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Advection on Graphs

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***Abstract*—This paper examines the dynamics of a networked, multi-agent system operating with an advection-based coordination algorithm. Flow advection is a close relative of diffusion whose discretized version forms the basis of the popular consensus dynamics. We endeavor to demonstrate in this paper that discretizing the continuous advection equation also forms an attractive set of system dynamics for coordinated control. The key advantage of advection-based algorithms over directed consensus is that the sum of the states is always conserved. This paper includes a formulation of the advection dynamics on directed graphs and a presentation of some of its characteristics, which are compared to the consensus dynamics. We also provide examples of the versatility of the advection dynamics: a formation control and sensor coverage example.**

***Index Terms*—Advection protocol; Consensus protocol; Networked control**

I. INTRODUCTION

Advection is the process where a distribution is actively transported by a flow field. A simple example of advection is oil dropped into a body of water. If the water is still, the oil will tend to remain concentrated, but if the water is flowing, for example in a river, the flow will cause a change in concentration. This paper introduces a discrete form of the advection process for application in networked, multi-agent systems. Here a flow field defines interactions between agents, and inter-agent dynamics are based on the advection process.

Advection shares many similarities to diffusion and may be interpreted as diffusion in a flow field. A discretized form of diffusion is the framework for consensus problems. Consensus provides an effective model for distributed information-sharing and control of networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see, for example, [1], [2], [3], [4]. An appeal of the consensus framework is that locally-based interaction dynamics can produce global network characteristics. Further, the performance characteristics are coupled to the underlying network structure. Advection dynamics are similarly coupled to the network, and in certain classes of networks, share identical dynamics with consensus.

At the heart of consensus problems is the diffusion model [5]. For undirected consensus, where the underlying agent interactions are Euclidean-based, the governing dynamics are precisely the discrete version of diffusion. The two core properties and attractions of discrete diffusion are (a) the interpretation of a directed edge existing from agent i to j is that agent i can "influence" agent j and (b) the state sum is always conserved. For directed graphs (digraphs), where every edge from i to j does not necessarily have a corresponding edge from j to i, generally one of these properties has to be sacrificed. Traditionally, graph literature [1], [6] preferentially chooses the "influence" property and the sum conservation property only holds true for balanced graphs, which corresponds to the "perfect" discretization of diffusion with both conditions. In this paper, we will instead preferentially choose the second conservation property and sacrifice at times the "influence" property and formulate our model accordingly. The sum conservation property has the effect of inducing a flow through the directed edges in the graph and in turn corresponds to the discrete version of the advection model. Advection has been used to model the spread of diseases [6], population migration [7], and supply and demand in economic systems [8].

The organization of the paper is as follows. We begin by defining advection dynamics and characterize its state matrix, dynamics and equilibrium, with a particular focus on the underlying graph structure. We then compare and contrast the advection features with traditional consensus dynamics and illustrate the strong link between them. Finally, we examine two advection problems.

II. BACKGROUND AND MODEL

We provide a brief background on constructs and models that will be used in this paper.

Firstly, we define the vectors $\mathbf{1} := [1, 1, \ldots, 1]^T$ and $\mathbf{0} := [0, 0, \ldots, 0]^T$. For column vector $v \in \mathbb{R}^p$, v_i denotes the ith element. For matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column.

The advection equation, also known as the transport equation, involves a scalar concentration u of a material affected by a flow field $\vec{v}$ and is conventionally [5] given by

$$\frac{\partial u}{\partial t} = -\nabla \cdot (\vec{v}u).$$

Here ∇ is the divergence operator defined on a continuously differentiable vector field $F = \sum_{i=1}^{k} U_i a_i$ with basis vectors a_i and coordinate frame $\{x_1, x_2, \ldots, x_k\}$ in $\mathbb{R}^k$, and defined as $\nabla \cdot F = \sum_{i=1}^{k} \frac{\partial U_i}{\partial x_i}$. The flux of the advection process is subsequently $F = \vec{v}u$.

In a discrete calculus analogue of the advection equation, we first define an interaction graph (directed and weighted) over nodes based on the flow $\vec{v}$. The flow vector $\vec{v}$ dictates the interactions between nodes by defining directed edges and edge weights. We then adopt a discretized view of the flux $\vec{v}u$ through an edge $i \to j$ as consisting of the flow w_{ji}

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prescribed by $\vec{v}$ at the edge modified by the concentration x_i perscribed by u at node i. The flow along edge $i \rightarrow j$ is consequently $w_{ij}x_i(t)$. The concentration at node i at time t is denoted $x_i(t)$. The rate of change of the concentration of node i is then the flow into the node minus the flow out of the node, i.e.,

$$\dot{x}_i(t) = -\sum_{\{\forall j | i \rightarrow k\}} w_{ki}x_i(t) + \sum_{\{\forall j | j \rightarrow i\}} w_{ij}x_j(t). \quad (1)$$

This problem is well suited to a graph theoretic analysis. As such, we proceed by presenting some graph theory background and rewrite the dynamics (1).

A weighted directed graph $\mathcal{G} = (V, E, W)$ is defined by a node set V with cardinality n, an edge set E comprised of pairs of nodes with cardinality m, and a weight set W with cardinality m, where information flows from node v_i to v_j if $(i,j) \in E$ with edge weight $w_{ji} \in W$. An undirected graph occurs when $(i,j) \in E$ implies $(j,i) \in E$ and $w_{ji} = w_{ij}$. The weighted out-degree and in-degree of node i is $d_i = \sum_{(i,k)\in E} w_{ki}$ and $\tilde{d}_i = \sum_{(j,i)\in E} w_{ij}$, respectively. A graph is called *balanced* if $d_i = \tilde{d}_i$ for all $i \in V$. The number of out-degree edges of node i is denoted by δ_i. The out-degree matrix $\Delta_{\text{out}}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with d_i at entry (i,i). The in-degree matrix $\Delta_{\text{in}}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is similarly defined with $\tilde{d}_i$ at entry (i,i). The adjacency matrix $\mathcal{A}(\mathcal{G})$ is an $n \times n$ matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = w_{ij}$ when $(v_j, v_i) \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The in-degree and out-degree Laplacian are defined as $L_{\text{out}}(\mathcal{G}) = \Delta_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ and $L_{\text{in}}(\mathcal{G}) = \Delta_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, respectively.

We can now rewrite our dynamics (1) using these graph concepts with the flow $\vec{v}$ generating the graph $\mathcal{G}(\vec{v}) = (V, E(\vec{v}), W(\vec{v}))$ as

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{(j,i)\in E} w_{ij}x_j(t). \quad (2)$$

For brevity, we will denote the graph as $\mathcal{G} = (V, E, W)$. The advection dynamics can therefore be written as

$$\dot{x}(t) = -L_{\text{out}}(\mathcal{G})x(t). \quad (3)$$

We now proceed to examine system characteristics of the advection dynamics and compare them with consensus dynamics. Specifically, we compare invariance properties, equilibrium, and flow and stochastic interpretations of the dynamics where $\mathcal{G}$ contains a rooted branching. A road map of these features are depicted in Figure 1.

III. Advection Properties

We will now proceed to examine some of the characteristics of advection and compare them with the more familiar consensus dynamics. The node dynamics for consensus are

$$\begin{aligned}\dot{x}_i(t) &= \sum_{(j,i)\in E} w_{ij}\left(x_j(t) - x_i(t)\right) \quad (4)\\ &= -\tilde{d}_i x_i(t) + \sum_{(j,i)\in E} w_{ij}x_j(t).\end{aligned}$$

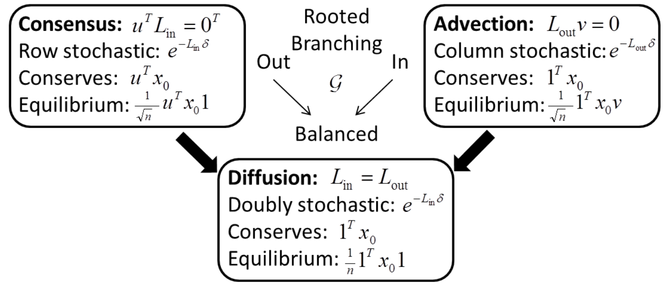

Figure 1. Relationship between rooted out-branching consensus, rooted in-branching advection and rooted in/out-branching balanced consensus (in the discretization sense - "perfect" diffusion - exhibiting properties of both consensus and advection).

In compact form, the consensus dynamics are

$$\dot{x}(t) = -L_{\text{in}}(\mathcal{G})x(t). \quad (5)$$

The first significant difference between the advection and consensus dynamics is the sum conservation property of advection, stated in the following proposition, which is generally not a property of directed consensus.

Proposition 1. *The advection dynamics* (3) *are (state) sum conservative, i.e.,* $\sum_{i=1}^n x_i(t) = \sum_{i=1}^n x_i(0)$ *for all time* t.

Proof: Directly from (2),

$$\begin{aligned}\sum_{i=1}^n \dot{x}_i(t) &= \sum_{i=1}^n \left(-\sum_{(i,k)\in E} w_{ki}x_i(t) + \sum_{(j,i)\in E} w_{ij}x_j(t)\right)\\ &= \sum_{(j,i)\in E} \left(-w_{ij}x_i(t) + w_{ij}x_i(t)\right) = 0.\end{aligned}$$

Therefore, $\sum_{i=1}^n x(t)$ is always conserved. ■

An alternate interpretation of the sum conservation property is that the mean of the states is always constant. Consensus also conserves a weighted sum of the initial states. This feature will be stated later in Proposition 9.

Another difference between these dynamics is that for consensus the interaction mechanics between neighboring agents where the dynamics of agent i can only "influence" agent j directly if there is an edge (i,j) in the graph $\mathcal{G}$. This is generally not the case for advection. The advantage of this "influence" property means that the consensus dynamics are only based on relative states between neighboring agents as represented in (4); an absolute reference frame is not required. Consequently, the consensus dynamics can be driven by sensors that can only measure relative states, e.g., if x_i is the position of agent i, distance sensors mounted on agent i can measure the relative position to agent j as $x_j - x_i$. For advection, unless $d_i = \tilde{d}_i$, the dynamics of agent i can not be represented as purely a sum of relative states and so agent i must have knowledge of its state in a global frame.

We hope to demonstrate that for some applications the cost of maintaining an absolute frame is minimal and worth the benefits of conservation exhibited by the advection dynamics. The first noteworthy application is for information networks where an agent's information state is communicated between agents, e.g., a wireless sensor network where x_i is the ith sensor's state and agent j transmits its state to agent i if $(i,j) \in E$. Provided agent i knows its weighted "influence" on its neighbors d_i, no further information is required for the advection dynamics. The advection dynamics also has an extension to heterogeneous networks with more complex agents, able to sense their own state, selecting arbitrary out-degree edge weights, and simpler agents without global state and/or d_i knowledge, selecting weights such that $\tilde{d}_i = d_i$. It will be shown in the following proposition that if the graph is balanced, then advection dynamics is the same as consensus dynamics. For unbalanced graphs, there can be as many as $n-2$ agents in the network with advection dynamics identical to consensus dynamics requiring only relative state measurements, with only the remaining two agents with non-identical advection dynamics requiring knowledge of their state in a global frame.

Proposition 2. *The advection dynamics* (3) *and consensus dynamics* (5) *are identical for all initial* $x(0)$ *if and only if the underlying graph* $\mathcal{G}$ *is balanced.*

Proof: The dynamics are identical for all initial $x(0)$ if and only if $L_{\text{in}}(\mathcal{G}) = L_{\text{out}}(\mathcal{G})$, if and only if $\Delta_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}) = \Delta_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, if and only if $\Delta_{\text{in}}(\mathcal{G}) = \Delta_{\text{out}}(\mathcal{G})$, if and only if $d_i = \tilde{d}_i$ for all $i,j \in V$, i.e., $\mathcal{G}$ is balanced. ∎

An interpretation of advection over balanced graphs is that the flow divergence at each node is always zero which reduces the dynamics to diffusion. A consequence of Proposition 2 is that one can consider balanced graphs as the "perfect" discretization of the diffusion dynamics with both the sum conservation and the "influence" property.

Out-degree Laplacian

Many of the properties of consensus dynamics (5) are well understood and it is useful to relate them to advection dynamics (3) using their respective state matrices $L_{\text{in}}(\mathcal{G})$ and $L_{\text{out}}(\mathcal{G})$. We proceed to accomplish this by using the notion of a reverse graph $\mathcal{G}^r = (V, E^r, W^r)$ formed by reversing the edges of a graph $\mathcal{G} = (V, E, W)$, i.e., $(i,j) \in E$ with $w_{ji} \in W \iff (j,i) \in E^r$ with $w_{ij} \in W^r$. We then have the following proposition:

Proposition 3. *For a graph* $\mathcal{G}$ *and corresponding reverse graph* $\mathcal{G}^r$, $L_{\text{out}}(\mathcal{G}) = L_{\text{in}}(\mathcal{G}^r)^T$.

Proof: Directly from the definition of the reverse graph, we note that the out-degree of node i in $\mathcal{G}$ is the in-degree of node i in $\mathcal{G}^r$, consequently $\Delta_{\text{in}}(\mathcal{G}) = \Delta_{\text{out}}(\mathcal{G}^r)$. Further, $[\mathcal{A}(\mathcal{G})]_{ij} = w_{ij} = [\mathcal{A}(\mathcal{G}^r)]_{ji}$, so $\mathcal{A}(\mathcal{G}) = \mathcal{A}(\mathcal{G}^r)^T$. Therefore, noting that $\Delta_{\text{in}}(\mathcal{G})$ is a symmetric matrix, $L_{\text{out}}(\mathcal{G}) = \Delta_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}) = \Delta_{\text{in}}(\mathcal{G}^r) - \mathcal{A}(\mathcal{G}^r)^T = (\Delta_{\text{in}}(\mathcal{G}^r) - \mathcal{A}(\mathcal{G}^r))^T = L_{\text{in}}(\mathcal{G}^r)^T$. ∎

Using Proposition 3, we adapt existing properties of $L_{\text{in}}(\mathcal{G})$ to properties of $L_{\text{out}}(\mathcal{G})$. Let us define a classifications of graphs which we use in subsequent propositions.

Definition 1. A graph $\mathcal{G}$ is a *rooted branching* if
(1) it does not contain a directed cycle and
(2) it has a node v_r (root) such that

(a) for every other node $v \in G$ there is a directed path from v_r to v, in which case it is called a *rooted out-branching* or

(b) for every other node $v \in G$ there is a directed path from v to v_r, in which case it is a *rooted in-branching*.

We now state some known properties of $L_{\text{in}}(\mathcal{G})$ and equivalent properties of $L_{\text{out}}(\mathcal{G})$ for the cases where $\mathcal{G}$ contains a rooted out-branching, rooted in branching, and/or when $\mathcal{G}$ is balanced. Let $L_{\text{in}}(\mathcal{G}) = P_{\text{in}} J_{\text{in}}(\Lambda_{\text{in}}) P_{\text{in}}^{-1}$ be the Jordan decomposition of $L_{\text{in}}(\mathcal{G})$, P_{in} a nonsingular matrix with normalized columns, and Λ_{in} the eigenvalues of $L_{\text{in}}(\mathcal{G})$. Similarly, $L_{\text{out}}(\mathcal{G}) = P_{\text{out}} J_{\text{out}}(\Lambda_{\text{out}}) P_{\text{out}}^{-1}$.

Proposition 4. *[6], [9] The matrix* $L_{\text{in}}(\mathcal{G})$ *exhibits the following properties relating to* $\mathcal{G}$*:*
(a) if $\mathcal{G}$ *contains a rooted out-branching then*

(i) $\boldsymbol{rank}\, L_{\text{in}}(\mathcal{G}) = n-1$ *with* $L_{\text{in}}(\mathcal{G})\mathbf{1} = \mathbf{0}$ *and*

(ii) $$\text{J}_{\text{in}}(\Lambda) = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & J_{\text{in}}(\lambda_2) & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & J_{\text{in}}(\lambda_m) \end{bmatrix},$$ *where* $Re(\lambda_i) > 0$, *and* $J_{\text{in}}(\lambda_i)$ *is the Jordan block associated with eigenvalues* λ_i, $i = 2, \ldots, m$ *and* $\lambda_1 = 0$.

(b) if $\mathcal{G}$ *is balanced then*

(i)$\mathbf{1}^T L_{\text{in}}(\mathcal{G}) = \mathbf{0}^T$ *and*

(ii) $L_{\text{in}}(\mathcal{G}) + L_{\text{in}}(\mathcal{G})^T$ *is positive semidefinite.*

Corollary 5. *The matrix* $L_{\text{out}}(\mathcal{G})$ *exhibits the following properties relating to* $\mathcal{G}$*:*
(a) if $\mathcal{G}$ *contains a rooted in-branching then*

(i) $\boldsymbol{rank}\, L_{\text{out}}(\mathcal{G}) = n-1$ *with* $\mathbf{1}^T L_{\text{out}}(\mathcal{G}) = \mathbf{0}^T$ *and*

(ii) $$J_{\text{out}}(\Lambda) = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & J_{\text{out}}(\lambda_2) & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & J_{\text{out}}(\lambda_m) \end{bmatrix},$$ *where* $Re(\lambda_i) > 0$, *and* $J_{\text{out}}(\lambda_i)$ *is the Jordan block associated with eigenvalues* λ_i, $i = 2, \ldots, m$ *and* $\lambda_1 = 0$.

(b) if $\mathcal{G}$ *is balanced then*

(i) $L_{\text{out}}(\mathcal{G})\mathbf{1} = \mathbf{0}$ *and*

(ii) $L_{\text{out}}(\mathcal{G}) + L_{\text{out}}(\mathcal{G})^T$ *is positive semidefinite.*

Proof: If $\mathcal{G}$ has a rooted out-branching, then $\mathcal{G}^r$ has a rooted in-branching. Applying Proposition 3, $\boldsymbol{rank}\, L_{\text{in}}(\mathcal{G}) = \boldsymbol{rank}\, L_{\text{in}}(\mathcal{G})^T = \boldsymbol{rank}\, L_{\text{out}}(\mathcal{G}^r)$ and $L_{\text{in}}(\mathcal{G})\mathbf{1} = L_{\text{out}}(\mathcal{G}^r)^T \mathbf{1} = (\mathbf{1}^T L_{\text{out}}(\mathcal{G}^r))^T$. Using Proposition 4(a.i), then (a.i) follows. Now, $L_{\text{in}}(\mathcal{G})^T = (P_{\text{in}} J_{\text{in}}(\Lambda) P_{\text{in}}^{-1})^T = (P_{\text{in}}^{-1})^T J_{\text{in}}(\Lambda)^T P_{\text{in}}^T$. Applying Proposition 3, $L_{\text{out}}(\mathcal{G}^r) = P_{\text{out}} J_{\text{out}}(\Lambda) P_{\text{out}}^{-1}$ has the properties $P_{\text{out}} = (P_{\text{in}}^{-1})^T$ and $J_{\text{in}}(\lambda_2) = J_{\text{out}}(\lambda_2)^T$ and so $Re(\lambda_i)$ of $J_{\text{in}}(\lambda_i)$ and $J_{\text{out}}(\lambda_i)$ for all $i = 1, \ldots, m$ are equal. Using Proposition 4(a.ii), (a.ii) follows. If $\mathcal{G}$ is balanced, from Proposition 2 $L_{\text{out}}(\mathcal{G}) = L_{\text{in}}(\mathcal{G})$, thus using Proposition 4(b.i) and (b.ii), (b.i) and (b.ii) follows. ∎

Dynamics and Agreement

We now examine the dynamics of (3). The equilibrium $\tilde{x}_i$ for all $i \in V$ satisfies

$$d_i \tilde{x}_i = \sum_{(j,i)\in E} w_{ij}\tilde{x}_j. \quad (6)$$

The flow interpretation of this equilibrium is that the divergence of flow at every node goes to zero.

One significant feature of this equilibrium can be found when we define $w_{ij} = \frac{1}{\delta_i}$ for each $i \in V$. Therefore, $d_i = 1$ and the equilibrium condition is $x_i = \sum_{(j,i)\in E} \frac{1}{\delta_j} x_j$ for all $i \in V$. This equilibrium is exactly the rank metric used in the PageRank algorithm underlying the Google search engine [10]. The premise of the ranking is that a node i should be high ranked if (a) the rank metric of nodes linking to node i are high and (b) should be low if node i has few incoming edges or if the nodes linking to node i have a low rank metric.

The dynamics of consensus and advection can be represented in closed form as $x(t) = e^{-L_{\text{in}}(\mathcal{G})t}x(0)$ and $x(t) = e^{-L_{\text{out}}(\mathcal{G})t}x(0)$, respectively. Many features of these dynamics are a direct consequence of the characteristics of the matrices $e^{-L_{\text{in}}(\mathcal{G})t}$ and $e^{-L_{\text{out}}(\mathcal{G})t}$. These features are later exploited in the following proposition to reason about invariance and convergence properties of these dynamics.

Proposition 6. *[6] For $\psi_{\text{in}}(\mathcal{G}) = e^{-L_{\text{in}}(\mathcal{G})\delta}$, where $\delta > 0$,*

(i) $\psi_{\text{in}}(\mathcal{G}) \geq 0$, $\psi_{\text{in}}(\mathcal{G})$ is a right stochastic matrix, i.e., $[\psi_{\text{in}}(\mathcal{G})]_{ij} \geq 0$ and $\psi_{\text{in}}(\mathcal{G})\mathbf{1} = \mathbf{1}$, and

(ii) $[\psi_{in}(\mathcal{G})]_{ij} > 0 \iff i = j$ or there is a directed path from j to i in $\mathcal{G}$.

Corollary 7. *For $\psi_{\text{out}}(\mathcal{G}) = e^{-L_{\text{out}}(\mathcal{G})\delta}$, where $\delta > 0$,*

(i) $\psi_{\text{out}}(\mathcal{G}) \geq 0$, $\psi_{\text{out}}(\mathcal{G})$ is a left stochastic matrix, i.e., $[\psi_{\text{out}}(\mathcal{G})]_{ij} \geq 0$ and $\mathbf{1}^T\psi_{\text{out}}(\mathcal{G}) = \mathbf{1}^T$, and

(ii) $[\psi_{\text{out}}(\mathcal{G})]_{ij} > 0 \iff i = j$ or there is a directed path from i to j in $\mathcal{G}$.

Proof: Applying Proposition 3, $\psi_{\text{in}}(\mathcal{G}^r) = e^{-L_{\text{in}}(\mathcal{G}^r)\delta} = e^{-L_{\text{out}}(\mathcal{G})^T\delta} = \left(e^{-L_{\text{out}}(\mathcal{G})\delta}\right)^T = \psi_{\text{out}}^T(\mathcal{G})$. Using Proposition 6, the Corollary follows. ■

The familiar property that a balanced graph $\mathcal{G}$ exhibits a doubly stochastic ψ_{in} is a consequence of Proposition 6, the related Corollary 7, and for balanced $\mathcal{G}$, $L_{\text{in}}(\mathcal{G}) = L_{\text{out}}(\mathcal{G})$ (Proposition 2). A result of this property is that, for consensus dynamics over a balanced graph, the state of the nodes, at any instant in time, is a convex combination of the values of all the nodes at a previous instant in time.

Proposition 8. *The advection dynamics are positively invariant over $x_i \geq 0$ for all $i \in V$, i.e., if $x_i(0) \geq 0$ for all $i \in V$ then $x_i(t) \geq 0$ for all $i \in N$ for all $t > 0$.*

Proof: From Corollary 7, $e^{-L_{\text{out}}(\mathcal{G})t}$ is a nonnegative matrix so $x(t) = e^{-L_{\text{out}}(\mathcal{G})t}x(0)$ is nonnegative for $x_i(0) \geq 0$ for all $i \in N$ and all $t > 0$. ■

The following Proposition 9 for the consensus dynamics characterizes the equilibrium where $\mathcal{G}$ contains a rooted out-branching. Equivalently, Proposition 10 for the advection dynamics characterizes the equilibrium where $\mathcal{G}$ contains a rooted in-branching.

Proposition 9. *[6] For a graph $\mathcal{G}$ containing a rooted out-branching, the consensus dynamics (5), initialized from $x(0) = x_0$, satisfies*

$$\lim_{t\to\infty} x(t) = \frac{1}{\sqrt{n}}\left(u^T x_0\right)\mathbf{1},$$

where $u = \bar{u}/\|\bar{u}\|$ and $\bar{u}^T L_{in}(\mathcal{G}) = 0$. Further, the quantity $u^T x_0$ is conserved and $u_i > 0$ for all $i \in V$ if and only if $\mathcal{G}$ is strongly connected[1].

Proposition 10. *For a graph $\mathcal{G}$ containing a rooted in-branching, the advection dynamics (3), initialized from $x(0) = x_0$, satisfies*

$$\lim_{t\to\infty} x(t) = \frac{1}{\sqrt{n}}\left(\mathbf{1}^T x_0\right)v,$$

where $v = \bar{v}/\|\bar{v}\|$ and $L_{\text{out}}(\mathcal{G})\bar{v} = 0$. Further, $v_i > 0$ for all $i \in V$ if and only if $\mathcal{G}$ is strongly connected.

Proof: Noting that $x(t) = e^{-L_{\text{out}}(\mathcal{G})t}x_0$ and $\lim_{t\to\infty} e^{-L_{\text{out}}(\mathcal{G})t} = vw^T$, from the rank condition and Jordan decomposition in Corollary 5, where the first column of P_{out} and first row of P_{out}^{-1} are v and w^T respectively (i.e., the normalized right and left eigenvectors associated with the zero eigenvalue). Choosing $w = \frac{1}{\sqrt{n}}\mathbf{1}$ from Corollary 5 one has

$$\lim_{t\to\infty} x(t) = \left(\frac{1}{\sqrt{n}} v\mathbf{1}^T\right)x_0 = \frac{1}{\sqrt{n}}\left(\mathbf{1}^T x_0\right)v.$$

Since $\mathcal{G}$ is strongly connected, for every node pair i and j there exists a directed path in $\mathcal{G}$, and so for every node pair j and i there exists a reverse directed path. Consequently, $\mathcal{G}^r$ is also strongly connected. From Proposition 9, the left null space u of $L_{\text{in}}(\mathcal{G}^r)$ has all positive elements, therefore, from Proposition 3, since $L_{\text{out}}(\mathcal{G}) = L_{\text{in}}(\mathcal{G}^r)^T$ the right null space v of $L_{\text{out}}(\mathcal{G})$ has all positive elements. ■

The following proposition relating to the equilibrium state of the consensus dynamics over a balanced, rooted in-branching $\mathcal{G}$ is a consequence of Propositions 9 and 10, the fact that the consensus and advection dynamics are identical (Proposition 2), and that if a graph is balanced and has a rooted in-branching then it also has a rooted out-branching.

Proposition 11. *[6] The consensus dynamics (and hence advection dynamics) over a graph $\mathcal{G}$ reaches $x(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x_0$ for every initial condition if and only if $\mathcal{G}$ is balanced and contains a rooted in-branching (and hence a rooted out-branching).*

The results of this section are summarized in Figure 1 which is structured to highlight the differences between the two sets of dynamics over rooted in/out-branching graphs and the culmination of these features in balanced graphs.

[1] A graph $\mathcal{G}$ is strongly connected if between every pair of distinct vertices there exists a directed path.

IV. Examples

Next we showcase two applications of advection dynamics to networked, multi-agent systems. We focus on two examples that of formation control and a sensor coverage problem.

A. Example 1 - Formation Control

The formation control problem where multi-agent teams form geometric patterns is a popular application for consensus [6]. With consensus, the formation is acquired by reaching a consensus on an origin in space and then each agent is coded with a reference position relative to this origin. We now proceed to perform the same task using advection where the geometry of the configuration is controlled by the weights of the edges.

We consider a swarm of n vehicles moving along the x-axis with velocity $\nu \in \mathbb{R}$. The objective is for the swarm to move in formation with a predefined shape. We assume each vehicle is aware of its global position and that each vehicle is able to measure the relative position to one other predesignated vehicle, thus acquiring its global position.

Let $\bar{x}(t) \in \mathbb{R}^n$ and $\bar{y}(t) \in \mathbb{R}^n$ be the position of vehicles at time t along the x and y axes, respectively. The origin of the coordinate system is selected such that $\mathbf{1}^T \bar{x}_i(0) \neq 0$ and $\mathbf{1}^T \bar{y}_i(0) \neq 0$. The advection-based dynamics of the vehicles are

$$\begin{aligned} \dot{x}(t) &= -L_{\text{out}}(\mathcal{G}_x)x(t) \\ \dot{y}(t) &= -L_{\text{out}}(\mathcal{G}_y)y(t), \end{aligned} \tag{7}$$

where $x(t) := \bar{x}(t) - \nu t\mathbf{1}$ and $y(t) := \bar{y}(t)$. The graphs $\mathcal{G}_x = (V, E, W_x)$ and $\mathcal{G}_y = (V, E, W_y)$ are directed cycle graphs[2] with positive weights $\left\{w^x_{i,j}\right\} \in W_x$ and $\left\{w^y_{i,j}\right\} \in W_y$, respectively. The edge set E corresponds to the inter-agent sensing where if vehicle i can measure the position of vehicle j then there is an edge $\{i, j\} \in E$. From (6), for the advection dynamics with underlying graph $\mathcal{G}^x$ the equilibrium $(x, y) = (\tilde{x}, \tilde{y})$ is $\tilde{x}_i = \alpha_x / w^x_{i,i+1}$ for all $i = 1, \ldots, n-1$ and $\tilde{x}_n = \alpha_x / w^x_{n,1}$ where $\alpha_x = \mathbf{1}^T x_0 / \sum_{\{j,i\} \in E} \left(1/w^x_{i,j}\right)$. As all weights are positive and $x_i(0) > 0$ then $\alpha_x > 0$. Similarly, for equilibrium $\tilde{y}$, constant $\alpha_y > 0$ and corresponding to graph $\mathcal{G}^y$. Therefore, the weight selection completely dictates the *shape* of the equilibrium with the scaling of the shape is dictated by α_x and α_y. From Proposition 1, the sum of states is always constant and so the centroid of the formation is $\left(\frac{1}{n}\mathbf{1}^T x(0) + vt, \frac{1}{n}\mathbf{1}^T y(0)\right)$.

We apply this advection formation technique to 6 vehicles moving along the x axis with velocity $\nu = 2m/s$ and a required constellation with shape defined in an arbitrary reference frame as $x_s = (1, 2, 1, 3, 2, 1)^T$ and $y_s = (1, 2, 0, 0, -2, -1)^T / \sqrt{2}$; see Figure 2 for shape. Let $\left[\begin{array}{cccccc} w^x_{12} & w^x_{23} & w^x_{34} & w^x_{45} & w^x_{56} & w^x_{61} \end{array}\right] = \left[\begin{array}{cccccc} 1 & \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{2} & 1 \end{array}\right]$ and $\left[\begin{array}{cccccc} w^y_{12} & w^y_{23} & w^y_{34} & w^y_{45} & w^y_{56} & w^y_{61} \end{array}\right] = \sqrt{2}\left[\begin{array}{cccccc} 1 & 1/2 & 0 & 0 & -1/2 & -1 \end{array}\right] + 2$ where this weight selection was chosen for ease of demonstration and is not the only selection that will satisfy a certain formation geometry. The vehicle trajectory initialized randomly in the first quadrant is displayed in Figure 2 with the desired final formation achieved.

[2]A directed cycle graph $\mathcal{G} = (V, E, W)$ is a n node graph with $\{i+1, i\} \in E$ and edge weight $w_{i,i+1} \in W$ for $i = 1, \ldots, n-1$ as well as $\{1, n\} \in E$ with edge weight $w_{n,1} \in W$.

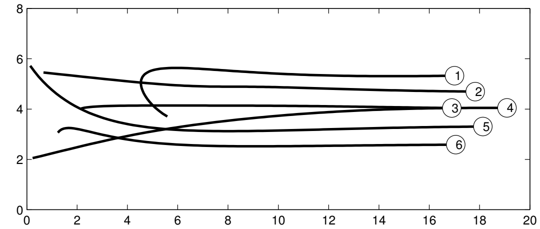

Figure 2. Advection formation control with initial random condition and position after 8sec marked with circles.

B. Example 2 - Sensor Network

We consider a sensor surveillance task, operating in a land corridor of length d_w, where the sensors are directional with a narrow cone of observation. A set of n sensors are randomly placed pointing west in the corridor, which is oriented east to west. The objective is to acquire good coverage of the corridor while satisfying a total sensor power constraint by trading power between neighboring sensors. Let $x_i(t) \geq 0$ be the fraction of total network power and $z_i(t) \geq 0$ be the coverage of sensor i at time t. We utilize a coverage model of the form $z_i(t) = \beta x_i(t)^3$ (single-ray range sensor or wedge-shaped sensor with fixed arc-length) for some $\beta > 0$. We subsequently require that the total power of the sensor network is constant, i.e., $\sum x_i(t) = 1$ for all t, while maintaining good coverage. We will now discuss our measure of "good coverage", which aims to minimize gaps between sensor observation areas along the east-west axis.

Assuming the minimal power is available to guarantee no gaps between sensors areas, the corresponding optimal power for each sensor i would be $x^*_i = \frac{\zeta_i}{d_w}$, where ζ_i is the distance to the closest sensor to the east of sensor i. Consider a local area around a point $\boldsymbol{p} \in \mathbb{R}^2$ in the corridor, with a set u^+ of δ^+_i sensors west of $\boldsymbol{p}$, and a set u^- of δ^-_i sensors east of $\boldsymbol{p}$. Our local coverage cost function is $c_i(\boldsymbol{p}) = \left|\sum_{k \in u+} z_k - \sum_{k \in u-} z_k\right|$. The cost function penalizes nonuniform coverage east and west of point $\boldsymbol{p}$. For an infinite corridor, $x^* = \text{argmin}_x \int_0^{d_w} c_i(\rho, 0)\, d\rho$. If the coverage for all $k \in u^-$ is approximately equal to z^-, $c_i = \left|\sum_{k \in u+} z_k - \delta^-_i z^-\right|$. Ideally then,

$$z^- = \frac{1}{\delta^-_i} \sum_{k \in u+} z_k, \tag{8}$$

and so if we were to place a sensor i at location $\boldsymbol{p}$ then we assume for "good coverage" $z_i \approx z^-$. An interpretation of this equilibrium is that coverage of an area west of the sensor i which is $\sum_{k \in u+} \tilde{z}_k$ should be maintained in an equal area east of sensor i where there are already located δ^-_i sensors,

[3]This work as also been extended to $z_i(t) = \beta x_i(t)^p$ for $p > 0$.

so sensor i is responsible for approximately $\frac{1}{\delta_i^-}$ of the area east of $\boldsymbol{p}$, i.e., $\frac{1}{\delta_i^-}\sum_{k\in u+}\tilde{z}_k$.

Let the position of sensor i be $(p_i, q_i) \in \mathbb{R}^2$ (where the x axis points east and the y axis north). We assume, since the sensors are in a land corridor, that $\max_{i\in N}(q_i) - \min_{i\in N}(q_i)$ is small, and subsequently, if the communication range on sensors is $d_c > 0$, $d_c \in \mathbb{R}$, that sensor i is in communication range of sensor j if $|p_i - p_j| \le d_c$. Additionally we assume that if sensors are sufficiently close to the east or west end of the corridor, communications can be relayed between them via some infrastructure at the ends of the corridor. A communication graph $\mathcal{G} = (V, E, W)$ is defined such that if sensor j is within communication range of sensor i and $p_j > p_i$ then $(i, j) \in E$ with the exception of the sensors at the ends of the corridor where if $p_j - p_i \ge d_w - d_c$ then $(j, i) \in E$. The graph is unweighted with $w_{ij} = 1$ for all $(j, i) \in E$.

Assuming d_c is large enough to form a rooted in-branching, and $\delta_i > 0$ for all $i \in V$, then $\mathcal{G}$ will be strongly connected. Applying the advection dynamics (3), our equilibrium power $\tilde{x}$ will satisfy

$$\tilde{x}_i = \frac{1}{\delta_i}\sum_{(k,i)\in E}\tilde{x}_k,$$

and $\tilde{x}_i > 0$ for all $i \in V$ if $\sum x_i(0) = \mathbf{1}^T x(0) > 0$ (Proposition 10). The corresponding equilibrium coverage $\tilde{z}$ is $\tilde{z}_i = \frac{1}{\delta_i}\sum_{(k,i)\in E}\tilde{z}_k$ for all $i \in V$. As previously discussed, this is our condition (8) for "good coverage" whereby the coverage of the area d_c west of the sensor maintained d_c east of sensor i. An interesting aspect of this formulation is that the unweighted network topology is being exploited to infer inter-sensor distance information and hence coverage density characteristics.

We assume that all sensors are initialized with a feasible power, i.e., $\sum_{i=1}^n x_i(0) = 1$ and $x_i(0) \ge 0$ for all $i \in V$. From Proposition 1, the total power of the sensor network will be conserved, i.e., $\sum_{i=1}^n x_i(t) = 1$ for all time t. As advection is positively invariant over the nonnegative $x_i(t)$, from Proposition 8, the power will always be nonnegative, i.e., $x_i(t) \ge 0$, for all $i \in V$.

We apply this approach to a $d_w = 40$m long land corridor containing 40 randomly placed sensors. The initial power fraction was assigned randomly and $d_c = 1.75$ m dictates the topology of the flow graph. The final equilibrium power $\tilde{x}(t)$ overlayed on the graph $\mathcal{G}$ is displayed in Figure 3. Figure 4 depicts the observation cones for a) the optimal power usage from all sensors, b) the equilibrium power usage obtained using an arc-length fixed to that of the corridor's width, and c) a uniform power usage for all sensors. We find that the minimum power requirement by the advection power distribution is within 1.25 times of the optimal power. This is compared to a uniform sensor power which required 2.5 times the optimal power.

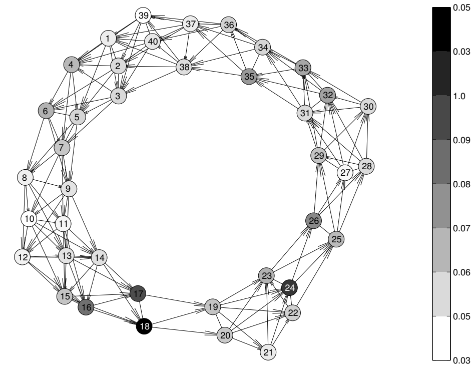

Figure 3. Sensor graph with grayscale gradations corresponding to equilibrium power $x(t)$. Nodes are numbered from west to east.

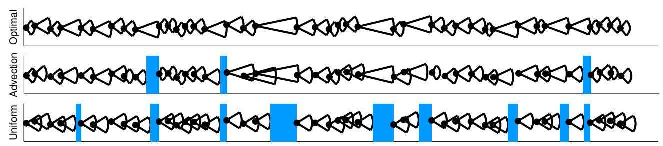

Figure 4. Optimal, advection dynamics and uniform sensor coverage of the land corridor. The shaded bands indicate areas not covered by the sensors.

V. Conclusion

This paper presents an advection-based approach to multi-agent cooperative control. We compare the advection dynamics to the popular consensus dynamics but also make comments on novel properties that are only held by advection. One property of particular significance is the conservation of the sum of the states. We demonstrate the utility of this property in such applications as sensor coverage where power can be traded through the network to optimize coverage. Because of the parallels with consensus dynamics, there is a large area of future advection research involving the application of advection to problems traditionally solved by consensus. One application of particular interest is the introduction of control nodes, which do not conform to advection, into an advection-based network.

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