\phi(t) = \sup_{s > 0} \frac{\phi(s\,t)}{\phi(s)}, \qquad t > 0,$$

see [KPS] and [KK]. Observe that $0 \leq i_\phi \leq I_\phi \leq \infty$. If $\varphi_\mathbb{X}$ is the fundamental function of a RIQBFS $\mathbb{X}$ then, $h_{\varphi_\mathbb{X}}(t)$ is the norm of the dilation operator D_t over the characteristic functions. Hence, $h_{\varphi_\mathbb{X}}(t) \leq h_\mathbb{X}(t)$ for all $t > 0$. Thus, the following relationship between indices hold

$$p_\mathbb{X} \leq \frac{1}{I_{\varphi_\mathbb{X}}} \leq \frac{1}{i_{\varphi_\mathbb{X}}} \leq q_\mathbb{X}.$$

If we consider the space $\mathbb{X} = \widetilde{\mathbb{M}}_\varphi$, from

$$\|D_t f\|_{\overline{\mathbb{X}}} = \sup_{s > 0} \varphi(s)\, f\left(\frac{s}{t}\right) = \sup_{s > 0} \varphi(s\,t)\, f(s) = \sup_{s > 0} \frac{\varphi(s\,t)}{\varphi(s)} \varphi(s)\, f(s) \leq h_\varphi(t)\, \|f\|_{\overline{\mathbb{X}}}$$

for $t > 0$, it follows that $h_\mathbb{X}(t) \leq h_\varphi(t)$, for $t > 0$. Since $\varphi = \varphi_\mathbb{X}$, we have

$$p_\mathbb{X} = \frac{1}{I_\varphi}, \qquad\qquad q_\mathbb{X} = \frac{1}{i_\varphi}, \qquad\qquad (2.15)$$

see [KPS, p. 99]. A similar computation establishes also the result for $\mathbb{M}_\varphi$ and Λ_φ. With a different argument, the same result holds also for Orlicz spaces, see [BS].

The dilation indices allow one to give a sufficient condition for (2.14), namely $0 < i_{\overline{\varphi}} \leq I_{\overline{\varphi}} < \infty$. In the case of Marcinkiewicz spaces $\mathbb{M}_\varphi$ the function $\overline{\varphi}$ is concave and so $I_{\overline{\varphi}} \leq 1$. Hence (2.14) holds whenever $i_{\overline{\varphi}} > 0$. Direct computation shows that this is equivalent to $I_\varphi < 1$, see [KPS, p. 53–57].

3. Main results on modular estimates

As we have shown before, the extrapolation method works any time we have reasonable Banach or quasi-Banach spaces. Somehow, we need to write the weighted estimates in $\mathbb{X}$ as certain integrals in such a way that the inequalities in $L^p(w)$ for some $0 < p < \infty$ and $w \in A_\infty$ can be used. When $\mathbb{X}$ is a Banach space this can be done via the dual space. Note that for quasi-Banach spaces we have assumed that, with some large power, there is a dual space. However, there are estimates in Harmonic analysis which are not associated with a Banach or quasi-Banach space. This is the case of some modular inequalities. For example, for the maximal function M the conditions on ϕ for which we have the modular inequality

$$\int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\, dx \leq C_\phi \int_{\mathbb{R}^n} \phi\big(C_\phi\,|f(x)|\big)\, dx \qquad\qquad (3.1)$$

are well known (see for instance [KK]). In order to extend the extrapolation technique to this context, we need to write the integrals above as estimates on weighted Lebesgue

spaces. Note that weighted modular estimates are not necessarily associated with Banach or quasi-Banach spaces and so the duality can not be used. As a substitute we will use Young's inequality with ϕ and its complementary function $\overline{\phi}$, provided ϕ is convex. If this is not the case, we will assume that $\phi(t^{r_0})^{s_0}$ is convex for some large exponents r_0, s_0. Let us point out the analogy with the RIQBFS case on which one assumes that $\mathbb{X}^r$ is a Banach space for some large r.

Before stating our main result on modular inequalities, we need to introduce some notation. The terminology used is taken from [KK] and [RR]. Let Φ be the set of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ which are nonnegative, increasing and such that $\phi(0+) = 0$ and $\phi(\infty) = \infty$. If $\phi \in \Phi$ is convex we say that ϕ is a Young function. An N-function (from *nice Young function*) ϕ is a Young function such that

$$\lim_{t \to 0^+} \frac{\phi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t \to \infty} \frac{\phi(t)}{t} = \infty.$$

We say that $\phi \in \Phi$ is quasi-convex if there exists a convex function $\widetilde{\phi}$ and $a_1 \geq 1$ such that

$$\widetilde{\phi}(t) \leq \phi(t) \leq a_1 \widetilde{\phi}(a_1 t), \qquad t \geq 0. \tag{3.2}$$

It may be useful the following characterization (see [KK, p. 4]): ϕ is *quasi-convex* if and only there is $a > 1$ such that

$$0 < s < t \qquad \text{implies} \qquad \frac{\phi(s)}{s} \leq a \frac{\phi(at)}{t}. \tag{3.3}$$

The function $\phi \in \Phi$ satisfies the Δ_2 condition, we will write $\phi \in \Delta_2$, if ϕ is doubling, that is, if

$$\phi(2t) \leq C \phi(t), \qquad t \geq 0.$$

Note that if ϕ quasi-convex, then $i_\phi \geq 1$ and that $\phi \in \Delta_2$ if and only if $I_\phi < \infty$.

Given $\phi \in \Phi$ we define the complementary function $\overline{\phi}$ by

$$\overline{\phi}(s) = \sup_{t > 0} \{s\, t - \phi(t)\}, \qquad s \geq 0.$$

By definition we have Young's inequality

$$s\, t \leq \phi(s) + \overline{\phi}(t), \qquad s, t \geq 0. \tag{3.4}$$

When ϕ is an N-function, then $\overline{\phi}$ is an N-function too, and we have the following

$$t \leq \phi^{-1}(t)\, \overline{\phi}^{-1}(t) \leq 2\, t, \qquad t \geq 0. \tag{3.5}$$

In [KK, p. 15] (see also the arguments given in the proof of Lemma 5.2 below) we can find following property: if ϕ is an N-function then

there exists $0 < \alpha < 1$ such that ϕ^α is quasi-convex $\qquad \Longleftrightarrow \qquad \overline{\phi} \in \Delta_2$,

where $\phi^\alpha(t) = \phi(t)^\alpha$.

Next, we state our main result in this section that allows us to extrapolate from estimates on weighted Lebesgue spaces to weighted modular inequalities. The proof of this result is given in Section 5 below.

Theorem 3.1. *Let $0 < p_0 < \infty$ and $\mathcal{F}$ be a family of couples of non-negative functions such that*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f, g) \in \mathcal{F}, \tag{3.6}$$

for all $w \in A_\infty$. Let $\phi \in \Phi$ be such that

(i) $\phi \in \Delta_2$, *equivalently,* $I_\phi < \infty$.

(ii) *There exist some exponents* $0 < r_0, s_0 < \infty$ *such that* $\phi(t^{r_0})^{s_0}$ *is quasi-convex.*

Then for all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} \phi\big(f(x)\big) \, w(x) \, dx \le C \int_{\mathbb{R}^n} \phi\big(g(x)\big) \, w(x) \, dx, \tag{3.7}$$

for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite. Furthermore, for all RIQBFS $\mathbb{X}$, p-convex for some $0 < p \le 1$ —or equivalently, $\mathbb{X}^r$ is Banach for some $r \ge 1$— and with upper Boyd index $q_\mathbb{X} < \infty$, and for all $w \in A_\infty$ we have

$$\big\| \phi(f) \big\|_{\mathbb{X}(w)} \le C \big\| \phi(g) \big\|_{\mathbb{X}(w)}, \tag{3.8}$$

for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite. In particular, we have the following weak-type modular inequality: for all $w \in A_\infty$,

$$\sup_\lambda \phi(\lambda) \, w\{x \in \mathbb{R}^n : f(x) > \lambda\} \le C \sup_\lambda \phi(\lambda) \, w\{x \in \mathbb{R}^n : g(x) > \lambda\}, \tag{3.9}$$

for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite.

Remark 3.2. We would like to emphasize the analogy between the hypotheses of Theorems 2.1 and 3.1. The assumptions $\mathbb{X}$ is p-convex (that is, $\mathbb{X}^r$ is Banach space), and $\phi(t^{r_0})^{s_0}$ is quasi-convex play the same role, since in both cases they allow us to use a duality argument: for $\mathbb{X}^r$ there is an associate space and $\phi(t^{r_0})^{s_0}$ has a complementary function. On the other hand, note that we have the same assumptions in the upper indices $q_\mathbb{X} < \infty$ and $I_\phi < \infty$ (equivalently, $\phi \in \Delta_2$). These are used in the proofs to ensure that the Hardy-Littlewood maximal function is bounded on the dual space in the RIQBFS case, or satisfies a modular inequality with respect to the complementary function in the second case.

Remark 3.3. Let us point out that one can reformulate the previous result replacing ϕ by the functions $\widetilde{\phi}(t) = \phi(t^q)^p$, for all $0 < p, q < \infty$. Both ways are equivalent since $\phi \in \Phi$ satisfies (i) and (ii) if an only if $\widetilde{\phi}$ does. This fact is the analog of Remark 2.5 in this modular case. Note also that the same can be done for the vector-valued estimates that we present in the next corollary.

Remark 3.4. Note that for $f \ge 0$ we have

$$\big\| \phi(f) \big\|_{L^{1,\infty}(w)} = \sup_\lambda \lambda \, w\{x : \phi\big(f(x)\big) > \lambda\} = \sup_\lambda \phi(\lambda) \, w\{x : f(x) > \lambda\}$$

and therefore (3.9) is a particular case of (3.8) with $\mathbb{X} = L^{1,\infty}$.

Corollary 3.5. *Under the same hypotheses of Theorem 3.1, we additionally have the following vector-valued estimates: for all $0 < q < \infty$ and all $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} \phi\left(\Big(\sum_j f_j(x)^q \Big)^{\frac{1}{q}} \right) w(x) \, dx \ \le \ C \int_{\mathbb{R}^n} \phi\left(\Big(\sum_j g_j(x)^q \Big)^{\frac{1}{q}} \right) w(x) \, dx, \tag{3.10}$$

$$\int_{\mathbb{R}^n} \Big(\sum_j \phi(f_j(x))^q \Big)^{\frac{1}{q}} w(x) \, dx \ \le \ C \int_{\mathbb{R}^n} \Big(\sum_j \phi\big(g_j(x)\big)^q \Big)^{\frac{1}{q}} w(x) \, dx, \tag{3.11}$$

for any $\{(f_j, g_j)\}_j \subset \mathcal{F}$ such that the left-hand sides are finite. Moreover, both esti-mates have their analogs on $\mathbb{X}(w)$ as in (3.8), and in particular they provide vector-valued weak-type modular extensions of the estimate (3.9).

As we did for the RIQBFS we can apply Theorem 3.1 to (1.5) and get the following weighted modular inequalities:

Theorem 3.6. *Let T be a Calderón-Zygmund operator with standard kernel and let M be the Hardy-Littlewood maximal function. Assume that $\phi \in \Phi$ satisfies (i) and (ii) in Theorem 3.1. Then for all $w \in A_\infty$, we have*

$$\int_{\mathbb{R}^n} \phi(|Tf(x)|)\, w(x)\, dx \;\leq\; C \int_{\mathbb{R}^n} \phi(Mf(x))\, w(x)\, dx, \qquad (3.12)$$

$$\sup_\lambda \phi(\lambda)\, w\{x : |Tf(x)| > \lambda\} \;\leq\; C \sup_\lambda \phi(\lambda)\, w\{x : Mf(x) > \lambda\}, \qquad (3.13)$$

Furthermore, one can also get estimates in $\mathbb{X}(w)$ and the corresponding vector-valued inequalities arising from Corollary 3.5.

This result can be used to prove weighted modular inequalities for T once we know them for M. In order to obtain such inequalities, we will need some convexity of the function ϕ which will be given from the growth properties of ϕ. Indeed, the lower and upper dilation indices i_ϕ and I_ϕ allow us to estimate this growth via power functions. In particular, if $0 < i_\phi < \infty$, for any small $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\phi(t\, s) \leq C_\epsilon\, t^{i_\phi - \varepsilon}\, \phi(s), \qquad \text{for } 0 \leq t \leq 1 \text{ and } s \geq 0.$$

We have an analog of Theorem 2.3 with modular inequalities.

Theorem 3.7. *Let T be a Calderón-Zygmund operator with standard kernel and let M be the Hardy-Littlewood maximal function. Let $\phi \in \Phi$ be such that ϕ is quasi-convex.*

(i) *Let $w \in A_{i_\phi}$. If $1 < i_\phi \leq \infty$ we have*

$$\int_{\mathbb{R}^n} \phi(Mf(x))\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi(C\, |f(x)|)\, w(x)\, dx,$$

and if $i_\phi = 1$

$$\sup_\lambda \phi(\lambda)\, w\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq C \int_{\mathbb{R}^n} \phi(C\, |f(x)|)\, w(x)\, dx.$$

(ii) *Let $\phi \in \Delta_2$ $(i.e., I_\phi < \infty)$ and $w \in A_{i_\phi}$. If $i_\phi > 1$ we have*

$$\int_{\mathbb{R}^n} \phi(|Tf(x)|)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi(|f(x)|)\, w(x)\, dx,$$

and if $i_\phi = 1$

$$\sup_\lambda \phi(\lambda)\, w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \leq C \int_{\mathbb{R}^n} \phi(|f(x)|)\, w(x)\, dx.$$

(iii) *Let $\phi \in \Delta_2$ $(i.e., I_\phi < \infty)$, $w \in A_{i_\phi}$ and $1 < q < \infty$. If $i_\phi > 1$ we have*

$$\int_{\mathbb{R}^n} \phi\!\left(\Big(\sum_j Mf_j(x)^q\Big)^{\frac{1}{q}}\right) w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\!\left(\Big(\sum_j f_j(x)^q\Big)^{\frac{1}{q}}\right) w(x)\, dx,$$

and if $i_\phi = 1$

$$\sup_\lambda \phi(\lambda)\, w\left\{ x : \left(\sum_j Mf_j(x)^q \right)^{\frac{1}{q}} > \lambda \right\} \leq C \int_{\mathbb{R}^n} \phi\left(\left(\sum_j f_j(x)^q \right)^{\frac{1}{q}} \right) w(x)\, dx,$$

Analogously, T satisfies the same estimates.

Remark 3.8. As mentioned before, $\phi \in \Delta_2$ is equivalent to $I_\phi < \infty$. So, for the strong inequalities in (ii) and (iii) the hypotheses can be written as $1 < i_\phi \leq I_\phi < \infty$ and this should be compared with Theorem 2.3.

Remark 3.9. When $i_\phi > 1$, we will see that $\phi(t^{1/r})^\alpha$ is quasi-convex for $1 < r < i_\phi$ and for $0 < \alpha < 1$ close enough to 1 and then the assumption ϕ quasi-convex is redundant. When $i_\phi = 1$, this assumption is necessary since for $w(x) \equiv 1 \in A_1$ the weak type estimate in (i) holds if and only if ϕ is quasi-convex, see [KK, p. 9].

Part (i) in Theorem 3.7 will be proved directly without using extrapolation. The key is that $i_\phi > 1$ implies some convexity of ϕ, see Lemma 5.2 below. This plus $\phi \in \Delta_2$ will allow us to extrapolate since (i) and (ii) in Theorem 3.1 hold. To prove the other conclusions we will use the extrapolation result Theorem 3.1 combined with Theorem 3.6. The proof will be presented in Section 5.

Part (i), under slightly stronger hypotheses (i.e., $\phi, \overline{\phi} \in \Delta_2$), was first considered in [KT]. The proof that we give below follows the lines of [KK, p. 33]. Conclusions (ii) and (iii) generalize some of the estimates obtained, by different methods, in [KK, Chapters 1, 2]. The reader is referred to this book for a complete account of modular inequalities.

4. Proof of the main results: RIQBFS

In this section we will present the proofs of Theorems 2.1 and 2.3

4.1. **Auxiliary results.** Let us recall that the Hardy-Littlewood maximal operator M is defined as

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all the balls B that contain $x \in \mathbb{R}^n$. It is well known that M is of weak type $(1,1)$ and thus bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. The lower Boyd index $p_{\mathbb{X}}$ characterizes the boundedness of M on RIQBFS as we can see in Theorem 1.1 in the Banach case or also in the following result from [Mon].

Theorem 4.1 ([Mon]). *Let $\mathbb{X}$ be a RIQBFS. Then the Hardy operator is bounded on $\overline{\mathbb{X}}$ if and only if $p_{\mathbb{X}} > 1$. Consequently, M is bounded on $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$.*

Next, we consider the weighted Hardy-Littlewood maximal operator M_w, given by

$$M_w f(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(y)|\, w(y)\, dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n).$$

We establish a weighted version of Lorentz-Shimogaki's theorem for M_w and $\mathbb{X}(w)$ where the Boyd index involved is $p_{\mathbb{X}}$ independently of the weight. By [AKMP] we have that M_w is of weak type $(1,1)$ with respect to the measure w if and only if

$$(M_w f)_w^*(t) \leq C\, f_w^{**}(t), \qquad t > 0, \tag{4.1}$$

where $f_w^{**}(t) = \frac{1}{t} \int_0^t f_w^*(s)\,ds$. Let us note that if $w \in A_\infty$, then $w(x)\,dx$ is a doubling measure and therefore M_w is of weak type $(1,1)$ with respect to w (see [GR, p. 144]). In this way we can use the previous estimate to get

$$\|M_w f\|_{\mathbb{X}(w)} = \left\|(M_w f)_w^*\right\|_{\overline{\mathbb{X}}} \leq C \left\|f_w^{**}\right\|_{\overline{\mathbb{X}}}$$

Also, observe that f_w^{**} is the Hardy operator acting over f_w^*. Thus, by Theorem 4.1 we get

$$\|M_w f\|_{\mathbb{X}(w)} \leq C \left\|f_w^{**}\right\|_{\overline{\mathbb{X}}} \leq C \left\|f_w^*\right\|_{\overline{\mathbb{X}}} = C \left\|f\right\|_{\mathbb{X}(w)}.$$

Therefore we have the following result.

Theorem 4.2. *Let $\mathbb{X}$ be a RIQBFS which is p-convex for some $0 < p \leq 1$. If $p_{\mathbb{X}} > 1$ and $w \in A_\infty$, then M_w is bounded on $\mathbb{X}(w)$.*

4.2. **Proof of Theorem 2.1.** Let $0 < p \leq 1$ be such that $\mathbb{X}$ is p-convex, or equivalently, that $\mathbb{Y} = \mathbb{X}^{\frac{1}{p}}$ is a Banach space. Then,

$$\|f\|_{\mathbb{X}(w)}^p = \|f^p\|_{\mathbb{Y}(w)} = \sup_h \int_{\mathbb{R}^n} f(x)^p \, h(x)\, w(x)\,dx$$

where the supremum is taken over all the functions $0 \leq h \in \mathbb{Y}'(w)$ with $\|h\|_{\mathbb{Y}'(w)} \leq 1$ —let us recall that $\mathbb{Y}(w)' = \mathbb{Y}'(w)$—. Fix such a function h. Since $q_{\mathbb{X}} < \infty$ we have that $q_{\mathbb{Y}} = q_{\mathbb{X}}/p < \infty$ and so $p_{\mathbb{Y}'} = (q_{\mathbb{Y}})' > 1$. Then by Theorem 4.2, M_w is bounded on $\mathbb{Y}'(w)$. Let us write $\|M_w\|$ for the norm of M_w as a bounded operator on $\mathbb{Y}'(w)$. Then we use the following version of Rubio de Francia's algorithm: We define

$$\mathcal{R}_w h(x) = \sum_{k=0}^{\infty} \frac{M_w^k h(x)}{2^k \|M_w\|^k},$$

where M_w^k is the operator M_w iterated k times for $k \geq 1$ and for $k = 0$ is just the identity. We have the following properties:

(a) $h(x) \leq \mathcal{R}_w h(x)$.

(b) $\|\mathcal{R}_w h\|_{\mathbb{Y}'(w)} \leq 2\|h\|_{\mathbb{Y}'(w)} \leq 2$.

(c) $M_w(\mathcal{R}_w h)(x) \leq 2 \|M_w\| \, \mathcal{R}_w h(x)$, and, consequently, $\mathcal{R}_w h \in A_1(w)$ (by this we mean that $\mathcal{R}_w h$ is a A_1-weight but with respect to the measure $w(x)\,dx$).

We need the following observation from [CMP].

Lemma 4.3. *If $w_1 \in A_r$, $1 \leq r \leq \infty$, and $w_2 \in A_1(w_1)$, then $w_1 w_2 \in A_r$.*

We apply this lemma to $\mathcal{R}_w h \in A_1(w)$ and $w \in A_\infty$ to get that $\mathcal{R}_w h\, w \in A_\infty$. Besides,

$$\int_{\mathbb{R}^n} f(x)^p \, \mathcal{R}_w h(x)\, w(x)\,dx \leq \|f^p\|_{\mathbb{Y}(w)} \|\mathcal{R}_w h\|_{\mathbb{Y}'(w)} \leq 2 \|f\|_{\mathbb{X}(w)}^p < \infty.$$

Thus we can apply Theorem 1.2 to obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} f(x)^p \, h(x)\, w(x)\,dx &\leq \int_{\mathbb{R}^n} f(x)^p \, \mathcal{R}_w h(x)\, w(x)\,dx \leq C \int_{\mathbb{R}^n} g(x)^p \, \mathcal{R}_w h(x)\, w(x)\,dx \\
&\leq C \|g^p\|_{\mathbb{Y}(w)} \|\mathcal{R}_w h\|_{\mathbb{Y}'(w)} \leq C \|g\|_{\mathbb{X}(w)}^p
\end{aligned}
$$

which, after taking the supremum on h, leads to the desired estimate.

The vector-valued inequalities arise in a very easy way. The ideas are taken from [CMP] and we include them here for completeness. Fix $0 < q < \infty$. By the monotone convergence theorem it suffices to show that the vector-valued inequalities hold only for finite sums. Fix $N \geq 1$ and set

$$f_q(x) = \Big(\sum_{j=1}^{N} f_j(x)^q \Big)^{\frac{1}{q}}, \qquad g_q(x) = \Big(\sum_{j=1}^{N} g_j(x)^q \Big)^{\frac{1}{q}},$$

where $\{(f_j, g_j)\}_{j=1}^{N} \subset \mathcal{F}$. Consider a new family $\mathcal{F}_q$ consisting of all these couples (f_q, g_q). Then, for every $w \in A_\infty$ and $(f_q, g_q) \in \mathcal{F}_q$ we have

$$\|f_q\|_{L^q(w)}^q = \sum_{j=1}^{N} \int_{\mathbb{R}^n} f_j(x)^q \, w(x) \, dx \leq C \sum_{j=1}^{N} \int_{\mathbb{R}^n} g_j(x)^q \, w(x) \, dx = C \, \|g_q\|_{L^q(w)}^q,$$

by Theorem 1.2. This inequality says that the hypotheses of Theorem 2.1 are fulfilled by $\mathcal{F}_q$ with $p_0 = q$. Then we can apply Theorem 2.1 to get as desired

$$\|f_q\|_{\mathbb{X}(w)} \leq C \, \|g_q\|_{\mathbb{X}(w)}.$$

4.3. Proof of Theorem 2.3.

4.3.1. *Part (i)*. We first consider the case $1 < p_{\mathbb{X}} < \infty$. Observe that if $w \in A_q$, $1 < q < \infty$, we have that for any ball $B \ni x$

$$\frac{1}{|B|} \int_B |f| = \frac{1}{|B|} \int_B |f| \, w^{\frac{1}{q}} \, w^{-\frac{1}{q}} \leq \Big(\frac{1}{|B|} \int_B |f|^q \, w \Big)^{\frac{1}{q}} \Big(\frac{1}{|B|} \int_B w^{1-q'} \Big)^{\frac{1}{q'}}$$

$$= \Big(\frac{1}{w(B)} \int_B |f|^q \, w \Big)^{\frac{1}{q}} \Big[\Big(\frac{1}{|B|} \int_B w \Big) \Big(\frac{1}{|B|} \int_B w^{1-q'} \Big)^{q-1} \Big]^{\frac{1}{q}}$$

$$\leq [w]_{A_q}^{\frac{1}{q}} \, M_w \big(|f|^q \big)(x)^{\frac{1}{q}},$$

where $[w]_{A_q}$ denotes the A_q-constant of $w \in A_q$. Thus, this standard computation shows that

$$M f(x) \leq [w]_{A_q}^{\frac{1}{q}} \, M_w \big(|f|^q \big)(x)^{\frac{1}{q}}, \qquad \text{for all } w \in A_q. \tag{4.2}$$

Our weight w belongs to $A_{p_{\mathbb{X}}}$, then by the reverse Hölder inequality, there exists $1 < q < p_{\mathbb{X}}$ such that $w \in A_q$. Then,

$$\|M f\|_{\mathbb{X}(w)} \leq [w]_{A_q}^{\frac{1}{q}} \, \big\| M_w \big(|f|^q \big)^{\frac{1}{q}} \big\|_{\mathbb{X}(w)} = [w]_{A_q}^{\frac{1}{q}} \, \big\| M_w \big(|f|^q \big) \big\|_{\mathbb{X}^{\frac{1}{q}}(w)}^{\frac{1}{q}}.$$

Note that $p_{\mathbb{X}^{\frac{1}{q}}} = \frac{p_{\mathbb{X}}}{q} > 1$ and we can use Theorem 4.2 to get

$$\|M f\|_{\mathbb{X}(w)} \leq C \, [w]_{A_q}^{\frac{1}{q}} \, \big\| |f|^q \big\|_{\mathbb{X}^{\frac{1}{q}}(w)}^{\frac{1}{q}} = C \, [w]_{A_q}^{\frac{1}{q}} \, \|f\|_{\mathbb{X}(w)}.$$

The case $p_{\mathbb{X}} = \infty$ is easier, it suffices to note that $w \in A_\infty$ implies $w \in A_q$ for some $1 < q < \infty$. Then we can repeat the argument above using that M_w is bounded on $\mathbb{X}(w)$ provided $p_{\mathbb{X}^{\frac{1}{q}}} > 1$, which is the case since $p_{\mathbb{X}^{\frac{1}{q}}} = \frac{p_{\mathbb{X}}}{q} = \infty$.

4.3.2. *Part (ii)*. We just need to use part (i) and (2.4) to get that T is bounded on $\mathbb{X}(w)$. For the unweighted estimate note that $w = 1 \in A_1 \subset A_{p_{\mathbb{X}}}$ since $p_{\mathbb{X}} > 1$.

4.3.3. *Part (iii).* We will prove an estimate better than (2.5); namely for $f = \{f_j\}_j$, and for all $w \in A_\infty$ we have

$$\left\| \left(\sum_j (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \leq C \left\| M(\|f\|_{\ell^q}) \right\|_{\mathbb{X}(w)}, \tag{4.3}$$

where $\|f\|_{\ell^q} = \left(\sum_j |f_j|^q \right)^{\frac{1}{q}}$. Assuming this inequality, by Part (i), if $w \in A_{p_\mathbb{X}}$ we get (2.5) as desired:

$$\left\| \left(\sum_j (Mf_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \leq C \left\| M(\|f\|_{\ell^q}) \right\|_{\mathbb{X}(w)} \leq C \left\| \|f\|_{\ell^q} \right\|_{\mathbb{X}(w)} = C \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)}.$$

Note that by Theorem 2.2 we get the same estimate for T.

To prove (4.3) we first need to introduce some notation: let φ be a smooth function such that $\chi_{[0,1]}(t) \leq \varphi(t) \leq \chi_{[0,2]}(t)$, and consider the following smoothed version of the Hardy-Littlewood maximal function

$$M_\varphi f(x) = \sup_{r>0} \frac{1}{r^n} \int_{\mathbb{R}^n} \varphi\left(\frac{|x-y|}{r} \right) |f(y)| \, dy = \sup_{r>0} (\varphi_r * |f|)(y).$$

Note that $Mf(x) \approx M_\varphi f(x)$ and therefore it suffices to show (4.3) for M_φ in place of M. For $f = \{f_j\}_j$, we use the notation

$$\overline{M}_{\varphi,q} f = \|\overline{M}_\varphi f\|_{\ell^q} = \left\| \{M_\varphi f_j\}_j \right\|_{\ell^q} = \left(\sum_j (M_\varphi f_j)^q \right)^{\frac{1}{q}}.$$

Let us recall the definition of the Fefferman-Stein sharp maximal function

$$M^\# f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy \approx \sup_{x \in B} \inf_c \frac{1}{|B|} \int_B |f(y) - c| \, dy,$$

where f_B stands for the average of f over B. Given $0 < \delta < 1$ we also consider $M_\delta^\# g(x) = M^\#(|g|^\delta)(x)^{\frac{1}{\delta}}$.

Inspired by [PTr2] we show the following pointwise estimate whose proof is given below.

Proposition 4.4. *Let* $1 < q < \infty$ *and* $0 < \delta < 1$. *Then there exists a constant* $C > 0$ *such that*

$$M_\delta^\# \left(\overline{M}_{\varphi,q} f \right)(x) \leq C_\delta \, M(\|f\|_{\ell^q})(x), \tag{4.4}$$

for any vector function $f = \{f_j\}_j$ *and for every* $x \in \mathbb{R}^n$.

Assuming for the moment this result, we use the well known C. Fefferman-Stein estimate [FS2] (see also [Duo]):

$$\int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx \leq C \int_{\mathbb{R}^n} M^\# f(x)^p \, w(x) \, dx,$$

for any A_∞-weight w, any p, $0 < p < \infty$ and for any function f such that left hand side is finite. Hence, if $0 < \delta < 1$,

$$\int_{\mathbb{R}^n} \overline{M}_{\varphi,q} f(x)^p \, w(x) \, dx = \int_{\mathbb{R}^n} \left(\overline{M}_{\varphi,q} f(x)^\delta \right)^{\frac{p}{\delta}} w(x) \, dx$$

$$\leq C \int_{\mathbb{R}^n} M_\delta^\# (\overline{M}_{\varphi,q} f)(x)^p \, w(x) \, dx$$

$$\leq \; C \int_{\mathbb{R}^n} M(\|f\|_{\ell^q})(x)^p \, w(x) \, dx.$$

Thus, we can apply Theorem 2.1 with the pairs $\left(\overline{M}_{\varphi,q}f, M(\|f\|_{\ell^q})\right)$ to deduce

$$\left\|\overline{M}_{\varphi,q}f\right\|_{\mathbb{X}(w)} \leq C \left\|M(\|f\|_{\ell^q})\right\|_{\mathbb{X}(w)},$$

for all $w \in A_\infty$, which implies (4.3).

Proof of Proposition 4.4. We adapt the proof contained in [PTr2]. We can assume that each $f_j \geq 0$. Fix $x \in \mathbb{R}^n$ and let B be a ball centered at x of radius r. We split $f = \{f_j\}_j$ as

$$f = f^1 + f^2 = f \, \chi_{5B} + f \, \chi_{\mathbb{R}^n \setminus 5B} = \{f_j \, \chi_{5B}\}_j + \{f_j \, \chi_{\mathbb{R}^n \setminus 5B}\}_j$$

Set

$$c = \|(\overline{M}_\varphi f^2)_B\|_{\ell^q} = \left(\sum_{j=1}^\infty |(M_\varphi f_j^2)_B|^q\right)^{1/q}.$$

Since $0 < \delta < 1$ we have

$$\left(\frac{1}{|B|} \int_B \left|\overline{M}_{\varphi,q}f(y)^\delta - c^\delta\right| dy\right)^{\frac{1}{\delta}} \leq \left(\frac{1}{|B|} \int_B \left\|\overline{M}_\varphi f(y) - (\overline{M}_\varphi f^2)_B\right\|_{\ell^q}^\delta dy\right)^{\frac{1}{\delta}}$$

$$\leq \; C_\delta \left[\left(\frac{1}{|B|} \int_B \left\|\overline{M}_\varphi f^1(y)\right\|_{\ell^q}^\delta dy\right)^{\frac{1}{\delta}} + \left(\frac{1}{|B|} \int_B \left\|\overline{M}_\varphi f^2(y) - (\overline{M}_\varphi f^2)_B\right\|_{\ell^q}^\delta dy\right)^{\frac{1}{\delta}}\right]$$

$$= \; C_\delta \, (I + II).$$

For I, by Kolmogorov's inequality, see [GR, p. 485]

$$I \; \leq \; \frac{C}{|B|} \|\overline{M}_{\varphi,q}f^1\|_{L^{1,\infty}} \leq \frac{C}{|B|} \left\|\left(\sum_j (Mf_j^1)^q\right)^{\frac{1}{q}}\right\|_{L^{1,\infty}} \leq \frac{C}{|B|} \left\|\left(\sum_j |f_j^1|^q\right)^{\frac{1}{q}}\right\|_{L^{1,\infty}}$$

$$= \; C \frac{1}{|5B|} \int_{5B} \|f(y)\|_{\ell^q} \, dy \leq C \, M(\|f\|_{\ell^q})(x), \tag{4.5}$$

where in the third estimate we have used the C. Fefferman-Stein inequality giving the vector-valued weak-type $(1,1)$ for M, see [FS1]. To estimate II we will be using standard techniques from the vector-valued theory of singular integrals (see [RRT] and [GR]). Indeed, by the smoothness of φ we have,

$$\sup_{r>0} |\varphi_r(x - y) - \varphi_r(x)| \leq C \frac{|y|}{|x|^{n+1}} \qquad |x| > 2 \, |y|,$$

and hence for any $y, z \in B$

$$\|\overline{M}_\varphi f^2(y) - \overline{M}_\varphi f^2(z)\|_{\ell^q} \leq \left\|\left\{\sup_{r>0} |\varphi_r * f_j^2(y) - \varphi_r * f_j^2(z)|\right\}_j\right\|_{\ell^q}$$

$$\leq \; \int_{\mathbb{R}^n \setminus 5B} \sup_{r>0} |\varphi_r(y - u) - \varphi_r(z - u)| \, \|f(u)\|_{\ell^q} \, du \leq C \, M(\|f\|_{\ell^q})(x),$$

in the usual way. This estimate yields

$$II \leq C \, M(\|f\|_{\ell^q})(x),$$

which, together with (4.5), completes the proof. $\qquad\square$

5. Proof of the main results: Modular inequalities

In this section we will prove Theorems 3.1 and 3.7.

5.1. **Auxiliary results.** As we mentioned before, there is a characterization of the modular inequalities for the Hardy-Littlewood maximal function (3.1) in terms of the function ϕ. Namely, in [KK] it is shown that for $\phi \in \Phi$, the modular estimate (3.1) holds if and only if ϕ^α is quasi-convex for some $0 < \alpha < 1$. In order to use Rubio de Francia's algorithm we need modular inequalities for the weighted Hardy-Littlewood maximal function.

Proposition 5.1. *Let $w \in A_\infty$ and $\phi \in \Phi$ be such that there exists $0 < \alpha < 1$ for which ϕ^α is a quasi-convex function. Then, there exists some constant a_2, depending on ϕ and w, such that*

$$\int_{\mathbb{R}^n} \phi\big(M_w f(x)\big)\, w(x)\, dx \le a_2 \int_{\mathbb{R}^n} \phi\big(a_2\, |f(x)|\big)\, w(x)\, dx. \tag{5.1}$$

The proof follows the ideas in [KK].

Proof. Since ϕ^α is quasi-convex, there is a convex function ψ such that the corresponding inequality to (3.2) holds. Then, by Jensen's inequality, for any cube Q,

$$\phi^\alpha\Big(\frac{1}{w(Q)} \int_Q |f(y)|\, w(y)\, dy\Big) \le a_1\, \psi\Big(\frac{1}{w(Q)} \int_Q a_1\, |f(y)|\, w(y)\, dy\Big)$$

$$\le\ a_1\, \frac{1}{w(Q)} \int_Q \psi\big(a_1\, |f(y)|\big)\, w(y)\, dy \le a_1\, \frac{1}{w(Q)} \int_Q \phi^\alpha\big(a_1\, |f(y)|\big)\, w(y)\, dy.$$

This yields

$$\phi\big(M_w f(x)\big) = \phi^\alpha\big(M_w f(x)\big)^{\frac{1}{\alpha}} \le a_1^{\frac{1}{\alpha}}\, M_w\big(\phi^\alpha(a_1\, |f|)\big)(x)^{\frac{1}{\alpha}},$$

and therefore

$$\int_{\mathbb{R}^n} \phi\big(M_w f(x)\big)\, w(x)\, dx \le a_1^{\frac{1}{\alpha}} \int_{\mathbb{R}^n} M_w\big(\phi^\alpha(a_1\, |f|)\big)(x)^{\frac{1}{\alpha}}\, w(x)\, dx$$

$$\le\ a_1^{\frac{1}{\alpha}}\, C \int_{\mathbb{R}^n} \phi^\alpha\big(a_1\, |f(x)|\big)^{\frac{1}{\alpha}}\, w(x)\, dx \le a_2 \int_{\mathbb{R}^n} \phi\big(a_2\, |f(x)|\big)\, w(x)\, dx,$$

where we have used that M_w is bounded on $L^{1/\alpha}(w)$ since $0 < \alpha < 1$ and $w \in A_\infty$. $\quad\square$

5.2. **Proof of Corollary 3.5.** We consider a new family of pairs of functions:

$$\mathcal{F}_\phi = \big\{\big(\phi(f), \phi(g)\big) : (f, g) \in \mathcal{F}\big\}.$$

By Theorem 3.1 we have (3.7), and this means that (1.10) holds for the family $\mathcal{F}_\phi$. Then by Theorem 1.2 we have (1.12) which turns out to be (3.11). In the same way applying Theorem 2.1 to $\mathcal{F}_\phi$ we get the corresponding estimates in $\mathbb{X}(w)$. To get (3.10) we define another family

$$\mathcal{F}_r = \Big\{\Big(\Big(\sum_j (f_j)^r\Big)^{\frac{1}{r}}, \Big(\sum_j (g_j)^r\Big)^{\frac{1}{r}}\Big) : \{(f_j, g_j)\}_j \subset \mathcal{F}\Big\},$$

which satisfies (1.12) in Theorem 1.2. Thus we have (3.6) for $\mathcal{F}_r$ and we can apply Theorem 3.1. Note that (3.10) is (3.7) for $\mathcal{F}_r$ and that (3.8) for $\mathcal{F}_r$ provides the corresponding estimates in $\mathbb{X}(w)$.

5.3. **Proof of Theorem 3.1.** First of all, note that it suffices to show (3.7): once this estimate holds, we can apply Theorem 2.1 to the family $\mathcal{F}_\phi$ introduced in the previous proof and we get (3.8). As mentioned in Remark 3.4, the modular weak type estimate (3.9) follows by taking $\mathbb{X} = L^{1,\infty}$. So we will focus on proving (3.7). We will do it in several steps:

Step 1. We first prove the theorem under stronger hypotheses, namely, if ψ is N-function and $\psi \in \Delta_2$ we will see that (3.6) implies

$$\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx, \tag{5.2}$$

for all $w \in A_\infty$ and for $(f,g) \in \mathcal{F}$ such that the left hand-side es finite. Fix an N-function $\psi \in \Delta_2$, $w \in A_\infty$ and let $(f,g) \in \mathcal{F}$ such that both the left-hand side and the right-hand side of (5.2) are finite. As we mentioned before, the fact that $\psi \in \Delta_2$ implies that $\overline{\psi}^\alpha$ is quasi-convex for some $0 < \alpha < 1$. Thus, for $0 < \theta < 1$ and $t \geq 0$ we have

$$\overline{\psi}(\theta\, t) = \overline{\psi}^\alpha(\theta\, t + (1 - \theta)\, 0)^{\frac{1}{\alpha}} \leq a_1^{\frac{1}{\alpha}}\, \theta^{\frac{1}{\alpha}}\, \overline{\psi}(a_1\, t).$$

On the other hand, since $w \in A_\infty$ and $\overline{\psi}^\alpha$ is quasi-convex then we can apply Proposition 5.1 and we have that M_w satisfies (5.1) with $\overline{\psi}$ in place of ϕ. Let $a_0 = \max\{a_1, a_1^{\frac{1}{\alpha}}, a_2\}$, and note that $a_0 \geq 1$, since $a_1 \geq 1$ by convexity. We have the following estimates that will be used later

$$\int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{M_w f(x)}{a_0}\Big)\, w(x)\, dx \;\; \leq \;\; a_0 \int_{\mathbb{R}^n} \overline{\psi}\big(|f(x)|\big)\, w(x)\, dx, \tag{5.3}$$

$$\overline{\psi}(\theta\, t) \;\; \leq \;\; a_0\, \theta^{\frac{1}{\alpha}}\, \overline{\psi}(a_0\, t). \tag{5.4}$$

Let $0 < \theta < 1$ to be chosen later, and define

$$0 \leq h(x) = \frac{\theta\, \psi\big(f(x)\big)}{a_0\, f(x)}$$

whenever $f(x) > 0$ and $h(x) = 0$ otherwise. We consider the following version of Rubio de Francia's algorithm:

$$\mathcal{R}_w h(x) = \frac{2\, a_0 - 1}{2\, a_0} \sum_{k=0}^{\infty} \frac{1}{(2\, a_0)^k}\, \frac{M_w^k h(x)}{a_0^k}.$$

We have the following properties:

(a) $h(x) \leq \frac{2\, a_0}{2\, a_0 - 1}\, \mathcal{R}_w h(x)$.

(b) $\displaystyle\int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx \leq \frac{2\, a_0 - 1}{a_0} \int_{\mathbb{R}^n} \overline{\psi}\big(h(x)\big)\, w(x)\, dx.$

(c) $M_w(\mathcal{R}_w h)(x) \leq 2\, a_0^2\, \mathcal{R}_w h(x)$, and, consequently, $\mathcal{R}_w h \in A_1(w)$ with constant independent of f.

Note that (a) is trivial since M_w^0 is the identity operator. For (b) we first use that $\overline{\psi}$ is convex: set $\theta_k = (2\, a_0 - 1)/(2\, a_0\, 2^k\, a_0^k)$ and then

$$\overline{\psi}\big(\mathcal{R}_w h(x)\big) = \overline{\psi}\Big(\sum_{k=0}^{\infty} \theta_k\, \frac{M_w^k h(x)}{a_0^k}\Big) \leq \sum_{k=0}^{\infty} \theta_k\, \overline{\psi}\Big(\frac{M_w^k h(x)}{a_0^k}\Big)$$

since $\sum_k \theta_k = 1$. Now we iterate (5.3)

$$
\begin{aligned}
\int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx \;\leq\;& \sum_{k=0}^{\infty} \theta_k \int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{M_w^k h(x)}{a_0^k}\Big)\, w(x)\, dx \\
=\;& \sum_{k=0}^{\infty} \theta_k \int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{M_w\big(M_w^{k-1}h/a_0^{k-1}\big)(x)}{a_0}\Big)\, w(x)\, dx \\
\leq\;& \sum_{k=0}^{\infty} \theta_k\, a_0 \int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{M_w^{k-1}h(x)}{a_0^{k-1}}\Big)\, w(x)\, dx \leq \ldots \\
\leq\;& \sum_{k=0}^{\infty} \theta_k\, a_0^k \int_{\mathbb{R}^n} \overline{\psi}\big(h(x)\big)\, w(x)\, dx \\
=\;& \frac{2\, a_0 - 1}{a_0} \int_{\mathbb{R}^n} \overline{\psi}\big(h(x)\big)\, w(x)\, dx.
\end{aligned}
$$

To see (c) we only need to use the sublinearity of M_w:

$$
M_w\big(\mathcal{R}_w h\big)(x) \leq \frac{2\, a_0 - 1}{2\, a_0} \sum_{k=0}^{\infty} \frac{1}{(2\, a_0)^k}\, \frac{M_w^{k+1} f(x)}{a_0^k} \leq 2\, a_0^2\, \mathcal{R}_w h(x).
$$

Once we have shown the properties of $\mathcal{R}_w$ we can continue with the proof. Recall that $0 < \theta < 1$ is a fixed number to be chosen. By (a) we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \;=\;& \frac{a_0}{\theta} \int_{\mathbb{R}^n} f(x)\, h(x)\, w(x)\, dx \\
\leq\;& \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta} \int_{\mathbb{R}^n} f(x)\, \mathcal{R}_w h(x)\, w(x)\, dx.
\end{aligned}
$$

Note that by (c) and by Lemma 4.3 we have that the weight $\mathcal{R}_w h(x)\, w(x) \in A_\infty$. On the other hand, by Theorem 1.2, the hypothesis (3.6) implies (1.11) for every $0 < p < \infty$, and in particular for $p = 1$. Then we have,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \;\leq\;& \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta} \int_{\mathbb{R}^n} f(x)\, \mathcal{R}_w h(x)\, w(x)\, dx \\
\leq\;& \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \int_{\mathbb{R}^n} g(x)\, \mathcal{R}_w h(x)\, w(x)\, dx, \qquad (5.5)
\end{aligned}
$$

provided the middle term is finite. If we use Young's inequality (3.4) we have

$$
\int_{\mathbb{R}^n} f(x)\, \mathcal{R}_w h(x)\, w(x)\, dx \leq \int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx + \int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx.
$$

Note that the first quantity is finite by hypothesis and we only need to work with the second one. By (b) and by (5.4) —since $0 < \theta < 1$— we observe that

$$
\begin{aligned}
\int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx \;\leq\;& \frac{2\, a_0 - 1}{a_0} \int_{\mathbb{R}^n} \overline{\psi}\big(h(x)\big)\, w(x)\, dx \\
=\;& \frac{2\, a_0 - 1}{a_0} \int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{\theta\, \psi\big(f(x)\big)}{a_0\, f(x)}\Big)\, w(x)\, dx \\
\leq\;& \frac{2\, a_0 - 1}{a_0}\, a_0\, \theta^{\frac{1}{\alpha}} \int_{\mathbb{R}^n} \overline{\psi}\Big(\frac{\psi\big(f(x)\big)}{f(x)}\Big)\, w(x)\, dx.
\end{aligned}
$$

On the other hand, (3.5) yields $\overline{\psi}\big(\psi(t)/t\big) \leq \psi(t)$ and therefore

$$\int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx \leq (2\, a_0 - 1)\, \theta^{\frac{1}{\alpha}} \int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx, \qquad (5.6)$$

which is finite. Thus we conclude that the middle term in (5.5) is finite as desired. Then, we continue with this estimate and following the same ideas we have

$$\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \leq \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \int_{\mathbb{R}^n} g(x)\, \mathcal{R}_w h(x)\, w(x)\, dx$$

$$\leq \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \left(\int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx + \int_{\mathbb{R}^n} \overline{\psi}\big(\mathcal{R}_w h(x)\big)\, w(x)\, dx \right)$$

$$\leq \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx + 2\, a_0^2\, (C+1)\, \theta^{(1-\alpha)/\alpha} \int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx.$$

We chose $\theta = \big(4\, a_0^2\, (C+1)\big)^{-\alpha/(1-\alpha)}$ which satisfies $0 < \theta < 1$ since $a_0 > 1$ and $0 < \alpha < 1$. Then, we conclude that

$$\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \leq \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx + \frac{1}{2} \int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx.$$

Note that by hypothesis the last term in the right-hand side is finite, and so we can subtract it. We eventually obtain

$$\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \leq 2\, \frac{2\, a_0^2}{(2\, a_0 - 1)\, \theta}\, C \int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx,$$

which completes **Step 1**.

Step 2. We show that there is $\psi \in \Delta_2$ an N-function such that

$$\psi(t) \leq \phi(t^{2\, r_0})^{s_0} \leq c\, \psi(c\, t). \qquad (5.7)$$

Since $\phi(t^{r_0})^{s_0}$ is quasi-convex, there is a convex function $\varphi(t)$ with

$$\varphi(t) \leq \phi(t^{r_0})^{s_0} \leq c_1\, \varphi(c_1\, t).$$

We take $\psi(t) = \varphi(t^2)$ and we have (5.7). It is clear that $\psi \in \Delta_2$ since $\phi \in \Delta_2$ and so is φ. Besides, since $\varphi(t)$ is convex, then ψ is convex. On the other hand, for $0 < t < 1$, since φ is convex,

$$\frac{\psi(t)}{t} = \frac{\varphi(t^2)}{t} = \frac{\varphi(t^2 \cdot 1 + (1 - t^2) \cdot 0)}{t} \leq \frac{t^2\, \varphi(1)}{t} = \varphi(1)\, t \longrightarrow 0, \qquad \text{as } t \to 0^+.$$

For $t > 1$, we use again the convexity of φ,

$$\varphi(t) = \varphi\Big(\frac{1}{t}\, t^2 + \Big(1 - \frac{1}{t}\Big) \cdot 0\Big) \leq \frac{1}{t}\, \varphi(t^2) = \frac{1}{t}\, \psi(t)$$

and hence

$$\frac{\psi(t)}{t} \geq \varphi(t) \longrightarrow \infty, \qquad \text{as } t \to \infty,$$

since $\phi(\infty) = \infty$. Thus we have seen that ψ is an N-function.

Step 3. We finally show (3.7) in full generality. Using **Step 1** and **Step 2**, for all $w \in A_\infty$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \phi\big(f(x)^{2\,r_0}\big)^{s_0}\, w(x)\, dx \;&\leq\; c \int_{\mathbb{R}^n} \psi\big(c\, f(x)\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \\
&\leq\; C \int_{\mathbb{R}^n} \psi\big(g(x)\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\big(g(x)^{2\,r_0}\big)^{s_0}\, w(x)\, dx,
\end{aligned}
$$

provided de third integral is finite, which is the case since by hypothesis:

$$
\int_{\mathbb{R}^n} \psi\big(f(x)\big)\, w(x)\, dx \leq \int_{\mathbb{R}^n} \phi\big(f(x)^{2\,r_0}\big)^{s_0}\, w(x)\, dx < \infty.
$$

Now we apply Theorem 1.2 to the family of pairs $\big(\phi(f^{2\,r_0}), \phi(g^{2\,r_0})\big)$ and with $p = 1$ to get

$$
\int_{\mathbb{R}^n} \phi\big(f(x)^{2\,r_0}\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\big(g(x)^{2\,r_0}\big)\, w(x)\, dx. \tag{5.8}
$$

Note that we have obtained this inequality staring with (3.6). But, again by Theorem 1.2, we have that the pairs $(f^{1/(2\,r_0)}, g^{1/(2\,r_0)})$ satisfy (3.6) as well. Then, we apply (5.8) to these pairs concluding as desired

$$
\int_{\mathbb{R}^n} \phi\big(f(x)\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\big(g(x)\big)\, w(x)\, dx.
$$

5.4. **Proof of Theorem 3.7.**

5.4.1. *Part* (i). Some of the cases of this part are done in a different way in [KT] and [KK, p. 33]. However, we will follow the ideas already used in the RIQBFS case, see the proof of Theorem 2.3. We start with $i_\phi > 1$ proving that ϕ satisfies the following convexity property.

Lemma 5.2. *If $i_\phi > 1$ then for every $1 < r < i_\phi$ there is some $0 < \alpha < 1$ which depends on r and i_ϕ such that the function $\phi(t^{1/r})^\alpha$ is quasi-convex.*

We will give the proof of this result below. If $i_\phi < \infty$, since $w \in A_{i_\phi}$ there exists $1 < r < i_\phi$ such that $w \in A_r$. When $i_\phi = \infty$ we can also find $1 < r < \infty = i_\phi$ such that $w \in A_r$. So we prove these two cases together. Set $\phi_r(t) = \phi(t^{1/r})$. By (4.2) we have

$$
\phi\big(Mf(x)\big) \leq \phi\Big([w]_{A_r}^{\frac{1}{r}} M_w(|f|^r)(x)^{\frac{1}{r}}\Big) = \phi_r\big(M_w \widetilde{f}(x)\big)
$$

where $\widetilde{f}(x) = [w]_{A_r} |f(x)|^r$. Using Lemma 5.2 there exists $0 < \alpha < 1$ such that $\phi_r(t)^\alpha = \phi(t^{1/r})^\alpha$ is quasi-convex. This allows us to use Proposition 5.1 and therefore

$$
\begin{aligned}
\int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\, w(x)\, dx \;&\leq\; \int_{\mathbb{R}^n} \phi_r\big(M_w \widetilde{f}(x)\big)\, w(x)\, dx \leq a_2 \int_{\mathbb{R}^n} \phi_r\big(a_2\, \widetilde{f}(x)\big)\, w(x)\, dx \\
&=\; C \int_{\mathbb{R}^n} \phi\big(C\, |f(x)|\big)\, w(x)\, dx.
\end{aligned}
$$

Let us do now the case $i_\phi = 1$. For any ball $x \in \mathbb{R}^n$ and any ball $B \ni x$, using that $w \in A_1$ we have

$$
\begin{aligned}
\frac{1}{|B|} \int_B |f(y)|\, dy \;&=\; \frac{1}{w(B)} \int_B |f(y)| \frac{w(B)}{|B|}\, dy \leq [w]_{A_1} \frac{1}{w(B)} \int_B |f(y)|\, w(y) dy \\
&\leq\; [w]_{A_1} M_w f(x).
\end{aligned}
$$

and, therefore, $Mf(x) \leq [w]_{A_1} M_w f(x)$. By hypothesis, ϕ is quasi-convex and, as we did in the proof of Proposition 5.1, we have that

$$\phi\big(Mf(x)\big) \leq \phi\big([w]_{A_1} M_w f(x)\big) \leq C\, M_w\big(\phi(C\,|f|)\big)(x).$$

Hence,

$$\begin{aligned}
\phi(\lambda)\, w\big\{x \in \mathbb{R}^n : Mf(x) > \lambda\big\} &= \phi(\lambda)\, w\big\{x \in \mathbb{R}^n : \phi\big(Mf(x)\big) > \phi(\lambda)\big\} \\
&\leq\ \phi(\lambda)\, w\big\{x \in \mathbb{R}^n : C\, M_w\big(\phi(C\,|f|)\big)(y) > \phi(\lambda)\big\} \\
&\leq\ C \int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\, w(x)\, dx,
\end{aligned}$$

where we have used that M_w is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ since $w \in A_1$ implies that w is a doubling measure.

Proof of Lemma 5.2. We first do the case $i_\phi < \infty$ following the ideas of [KK, p. 40]. Let $\varepsilon > 0$, then there is a constant $C_\varepsilon > 0$ such that

$$\phi(t\,s) \leq C_\epsilon\, t^{i_\phi - \varepsilon}\, \phi(s), \qquad \text{for } 0 \leq t \leq 1 \text{ and } s \geq 0. \tag{5.9}$$

Using this estimate and taking $0 < \varepsilon < i_\phi - r$, for $0 < s_1 < s_2 < \infty$ we can write

$$\phi\big(s_1^{1/r}\big) = \phi\big((s_1/s_2)^{1/r}\, s_2^{1/r}\big) \leq C_\varepsilon\, (s_1/s_2)^{\frac{i_\phi - \varepsilon}{r}}\, \phi\big(s_2^{1/r}\big).$$

Thus, choosing α such that $r/(i_\phi - \varepsilon) < \alpha < 1$ we have

$$\frac{\phi\big(s_1^{1/r}\big)^\alpha}{s_1} \leq C_\varepsilon^\alpha \left(\frac{s_1}{s_2}\right)^{\alpha\,\frac{i_\phi - \varepsilon}{r} - 1} \frac{\phi\big(s_2^{1/r}\big)^\alpha}{s_2} \leq C_\varepsilon^\alpha\, \frac{\phi\big(s_2^{1/r}\big)^\alpha}{s_2}.$$

This implies that $\phi\big(s_1^{1/r}\big)^\alpha$ is quasi-convex by the characterization given in (3.3).

For the case $i_\phi = \infty$, we observe that $i_\phi - \varepsilon$ can be replaced in (5.9) by any number T larger than r. Then we repeat the same computations choosing $r/T < \alpha < 1$. $\square$

5.4.2. *Part (ii).* We just need to use Part (i) and (3.12) or (3.13). Note that ϕ satisfies the required hypotheses since $\phi \in \Delta_2$ and ϕ is itself quasi-convex.

5.4.3. *Part (iii).* First of all, note that one only needs to prove the vector-valued estimates for M as we have just done in Part (ii). To get them, we observe that in the proof of Part (iii) in Theorem 2.3 we showed that

$$\left\|\left(\sum_j (Mf_j)^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \leq C\,\big\|M(\|f\|_{\ell^q})\big\|_{L^p(w)},$$

for any $f = \{f_j\}$ and for all $w \in A_\infty$. Therefore, as we know that ϕ is quasi-convex and satisfies the Δ_2 condition, we can apply Theorem 3.1 to this inequality and we get

$$\int_{\mathbb{R}^n} \phi\left(\left(\sum_j Mf_j(x)^q\right)^{\frac{1}{q}}\right) w(x)\, dx\ \leq\ C \int_{\mathbb{R}^n} \phi\big(M(\|f\|_{\ell^q})(x)\big)\, w(x)\, dx,$$

$$\sup_\lambda \phi(\lambda)\, w\left\{x : \left(\sum_j Mf_j(x)^q\right)^{\frac{1}{q}} > \lambda\right\}\ \leq\ C \sup_\lambda \phi(\lambda)\, w\big\{x : M(\|f\|_{\ell^q})(x) > \lambda\big\},$$

for any $w \in A_\infty$. The proof will be completed by using the weighted modular inequalities for M obtained in (i).

6. Applications

In this section we give a number of applications to show how our extrapolation results can be used to derive estimates in RIQBFS and also of modular type.

6.1. Commutators with Calderón-Zygmund operators.

In this section we apply our results to derive endpoint estimates for commutators of Calderón-Zygmund operators with BMO functions. Let T be any Calderón-Zygmund operator with standard kernel and let $b \in \mathrm{BMO}$, define the commutators C_b^m by setting

$$C_b^1 f(x) = [b, T]f(x) = b(x)\, Tf(x) - T(b\, f)(x),$$

and for $m \geq 2$, $C_b^m f(x) = [b, C_{m-1}]f(x)$, or

$$C_b^m f(x) = \int_{\mathbb{R}^n} \big(b(x) - b(y)\big)^m K(x, y)\, f(y)\, dy.$$

The maximal operator that controls the commutator C_b^m is M^{m+1} which is the Hardy-Littlewood maximal function iterated $m + 1$-times. Namely, in [Pe3] it is shown that

$$\int_{\mathbb{R}^n} |C_b^m f(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} M^{m+1} f(x)^p\, w(x)\, dx \tag{6.1}$$

for every $0 < p < \infty$ and $w \in A_\infty$. As mentioned in the introduction, the extrapolation results developed in [CMP] are not suitable to deal with the endpoint estimates associated with this operators. Using the results that we have obtained in the present paper we can derive endpoints estimates on RIQBFS and also of modular type.

6.1.1. *Endpoint estimates: Marcinkiewicz spaces.* We would like to use Theorem 2.1 to get endpoint estimates for the commutators in some appropriate RIQBFS. Namely, we will work with the Marcinkiewicz spaces introduced in Section 2.2. We take the function

$$\varphi_m(t) = \frac{t}{(1 + \log^+ t)^m}$$

which is increasing, quasi-concave and satisfies that $\varphi_m(0) = 0$. Let us recall the definition of the Marcinkiewicz type spaces $\mathbb{M}_{\varphi_m}$ and $\widetilde{\mathbb{M}}_{\varphi_m}$ which are given by the function norm or quasi-norm

$$\|f\|_{\mathbb{M}_{\varphi_m}} = \sup_t \frac{\varphi_m(t)}{t} \int_0^t f^*(s)\, ds, \qquad \|f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} = \sup_t \varphi_m(t)\, f^*(t).$$

As mentioned, $\mathbb{M}_{\varphi_m}$ is a Banach space. However, φ_m does not satisfy (2.13). Therefore $\widetilde{\mathbb{M}}_{\varphi_m}$ is a RIQBFS which does not coincide with $\mathbb{M}_{\varphi_m}$, and we have $\mathbb{M}_{\varphi_m} \subsetneq \widetilde{\mathbb{M}}_{\varphi_m}$.

As explained in Section 2.2 when we treated the Marcinkiewicz spaces, setting $\mathbb{X} = \widetilde{\mathbb{M}}_{\varphi_m}$ we have:

- $\mathbb{X}^r$ is a Banach space for any $1 < r < \infty$, since $\mathbb{X}^r = \widetilde{\mathbb{M}}_{(\varphi_m)^{1/r}} = \mathbb{M}_{(\varphi_m)^{1/r}}$.

- $q_{\mathbb{X}} = 1$, namely, the submultiplicativity of the function $(1 + \log^+ s)$ yields

$$h_{\varphi_m}(t) = \sup_{s>0} \frac{\varphi_m(s\, t)}{\varphi_m(s)} = t \left(1 + \log^+ \frac{1}{t}\right)^m.$$

This and (2.15) provides

$$q_{\mathbb{X}} = \frac{1}{i_{\varphi_m}} = \lim_{t \to 0^+} \frac{\log t}{\log h_{\varphi_m}(t)} = 1.$$

Remark 6.1. We would like to point out that in this case one can easily show that $p_{\mathbb{X}} = q_{\mathbb{X}} = 1$. Indeed, the fact that $\mathbb{X}^r$ is a Banach space for every $r > 1$ implies that $1 \leq p_{\mathbb{X}^r} = p_{\mathbb{X}} \cdot r$. Therefore, $1 \leq p_{\mathbb{X}} \leq q_{\mathbb{X}} = 1$.

Using Theorem 2.1 we prove the following estimate for the commutators C_b^m.

Theorem 6.2. *Let* $\varphi_m(t) = \dfrac{t}{(1 + \log^+ t)^m}$. *Then*

$$C_b^m : L(\log L)^m \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}$$

Proof. Observe that we can use Theorem 2.1 with $\mathbb{X} = \widetilde{\mathbb{M}}_{\varphi_m}$ and with the pairs $(|C_b^m f|, M^{m+1} f)$, which satisfy (6.1). Then,

$$\|C_b^m f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} \leq C \, \|M^{m+1} f\|_{\widetilde{\mathbb{M}}_{\varphi_m}}$$

where we have taken the weight $w(x) \equiv 1$. Therefore, it suffices to show that M^{m+1} maps $L(\log L)^m$ into $\widetilde{\mathbb{M}}_{\varphi_m}$. We recall that

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) \, ds.$$

which implies

$$(M^2 f)^*(t) \approx \frac{1}{t} \int_0^t (Mf)^*(s) \, ds \approx \frac{1}{t} \int_0^t \frac{1}{s} \int_0^s f^*(u) \, du \, ds = \frac{1}{t} \int_0^t f^*(s) \log \frac{t}{s} \, ds.$$

By iterating we get

$$(M^{m+1} f)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) \left(\log \frac{t}{s} \right)^m ds. \tag{6.2}$$

The submultiplicativity of the function $(1 + \log^+ s)$ yields

$$\begin{aligned}
(M^{m+1} f)^*(t) &\leq \frac{C}{t} \int_0^\infty f^*(s) \left(1 + \log^+ \frac{t}{s} \right)^m ds \\
&\leq C \frac{(1 + \log^+ t)^m}{t} \int_0^\infty f^*(s) \left(1 + \log^+ \frac{1}{s} \right)^m ds,
\end{aligned}$$

and hence

$$\|M^{m+1} f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} \leq C \int_0^\infty f^*(s) \left(1 + \log^+ \frac{1}{s} \right)^m ds.$$

This last expression defines a RIBFS which coincides with $L(\log L)^m$ (this can be seen repeating the computations in [BS, p. 244]). Therefore, their norms are equivalent, see [BS, p. 7]. □

Remark 6.3. As mentioned before, we have that $\mathbb{M}_{\varphi_m} \subsetneq \widetilde{\mathbb{M}}_{\varphi_m}$ and it is natural to wonder whether the target space can be replaced by $\mathbb{M}_{\varphi_m}$. We can easily show that M^{m+1} does not map $L(\log L)^m$ into $\mathbb{M}_{\varphi_m}$. To see it, we first observe that

$$\|f\|_{\mathbb{M}_{\varphi_m}} = \sup_{t > 0} \frac{\varphi_m(t)}{t} \int_0^t f^*(s) \, ds \approx \sup_{t > 0} \varphi_m(t) \, (Mf)^*(t) = \|Mf\|_{\widetilde{\mathbb{M}}_{\varphi_m}}.$$

In $\mathbb{R}$, taking $f(t) = \chi_{[0,1]}(t) \in L(\log L)^m$, by (6.2), we have

$$\|M^{m+1} f\|_{\mathbb{M}_{\varphi_m}} = \|M^{m+2} f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} = \sup_t \varphi_m(t) \, (M^{m+2} f)^*(t)$$

$$\approx \ \sup_{t} \frac{\varphi_m(t)}{t} \int_0^t f^*(s) \left(\log \frac{t}{s} \right)^{m+1} ds$$

$$= \ \sup_{t} \varphi_m(t) \int_0^{\min\{1,1/t\}} \left(\log \frac{1}{s} \right)^{m+1} ds$$

$$\geq \ \sup_{t \geq 1} \varphi_m(t) \frac{1}{t} (\log t)^{m+1} = \infty.$$

Remark 6.4. The same ideas can be applied to obtain the following weighted estimate

$$C_b^m : L(\log L)^m(w) \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}(w),$$

for every $w \in A_1$. As before, it suffices to show that M^{m+1} is bounded operator between these spaces. We repeat the computations of the unweighted case using (4.1) and that $w \in A_1$ implies that $Mf(x) \leq C\, M_w f(x)$ for $x \in \mathbb{R}^n$:

$$\|M^{m+1}f\|_{\widetilde{\mathbb{M}}_{\varphi_m}(w)} \ \leq \ C\, \|M_w^{m+1}f\|_{\widetilde{\mathbb{M}}_{\varphi_m}(w)} = C\, \sup_{t} \varphi_m(t)\, (M_w^{m+1}f)_w^*(t)$$

$$\leq \ C\, \sup_{t} \varphi_m(t) \frac{1}{t} \int_0^t f_w^*(s) \left(\log \frac{t}{s} \right)^m ds$$

$$\leq \ C\, \sup_{t} \varphi_m(t) \frac{(1 + \log^+ t)^m}{t} \int_0^\infty f_w^*(s) \left(1 + \log^+ \frac{1}{s} \right)^m ds$$

$$= \ C \int_0^\infty f_w^*(s) \left(1 + \log^+ \frac{1}{s} \right)^m ds \leq C\, \|f\|_{L\,(\log L)^m(w)}.$$

6.1.2. *Endpoint estimates: Modular inequalities.* Estimate (6.1) says that M^{m+1} controls the m-order commutator. The right endpoint modular inequality for M^{m+1} is given by

$$\left| \{ x \in \mathbb{R}^n : M^{m+1}f(x) > \lambda \} \right| \leq C \int_{\mathbb{R}^n} \psi_m \left(\frac{|f(x)|}{\lambda} \right) dx, \qquad (6.3)$$

where $\psi_m(t) = t\,(1 + \log^+ t)^m$. Thus, once can expect that the same estimate holds for C_b^m. This is indeed the case as we can find in [Pe1]. Our goal is to derive it by extrapolation as a consequence of (6.1).

We first outline the way to prove (6.3). We introduce the maximal operator associated to an Orlicz space. Given a Young function ψ, as done in Section 2.2, we can consider the Orlicz space L^ψ. For every cube Q, a localized and averaged version of the norm $\| \cdot \|_{L^\psi}$ is given by

$$\|f\|_{\psi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \psi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}. \qquad (6.4)$$

Associated with ψ, we define the maximal operator

$$M_\psi f(x) = \sup_{Q \ni x} \|f\|_{\psi,Q}.$$

For instance, if $\psi(t) = t^r$ then $L^\psi = L^r$ and the maximal operator associated with this space is $M_\psi f(x) = M_r f(x) = M(|f|^r)(x)^{1/r}$. We remit to [Pe2] for more information about these operators.

Using standard arguments, namely a Vitali covering lemma, one can show the following endpoint modular estimate for M_ψ

$$\left|\{x \in \mathbb{R}^n : M_\psi f(x) > \lambda\}\right| \le C \int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) dx. \tag{6.5}$$

The same argument can be modified in order to introduce weights: if $w \in A_1$ we have

$$w\{x \in \mathbb{R}^n : M_\psi f(x) > \lambda\} \le C \int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) w(x)\, dx. \tag{6.6}$$

The Young function $\psi_m(t) = t\,(1 + \log^+ t)^m$ defines the Orlicz space $L\,(\log L)^m$. On the other hand, as we are going to see next $M_{\psi_m} f(x) \approx M^{m+1} f(x)$. Hence, (6.5) implies (6.3) and we also have the corresponding weighted version for $w \in A_1$.

We show that $M_{L\,(\log L)^m} f(x) \approx M^{m+1} f(x)$. This estimate was already obtained in [Pe1]. However, we can get it with no much effort using what we obtained before for $(M^{m+1} f)^*$ and taking some ideas from [LN]. Given a cube Q, we consider the function $g_Q(x) = (f \cdot \chi_Q)(x\,\ell(Q) + x_Q)$ where x_Q denotes the center of Q. Note that $\operatorname{supp} g_Q \subset Q_0 = [-\frac{1}{2}, \frac{1}{2}]^n$. By the translation and dilation properties of the Lebesgue measure we have that $(f \cdot \chi_Q)^*(s\,|Q|) = g_Q^*(s)$ since

$$\left|\{x : |g_Q(x)| > \lambda\}\right| = \frac{1}{|Q|}\left|\{x : |(f \cdot \chi_Q)(x)| > \lambda\}\right|.$$

Thus, as in [BS, p. 244], we conclude that

$$\begin{aligned}
\|f\|_{L\,(\log L)^m, Q} &= \|g_Q\|_{L\,(\log L)^m, Q_0} \approx \int_0^1 (g_Q)^*(s) \left(\log \frac{1}{s}\right)^m ds \\
&= \int_0^1 (f \cdot \chi_Q)^*(s\,|Q|) \left(\log \frac{1}{s}\right)^m ds,
\end{aligned} \tag{6.7}$$

where the constants do not depend on Q. This inequality and (6.2) give

$$\frac{1}{|Q|} \int_Q M^m(f \cdot \chi_Q)(x)\, dx \le \frac{1}{|Q|} \int_0^{|Q|} \left(M^m(f \cdot \chi_Q)\right)^*(t)\, dt \approx \left(M^{m+1}(f \cdot \chi_Q)\right)^*(|Q|)$$

$$\approx \frac{1}{|Q|} \int_0^{|Q|} (f \cdot \chi_Q)^*(s) \left(\log \frac{|Q|}{s}\right)^m ds \approx \|f\|_{L\,(\log L)^m, Q}.$$

To show that $M^{m+1} f(x) \le C\, M_{L\,(\log L)^m} f(x)$, we proceed by induction. If $m = 1$,

$$\begin{aligned}
\frac{1}{|Q|} \int_Q M f(y)\, dy &\le \frac{3^n}{|3Q|} \int_{3Q} M(f \cdot \chi_{3Q})(y)\, dy + \frac{1}{|Q|} \int_Q M(f \cdot \chi_{\mathbb{R}^n \setminus 3Q})(y)\, dy \\
&\le C\,\|f\|_{L\,(\log L), 3Q} + C\, M f(x) \le C\, M_{L\,(\log L)} f(x).
\end{aligned}$$

where we have used that $M(f \cdot \chi_{\mathbb{R}^n \setminus 3Q})(y) \approx M(f \cdot \chi_{\mathbb{R}^n \setminus 3Q})(z)$ for $y, z \in Q$ (see [GR, p. 159]). Taking the supremum over all the cubes we get $M^2 f(x) \le C\, M_{L\,(\log L)} f(x)$.

Now suppose that the case $m - 1$ is proved and we show the estimate for m: given $Q \ni x$ we observe that

$$\begin{aligned}
\frac{1}{|Q|} \int_Q M^m f(y)\, dy &\le \frac{3^n}{|3Q|} \int_{3Q} M^m(f \cdot \chi_{3Q})(y)\, dy + \frac{1}{|Q|} \int_Q M^m(f \cdot \chi_{\mathbb{R}^n \setminus 3Q})(y)\, dy \\
&\le C\,\|f\|_{L\,(\log L)^m, 3Q} + \frac{1}{|Q|} \int_Q M_{L\,(\log L)^{m-1}}(f \cdot \chi_{\mathbb{R}^n \setminus 3Q})(y)\, dy
\end{aligned}$$

$$\leq \quad C\,M_{L\,(\log L)^m}f(x) + C\,M_{L\,(\log L)^{m-1}}f(x) \leq C\,M_{L\,(\log L)^m}f(x)$$

where we have used the estimate

$$M_{L\,(\log L)^{m-1}}(f\cdot\chi_{\mathbb{R}^n\setminus 3Q})(y) \approx M_{L\,(\log L)^{m-1}}(f\cdot\chi_{\mathbb{R}^n\setminus 3Q})(z), \qquad y, z \in Q,$$

which can be proved as for the Hardy-Littlewood maximal function, see [GR, p. 159]. Taking the supremum over all cubes $Q \ni x$ we conclude as desired $M^{m+1}f(x) \leq C\,M_{L\,(\log L)^m}f(x)$.

For the converse estimate, we show that for every $m \geq 1$ and for every cube Q

$$\|f\|_{L\,(\log L)^m,Q} \leq C\,\|Mf\|_{L\,(\log L)^{m-1},Q} \tag{6.8}$$

where C is independent of Q and f, and for $m = 1$ we write $L\,(\log L)^{m-1} = L^1$. Iterating this estimate and taking the supremum on $Q \ni x$, we conclude $M_{L\,(\log L)^m}f(x) \leq C\,M^{m+1}f(x)$, as desired.

Let us show (6.8). We fix $f \geq 0$ and set $\lambda(t) = (M_Q f)^*(t\,|Q|)$, where M_Q is the Hardy-Littlewood maximal function localized to Q and with the supremum restricted to those cubes contained in Q. Write $f_0 = f\,\chi_{Q\cap\{f>\lambda(t)\}}$ and $f_1 = f_0 - f\,\chi_Q$. We observe that $\|f_1\|_{L^\infty} \leq \lambda(t)$ and also that the Calderón-Zygmund decomposition yields as in [GR, p. 145]

$$\|f_0\|_{L^1} = \int_{Q\cap\{f>\lambda(t)\}} f(x)\,dx \leq C\,\lambda(t)\,\big|\{x \in Q : M_Q f(x) > \lambda(t)\}\big| \leq C\,\lambda(t)\,t\,|Q|.$$

Using the linearity of Hardy's operator we have

$$\frac{1}{t}\int_0^t (f\cdot\chi_Q)^*(s\,|Q|)\,ds \leq \frac{\|f_0\|_{L^1}}{t\,|Q|} + \|f_1\|_\infty \leq C\,\lambda(t) \leq C\,(Mf\cdot\chi_Q)^*(t\,|Q|).$$

Thus, by (6.7) we conclude as desired

$$
\begin{aligned}
\|f\|_{L\,(\log L)^m,Q} &\approx \int_0^1 (f\cdot\chi_Q)^*(s\,|Q|)\left(\log\frac{1}{s}\right)^m ds \\
&= m\int_0^1 \left[\frac{1}{t}\int_0^t (f\cdot\chi_Q)^*(s\,|Q|)\,ds\right]\left(\log\frac{1}{t}\right)^{m-1} dt \\
&\leq C\int_0^1 (Mf\cdot\chi_Q)^*(t\,|Q|)\left(\log\frac{1}{t}\right)^{m-1} dt \\
&\approx \|Mf\|_{L\,(\log L)^{m-1},Q}.
\end{aligned}
$$

Once we have obtained that $M_{\psi_m}f(x) \approx M^{m+1}f(x)$, it follows (6.3) by (6.5). Let us prove the same estimate for the commutator C_b^m. Take

$$\phi_m(t) = \frac{1}{\psi_m(\frac{1}{t})} = \frac{t}{(1 + \log^+\frac{1}{t})^m}.$$

Note that $\phi_m \in \Delta_2$ and $\phi(t^r)$ is quasi-convex for r large enough. Therefore, we can apply Theorem 3.1 with the pairs $(|C_b^m f|, M^{m+1}f)$ for $f \in C_0^\infty$, which satisfy the starting estimate (6.1). Thus, equation (3.9) with $w(x) \equiv 1$ implies

$$
\begin{aligned}
\big|\{x \in \mathbb{R}^n : |C_b^m f(x)| > 1\}\big| &\leq \sup_\lambda \phi_m(\lambda)\,\big|\{x \in \mathbb{R}^n : |C_b^m f(x)| > \lambda\}\big| \\
&\leq C\sup_\lambda \phi_m(\lambda)\,\big|\{x \in \mathbb{R}^n : M^{m+1}f(x) > \lambda\}\big|. \quad (6.9)
\end{aligned}
$$

On the other hand, note that by (6.3) and since ψ_m is submultiplicative we have

$$\phi_m(\lambda)\left|\left\{x \in \mathbb{R}^n : M^{m+1}f(x) > \lambda\right\}\right| \le C\,\phi_m(\lambda) \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right)\,dx$$

$$\le\ C\,\phi_m(\lambda)\,\psi_m\left(\frac{1}{\lambda}\right) \int_{\mathbb{R}^n} \psi_m\big(|f(x)|\big)\,dx = C \int_{\mathbb{R}^n} \psi_m\big(|f(x)|\big)\,dx.$$

This inequality and (6.9) yields, after using the homogeneity of the estimate,

$$\left|\left\{x \in \mathbb{R}^n : |C_b^m f(x)| > \lambda\right\}\right| \le C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right)\,dx.$$

Furthermore, the same argument can be repeated with $w \in A_1$, since we have (6.6). Thus, for any $w \in A_1$ we obtain

$$w\big\{x \in \mathbb{R}^n : |C_b^m f(x)| > \lambda\big\} \le C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right)\,w(x)\,dx.$$

6.2. **Multilinear commutators.** In this section we are going to consider the following operator

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^{m}\big(b_j(x) - b_j(y)\big)\right] K(x,y)\,f(y)\,dy,$$

where K is any Calderón-Zygmund kernel and the vector "symbol" $\vec{b} = (b_1, \ldots, b_m)$ is formed by locally integrable functions. Note that for $b_1 = \cdots = b_m = b$ we have $T_{\vec{b}} = C_b^m$. These operators have been considered in [PTr1] and can be seen as multilinear extensions of the commutators of Coifman-Rochberg-Weiss [CRW] defined before. We define the following version of the mean oscillation of a function b by the expression

$$\|b\|_{\mathrm{osc}_{\exp(L^r)}} = \sup_{Q} \|b - b_Q\|_{\exp L^r, Q}$$

where the supremum is taken over all the cubes Q and $b_Q = |Q|^{-1} \int_Q b(x)\,dx$. Note that in the latter expression the Orlicz norm is given by the Young function $\psi(t) = \exp(t^r) - 1$. When $r = 1$ this corresponds to the BMO space of John-Nirenberg. We use the notation

$$\|\vec{b}\| = \prod_{j=1}^{m} \|b_j\|_{\mathrm{osc}_{\exp(L^{r_j})}}$$

and

$$\psi_r(t) = t\,(1 + \log^+ t)^{1/r}; \qquad \frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}, \qquad r_1, \ldots, r_m \ge 1.$$

In order to apply Theorem 2.1 to these operators we will be using as starting point the following result from [PTr1]: for all $w \in A_\infty$ and $0 < p < \infty$ we have

$$\int_{\mathbb{R}^n} |T_{\vec{b}}f(x)|^p\,w(x)\,dx \le C\,\|\vec{b}\|^p \int_{\mathbb{R}^n} M_{\psi_r}f(x)^p\,w(x)dx,$$

for any function smooth f such that the left hand side is finite. This is the initial extrapolation hypothesis from which we can derive the estimates on RIQBFS and also modular inequalities for $T_{\vec{b}}f$.

Theorem 6.5. *Let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \le 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for any $w \in A_\infty$, $T_{\vec{b}}$ satisfies*

$$\|T_{\vec{b}}f\|_{\mathbb{X}(w)} \le C \, \|\vec{b}\| \, \|M_{\psi_r}f\|_{\mathbb{X}(w)},$$

and also

$$\left\| \left(\sum_j |T_{\vec{b}}f_j|^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \le C \, \|\vec{b}\| \, \left\| \left(\sum_j (M_{\psi_r}f_j)^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)},$$

for all $0 < q < \infty$. As a consequence, if it is also assumed that $p_{\mathbb{X}} > 1$, then $T_{\vec{b}}$ is bounded on $\mathbb{X}(w)$ for every $w \in A_{p_{\mathbb{X}}}$ and it satisfies the following weighted vector-valued inequality

$$\left\| \left(\sum_j |T_{\vec{b}}f_j|^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \le C \, \|\vec{b}\| \, \left\| \left(\sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)},$$

for all $1 < q < \infty$ and all $w \in A_{p_{\mathbb{X}}}$. In particular, $T_{\vec{b}}$ is bounded on $\mathbb{X}$ and satisfies the corresponding unweighted vector-valued inequalities on $\mathbb{X}$.

Similarly we can obtain modular inequalities:

Theorem 6.6. *If $\phi \in \Phi$ satisfies (i) and (ii) in Theorem 3.1, then*

$$\int_{\mathbb{R}^n} \phi\big(|T_{\vec{b}}f(x)|\big) \, w(x) \, dx \le C \int_{\mathbb{R}^n} \phi\big(M_{\psi_r}f(x)\big) \, w(x) \, dx,$$

and

$$\sup_{\lambda > 0} \phi(\lambda) \, w\big\{y \in \mathbb{R}^n : |T_{\vec{b}}f(y)| > \lambda\big\} \le C \sup_{\lambda > 0} \phi(\lambda) \, w\big\{y \in \mathbb{R}^n : M_{\psi_r}f(y) > \lambda\big\}. \quad (6.10)$$

As a consequence, if it is also assumed that $i_\phi > 1$ we obtain

$$\int_{\mathbb{R}^n} \phi\big(|T_{\vec{b}}f(x)|\big) \, w(x) \, dx \le C \int_{\mathbb{R}^n} \phi\big(|f(x)|\big) \, w(x) \, dx, \quad (6.11)$$

for all $w \in A_{i_\phi}$. On the other hand, it also follows that

$$w\big\{y \in \mathbb{R}^n : |T_{\vec{b}}f(y)| > \lambda\big\} \le C \int_{\mathbb{R}^n} \psi_r\left(\frac{|f(x)|}{\lambda} \right) \, w(x) \, dx. \quad (6.12)$$

for all $w \in A_1$ and in particular for $w(x) \equiv 1$.

In Theorem 6.5, to obtain that $T_{\vec{b}}$ is bounded on $\mathbb{X}(w)$, we observe that since $r_i \ge 1$ for each i, it follows that M_{ψ_r} is pointwise smaller than $M_{L\,(\log L)^m}$ which, as mentioned in the previous section, is equivalent to M^{m+1}. Note that by Theorem 2.3, M^{m+1} is bounded on $\mathbb{X}(w)$ provided $p_{\mathbb{X}} > 1$ and $w \in A_{p_{\mathbb{X}}}$. The last vector-valued inequality in Theorem 6.5 follows in the same way.

In Theorem 6.6, the modular estimate (6.11) follows by using Theorem 3.7, since, as before, $M_{\psi_r}f(x) \le C \, M^{m+1}f(x)$. On the other hand, (6.12) can be proved using (6.10) with the function $\phi(t) = \psi_r(t^{-1})^{-1}$ and following the ideas given in Section 6.1.2.

As in the previous section we derive an endpoint estimate within the context of Marcinkiewicz spaces:

Theorem 6.7. *Let $T_{\vec{b}}$ be the multilinear commutator as above with symbol $\vec{b}$ and $\varphi_r(t) = \dfrac{t}{(1 + \log^+ t)^{\frac{1}{r}}}$. Then T maps $L(\log L)^{\frac{1}{r}}$ to $\widetilde{\mathbb{M}}_{\varphi_r}$.*

Proof. We set $\mathbb{X} = \widetilde{\mathbb{M}}_{\varphi_r}$ which, as in the previous section, satisfies the hypotheses of Theorem 6.5. Thus, it suffices to show that M_{ψ_r} maps $L(\log L)^{\frac{1}{r}}$ to $\widetilde{\mathbb{M}}_{\varphi_r}$. Let $f \in L(\log L)^{\frac{1}{r}}$ which can be taken with $\|f\|_{L(\log L)^{\frac{1}{r}}} = 1$ and so

$$\int_{\mathbb{R}^n} \psi_r\big(|f(x)|\big)\, dx = \int_0^\infty \psi_r\big(f^*(t)\big)\, dt = 1. \qquad (6.13)$$

In this way we have to show that $\|M_{\psi_r} f\|_{\widetilde{\mathbb{M}}_{\varphi_r}} \leq C$. We define the following generalized Hardy type operator

$$f_{\psi_r}^{**}(t) = \|f^*\|_{\psi_r, [0,t]} = \inf\left\{\lambda > 0 : \frac{1}{t}\int_0^t \psi_r\left(\frac{f^*(s)}{\lambda}\right) ds \leq 1\right\}, \qquad t > 0.$$

Taking some ideas from [BP] we next show that

$$\big(M_{\psi_r} f\big)^*(t) \leq C\, f_{\psi_r}^{**}(t). \qquad (6.14)$$

Using a Vitali covering lemma one obtains

$$|E_\lambda| = |\{x \in \mathbb{R}^n : M_{\psi_r} f(x) > \lambda\}| \leq C_0 \int_{E_\lambda} \psi_r\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Then, by the convexity of ψ_r we have that for $\widetilde{E}_\lambda = E_{2\,C_0\,\lambda}$

$$|\widetilde{E}_\lambda| \leq C_0 \int_{\widetilde{E}_\lambda} \psi_r\left(\frac{|f(x)|}{2\,C_0\,\lambda}\right) dx \leq \frac{1}{2}\int_{\widetilde{E}_\lambda} \psi_r\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Let $\lambda > f_{\psi_r}^{**}(t)$, and assume that $|\widetilde{E}_\lambda| > t$. Then

$$\frac{1}{t}\int_0^t \psi\left(\frac{f^*(s)}{\lambda}\right) ds \leq 1 \leq \frac{1}{2\,|\widetilde{E}_\lambda|}\int_{\widetilde{E}_\lambda} \psi\left(\frac{f(x)}{\lambda}\right) dx \leq \frac{1}{2\,|\widetilde{E}_\lambda|}\int_0^{|\widetilde{E}_\lambda|} \psi\left(\frac{f^*(s)}{\lambda}\right) ds$$

$$\leq \frac{1}{2\,|\widetilde{E}_\lambda|}\int_0^{|\widetilde{E}_\lambda|} \psi_r\left(\frac{f^*(s\,t/|\widetilde{E}_\lambda|)}{\lambda}\right) ds = \frac{1}{2\,t}\int_0^t \psi_r\left(\frac{f^*(s)}{\lambda}\right) ds,$$

which yields a contradiction. Thus, $|\widetilde{E}_\lambda| \leq t$ and $\big(M_\psi f\big)^*(t) \leq 2\,C_0\,\lambda$. Since this holds for any $\lambda > f_{\psi_r}^{**}(t)$, we obtain (6.14) as desired.

Using (6.14) we have

$$\|M_{\psi_r} f\|_{\widetilde{\mathbb{M}}_{\varphi_r}} = \sup_{t>0} \varphi_r(t)\big(M_{\psi_r} f\big)^*(t) \leq C \sup_{t>0} \varphi_r(t) f_{\psi_r}^{**}(t)$$

Besides, by the submultiplicativity of ψ_r and using (6.13)

$$\frac{1}{t}\int_0^t \psi_r\big(\varphi_r(t)\, f^*(s)\big)\, ds \leq \frac{\psi_r \circ \varphi_r(t)}{t}\int_0^t \psi_r\big(f^*(s)\big)\, ds \leq \int_0^\infty \psi_r\big(f^*(s)\big)\, ds \leq 1,$$

since $\psi_r \circ \varphi_r(t) \leq t$ —indeed, the inverse of ψ_r is (essentially) φ_r—. This shows that $f_{\psi_r}^{**}(t) \leq 1/\varphi_r(t)$, which implies that $\|M_{\psi_r} f\|_{\widetilde{\mathbb{M}}_{\varphi_r}} \leq C$. $\qquad \square$

6.3. **Fractional integrals and commutators.** We define the potential operators by

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) \, dy,$$

where the kernel satisfies the following size estimate

$$|k(x,y)| \le C \, |x-y|^{\alpha-n} \tag{6.15}$$

where $0 < \alpha < n$. Note that $|Kf(x)| \le C \, I_\alpha(|f|)(x)$ where I_α is the classical fractional integral of order α. We also define the closely related fractional maximal operator by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| \, dy.$$

The following inequality holds: for every $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Kf(x)|^p \, w(x) \, dx \le C \int_{\mathbb{R}^n} (M_\alpha f(x))^p \, w(x) \, dx. \tag{6.16}$$

In the classical situation $K = I_\alpha$, this inequality is due to Muckenhoupt and Wheeden [MW]. Their proof is based on a good-λ inequality relating I_α and M_α and the fact that I_α is of the weak type $(1, \frac{n}{n-\alpha})$ plays a key role. However, it is pointed out in [CMP] that it is possible to avoid such good-λ inequality using ideas from [Pe3]. The key is to get (6.16) with $p = 1$ and then to extrapolate to recover the full range of exponents $0 < p < \infty$. The fact that $p = 1$ is basic to get such an inequality, since the method relies in some discretization of I_α. We would like to point out that no boundedness of the operator is used to derive (6.16) by that discretization technique. Combining (6.16) with our extrapolation result we get estimates in RIQBFS.

Theorem 6.8. *Let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \le 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for any $w \in A_\infty$, we have*

$$\|Kf\|_{\mathbb{X}(w)} \le C \, \|M_\alpha f\|_{\mathbb{X}(w)},$$

as well as the corresponding vector-valued inequalities.

We can also get estimates for commutators of fractional integrals. Let K be an operator as above with kernel k satisfying the size condition (6.15) and let be b any measurable function. We define the commutator

$$[b,K]f(x) = b(x) \, Kf(x) - K(b\,f)(x) = \int_{\mathbb{R}^n} k(x,y)(b(x) - b(y)) \, f(y) \, dy.$$

These commutators are intimately related to following fractional Orlicz maximal operator defined similarly as above: given a Young function ψ, let

$$M_{\psi,\alpha} f(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \, \|f\|_{\psi,Q},$$

where $\| \cdot \|_{\psi,Q}$ is defined in (6.4). Let $0 < \alpha < n$, $b \in \mathrm{BMO}$, $w \in A_\infty$ and $\psi(t) = t(1 + \log^+ t)$. Then,

$$\int_{\mathbb{R}^n} \left| [b,K]f(x) \right| w(x) \, dx \le C \int_{\mathbb{R}^n} M_{\psi,\alpha} f(x) \, w(x) \, dx.$$

The proof of this estimate was given in [CMP] by means of a discretization argument that avoids again the good-λ method and does not use any boundedness of the commutator. As above, the fact that the exponent is one plays an important role in the proof. Then, Theorem 2.1 and Theorem 3.1 yield the following result.

Theorem 6.9. *Given $0 < \alpha < n$ and $b \in$ BMO, let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \leq 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then, for all $w \in A_\infty$*

$$\big\|[b, K]f\big\|_{\mathbb{X}(w)} \leq C\, \|M_{\psi,\alpha}f\|_{\mathbb{X}(w)}. \tag{6.17}$$

Similarly, if ϕ satisfies (i) and (ii) in Theorem 3.1, then

$$\int_{\mathbb{R}^n} \phi\big(\big|[b,K]f(x)\big|\big)\, w(x)\, dx \leq C \int_{\mathbb{R}^n} \phi\big(M_{\psi,\alpha}(f)(x)\big)\, w(x)\, dx, \tag{6.18}$$

and

$$\sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : \big|[b,K]f(y)\big| > \lambda\} \leq C \sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : M_{\psi,\alpha}(f)(y) > \lambda\}.$$

Next, we show how to derive the main result in [CUF] using our modular extrapolation result. Indeed, it is proved in that paper that the following estimate holds:

$$\big|\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > \lambda\}\big| \leq C\, \varphi\left(\int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) dx\right), \tag{6.19}$$

where $\psi(t) = t\,(1 + \log^+ t)$ and $\varphi(t) = \psi(t)^{\frac{n}{n-\alpha}}$.

In order to apply the later result to this example we define $\phi(t) = \frac{1}{\varphi \circ \psi(\frac{1}{t})}$. Observe that both ψ and φ are submultiplicative (with constant one) and so ϕ is doubling. Also, observe that ϕ is quasi-convex since ψ and φ are. Therefore, we can apply Theorem 3.1, obtaining

$$\sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : \big|[b,K]f(y)\big| > \lambda\} \leq C \sup_{\lambda>0} \phi(\lambda)\, w\{y \in \mathbb{R}^n : M_{\psi,\alpha}(f)(y) > \lambda\}.$$

Since (6.19) is homogeneous in f, we can assume that $\lambda = 1$. Hence

$$\begin{aligned}
\big|\{x \in \mathbb{R}^n : \big|[b,K]f(x)\big| > 1\}\big| &\leq \sup_{\lambda>0} \phi(\lambda)\, \big|\{y \in \mathbb{R}^n : \big|[b,K]f(y)\big| > \lambda\}\big| \\
&\leq C \sup_{\lambda>0} \phi(\lambda)\, \big|\{y \in \mathbb{R}^n : M_{\psi,\alpha}f(y) > \lambda\}\big|.
\end{aligned}$$

Since ψ and φ are submultiplicative, by means of a covering lemma, one can show that

$$\begin{aligned}
\big|\{y \in \mathbb{R}^n : M_{\psi,\alpha}f(y) > \lambda\}\big| &\leq C\, \varphi\left(\int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) dx\right) \\
&\leq C\, \varphi \circ \psi\left(\frac{1}{\lambda}\right) \varphi\left(\int_{\mathbb{R}^n} \psi\left(|f(x)|\right) dx\right) \\
&= \frac{C}{\phi(\lambda)}\, \varphi\left(\int_{\mathbb{R}^n} \psi\left(|f(x)|\right) dx\right).
\end{aligned}$$

and thus

$$\big|\{x \in \mathbb{R}^n : \big|[b,K]f(x)\big| > 1\}\big| \leq C\, \varphi\left(\int_{\mathbb{R}^n} \psi\left(|f(x)|\right) dx\right).$$

Note that this generalizes (6.19) to more general potential operators K.

Besides the classical fractional integrals, an example of such an operator K is given as follows. Let L be a linear operator on $L^2(\mathbb{R}^n)$ such that $(-L)$ generates an analytic semigroup e^{-tL}. We suppose that this semigroup has a kernel $p_t(x, y)$ which satisfies

$$|p_t(x, y)| \leq \frac{C}{t^{\frac{n}{2}}} e^{-c \frac{|x-y|^2}{t}}, \qquad \text{for all } x, y \in \mathbb{R}^n;\, t > 0.$$

We consider the generalized fractional integrals,

$$L^{-\frac{\alpha}{2}} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tL} f(x)\, t^{\frac{\alpha}{2}}\, \frac{dt}{t}, \qquad 0 < \alpha < n,$$

and the corresponding commutator

$$[b, L^{-\frac{\alpha}{2}}] f(x) = b(x)\, L^{-\frac{\alpha}{2}} f(x) - L^{-\frac{\alpha}{2}}(b\, f)(x), \qquad b \in \text{BMO}.$$

Note that if $L = -\Delta$ in $\mathbb{R}^n$, then $L^{-\frac{\alpha}{2}}$ is the classical fractional integral I_α. It is easy to show that the bound on the kernel of the semigroup implies that $|k_\alpha(x, y)| \leq C\,|x - y|^{\alpha - n}$, where k_α is the kernel of $L^{-\frac{\alpha}{2}}$. In particular, $|L^{-\frac{\alpha}{2}} f(x)| \leq C\, I_\alpha(|f|)(x)$, and thus estimates for I_α also yield similar results for $L^{-\frac{\alpha}{2}}$. We can apply Theorem 6.9 to get inequalities in weighted RIQBFS for $[b, L^{-\frac{\alpha}{2}}]$. These commutators have been previously studied in [DY] by a different method. Namely, the authors use a new sharp maximal function introduced in [Mar] and obtain the boundedness of $[b, L^{-\frac{\alpha}{2}}]$ on unweighted Lebesgue spaces. Compare to [DY], we establish weighted and vector-valued inequalities. Note also that our only requirement is the size estimate of the kernel, and so we do not use any other property of the semigroup e^{-tL}.

6.4. Multilinear Calderón-Zygmund operators.

Let T be a multilinear Calderón-Zygmund operator, that is, T is an m-linear operator mapping continuously $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $1 < q_1, \ldots, q_m < \infty$, $0 < q < \infty$ and

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}. \tag{6.20}$$

The operator T is associated with a Calderón-Zygmund kernel K by

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m)\, f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m,$$

whenever $f_1, \ldots, f_m$ are in C_0^∞ and $x \notin \bigcap_{j=1}^m \text{supp}\, f_j$. We assume that K satisfies the appropriate decay and smoothness conditions (see [GT1] and [GT2] for full details). Such an operator T turns out to be bounded on any other product of Lebesgue spaces with exponents $1 < q_1, \ldots, q_m < \infty$, $0 < q < \infty$ satisfying (6.20). Further, it verifies weak endpoint estimates when some of the q_i's are equal to one. There are also weighted norm inequalities for multilinear Calderón-Zygmund, these were first proved in [GT2] using a good-λ inequality, and later in [PTo] using the sharp maximal function. They showed that for $0 < p < \infty$ and for all $w \in A_\infty$,

$$\|T(f_1, \ldots, f_m)\|_{L^p(w)} \leq C\, \Big\| \prod_{j=1}^m M f_j \Big\|_{L^p(w)}.$$

The same inequality also holds with T replaced by T_*, which is the supremum of the truncated integrals. We apply Theorem 2.1 with (2.1) given by the latter inequality with the pairs $\big(T(f_1, \ldots, f_m), \prod_{j=1}^m M f_j\big)$, or analogously for T_* replacing T.

Theorem 6.10. *Consider a multilinear Calderón-Zygmund operator T and let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \leq 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then,*

$$\|T(f_1, \ldots, f_m)\|_{\mathbb{X}(w)} \leq C \Big\| \prod_{j=1}^{m} Mf_j \Big\|_{\mathbb{X}(w)},$$

for every $0 < p < \infty$ and all $w \in A_\infty$. Moreover, vector-valued inequalities as (2.3) also hold in the same manner and T_* can be placed in place of T at any of the previous inequalities.

We define the m-product operator

$$P_m(f_1, f_2, \ldots, f_m)(x) = \prod_{j=1}^{m} f_j(x).$$

As a consequence of both the latter result and Theorem 2.3 we can prove that T is bounded on some RIQBFS.

Corollary 6.11. *Consider a multilinear Calderón-Zygmund operator T and let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \leq 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Let $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_m$ be RIQBFS such that each of them is p-convex for some $0 < p \leq 1$. Assume that P_m maps continuously $\overline{\mathbb{X}}_1 \times \cdots \times \overline{\mathbb{X}}_m$ to $\overline{\mathbb{X}}$. If $\min\{p_{\mathbb{X}_1}, \ldots, p_{\mathbb{X}_m}\} > 1$ and $w \in A_{\min\{p_{\mathbb{X}_1}, \ldots, p_{\mathbb{X}_m}\}}$ then*

$$\|T(f_1, \ldots, f_m)\|_{\mathbb{X}(w)} \leq C \prod_{j=1}^{m} \|f_j\|_{\mathbb{X}_j(w)}, \tag{6.21}$$

and in particular T maps continuously $\mathbb{X}_1 \times \cdots \times \mathbb{X}_m$ to $\mathbb{X}$. Additionally, the following weighted vector-valued inequalities hold:

$$\Big\| \Big(\sum_k |T(f_1^k, \ldots, f_m^k)|^q \Big)^{\frac{1}{q}} \Big\|_{\mathbb{X}(w)} \leq C \prod_{j=1}^{m} \Big\| \Big(\sum_k |f_j^k|^{q_j} \Big)^{\frac{1}{q_j}} \Big\|_{\mathbb{X}_j(w)},$$

whenever $1 < q_1, \ldots, q_m < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots \frac{1}{q_m}$.

Next, we are going to present a collection of examples on which this result can be used.

- **Spaces in the same scale:** Let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \leq 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Consider $1 < p_1, \ldots, p_m < \infty$ such that $1 = \frac{1}{p_1} + \cdots \frac{1}{p_m}$ and set $p_0 = \min\{p_1, \ldots, p_m\}$. If $p_0 \cdot p_{\mathbb{X}} > 1$, we have that T is bounded from $\mathbb{X}^{p_1}(w) \times \cdots \times \mathbb{X}^{p_m}(w)$ to $\mathbb{X}(w)$ for all $w \in A_{p_0 \cdot p_{\mathbb{X}}}$ and in particular for $w = 1$. The only thing to be shown is the boundedness of P_m. Namely, taking $1 < r < \infty$ such that $\mathbb{Y} = \mathbb{X}^r$ is Banach, and for $h_1, \ldots, h_m$ non-negative functions we have

$$
\begin{aligned}
\|P_m(h_1, \ldots, h_m)\|_{\overline{\mathbb{X}}} &= \Big\| \Big(\prod_{j=1}^{m} h_j \Big)^{\frac{1}{r}} \Big\|_{\overline{\mathbb{Y}}}^{r} = \sup_h \Big(\int_0^\infty \prod_{j=1}^{m} h_j(t)^{\frac{1}{r}} h(t)\, dt \Big)^r \\
&\leq \sup_h \prod_{j=1}^{m} \Big(\int_0^\infty h_j(t)^{\frac{p_j}{r}} h(t)\, dt \Big)^{\frac{r}{p_j}} \leq \prod_{j=1}^{m} \|h_j\|_{\overline{\mathbb{X}}^{p_j}},
\end{aligned}
$$

where the supremum is taken over all $0 \leq h \in \overline{\mathbb{Y}}'$ with norm equal to 1. Some examples of spaces in the same scale on which T is bounded are:

– $T : L^{p_1}(w) \times \cdots \times L^{p_m}(w) \longrightarrow L^p(w)$ with $1 < p_1, \ldots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $w \in A_{\min\{p_1,\ldots,p_m\}}$.

– $T : L^{p_1}(\log L)^\alpha(w) \times \cdots \times L^{p_m}(\log L)^\alpha(w) \longrightarrow L^p(\log L)^\alpha(w)$ with $\alpha \in \mathbb{R}$, $1 < p_1, \ldots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $w \in A_{\min\{p_1,\ldots,p_m\}}$.

– $T : (L^\Psi)^{p_1}(w) \times \cdots \times (L^\Psi)^{p_m}(w) \longrightarrow (L^\Psi)^p(w)$ where $\mathbb{X} = L^\Psi$ is any Orlicz space (that is Banach by definition); $q_{\mathbb{X}} < \infty$; $0 < p, p_1, \ldots, p_m < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\min\{p_1, \ldots, p_m\} \cdot p_{\mathbb{X}} > 1$ and $w \in A_{\min\{p_1,\ldots,p_m\} \cdot p_{\mathbb{X}}}$.

- **Lorentz spaces:** For $1 < p_1, \ldots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$; $0 < r, r_1, \ldots r_m \leq \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$ and $w \in A_{\min\{p_1,\ldots,p_m\}}$ we have $T : L^{p_1,r_1}(w) \times \cdots \times L^{p_m,r_m}(w) \longrightarrow L^{p,r}(w)$. We only have to check that P_m is bounded. We do the case $r_1, \ldots, r_m > \infty$, leaving the other cases to the reader:

$$
\begin{aligned}
\|P_m(h_1, \ldots, h_m)\|_{L^{p,r}} &\leq \Big(\int_0^\infty \prod_{j=1}^m \big(h_j^*(s)\, s^{1/p_j} \big)^r \frac{ds}{s} \Big)^{\frac{1}{r}} \\
&\leq \prod_{j=1}^m \Big(\int_0^\infty \big(h_j^*(s)\, s^{1/p_j} \big)^{r_j} \frac{ds}{s} \Big)^{\frac{1}{r_j}} \\
&= \prod_{j=1}^m \|h_j\|_{L^{p_j,r_j}}
\end{aligned}
$$

Let us observe that although in this computations the underlying measure space is $(\mathbb{R}^n, dx)$, the same can be done in general measure spaces, and in particular in $(\mathbb{R}^+, dt)$. Since $\overline{L^{p,r}(\mathbb{R}^n, dx)} = L^{p,r}(\mathbb{R}^+, dt)$, we conclude that P_m is bounded between $\overline{L^{p_1,r_1}} \times \cdots \times \overline{L^{p_m,r_m}}$ and $\overline{L^{p,r}}$.

- **Orlicz Spaces:** Let $\Psi_0, \Psi_1, \ldots, \Psi_m$ be Young functions such that

$$
\Psi_0(\alpha\, x_1 \cdots x_m) \leq \Psi_1(x_1) + \cdots + \Psi_m(x_m), \qquad 0 \leq x_1, \ldots, x_m < \infty.
$$

This condition is implied, for example, by

$$
\Psi_1^{-1}(x) \cdots \Psi_m^{-1}(x) \leq \beta\, \Psi_0^{-1}(x), \qquad x \geq 0.
$$

Then $\|f_1 \cdots f_m\|_{L^{\Psi_0}} \leq C\, \|f_1\|_{L^{\Psi_1}} \cdots \|f_1\|_{L^{\Psi_m}}$ (see [RR, p. 179] or [PTr1]), and as before the same holds for the corresponding spaces in $(\mathbb{R}^+, dt)$. Assume that $q_{L^{\Psi_0}} < \infty$ and set $p_0 = \min\{p_{L^{\Psi_1}}, \ldots, p_{L^{\Psi_m}}\}$. Note that $p_0 \geq 1$ since all of these spaces are Banach. If $0 < p < \infty$ is such that $p \cdot p_0 > 1$ then

$$
T : (L^{\Psi_1})^p(w) \times \cdots \times (L^{\Psi_m})^p(w) \longrightarrow (L^{\Psi_0})^p(w)
$$

for all $w \in A_{p \cdot p_0}$.

- **Lorentz and Marcinkiewicz spaces:** Let $\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_m$ be RIBFS with fundamental functions $\varphi_1, \varphi_2, \ldots, \varphi_m$. Then $\varphi(t) = \prod_{j=1}^m \varphi_j(t)$ is increasing and $\varphi(0) = 0$. We will assume that φ is concave, hence we can consider the Lorentz

space Λ_φ. Then

$$P_m \colon \Lambda_{\varphi_1} \times \cdots \times \Lambda_{\varphi_m} \longrightarrow \Lambda_\varphi.$$

This follows from:

$$
\begin{aligned}
\int_0^\infty (f_1 f_2 \cdots f_m)^*(s) \, d\varphi(s) &\le \int_0^\infty f_1^*(s) f_2^*(s) \ldots f_m^*(s) \, d\varphi(s) \\
&= \sum_{j=1}^m \int_0^\infty \Big(\prod_{i \ne j} f_i^*(s) \, \varphi_i(s) \Big) f_j^*(s) \, d\varphi_j(s) \\
&\le \sum_{j=1}^m \Big(\prod_{i \ne j} \|f_i\|_{\mathbb{M}_{\varphi_i}} \Big) \|f_j\|_{\Lambda_{\varphi_j}} \le m \prod_{i=1}^m \|f_i\|_{\Lambda_{\varphi_i}},
\end{aligned}
$$

since $f_i^*(s) \, \varphi_i(s) \le \|f_i\|_{\mathbb{M}_{\varphi_i}} \le \|f_i\|_{\Lambda_{\varphi_i}}$.

If we assume an extra condition namely, there exists j_0 with $i_{\varphi_{j_0}} > 0$, which implies (2.14), that is $\varphi_{j_0}(t) \sim \int_0^t \frac{\varphi_{j_0}(s)}{s} \, ds$, then

$$P_m \colon \mathbb{M}_{\varphi_1} \times \cdots \times \mathbb{M}_{\varphi_{j_0-1}} \times \Lambda_{\varphi_{j_0}} \times \mathbb{M}_{\varphi_{j_0+1}} \times \cdots \times \mathbb{M}_{\varphi_m} \longrightarrow \mathbb{M}_\varphi.$$

This follows from:

$$
\begin{aligned}
\frac{\varphi(t)}{t} \int_0^t (f_1 f_2 \cdots f_m)^*(s) \, ds &\le \int_0^t \frac{\varphi_{j_0}(s) f_{j_0}^*(s)}{s} \prod_{j \ne j_0} \varphi_j(s) f_j^*(s) \, ds \\
&\le \Big(\prod_{j \ne j_0} \|f_j\|_{\mathbb{M}_{\varphi_i}} \Big) \int_0^t \frac{\varphi_{j_0}(s)}{s} f_{j_0}^*(s) \, ds \\
&\le \Big(\prod_{j \ne j_0} \|f_j\|_{\mathbb{M}_{\varphi_i}} \Big) \|f_{j_0}\|_{\Lambda_{\varphi_{j_0}}} \Big\| \frac{1}{\overline{\varphi}_{j_0}} \Big\|_{(\Lambda_{\varphi_{j_0}})'}
\end{aligned}
$$

where $\overline{\varphi}_{j_0}(s) = s/\varphi_{j_0}(s)$ and we have used that $\varphi(t)/t$ is decreasing as φ is concave. To end the proof we just have to check that $1/\overline{\varphi}_{j_0} \in (\Lambda_{\varphi_{j_0}})' = \mathbb{M}_{\overline{\varphi}_{j_0}}$, but this is so, precisely by the assumption we have made on φ_{j_0}.

Furthermore, if we assume that $I_\varphi < 1$ we have (2.14), that is $\overline{\varphi}(t) \sim \int_0^t \frac{\overline{\varphi}(s)}{s} \, ds$, and then

$$P_m \colon \mathbb{M}_{\varphi_1} \times \cdots \times \mathbb{M}_{\varphi_m} \longrightarrow \mathbb{M}_\varphi,$$

since we observe that

$$
\begin{aligned}
\frac{\varphi(t)}{t} \int_0^t (f_1 f_2 \cdots f_m)^*(s) \, d(s) &\le \frac{\varphi(t)}{t} \int_0^t \frac{\|f_1\|_{\mathbb{M}_{\varphi_1}}}{\varphi_1(s)} \cdots \frac{\|f_m\|_{\mathbb{M}_{\varphi_m}}}{\varphi_m(s)} \, ds \\
&= \Big(\prod_{j=1}^m \|f_j\|_{\mathbb{M}_{\varphi_j}} \Big) \frac{1}{\overline{\varphi}(t)} \int_0^t \frac{\overline{\varphi}(s)}{s} \, ds,
\end{aligned}
$$

and this last quantity is bounded due to our assumption on $\overline{\varphi}$. In this case, since $\mathbb{X}_i \hookrightarrow \mathbb{M}_{\varphi_i}$, we also have

$$P_m \colon \mathbb{X}_1 \times \cdots \times \mathbb{X}_m \longrightarrow \mathbb{M}_\varphi.$$

The above computations hold for general measure spaces, hence for $(\mathbb{R}^+, dt)$, so the continuity of P_m is also established when substituting the spaces Λ_φ and $\mathbb{M}_\varphi$ by $\overline{\Lambda_\varphi}$ and $\overline{\mathbb{M}_\varphi}$. Since the spaces involved are all RIBFS –hence 1-convex–, the only

condition needed in order to fulfill the conditions of Corollary 6.11 is that $q_{\mathbb{X}} < \infty$ when $\mathbb{X} = \Lambda_\varphi$ or $\mathbb{X} = \mathbb{M}_\varphi$ (in this later case the condition is precisely $i_\varphi > 0$).

Once we have shown that P_m is bounded we can obtain estimates for a multilinear Calderón-Zygmund operator T. Under the corresponding hypothesis for each case we have

$$T \colon \Lambda_{\varphi_1}(w) \times \cdots \times \Lambda_{\varphi_m}(w) \longrightarrow \Lambda_\varphi(w)$$
$$T \colon \mathbb{M}_{\varphi_1}(w) \times \cdots \times \Lambda_{\varphi_{j_0}}(w) \times \cdots \times \mathbb{M}_{\varphi_m}(w) \longrightarrow \mathbb{M}_\varphi(w)$$
$$T \colon \mathbb{M}_{\varphi_1}(w) \times \cdots \times \mathbb{M}_{\varphi_m}(w) \longrightarrow \mathbb{M}_\varphi(w)$$
$$T \colon \mathbb{X}_1(w) \times \cdots \times \mathbb{X}_m(w) \longrightarrow \mathbb{M}_\varphi(w)$$

provided $\min\{p_{\mathbb{Y}_1}, \ldots, p_{\mathbb{Y}_m}\} > 1$ and $w \in A_{\min\{p_{\mathbb{Y}_1},\ldots,p_{\mathbb{Y}_m}\}}$, where $\mathbb{Y}_j$ is, depending on the case, Λ_{φ_j}, $\mathbb{M}_{\varphi_j}$ or $\mathbb{X}_j$.

6.5. **Exotic maximal operators.** We mentioned in the introduction that there are examples in Harmonic Analysis whose behavior is unusual since they are bounded on $L^p(w)$ for any $0 < p < \infty$ and any $w \in A_\infty$. The first example is the so called geometric maximal operator defined by

$$M_0 f(x) = \sup_{Q \ni x} \exp\Big(\frac{1}{|Q|} \int_Q \log|f(y)|\,dy\Big).$$

This operator has been studied in the literature and we refer to [CUN2] and the references therein. Also, in this paper it is shown the relationship with the following operator:

$$M_0^\star f(x) = \lim_{r \to 0} M\big(|f|^r\big)(x)^{\frac{1}{r}}.$$

Observe that pointwise $M_0 f(x) \le M_0^\star f(x)$ and it is shown in [CUN2] that for many functions they coincide. The observation is that for any $0 < p < \infty$ and any $w \in A_\infty$:

$$\int_{\mathbb{R}^n} M_0^\star f(x)^p\, w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\,dx, \qquad (6.22)$$

To prove such inequalities we first observe that it suffices to deal with the case $p = 1$, since for any $0 < p < \infty$ by definition of the operators: $(M_0 f)^p = M_0(|f|^p)$ and $(M_0^\star f)^p = M_0^\star(|f|^p)$. Besides, using that $w \in A_\infty$ implies $w \in A_q$ for some $1 < q < \infty$ and that $M_0^\star f(x) \le M\big(|f|^{\frac{1}{q}}\big)(x)^q$, the case $p = 1$ follows since M is bounded on $L^q(w)$. Note that, as a consequence, M_0 satisfies the same estimate. However, this can be seen in a different way using the ideas of Section 5. As before, it suffices to consider the case $p = 1$. We set $\phi(t) = e^t$ and $f_1(x) = |f(x)|\, \chi_{\{x:|f(x)|>1\}}(x)$. Unfortunately, $\phi \notin \Phi$ so we can not use Theorem 3.7. Nevertheless, the proof can be adapted in the following way. Take $0 < \alpha < 1$ such that $w \in A_{1/\alpha}$ and notice that $\phi^\alpha(t) = \phi(t)^\alpha = e^{\alpha t}$ is convex. Then as in the proof of Proposition 5.1 we have

$$M_0 f(x) \le M_0 f_1(x) \le \phi\big(M(\log f_1)(x)\big) \le M\big(\phi^\alpha(\log f_1)\big)(x)^{\frac{1}{\alpha}} = M\big(f_1^\alpha\big)(x)^{\frac{1}{\alpha}}.$$

Therefore, as M is bounded on $L^{1/\alpha}(w)$,

$$\int_{\mathbb{R}^n} M_0 f(x)\, w(x)\,dx \le C \int_{\mathbb{R}^n} f_1(x)\, w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|\, w(x)\,dx.$$

Once we know that both $M_0^\star$ and M_0 satisfy (6.22) we can apply Theorem 2.1 and Theorem 3.1 with the $(M_0^\star f, |f|)$ and $(M_0 f, |f|)$

Theorem 6.12. *Let $\mathbb{X}$ be a RIQBFS, p-convex for some $0 < p \leq 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for any $w \in A_{\infty}$ and $0 < q < \infty$,*

$$\|M_0^{\star} f\|_{\mathbb{X}(w)} \leq C \|f\|_{\mathbb{X}(w)}, \qquad \left\|\left(\sum_j (M_0^{\star} f_j)^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}(w)} \leq C \left\|\left(\sum_j |f_j|^q\right)^{\frac{1}{q}}\right\|_{\mathbb{X}(w)}.$$

Similarly, if $\phi \in \Phi$ satisfies (i) and (ii) in Theorem 3.1, then

$$\int_{\mathbb{R}^n} \phi\big(|M_0^{\star} f(x)|\big) w(x) \, dx \leq C \int_{\mathbb{R}^n} \phi\big(|f(x)|\big) w(x) \, dx,$$

and

$$\sup_{\lambda > 0} \phi(\lambda) \, w\big\{x \in \mathbb{R}^n : |M_0^{\star} f(x)| > \lambda\big\} \leq C \sup_{\lambda > 0} \phi(\lambda) \, w\big\{x \in \mathbb{R}^n : |f(x)| > \lambda\big\}.$$

The same estimates hold with M_0 in place of $M_0^{\star}$.

Another related and more interesting operator is the minimal operator introduced in [CUN1]:

$$\mathfrak{M} f(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

The L^p behavior of this operator is very surprising, since it satisfies the following: for any $0 < p < \infty$ and any $w \in A_{\infty}$,

$$\int_{\mathbb{R}^n} \frac{1}{\mathfrak{M} f(x)^p} \, w(x) \, dx \leq C \int_{\mathbb{R}^n} \frac{1}{|f(x)|^p} \, w(x) \, dx.$$

The proof of this estimate is given in [CUN1]. We present here a slightly different approach, which establishes a two-weight inequality for $\mathfrak{M}$ implying the desired estimate by Remark 1.3. Let $0 < p < \infty$ and $0 \leq w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be an arbitrary weight, by this mean that w is not necessarily in A_{∞}. For any $1 < q < \infty$ we consider the function $\phi(t) = t^{-\frac{p}{q}}$ which is convex. Then, by Jensen's inequality for any Q we have

$$\left(\frac{1}{|Q|} \int_Q |f(x)|^{-\frac{p}{q}} \, dx\right)^{-\frac{q}{p}} \leq \frac{1}{|Q|} \int_Q |f(x)| \, dx$$

and therefore $M\big(|f|^{-\frac{p}{q}}\big)(x)^{-\frac{q}{p}} \leq \mathfrak{M} f(x)$. Thus, we obtain the following two-weight inequality for the minimal operator

$$\int_{\mathbb{R}^n} \frac{1}{\mathfrak{M} f(x)^p} \, w(x) \, dx \leq \int_{\mathbb{R}^n} M\big(|f|^{-\frac{p}{q}}\big)(x)^q \, w(x) \, dx \leq C \int_{\mathbb{R}^n} \frac{1}{|f(x)|^p} \, M w(x) \, dx.$$

If we assume that $w \in A_1$, which means $M w(x) \leq C \, w(x)$, we conclude that

$$\int_{\mathbb{R}^n} \frac{1}{\mathfrak{M} f(x)^p} \, w(x) \, dx \leq C \int_{\mathbb{R}^n} \frac{1}{|f(x)|^p} \, w(x) \, dx,$$

for every $w \in A_1$, and every $0 < p < \infty$. Therefore, by Remark 1.3, we can use the extrapolation results in [CMP] to establish the same estimate for every $w \in A_{\infty}$.

In this way we can apply Theorems 2.1 and 3.1 with the pairs $\left(\frac{1}{\mathfrak{M} f}, \frac{1}{|f|}\right)$.

Theorem 6.13. *Let* $\mathbb{X}$ *be a RIQBFS, p-convex for some $0 < p \le 1$ and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for any $w \in A_\infty$, and all $0 < q < \infty$,*

$$\left\| \frac{1}{\mathfrak{M}f} \right\|_{\mathbb{X}(w)} \le C \left\| \frac{1}{f} \right\|_{\mathbb{X}(w)}, \qquad \left\| \left(\sum_j \frac{1}{(\mathfrak{M}f_j)^q} \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)} \le C \left\| \left(\sum_j \frac{1}{|f_j|^q} \right)^{\frac{1}{q}} \right\|_{\mathbb{X}(w)}.$$

Similarly, if $\phi \in \Phi$ satisfies (i) and (ii) in Theorem 3.1, then

$$\int_{\mathbb{R}^n} \phi\left(\frac{1}{\mathfrak{M}f(x)} \right) w(x)\,dx \le C \int_{\mathbb{R}^n} \phi\left(\frac{1}{|f(x)|} \right) w(x)\,dx,$$

and

$$\sup_{\lambda>0} \phi(\lambda)\, w\left\{ x \in \mathbb{R}^n : \frac{1}{\mathfrak{M}f(x)} > \lambda \right\} \le C \sup_{\lambda>0} \phi(\lambda)\, w\left\{ x \in \mathbb{R}^n : \frac{1}{|f(x)|} > \lambda \right\}.$$

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