Adverse Selection in the Annuity Market and the Role for Social Security[*]

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Abstract

This paper studies the role of social security in providing annuity insurance. I calculate the welfare cost of adverse selection in the annuity market using a life cycle model in which individuals have private information about their mortality. I calibrate the model to the current U.S. social security replacement ratio, fraction of annuitized wealth and mortality heterogeneity in the Health and Retirement Study. My findings are as follows. First, in the absence of social security, individuals (on average) maintain about the same fraction of annuitized wealth as they do in the presence of social security, despite the fact that prices in the market are actuarially unfair. As a result, the welfare loss of abolishing social security is only 0.15 percent (in terms of consumption). Second, there is an ex ante gain of 0.51 percent from implementing the ex ante efficient allocations, which comes from redistributing resources from high mortality types to low mortality types. Individuals with high mortality (who will die soon and do not have demand for longevity insurance) incur large welfare losses from mandatory participation. These losses offset the benefits of providing insurance to low mortality types, leaving the overall ex ante welfare gain small.

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1   Introduction

Mandatory annuitization is a key feature of the current U.S. social security system. Its value is derived from its ability to overcome potential inefficiencies due to adverse selection in the annuity market.[1] The purpose of this paper is to determine the value of mandatory annuitization in the current U.S. social security system using a quantitative framework in which informational frictions in the annuity market are explicitly modeled. I will bias this exercise toward finding an upper bound for the value of mandatory annuitization, and I conclude that this value is small.

To do this, I develop a dynamic life cycle model in which individuals have private information about their mortality. Uncertainty about time of death generates demand for longevity insurance. In this environment individuals can purchase annuity and life insurance contracts at linear prices. I assume that contracts are non-exclusive and insurers cannot observe individuals' trades. The lack of observability implies that insurers cannot classify individuals by their risk type. As a result, the unit price of insurance coverage is identical for all agents. Individuals with higher mortality (who on average die earlier) demand little insurance (or nothing at all). This makes lower mortality types (types with higher risk of survival) more represented in the market. That in turn leads the equilibrium price of annuities to be higher than the overall actuarially fair value of their payment.

In this environment, I define and characterize the set of ex ante efficient allocations. I show that these allocations are independent of individuals' mortality risk type and are only contingent on survival, which is publicly observed. This feature implies that ex ante efficient allocations can be implemented by a system of mandatory annuitization in which every individual is taxed, lump sum, before the retirement and receives a benefit contingent on survival after retirement.[2] The ex ante efficient allocation will be the benchmark for the best outcome that any social security system can achieve.

The environment I study has three important features. First, it abstracts from any heterogeneity other than mortality types (e.g., in tastes, bequest motives, abilities, income shocks, etc.). It also abstracts from any distortionary effects of policy on labor supply and

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[1] Existence of adverse selection is well-documented by Finkelstein and Poterba (2002, 2004, 2006) and by Friedman and Warshawsky (1990) and McCarthy and Mitchell (2003) among others.

[2] The ex ante efficient allocation is achieved by forcing individuals with higher mortality (who on average die earlier) to pool with those of higher mortality (who on average die later). In a decentralized environment in which the choice of participation in the insurance pool is not mandatory, the participation of higher mortality types is always less than efficient.

retirement decisions. This enables me to focus only on the inefficiencies caused by adverse selection. Furthermore, it implies that optimal policies are uniform across individuals. This gives a uniform mandatory annuitization policy the best chance to produce large welfare gains. Second, I assume individuals know all the information about their mortality risk type at the beginning of time. This assumes away any possibility of insuring against the realization of risk type in the market, exaggerates the effect of adverse selection, and hence provides an upper bound of the usefulness of policy. Finally, studying the annuitization over the life cycle, as opposed to a decision at retirement, enables me to highlight how individuals make decisions over their life and prepare for retirement based on their private expectation about the time of death.

The quantitative exercise of this paper consists of welfare comparisons between three economies: 1) an economy with no social security in which individuals share their longevity risk only through the annuity and life insurance markets, 2) the same economy with the addition of a social security system that is calibrated to the current U.S. system, and 3) an economy in which ex ante efficient allocations are implemented.

The key quantitative object in the model is the distribution of mortality risk types. This distribution determines the extent of private information in the economy. Following the demography literature, I model heterogeneity in mortality risk as a *frailty* parameter that shifts the force of mortality.[3]This parameter, once realized at birth, stays constant throughout one's lifetime. Individuals with a higher frailty parameter are more likely to die at any given age. I parameterize the initial distribution of mortality types (frailty) and use the data on subjective survival probabilities in the Health and Retirement Study (HRS) to estimate those parameters.[4]I calibrate the model to the current U.S. social security replacement ratio and choose the preference parameters to match the fraction of annuitized wealth through social security and pension at retirement in the HRS.

The three main findings of the paper as follows: 1) the overall welfare loss from removing social security is 0.15 percent of consumption; 2) a considerable fraction of low mortality types (20 percent of population) prefer the economy without social security to the one that has current the U.S. system's replacement ratio. This result is contrary to the conventional wisdom that mandatory insurance is beneficial to high risk individuals (those who are at risk

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[3]See, for example, Butt and Haberman (2004), Vaupel et al. (1979), and Manton et al. (1981).

[4]Hurd and McGarry (1995, 2002) and Smith et al. (2001) document that these probabilities are consistent with life tables and ex post mortality experience. They argue that they are good predictors of individuals' mortality.

of survival to old age); 3) the overall welfare gain from ex ante efficient allocation over the market equilibrium without social security is 0.51 percent.

To understand the intuition for these results, I look at the effect of the presence of social security on individuals' participation in the annuity market and on equilibrium prices. In the presence of social security, half of the population (those with higher than average mortality) are not active in the annuity market.[5] These individuals get more annuitization than they need from social security. On the other hand, individuals with lower than average mortality, expecting longer life spans, accumulate more assets and have higher demand for annuitized wealth. These individuals purchase annuities in the market. However, since higher mortality types (good risk types) are not in the market, the equilibrium price is high.[6]

Two groups at the tails of mortality type distribution gain by removing social security. The first group consists of individuals with the highest mortality type (about 7 percent of the population). These individuals die very early, and the majority of them do not survive long enough to collect significant benefits from social security. They prefer an economy without social security because it gives them the freedom to choose low levels of annuitization. They gain up to 2 percent of consumption from removing social security (even though they get a lower return for their money than the implicit rate of return under social security). The second group are individuals with the lowest mortality type (20 percent of the population). These individuals annuitize a significant fraction of their wealth in the private annuity market even in the presence of social security. They gain up to 1 percent (in consumption terms) from removing social security because of the improved quality of the risk sharing of the pool in the private annuity market. This improvement results from the increased participation of high mortality types. These changes in the overall quality of the pool lower the price of annuities, resulting in a welfare gain for the lowest mortality types.

On the other hand, individuals in the middle of mortality distribution (the middle 73 percent) can lose up to 0.5 percent from removing social security. In the presence of social security, these individuals receive considerable transfers from higher mortality types. In the absence of social security, they participate in the annuity market and are outlived by individuals with lower mortality. As a result, they are worse off in an exclusively private

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[5]This is consistent with the evidence in the HRS data that only 43 percent of all adults hold pensions in their own names (Johnson et al. (2004)).

[6]The effect of social security on annuity prices was first studied by Abel (1986) and in more detail by Hosseini (2007). Walliser (2000) has investigated this issue quantitatively and found that removing social security can reduce the price of annuities up to 3 percent.

system. These two effects make the economy with social security more appealing to them. Overall, the gains and losses across these three groups almost offset each other, resulting in a loss of 0.15 percent from removing social security. This gain comes from forcing the highest mortality types to pool risk with lower mortality types–something they would never do on their own if given the choice.

In addition to the main results, I perform several sensitivity checks. I find that, contrary to common wisdom, increasing the degree of risk aversion in preferences does not lead to higher welfare gains from mandatory annuitization. At higher risk aversions, an individual of high mortality demands more insurance and at any given price is willing to participate more in the market. On the other hand, lower mortality types (who are generally over-insured) have a stronger preference for a smooth path of consumption. They increase their consumption in earlier periods by reducing their demand for annuities. This results in a flatter profile of the annuity purchase and a lower equilibrium price. Therefore, it is true that at higher levels of risk aversion the social value of insurance is higher, but at the same time there is better insurance available in the market, and the distance between equilibrium allocations and ex ante efficient allocations is reduced. Consequently, even when I repeat the welfare comparison with assuming a high degree of risk aversion, the welfare gains from mandatory annuitization are not large.

1.1 Related literature

This paper is related to three strands of literature. The first strand of the related literature focuses on the potential welfare-improving role for mandatory insurance in an environment with adverse selection. This was pioneered by Akerlof (1970), Rothschild and Stiglitz (1976) and Wilson (1977) in their seminal contribution that started the literature. [7] The role of mandatory annuitization in the annuity market with adverse selection was first studied by Eckstein et al. (1985) and Eichenbaum and Peled (1987). The contribution of this paper is the quantitative assessment of the welfare gains due to mandatory annuitization.

Second, this paper is related to the large literature measuring the insurance value of annuitization for representative life cycle consumers (e.g., Kotlikoff and Spivak (1981), Mitchell et al. (1999), Brown (2001), Brown et al. (2005)).[8] The exercise in these articles is to determine how much incremental, nonannnuitized wealth would be equivalent to providing access

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[7]See Dionne et al. (2000) for an excellent survey on theories of insurance markets with adverse selection.
[8]Brown (2003) and Gong and Webb (2006) consider observable heterogeneous types and study the redistribution impact of mandatory annuitization.

to actuarially fair annuity markets. A robust finding of this approach is that a 65-year-old adult with population average mortality gains up to 30 to 50 percent of his retirement wealth from access to actuarially fair insurance.[9] A key feature of all these studies is the static comparison between full insurance and no insurance at all.[10] In contrast, in this paper I allow for the annuitization through private annuity markets over the life cycle.[11] This allows me to distinguish between risk sharing that is provided by the market and to study how it changes in response to changes in publicly provided insurance.

Welfare gains from provision of annuity insurance in social security is also studied by Hubbard and Judd (1987), İmrohoroğlu et al. (1995) and Hong and Ríos-Rull (2007). None of these papers study an environment with adverse selection, which is the friction that makes mandatory annuitization valuable in my framework. The present paper is the first study that attempts to evaluate the welfare gain of mandatory annuitization in a dynamic environment in which adverse selection is explicitly modeled.[12]

Social security is a large program with many purposes. [13] On the normative side, its role is broadly categorized by Diamond (1977) into income redistribution, provision of insurance (when there is market failure) and paternalism toward irrational savings by individuals. These aspects of social security have been studied extensively in the literature. [14] For tractability reasons, in this paper I abstract from many features of social security and focus only on one of the many possible benefits, i.e., mandatory annuitization. This is an obvious advantage that a government system has (in imposing participation by everyone) that no

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[9]Gong and Webb (2006) did this exercise allowing for pre-annuitized wealth (e.g., through defined benefit pensions) and still found a big welfare gain (9 percent).

[10]A large part of this welfare gain comes from the fact that in the absence of any longevity insurance, individuals should rely on their savings in liquid assets. Therefore, with positive probability each individual dies with positive assets. Upon their death their assets evaporate from the economy. On the other hand, under full insurance these assets are annuitized. Upon individual's death, his/her assets are transfered to those who survived (the assets do not leave the economy).

[11]Butrica and Mermin (2006), Dushi and Webb (2002), Johnson et al. (2004), Moore and Mitchell (1997), and Poterba et al. (2003) document that a significant fraction of the wealth of retired adults is annuitized. In particular, Butrica and Mermin (2006) find that 10 percent of the wealth at retirement is annuitized through private annuities and pensions and about 45 percent through social security.

[12]A recent paper by Einav et al. (2007) assumes heterogeneity in mortality as well as preferences and measures the welfare cost of asymmetric information in the annuity market by using a structural estimation method to estimate a model of guarantee choice using data on the U.K. annuity market.

[13]See Mulligan and Sala–i–Martin (1999a,b) for an extensive survey on normative and positive theories of social security.

[14]For example, Golosov and Tsyvinski (2006) study the disability insurance aspect of social security and Gottardi and Kubler (2006) evaluate the role of social security in improving intergenerational risk sharing. Finally, Emre (2006) points out to the positive role of mandatory savings in social security when there is lack of commitment by the government.

private market system can mimic. I study an environment in which this is the only role for social security.

The paper is organized as follows. Section 2 describes the environment, defines and characterizes efficient allocations, and introduces the equilibrium notion. Section 3 contains a two-period example that highlights some features of the environment. Section 4 contains the parametric specification and calibrations. Section 5 reports the results, and Section 6 concludes.

2 Model

In this section I first describe the environment. Then, I describe two separate allocation mechanisms. The first one is the ex ante efficient allocation in which a social planner chooses the allocation to maximize ex ante welfare subject to informational and feasibility constraints. After describing that, I introduce market for survival contingent contracts and define a competitive equilibrium, which is the second allocation mechanism I consider. The ultimate goal of the paper is to compare the welfare under these two mechanisms.

2.1 Information

Consider a continuum economy with atom-less measure space of agents' labels $(I, \mathcal{I}, \iota)$. The economy starts at date zero and ends at date $\infty > T \geq 1$. Individuals are born at the beginning of period zero and face an uncertain life span. An individual who survives to age t faces the uncertainty of surviving to age $t + 1$ or dying at the end of age t. Anyone who survives to age T will die at the end of that age. I index the survival state at date t by $s_t \in S = \{0, 1\}$, in which 1 means the individual has survived to age t and 0 means the individual has died before age t. Agents' survival is an ex post state of the world in the sense that it is realized after all trade decisions are made. There is also a set of possible individual types or characteristics, Θ. Individuals' type, $\theta \in \Theta$, determines their likelihood of survival in each period. I assume that $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq \mathbb{R}_+$ and $\theta \in \Theta$ is the index of frailty. Individuals with lower θ have a higher probability of survival (and a longer expected lifetime). I assume θ is private information and known only by the individual. Furthermore, I assume $\theta \in \Theta$ is an ex ante state of the world and is realized before any transaction takes place.

To sum up, there are three sets of possible economic agents. First, there is a set I of labels (without loss of generality, we can assume this is a unit interval). Second, there is

the set $I \times \Theta$ of possible type-contingent agents, indexed with their label and their ex ante (private) types. And finally, there is the set $I \times \Theta \times S^T$ of possible types and survival contingent individuals indexed by their label $i \in I$, ex ante (private) types $\theta \in \Theta$ and ex post survival state $s_t \in S^t$. Note that if $s_t = 0$, then $s_{t'} = 0$ for all $t' > t$.

Suppose there is a well-defined distribution $G_0 \in \Delta(\Theta)$ with full support. Suppose each type realization $\theta \in \Theta$ determines the conditional probability of survival to date t in period zero (i.e., conditional probability that $s_t = 1$). I denote this conditional probability by $P_t(s_t = 1|\theta)$, or $P_t(\theta)$ for short. Therefore, the joint probability that an individual's type is in the set $Z \subseteq \Theta$ and survives to period t is $\mu_t(Z, s_t = 1) = \int_{\theta \in Z} P_t(\theta) dg_0(\theta)$.

Type realization and survival (conditional on types) are i.i.d. and there is no aggregate uncertainty about distribution of types and actual survival for the agents in any subset of I.[15] In other words, let $A \subseteq I$ be any non-zero measure subset of I. Then exactly $\iota(A)$ fraction of agents have labels in A. Furthermore, fraction $G_0(Z)$ of agents with label in A have type $\theta \in Z \subseteq \Theta$, and out of this fraction exactly $\mu_t(Z, s = 1)$ will survive through period t (conditional on being alive in period zero).

2.2 Preferences

Individuals have time separable utility over consumption, $u(\cdot)$, as long as they live. They also get utility from leaving a bequest at the time of death, $v(\cdot)$. These functions are assumed to be twice continuously differentiable with $u', v' > 0$ and $u'', v'' < 0$ and satisfy the usual INADA conditions. Let $x_t(\theta) = \frac{P_{t+1}(\theta)}{P_t(\theta)}$ be one-period conditional survival probability for type θ (probability of surviving to age $t+1$ condition on being alive at t). Then type θ's utility out of a given sequence of consumption, c_t, and bequest, b_t, is

$$\sum_{t=0}^{T} P_t(\theta) \beta^t [u(c_t) + (1 - x_{t+1}(\theta)) \beta v(b_t)], \quad 0 < \beta \leq 1.$$

Preference for bequest can be motivated and interpreted in several ways. I follow Abel and Warshawsky (1988) and interpret it as altruism towards future generation (or surviving spouse) whereby, $v(b)$ stands for reduced form lifetime value function of a child that is born after the individual's death and receives bequest b (or simply the surviving spouse that lives for fix number of periods after the agent dies).[16]

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[15]Subject to the usual caveat on continuum of i.i.d. random variables. See Judd (1985) and Uhlig (1996).

[16]Look also at Braun and Muermann (2004) for an interesting interpretation based on *regret* motive.

Each individual is endowed with a unit of labor endowment which is inelastically supplied for constant wage w in every period $t \leq J < T$ (after period J the individual cannot work). There is also a saving technology with gross rate $R \geq \frac{1}{\beta}$.

An *allocation* is a map from agents' label, type, and survival state to positive real line, i.e.,

$$c_t : I \times \Theta \times S^t \to \mathbb{R}_+ \quad 0 \leq t \leq T$$

$$b_t : I \times \Theta \times S^{t+1} \to \mathbb{R}_+ \quad 0 \leq t \leq T.$$

I will focus on symmetric allocations that depend only on type θ and survival and not an individual's label. Furthermore, since the agents do not care about consumption in the state in which they are dead (and about bequests in the state in which they are alive), I will drop the realization of the survival state from the argument of the allocation function. Therefore, it is understood that $c_t(\theta)$ is the consumption of all θ type individuals condition on their survival at age t (and similarly $b_t(\theta)$ is the bequest that type θ leaves if he dies at the end of age t). An allocation is *feasible* if

$$\int \sum_{t=0}^{T} \frac{P_t(\theta)}{R^t} \left[c_t(\theta) + \frac{(1 - x_{t+1}(\theta))}{R} b_t(\theta) \right] dG_0(\theta) = w \int \sum_{t=0}^{J} \frac{P_t(\theta)}{R^t} dG_0(\theta). \tag{1}$$

In the environment described above, the agents face the risk of outliving their assets. Also, from the ex ante point of view (before birth), agents face the risk of their type realization. Individuals whose type θ imply a higher survival probability need more resources to finance consumption through their lifetime relative to those types who have lower survival. Therefore, there is a need for insurance against these risks. Next, we study the ex ante efficient allocations as a benchmark that provides perfect insurance against both types of risks.

2.3 Ex ante efficient allocations

Consider the problem of a social planner who maximizes the expected discounted utility of agents behind the veil of ignorance, i.e., before agents are born.

$$\max_{c_t(\theta), b_t(\theta) \geq 0} \int \sum_{t=0}^{T} P_t(\theta) \beta^t [u(c_t(\theta)) + (1 - x_{t+1}(\theta)) \beta v(b_t(\theta))] dg G_0(\theta)$$

subject to (1)

It is straightforward to verify that the allocations that solve the above problem must satisfy

$$c_t(\theta) = c_t(\theta') = c_t \text{ for all } \theta, \theta' \in \Theta, \forall t$$

$$b_t(\theta) = b_t(\theta') = b_t \text{ for all } \theta, \theta' \in \Theta, \forall t$$

and

$$u'(c_t) = \beta R u'(c_{t+1}) = \beta R v'(b_t).$$

As is evident from the above equations, the allocations do not depend on individuals' type θ. The intuition for this result is the following. In this environment, individuals are heterogeneous ex ante (differ in the risk of survival) but identical ex post. There is no difference among dead individuals. There is also no difference among people who survive. Therefore, there is no reason that the planner should discriminate between them ex post.

The fact that allocations are independent of heterogeneous risk type means that a "one size fits all" identical allocation not only is ex ante efficient under full information, but also is incentive compatible and hence implementable even if risk type θ is private information. This means that the efficient allocation can be implemented by lump-sum tax and transfer.

Two key assumptions drive this result. One is that mortality risk is the only heterogeneity in this environment. If individuals are heterogeneous in other characteristics (such as ability or taste), then the efficient allocation would be type specific and therefore incentive compatibility constraints would not be trivially satisfied. Einav et al. (2007) estimate a model in which there is heterogeneity in mortality and preference for bequest, and they find that they are positively correlated. There are also numerous studies that document a negative correlation between status (such as education, income, etc.) and mortality (see, for example, Deaton and Paxson (1999)). Because of the two-dimensional private information in these environments, characterizing ex ante efficient allocations is difficult and is left for future research. The other key assumption is that the planner (as well as individuals) is expected utility maximizer. Removing this assumption also leads to efficient allocations that are type specific.

In the next section I describe a decentralized environment in which individuals can share the risk of their longevity in private life insurance and an annuity market, and possibly through a social security system.

2.4 Competitive equilibrium with asymmetric information

2.4.1 Survival contingent contracts

There are two types of survival contingent contracts: life insurance and annuity. One unit of life insurance coverage purchased at age t pays one unit of consumption good to the individual if the individual dies at the end of age t. One unit of annuity contract purchased at age t pays one unit of consumption good contingent on survival for as long as the agent survives starting at age $J+1$ (in other words, the annuity contract can be purchased at any age, but it pays only for post-retirement period $t > J$).

Contracts are assumed to be non-exclusive and cannot be contingent on the agent's past trades or volume of the transaction. Contracts are linear in the sense that to purchase l_t units of life insurance coverage, the individual pays $q_t^l l_t$ in premiums. Similarly, to purchase a_t unit of annuity coverage, the individual pays $q_t^a a_t$. Contract prices q_t^l and q_t^a are only contingent on the individual's age.

2.4.2 Consumer problem

Let s_t be the amount of non-contingent saving by the individual and $A_t = a_0 + \cdots + a_{t-1}$ be the total sum of annuity coverage purchased up to age t. Let $V_t(s_t, A_t; \theta)$ be the lifetime utility of an individual of type θ at age t who holds s_t units of saving and has total annuity coverage of A_t. The optimization problem faced by this individual is

$$V_t(s_t, A_t; \theta) = \max_{c_t, b_t, s_{t+1}, l_t, a_t \geq 0} u(c_t) + (1 - x_{t+1}(\theta))\beta v(b_t) + x_{t+1}(\theta)\beta V_{t+1}(s_{t+1}, A_{t+1}; \theta)$$

subject to

$$c_t + s_{t+1} + q_t^l l_t + q_t^a a_t \leq Rs_t + (1 - \tau)w \quad \text{for } t \leq J \tag{2}$$

$$c_t + s_{t+1} + q_t^l l_t + q_t^a a_t \leq Rs_t + A_t + z \quad \text{for } t > J$$

$$b_t = Rs_{t+1} + l_t$$

$$A_{t+1} = A_t + a_t,$$

in which l_t and a_t denote life insurance and annuity coverage purchased at age t. τ is social security tax, and z is social security benefit. Note that the individual faces short sale constraints on life insurance and annuity as well as saving. Note also that $x_{T+1}(\theta) = 0$ for all θ.

2.4.3 Social security

There is a fully funded social security system that taxes individuals at ages 0 to J at constant rate τ (since labor is inelastically supplied, this is in fact a lump-sum tax) and transfers constant social security benefit z to everyone at ages $t > J$ for as long as they are alive. The social security, therefore, is in fact a mandatory annuity insurance.

Let SSA_t denote the social security assets at date t. During periods $0 \leq t \leq J$, the mandatory contributions are collected from whoever is alive, and social security assets accumulate

$$SSA_{t+1} = R(SSA_t + \tau e \int P_t(\theta)dg_0(\theta)) \quad t \leq J \tag{3}$$

$$SSA_{t+1} = R(SSA_t - z \int P_t(\theta)dg_0(\theta)) \quad t > J,$$

in which $SSA_0 = SSA_{T+1} = 0$. Note that $\int P_t(\theta)dG_0(\theta)$ is the total fraction of people (of all types) that survive to age t.

2.4.4 Insurers

In this economy, the insurance contract is sold through agents called *insurers*. There are two types of insurers in this economy, detailed as follows.

Life Insurers

For each age, t, there are large number of insurers who sell life insurance to individuals of age t. The life insurer of type t chooses how many units of coverage, y_t^l, it sells. It collects the total premium charged, $q_t^l y_t^l$, invests it in saving technology, and makes payments to anyone who purchased life insurance at age t and did not survive to age $t + 1$. However, since risk of survival of the life insurance customer is unobservable, insurers must form expectations about the likelihood of payment. In other words, they have to form expectations about how each unit of coverage that they sell splits over different risk types. This determines the expected value of payments they have to make in period $t + 1$. Let $f_t^l(\cdot)$ be the *anticipated distribution of payouts*. $f_t^l(\theta)$ determines what fraction of every unit of life insurance sold in period t is sold to type θ individuals. Taken as given q_t^l and $f_t^l(\cdot)$, the life insurer of type t solves

$$\max_{y_t^l, k_t^l \geq 0} q_t^l y_t^l - k_t^l \tag{4}$$

subject to

$$y_t^l \int (1 - x_{t+1}(\theta)) df_t^l(\theta) \le Rk_t^l.$$

Annuity Insurers

For each age, t, there are large number of insurers who sell life annuity contracts to individuals of age t. Taken as given the annuity price q_t^a and the anticipated distribution of payouts $f_t^a(\cdot)$, the annuity insurer chooses the number of contracts it sells, y_t^a,

$$\max_{y_t^a, (k_s^a)_{s \ge t} \ge 0} q_t^a y_t^a - k_t^a \qquad (5)$$

subject to

$$y_t^a d_s \int \frac{P_s(\theta)}{P_t(\theta)} df_t^a(\theta) + k_{s+1}^a \le Rk_s^a \quad t \le s \le T,$$

and $d_s = 1$ if $s > J$ and zero otherwise.

In the equilibrium –which I will define shortly– $f_t^l(\cdot)$ and $f_t^a(\cdot)$ are required to be consistent with individuals' demand for life insurance and annuity.[17]

2.4.5 Competitive equilibrium

Before defining equilibrium, I need to introduce one more piece of notation: the distribution of type θ's that are alive at date t. Note that individuals who die exit the economy. Therefore, in each period the distribution of types (conditional on survival) becomes more skewed toward the higher survival (lower θ) types. Let G_t be the distribution of types conditional on survival to date t; then, the fraction of people with type in any set $Z \subseteq \Theta$ is

$$G_t(Z) = \frac{\int_{z \in Z} P_t(z) dG_0(z)}{\int_{\theta \in \Theta} P_t(\theta) dG_0(\theta)} \quad \forall Z \subseteq \Theta.$$

Competitive equilibrium is defined as follows.

Definition 1 *A competitive equilibrium with asymmetric information is the sequence of consumers' allocations, $(c_t^*(\theta), b_t^*(\theta), l_t^*(\theta), a_t^*(\theta), A_t^*(\theta), s_{t+1}^*(\theta))_{\theta \in \Theta}$, life insurer and annuity insurer decisions, (y_t^{l*}, k_t^{l*}) and $(y_t^{a*}, k_{s \ge t}^{a*})$, for each insurer's type t, prices (q_t^{a*}, q_t^{l*}), anticipated distribution of payouts by insurers, (f_t^{l*}, f_t^{a*}), and social security policy (τ, z, SSA_{t+1}) such that:*

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[17]An alternative way of modeling annuity contracts is to assume that each unit of annuity purchased at age t only pays at age $t + 1$. Since in this environment the annuity contracts are not actuarially fair priced for each type, these details about the timing do make a difference. However, the quantitative results are robust to these details regarding timing.

1. $(c_t^*(\theta), b_t^*(\theta), l_t^*(\theta), a_t^*(\theta), A_t^*(\theta), s_{t+1}^*(\theta))_{\theta \in \Theta}$ *solves consumer's problem for all* $\theta \in \Theta$ *given prices* (q_t^{a*}, q_t^{l*}).

2. (y_t^{l*}, k_t^{l*}) *solves life insurer of type t's problem given* q_t^{l*} *and* f_t^{l*}.

3. $(y_t^{a*}, k_{s \geq t}^{a*})$ *solves annuity insurer of type t's problem given* q_t^{a*} *and* f_t^{a*}.

4. *Allocations are feasible*

$$\int a_t^*(\theta) dg_t(\theta) = y_t^{a*} \tag{6}$$

$$\int l_t^*(\theta) dg_t(\theta) = y_t^{l*} \tag{7}$$

$$\int \sum_{t=0}^{T} \frac{P_t(\theta)}{R^t} \left[c_t(\theta) + \frac{(1 - x_{t+1}(\theta))}{R} b_t(\theta) \right] dG_0(\theta) = w \int \sum_{t=0}^{J} \frac{P_t(\theta)}{R^t} dG_0(\theta). \tag{8}$$

5. f_t^{l*} *and* f_t^{a*} *are consistent with consumers' choices, i.e., the fraction of total life insurance coverage bought by individuals with type in* $Z \subseteq \Theta$ *at age t is*

$$f_t^{l*}(Z) = \frac{\int_{\theta \in Z} l_t^*(\theta) dG_t(\theta)}{\int_{\theta \in \Theta} l_t^*(\theta) dG_t(\theta)} \quad \text{and with positive mass only on } \bar{\theta} \text{ if } l_t^*(\theta) = 0 \quad \forall \theta$$

and fraction of total annuity coverage bought by individuals with type in $Z \subseteq \Theta$ *at age t is*

$$f_t^{a*}(Z) = \frac{\int_{\theta \in Z} a_t^*(\theta) dG_t(\theta)}{\int_{\theta \in \Theta} a_t^*(\theta) dg G_t(\theta)} \quad \text{and with positive mass only on } \underline{\theta} \text{ if } a_t^*(\theta) = 0 \quad \forall \theta.$$

6. *Social security budget balances (equation (3)).*

The equilibrium notion is very similar to Bisin and Gottardi (1999, 2003) and also Dubey and Geanakoplos (2001). In Hosseini (2007) the existence result is established for a two-period version of this economy. In this paper contracts are assumed to be linear. Bisin et al. (1998) prove that almost linear contracts emerge as the result of non-exclusivity in a moral hazard environment, and they conjecture that the same result would hold in an environment with adverse selection.[18] The idea behind the non-exclusivity is that the insurers cannot observe and monitor individuals' trades. People may buy multiple insurance contracts from multiple insurers. Empirical evidence suggests that in the annuity market, insurers do not attempt to use menus of prices to classify individuals based on risk characteristics, even

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[18]Formally establishing this conjecture for the environment studied here is work in progress.

when they can condition prices on observable characteristics that correlate with mortality (see Finkelstein and Poterba (2006) for more details).

Insurers' constant return to scale technology implies that they make zero profit. Using zero profit condition, market clearing, and consistency conditions (condition 5 in equilibrium definition), we can get the equation for equilibrium prices

$$q_t^a \int a_{t+1}(\theta) dG_t(\theta) = \int a_{t+1}(\theta) \sum_{s=t+1}^{T} \left(\frac{P_{t+s}(\theta)}{P_t(\theta)} \frac{1}{R^{s-t}} \right) dG_t(\theta) \tag{9}$$

$$q_t^l \int l_{t+1}(\theta) dG_t(\theta) = \int \frac{(1 - x_{t+1}(\theta))}{R} l_{t+1} dG_t(\theta). \tag{10}$$

3 Two-period Example

Deriving qualitative results in the general case is difficult. To gain insights about some properties of equilibrium prices and allocations, I study a two-period example. Also, this example is useful in understanding what are the key factors/parameters in determining the size of inefficiencies.[19]

The economy lasts for two periods. All individuals live through the first period. They are alive in the second period with probability $P(\theta)$. θ is non-negative number, has distribution $G(\cdot)$ (with density $g(\cdot)$), and indexes individuals' frailty. $P(\cdot)$ is a decreasing function of θ. Individuals enjoy consumption while they are alive and leave bequests when they die. Period utility over consumption and bequest are CRRA with coefficient of risk aversion γ. There is no discounting. ξ is the weight of bequest in the utility function and is identical across all individuals:

$$u(c) = \frac{c_0^{1-\gamma}}{1-\gamma} \text{ and } v(b) = \xi \frac{b^{1-\gamma}}{1-\gamma}.$$

The timing is the following: 1) At the beginning of period 0 before any decision is made, individuals realize their θ (therefore, they know the probability that they will be alive in the second period, $P(\theta)$); 2) They make decisions about consumption, bequest and saving, and

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[19]All of these results have been derived for a two-period economy with a general class of utility functions in Hosseini (2007).

trade annuity and life insurance. The consumer problem is

$$\max \frac{c_0^{1-\gamma}}{1-\gamma} + (1 - P(\theta))\xi \frac{b_0^{1-\gamma}}{1-\gamma} + P(\theta)\left(\frac{c_1^{1-\gamma}}{1-\gamma} + \xi \frac{b_1^{1-\gamma}}{1-\gamma}\right)$$

subject to

$$c_0 + s + q^a a + q^l l \leq w(1-\tau)$$

$$c_1 + b_1 \leq s + a + z$$

$$b_0 = s + l.$$

Note that there is no discounting and $R = 1$. The decision problem in the second period is trivial since all the uncertainty is resolved. The problem can be simplified by replacing for the solution of the second period

$$\max \frac{c_0^{1-\gamma}}{1-\gamma} + (1 - P(\theta))\xi \frac{b_0^{1-\gamma}}{1-\gamma} + P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma \frac{(s + a + z)^{1-\gamma}}{1-\gamma}$$

subject to

$$c_0 + s + q^a a + q^l l \leq w(1-\tau).$$

The goal is to establish the following results: 1) Household decision over purchase of annuity and life insurance is monotone in their type. Individuals with higher $P(\theta)$ (higher probability of survival) purchase more annuity and less life insurance, and vice versa. 2) Equilibrium prices are unfair (they are above average actuarially fair prices). After these results are established, I perform some comparative statics. In particular I show that increasing the social security taxes increases equilibrium price of annuity. The first proposition establishes the unfairness in equilibrium prices.

Proposition 1 *For any set of prices such that $q^a + q^l > 1$, annuity purchase $a(q^a, q^l; P(\theta))$ is a monotone increasing function of $P(\theta)$ and life insurance purchase $l(q^a, q^l; P(\theta))$ is a monotone decreasing function of $P(\theta)$. Furthermore, in any equilibrium prices satisfy $q^a + q^l > 1$.*

Proof. See Appendix A. ∎

This proposition highlights the effect of adverse selection in increasing the price of insurance above an actuarially fair price in this environment. Individuals with a higher probability of survival demand more annuity insurance at any price. They also survive to the second

period with higher probability and therefore are more likely to claim the insurance they have purchased. Any unit of coverage that is sold to these individuals is more risky from the insurers' point of view. On the other hand, individuals with a lower probability of survival are less risky for insurers since they are less likely to survive and claim the insurance coverage. However, since they are less likely to survive, they purchase less insurance (relative to high survival types). As a result, the insurers are left with a pool of claims that are more likely to be materialized than the average probability of survival in the population. The risk in each insurers, pool is higher than what is implied by average risk of survival by individual agents in the economy. As a result, the equilibrium price of annuity is higher than the actuarially fair value of its payout. This is the essence of adverse selection in this environment.

Next, I show that increasing social security taxes leads to an increase in the price of annuity. Social security is a substitute for annuity that is purchased in the market. An increase in social security taxes causes everyone to reduce their demand in the annuity market. However, it has a larger effect on the demand for annuity by lower survival types. An increase in tax and transfer has an income effect and a substitution effect. The substitution effect is negative and is the same for all types. However, the income effect is positive. Higher survival types spend a larger share of their income on purchasing annuity, so the positive income effect is larger for them. Therefore, for higher survival types the overall (negative) effect is smaller. As a result, increasing social security makes the risk in the annuity pool higher. This is turn leads the equilibrium price to increase. Theorem 1 provides a formal proof of this argument.

Theorem 1 *Equilibrium price in the annuity market is an increasing function of social security tax, τ.*

Proof. See Appendix B. ∎

Although increasing social security taxes increases the price in the annuity market, its effect on welfare is not negative. In fact, as we see in Figure 1, increasing social security taxes improves welfare while increasing the equilibrium price of annuity. Higher social security taxes forces more lower survival types to join the overall pool of mandatory annuity insurance and provides better insurance for higher survival types.
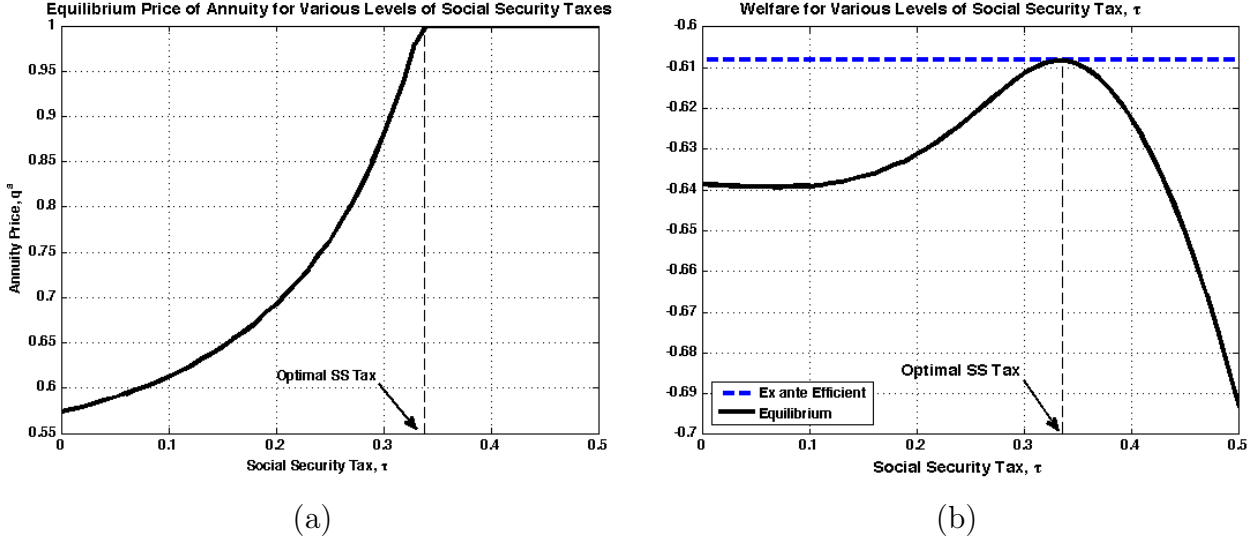
FIGURE 1: Changes in equilibrium prices (panel(a)) and welfare (panel (b)) as function of social security tax

4 Quantitative exercise

This section contains the parametric specifications of the quantitative model, as well as a description of the data and calibration procedure.

4.1 Parametric specifications

Preferences. Individuals have CRRA utility function with coefficient of risk aversion γ over consumption and bequest:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } v(b) = \xi \frac{b^{1-\gamma}}{1-\gamma}.$$

$\xi > 0$ is the weight on bequest in the utility function and is identical for every individual.[20] The higher ξ is, the higher is the value of bequest for individuals. I choose the parameters γ and ξ so that the fraction of annuitized wealth through social security and annuity purchase matches with the ones in the HRS data. More details on data and calibration are laid out in the next section.

Demographics. In what follows, I model aging as a continuous time process, and later on

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[20]See Abel and Warshawsky (1988) for relation between this *joy of giving* parameter and altruism. This exact parametric form arises if we assume child (or spouse) has the same CRRA utility function and lives for fixed number of periods after the agents death. Modeling the exact details of this inter-family/intergenerational link is left for future research. However, the quantitative results regarding welfare gains are robust to a wide range of values for ξ (and even to removing the altruism altogether).

I derive the age-specific probabilities.

Individuals are indexed by their *frailty* type, $\theta \in \mathbb{R}_+$. Let $h_t(\theta)$ be the force of mortality of an individual at age t with a *frailty* of θ. The *frailty* can be modeled in many ways. Here I follow Vaupel et al. (1979) and Manton et al. (1981) and assume the following:

$$\frac{h_t(\theta)}{h_t(\theta')} = \frac{\theta}{\theta'} \tag{11}$$

or alternatively

$$h_t(\theta) = \theta h_t.$$

An individual with frailty of 1 might be called a *standard* individual. I denote the force of mortality of standard individual by h_t (note that this is, in general, different from the average population force of mortality). The frailty index shifts the force of mortality. Furthermore, an individual's frailty does not depend on age. Therefore, $\theta > \theta'$ means that an individual with frailty θ has a higher likelihood of death at any age t than an individual with frailty θ' condition that they are both alive at age t. Let $H_t(\theta)$ be the cumulative mortality hazard; that is,

$$H_t(\theta) = \int_0^t h_s(\theta)ds = \theta \int_0^t z_s ds = \theta H_t. \tag{12}$$

Once again, H_t is the cumulative mortality hazard for standard individual. Finally, the probability that an individual of type θ survives to age t is

$$P_t(\theta) = \exp(-H_t(\theta)) = \exp(-\theta H_t). \tag{13}$$

Therefore, if an individual of θ has a 50 percent chance of survival to age t, an individual of type 2θ has a 25 percent chance of survival to the same age.[21]

Let $g_0(\theta)$ be the density of frailty at birth; that is, at age $t = 0$. Also let $\bar{P}_t$ be the overall survival probability in the population. $\bar{P}_t$ corresponds to the data that can be calculated using a life table, and it is the fraction of all individuals (across all θ types) who survive to

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[21]θ encompasses all of the factors affecting human mortality other than age. Needless to say, it is also possible to model the heterogeneity as factors that directly scale the probability of survival. However, given the fact that survival and death probabilities are naturally bounded above, the model becomes complicated. Modeling frailty as it is done here is convenient because it allows more flexibility in choosing a parametric class of distributions for heterogeneity.

age t. Therefore, the relationship between $\bar{P}_t$ and $P_t(\theta)$ is the following:

$$\bar{P}_t = \int_0^\infty P_t(\theta)g_0(\theta)d\theta. \tag{14}$$

Note that individuals with higher values of frailty θ will have a higher probability of dying and are more likely to die earlier. This leads to a selection effect that changes the distribution of frailty types who are alive at each age t. The conditional density of type θ who survive to age t can be found by applying *Bayes' rule*:

$$g_t(\theta) = \frac{P_t(\theta)g_0(\theta)}{\int_0^\infty P_t(\theta)g_0(\theta)d\theta} = \frac{P_t(\theta)g_0(\theta)}{\bar{P}_t}. \tag{15}$$

As the population ages, the distribution of frailty types who survive tilts toward the lower value of θ. This implies that the overall average mortality hazard in the population does not correspond to individuals' mortality hazard. The relationship between average population mortality hazard, $\bar{h}_t$, and individual mortality hazard, $h_t(\theta)$, can be established by the following equation:

$$\bar{h}_t = \int_0^\infty \theta h_t g_t(\theta)d\theta = h_t \int_0^\infty \theta g_t(\theta)d\theta = h_t \mathrm{E}[\theta|t] \tag{16}$$

in which $\mathrm{E}[\theta|t]$ is the mean frailty among survivors to age t. Note that since individuals with higher frailty die earlier and the distribution of types becomes skewed toward lower values of θ as the population ages, the mean frailty in the population decreases, i.e., $\mathrm{E}[\theta|t]$ is a decreasing function of t. This implies that overall, the population at each age t dies at a slower rate than individuals (unless g_0 is degenerate). Consequently, knowing the overall mortality rate, $\bar{h}_t$, which can be computed from life tables, is not enough to find individuals' mortality hazard rate. To uncover the individuals' mortality hazard rates, we need to make further assumptions on the shape of distribution g_0.

Following Vaupel et al. (1979), I assume the initial distribution of individual frailty, θ, is the gamma distribution with unit mean and variance $\sigma_\theta^2 = \frac{1}{k}$.[22]

$$g_0(\theta) \sim \mathcal{G}(\frac{1}{k}, k) = k^k \theta^{k-1} \frac{\exp(-k\theta)}{\Gamma(k)}.$$

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[22]The general formula for gamma distribution is

$$\mathcal{G}(m, k) = \theta^{k-1} \frac{\exp(-\theta/m)}{m^k \Gamma(k)}$$

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Aside from its flexible shape, a useful feature of gamma distribution is that the frailty among survivors at any age t is itself a gamma distribution. This keeps the evolution of type distribution across ages analytically tractable and convenient for computation. To see this, first replace for $P_t(\theta)$ from equation (13) into equation (14) and the formula for gamm distribution to simplify the equation. We get the following relation between H_t and $\bar{H}_t$

$$H_t = \frac{\exp(\sigma_\theta^2 \bar{H}_t) - 1}{\sigma_\theta^2}. \tag{17}$$

We can use this relation in the equation (15) and derive the formula for $g_t(\theta)$, which is itself a gamma distribution.

$$g_t(\theta) \sim \mathcal{G}(\frac{1}{k + H_t}, k) = (k + H_t)^k \theta^{k-1} \frac{\exp(-(k + H_t)\theta)}{\Gamma(k)} \tag{18}$$

Therefore, not only do we know the shape of distribution of frailty types at each age, we also know how the cumulative mortality hazard for standard type, H_t, is related to the population cumulative mortality hazard, $\bar{H}_t = -\log(\bar{P}_t)$.

The values for $\bar{P}_t$ at each age can be calculated from cohort life tables. In the model I assume that a period is 5 years, that individuals enter the economy at the age of 30, and that everyone dies at or before age 100. Given the variance of the initial distribution of frailty at birth , σ_θ^2, equation (17), together with our assumption about the frailty (equations (11) and (13)), can be used to uncover individuals' survival probabilities at each age t. These survival probabilities by construction are consistent with the life table data. That means, for any variance of initial distribution, σ_θ^2, overall population survival in the model is exactly equal to survival probabilities calculated from the life table. Therefore, we need an extra source of information to estimate the variance of initial distribution. I use the data on subjective survival probabilities in the Health and Retirement Study (HRS) to calibrate σ_θ^2. The details of calibration are laid out in the next section.

An alternative approach taken by Butt and Haberman (2004) and Einav et al. (2007) is to make a parametric assumption on H_t as well as the initial distribution of θ and estimate these parameters using only the life table data. The parametric assumption puts restriction

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in which m and k are the scale and shape parameters. The mean and variance of this distribution are

$$\begin{aligned} \mu_\theta &= km \\ \sigma_\theta^2 &= km^2. \end{aligned}$$

Normalizing the mean to one implies that $m = 1/k$ and $k = \sigma_\theta^{-2}$.

on the variance of θ (this can be seen by looking at equation (17)). Consequently, the extent of heterogeneity will depend on the details of the parametric form assumed for H_t. A novelty of the approach taken in this paper is that instead of identifying the extent of heterogeneity by functional forms, I let individuals' assessment about their mortality guide me in choosing the extent of heterogeneity.

4.2   Data and calibration

Individual survival probabilities

In order to calibrate the parameter of the initial distribution of frailty at birth, I use individual subjective survival probabilities from the Health and Retirement Study (HRS). The HRS is a biennial panel survey of individuals born in the years 1931-1941, along with their spouses. In 1992, when the first round was conducted, the sample was representative of the community-based U.S. population aged 51 to 61. The baseline sample contains 12,652 observations. The survey has been conducted every two years since. The HRS collects extensive information about health, cognition, economic status, work, and family relationships, as well as data on wealth and income. The particular observation on survival probabilities that I am going to use comes from the following question:

> Using any number from 0 to 10 where 0 equals *absolutely no chance* and 10 equals
> *absolutely certain*, what do you think are the chances you will live to be 75 and
> more?[23]

Hurd and McGarry (1995, 2002) analyzed HRS data on subjective survival probabilities and found that the responses aggregated quite closely to the predictions of life tables and varied appropriately with known risk factors and determinants of mortality. Also, Smith et al. (2001) found that subjective survival probabilities are also good predictors of actual survival and death.

Although the above-mentioned studies point to the potential usefulness of these responses as probabilities, there is a drawback. Gan et al. (2003) noticed the existence of focal points (0 or 1) in responses.[24] They propose a Bayesian updating procedure for recovering subjective survival probabilities (they study older respondents who were born before 1924). They assumed that individuals' true beliefs regarding their survival probability are unknown to

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[23]The question was repeated with the target age of 85, too. From wave 2 onward the respondents were asked to report a number between 0 to 100.

[24]They report that 30 percent of responses in wave 1 and 19 percent of responses in wave 2 are 0's or 1's.

the econometrician. However, the distribution of beliefs is known (which is taken as Bayesian prior). The individual reports a survival probability based on his true beliefs. The difference between his or her true beliefs and reported probabilities is modeled as measurement a error. Gan et al. (2003) use the self-reported probabilities to update the prior distribution and to obtain posterior distribution. They then apply the posterior distribution of survival probabilities to observed mortality among panel to estimate parameter values that best characterize each individual's belief about his survival probabilities. They also provide the estimate for variance of the hazard scaling parameter. This parameter in their model corresponds to θ in mine (they call this parameter 'the optimistic index'). For variance of initial distribution of frailty, I use their estimated value for variance of the hazard scaling parameter.[25]
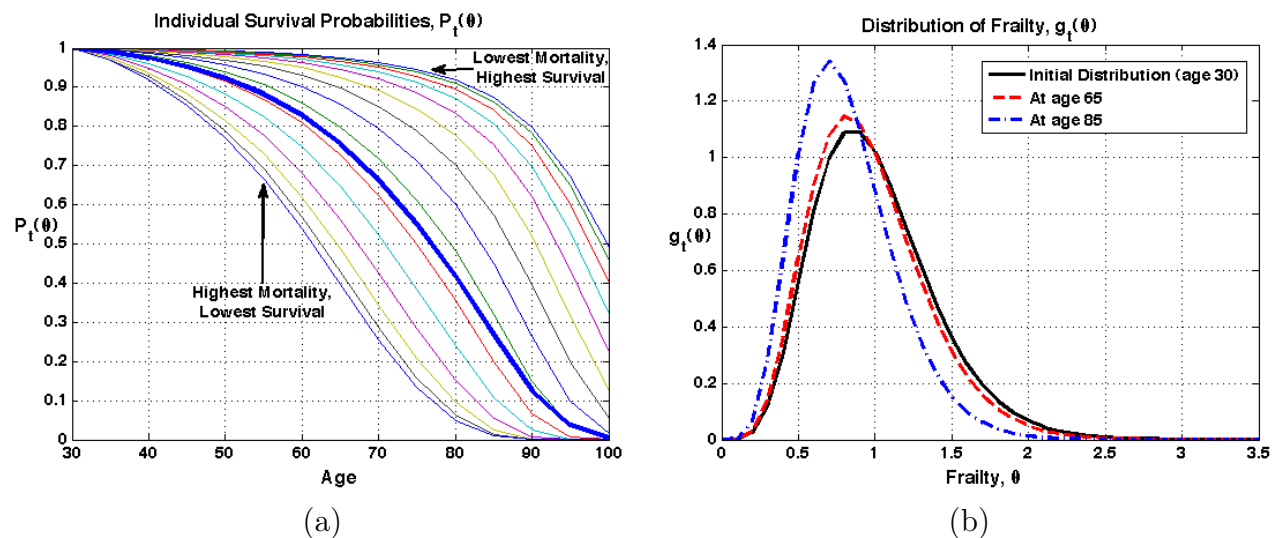

FIGURE 2: Panel (a) shows the calculated survival probabilities for each type. The thick line is the overall population survival probabilities (the life table data). Panel (b) shows the evolution of type distribution as the population ages. As argued in the text, since individuals with higher frailty (θ) have a higher likelihood of death at any age, the distribution of types who survive to age t each becomes skewed toward the lower value of θ as the population ages.

Once the variance of initial distribution of frailty is known, I use equation (17) to compute the cumulative mortality hazard of standard type, H_t. $\bar{H}_t = -\log(\bar{P}_t)$ and $\bar{P}_t$ is calculated using *Cohort Life Tables for the Social Security Area by Year of Birth and Sex* for males of 1930 birth cohort.[26] Then, equation (13) can be used to compute individuals' survival probabilities $P(\theta)$. Computed survival probabilities are plotted in Figure (2) (panel (a)). Panel (b) of Figure (2) shows the evolution of type distribution as the population ages.

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[25]I used their procedure to estimate the variance of the initial distribution of frailty, θ, with the parametric assumptions of my model. The results are very close. See the Appendix D.

[26]Table 7 in Bell and Miller (2005).

Fraction of annuitized wealth

Butrica and Mermin (2006) use the HRS data on household wealth and income to construct the measures of fraction of annuitized wealth at old ages. In their measure of wealth they include financial assets, housing equity, and other assets. Financial assets include IRA balances; stock and mutual fund values; bond funds; checking, savings, money market, and certificates of deposit account balances; and trusts, less unsecured debt. Housing equity is the value of home less mortgages and home loans. Other assets include the net value of other estates; vehicles; and businesses. To construct total retirement wealth they add the present discounted value of expected future stream of payment from social security and pension. The results of their calculation are presented in Table 1.

TABLE 1: Wealth Among Adults Ages 65+ with Median Expenditure

		Financial Assets	Housing/ Other	Social Security(1)	Pensions/ Annuities(2)	SSI (3)	Annuitized Wealth(1)+(2)+(3)
All Married		19%	26%	41%	14%	0%	55%
Age							
	65-69	16	23	43	17	0	60
	70-79	17	28	41	14	0	55
	>=80	30	33	30	7	0	37
All Unmarried		14%	26%	48%	10%	1%	59%
Age							
	65-69	4	26	59	11	0	70
	70-79	16	29	45	10	0	55
	>=80	26	25	40	7	2	49

Note: These are adults who collected either social security or social security disability insurance in 2002/2004 HRS. All percentages are computed as the mean ratio.
Source: Tables 2 and 3 in Butrica and Mermin (2006).

The upper section of Table 1 shows the result of the calculations for married adults who have median expenditure.[27] The lower section contains the same information for unmarried couples. The table shows that typical adults older than 65 have a significant percentage of their retirement wealth annuitized. For unmarried adults between 70 to 79, 45 percent of the wealth is annuitized through social security. These individuals, on average, choose to hold an extra 11 percent of their wealth in the form of an annuitized payment stream through either defined benefit pension plans or annuities purchased in the private market. In the model I treat these pension holdings as annuity that is purchased in the market.

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[27]These are averages over households with expenditure between the 45th and 55th percentile.

The calibration procedure is the following. I assume 3 percent annual real rate of return on liquid assets ($R = 1.03^5$) and assume no growth so that $\beta = \frac{1}{R}$. Next, I choose social security taxes τ to match the average U.S. social security replacement ratio of 45 percent. I then choose the coefficient of risk aversion, γ, and weight on bequest, ξ, in such a way that the percentage of annuitized wealth through social security and annuity matches the corresponding numbers in the lower section of Table 1 for 70-year-old unmarried adults. The summary of calibration results is presented in Table 2. Individuals' income is constant and normalized to 1 for the pre-retirement period (age 30 to 65), and afterward it is zero. Figure 3 shows the fit of the model in matching the average fraction of annuitized wealth at retirement.

TABLE 2: Calibration Summary

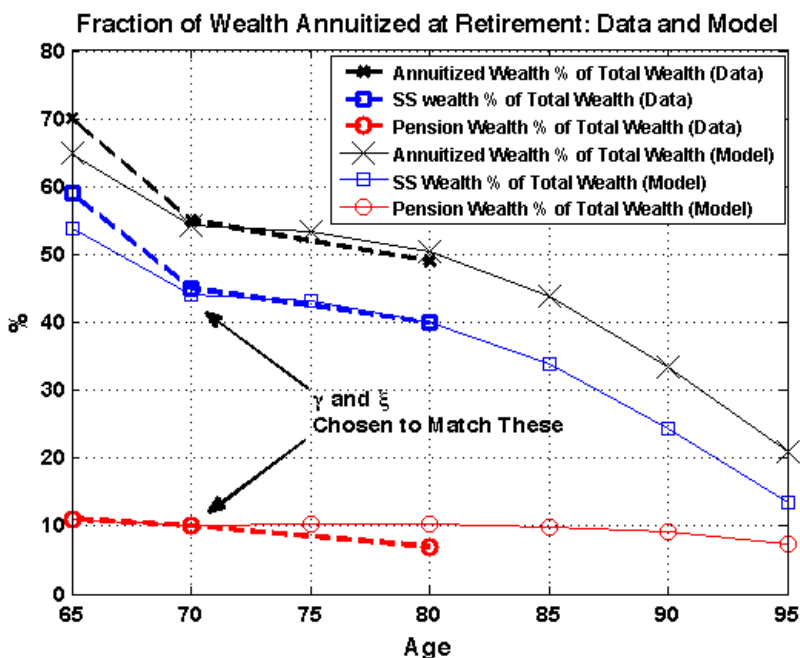| Parameter	Value	Note
γ (risk aversion)	1.09	Match fraction of social security wealth at 70
ξ (weight on bequest)	0.52	Match fraction of pension wealth at 70
β	0.97^5	
R	1.03^5	
τ	0.05	Match U.S. replacement ratio (45 percent)
σ_θ^2	0.14	From Gan et al. (2003)

FIGURE 3: Thick dashed lines are data from the lower section of Table 1. Thin lines are data generated by model.

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5 Results

In this section I report the results. First I present and describe the life cycle profile of allocations for the calibrated economy with the current U.S. social security replacement ratio. Next, I present and describe the allocations for the same economy but with no social security. I compare the annuitization decision and welfare in two economies and also report the welfare gain from implementing the ex ante efficient allocations. Finally, I report this welfare for various values of key parameters to check for robustness.

5.1 Model with current U.S. social security

Figure 4 shows the profile of consumption and holdings of liquid assets for different mortality types. Individuals whose frailty type is in the bottom 5 percent of initial type distribution (lowest 5 percent of mortality) start the life cycle with a low level of consumption (dashed line with ×). These individuals expect to survive to a very old age and therefore accumulate assets when they are young up to age 55, where they spend a large fraction of their accumulated wealth to purchase annuity (Figure 5 (b)). After they purchase annuity, they increase their consumption. On average, these individuals outlive the participants in the annuity market. The market price that they pay for annuity insurance is much lower than the value they get from the stream of payment (since they evaluate the payment stream by their own high probability of survival). Therefore, once they purchase annuity, the future consumption becomes cheap for them and they increase their consumption.

On the other hand, individuals with frailty type in the top 5 percent of initial type distribution expect shorter life spans and start the life cycle with higher consumption level (dashed line with +). These individuals accumulate liquid assets at a slower rate (panel (b) in Figure 4). They also do not spend those assets on annuity purchase. Since they have a high likelihood of dying at each age, the expected value of leaving a bequest is higher for them than the expected value of feature consumption. Consequently, they accumulate assets up to retirement (age 65) and run down their assets afterward until they die.

The average profile of consumption (thick line in Figure 4) is the average over consumption of the individuals who are alive in each age. The average profile is almost constant up to retirement, at which point it starts to go down up to age 85. The graph shows a big increase in the profile of consumption at older ages. The reason lies in the fact that as the population gets older, high frailty types (who have low consumption) die and exit the economy. Therefore, the majority of individuals who survive to very old ages are those who have low frailty (θ). These individuals also have a high level of consumption at old ages.
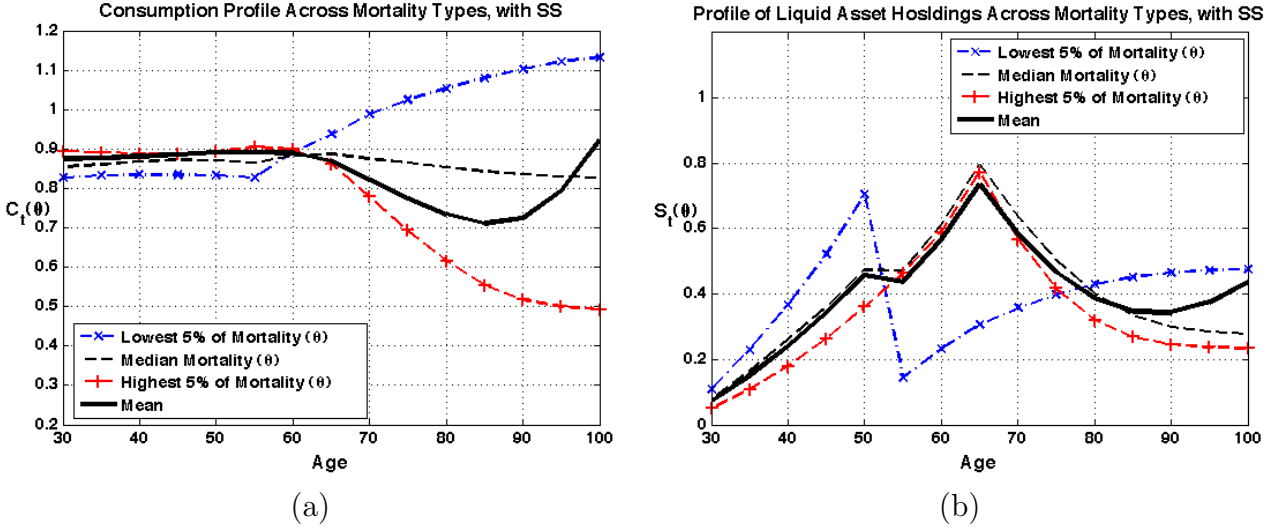
FIGURE 4: Life cycle profile of consumption (panel (a)) and holdings of liquid assets (panel (b)) across mortality types with U.S. social security replacement ratio.

This leads to a rise in consumption profile at old ages.[28]

Figure 5 shows the profile of annuity and life insurance purchases. Individuals purchase life insurance early in the life cycle, and individuals with higher mortality types who expect to die earlier purchase more relative to individuals with lower mortality. There is no purchase of annuity and life insurance in the same age. Those who purchase annuity do so at age 55. High mortality individuals do not want to postpone the purchase of annuity to a later period. Their participation in later periods will lead to a better insurance pool and reduces the price. This will encourage lower mortality types to purchase at a later period, too, which in turn causes the price to go up. In short, higher mortality individuals know that if they purchase at older ages, lower mortality people will also buy, keeping the price high; therefore, they do not buy annuity after 55.

5.2 Model without social security

Figure 6 shows the allocations over the life cycle in the same economy but with no social security. In this economy individuals purchase all the annuity insurance they need from the market. Panel (a) shows that there is little change in the life cycle profile of low mortality types. However, high mortality types significantly reduce their consumption at old ages. They also consume slightly more in pre-retirement periods (relative to the economy with

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[28]This prediction of the model can be checked by investigating the Consumer Expenditure Survey (CEX) data across different education levels. However, one difficulty is poor data in the old ages. This investigation is work in progress.
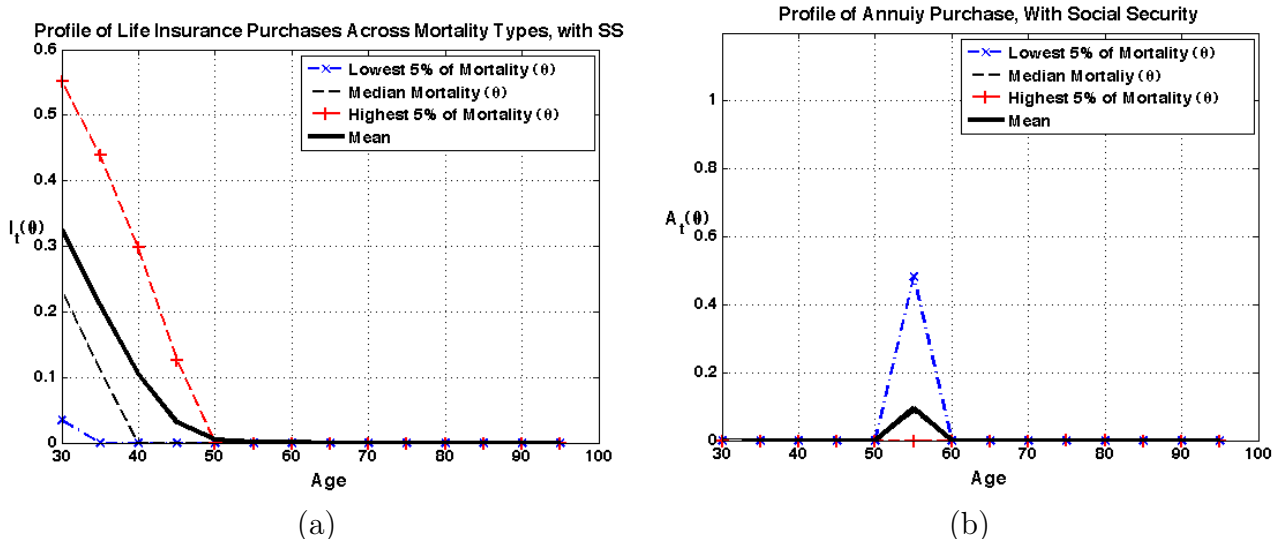
FIGURE 5: Profile of life insurance (panel (a)) and annuity (panel (b)) purchases over life cycle across mortality types when there social security with current US replacement ratio.

social security). The reason that this increase is not large is that although no social security taxes means they have more income to spend, they also have to increase their savings (holdings of liquid asset) to finance retirement.

Figure 7 shows that in the absence of social security, the purchase of life insurance in early ages is slightly reduced. In particular, the individuals with lowest mortality no longer purchase life insurance at age 30 (the dashed line with +). On the other hand, the purchase of annuity is significantly increased. Almost everyone purchases annuity at age 55. The average annuity purchase goes up by an order of 6. Another feature of this environment is that almost everyone purchases life insurance and annuity at the same age (age 55). In the absence of social security, everyone has to buy annuity to insure against the risk of longevity. In particular, individuals with lower mortality would like to purchase a large amount of coverage. They do not spread out their purchase over time, because they want to enjoy the benefit of pooling with the individuals with higher mortality (who have a lower risk of survival from the insurer's point of view and cause the price in the market to be low). Therefore, the lowest mortality types spend almost all of their liquid assets on annuity purchase. Since they also want to leave a bequest in the (unlikely) event of their death, they buy life insurance at the same time. Purchase of life insurance by low mortality types improves the pool of risk in that insurance market and also encourages higher mortality types to purchase life insurance. As a result, a spike in the life insurance purchase occurs at age 55.
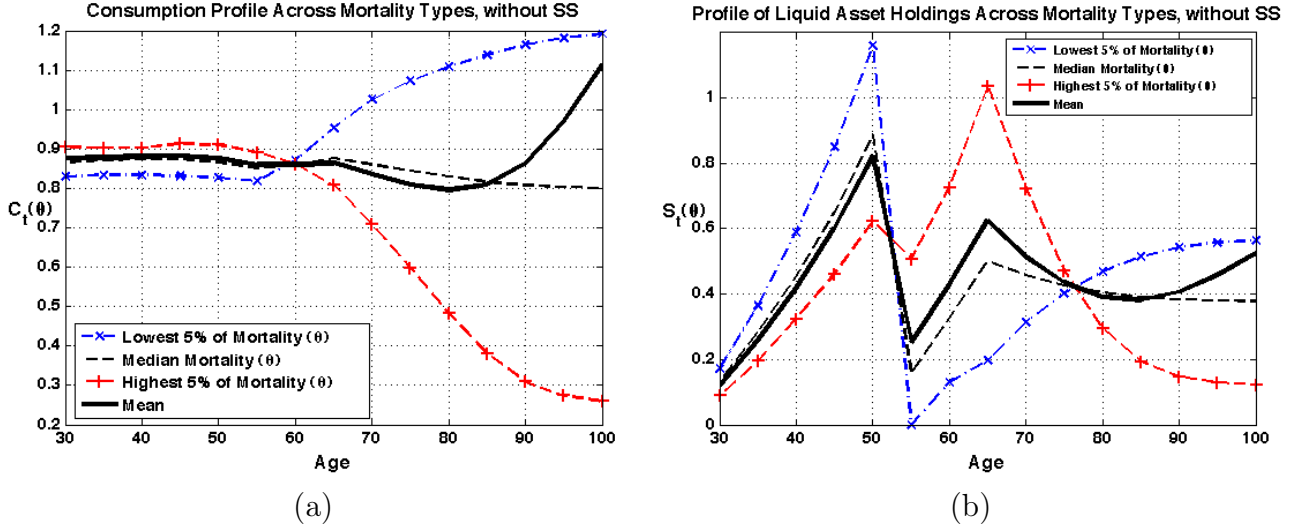
FIGURE 6: Life cycle profile of consumption (panel (a)) and holdings of liquid assets (panel (b)) across mortality types without social security

5.3 Comparing two environments

In this section I compare the annuitization decision of individuals in two environments. Panel (a) in Figure 8 shows the profile of annuity purchase across different types. Only half of the population with lower mortality (higher survival risk) purchases annuity. This is consistent with the findings of Johnson et al. (2004) who report that only 43 percent of all adults in HRS hold pensions in their own names (52 percent for males). Because the higher mortality types do not participate in the market, the quality of risk sharing in the annuity market is low. The price of annuity in the market is 30 percent higher than the overall actuarially fair value of the payments (evaluated using average survival probabilities from the life table). In the economy without social security almost, everyone purchases annuity in the market, even the individuals with the highest mortality types. As a result, the price is 15 percent lower relative to the economy with social security.[29]

In Figure 9, I present the same fact using the fraction of wealth annuitized at age 70. Panel (a) shows the fraction of wealth annuitized in the economy with social security. The plain dashed line is the fraction of total retirement annuitized through social security. Notice that this fraction is increasing in the frailty type (θ). In this economy every individual receives the same payment from social security. However, individuals with lower mortality types expecting longer life spans accumulate more assets. As a result, social security is a smaller fraction of their total wealth at retirement, and they turn to the annuity market to purchase more annuity insurance (the dot-dashed line). On the other hand, high mortality individuals

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[29]This is also pointed out by Walliser (2000), although he reports much smaller numbers (2-3 percent).
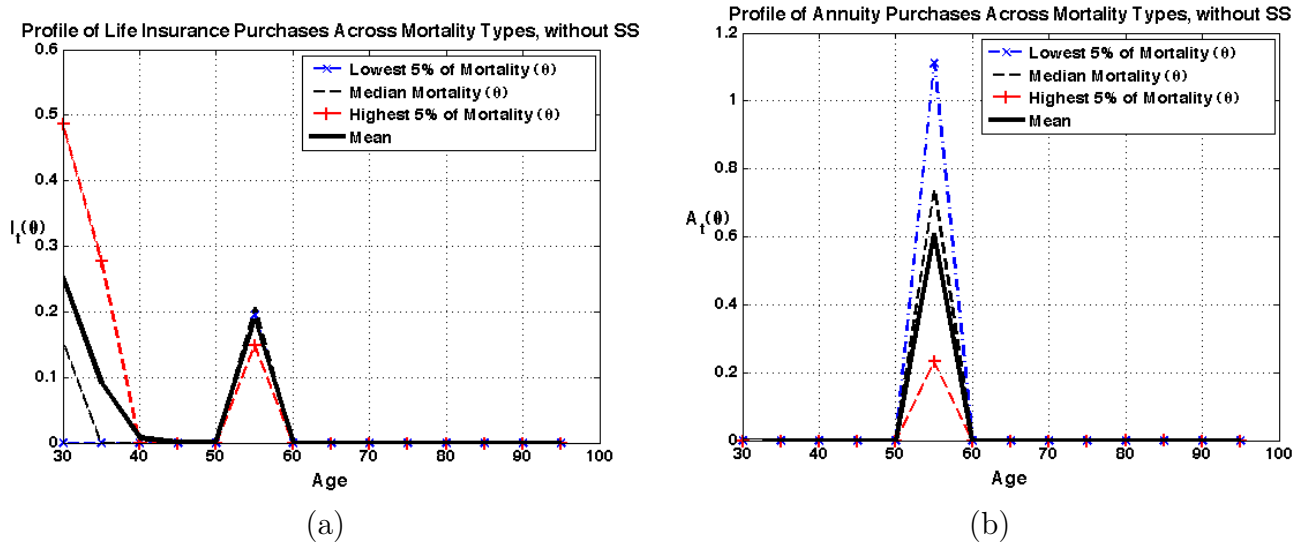
FIGURE 7: Profile of life insurance (panel (a)) and annuity (panel (b)) purchases over the life cycle across mortality types without social security.
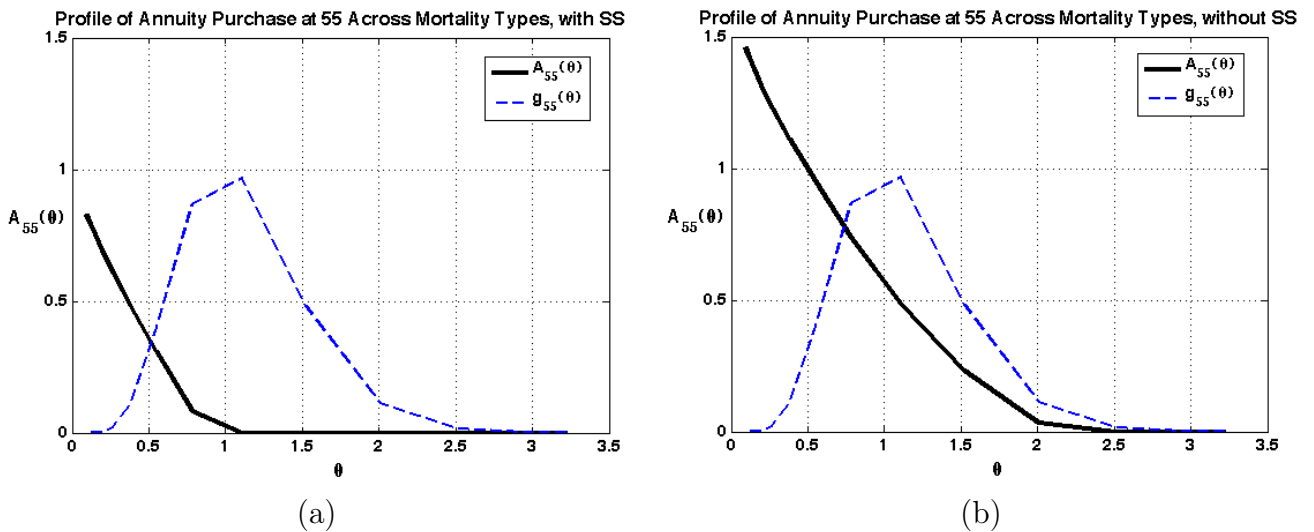

FIGURE 8: Profile of annuity purchase at age 55 across mortality types. Panel (a) shows the amount of annuity insurance individuals buy when there is social security with the current U.S. system's replacement ratio. Half of the population choose to buy zero. Panel (b) shows the annuity purchases in the same economy but with no social security.

expect a shorter life, and so they accumulate fewer assets. Social security constitutes a large fraction of their retirement wealth. They have too much wealth annuitized. This further suppresses their demand for annuitized assets, and they choose to buy zero annuity in the market.

Panel (b) in Figure 9 shows the fraction of wealth annuitized when there is no social security. Low mortality types hold almost the same level of annuitized wealth compared
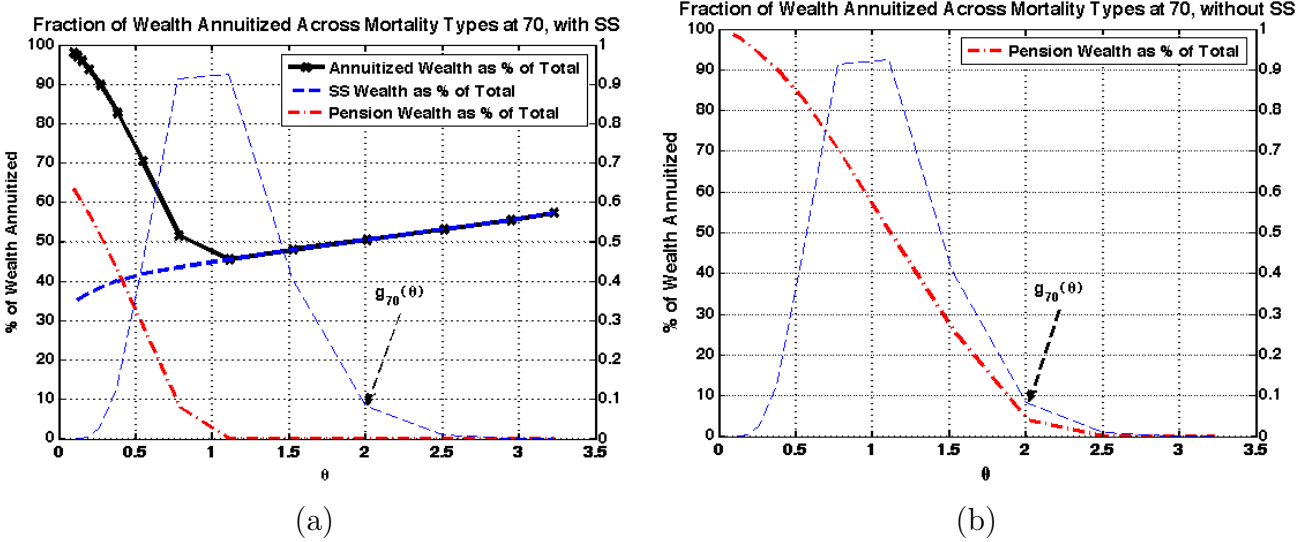
FIGURE 9: Fraction of total wealth that is annuitized. Panel (a) shows the result when there is social security with the current U.S. system replacement ratio. Panel (b) shows the result in the same economy, but without social security. The dashed line is the type distribution for those who are alive at 70. The numbers are in percentages.

to the economy with social security. On the other hand, although higher mortality types purchase a positive amount of annuity insurance, they choose a significantly lower level of annuity coverage. When they are given the choice, they choose to annuitize a much smaller fraction of their income (compared to the fraction that is annuitized through the current U.S. social security system).

5.4 Welfare

The central results of this paper are presented in Figure 10. The dashed line presents the welfare gain of the current U.S. social security replacement ratio over an economy with no social security. Two groups incur loss under the current U.S. social security replacement ratio. One group comprises of the individuals with the highest mortality types (top 8 percent). These individuals do not survive long enough to enjoy the benefits from annuitized income provided by social security. As we saw earlier, they have too much of their wealth annuitized under the current U.S. social security system. They prefer the economy with no social security because they will have the freedom to choose their desired level of annuity coverage.

The second group that loses under the current U.S. social security replacement ratio comprises of the individuals with the lowest morality types (bottom 20 percent). As indicated in Figure 9, these individuals rely on the annuity market for a significant proportion of their

annuitized wealth at retirement. However, when there is social security the quality of the pool of annuitant is not good enough to provide good risk sharing for these types. As a result, they end up paying a very high price for their annuity needs. With no social security, higher mortality types (good risk types) join the market and the price in the market is lower. Since the individuals at the lower tail of the mortality type distribution outlive the market, they benefit the most from these reduced prices. These individuals prefer no social security.[30]

Finally, individuals in the middle of type distribution (middle 72 percent) prefer having the current U.S. social security system over no social security. With no social security they are outlived by individuals to their left (lower mortality), and they pay an unfair price. Under social security, they receive transfer from individuals to their right (higher mortality) and the implicit rate of return in the social security is just about right for them. Overall, ex ante welfare gain is 0.15 (although there are big winners and losers).

The dashed line in Figure 10 is the welfare gain from implementing ex ante efficient allocation over an economy with no social security. To provide perfect risk sharing, everyone is taxed at a significantly higher rate (almost double), and the replacement ratio is 0.9 (double the average replacement ratio in the current U.S. system). Individuals with high mortality types are big losers under this system (top 18 percent of mortality distribution lose up to 5 percent). They pay a very high tax and don't live long enough to enjoy the benefits. On the other hand, since the amount of insurance provided is now much more, only about a tenth of a percent of low mortality types prefer no social security. Overall, the ex ante welfare gain is 0.51 percent. Table 3 summarizes the discussion above.

TABLE 3: Fraction of winners and losers under each policy

	Fraction of Losers		Fraction of	Overall
	Upper Tail	Lower Tail	Winners (middle)	Gain
SS with Replacement Ratio 0.45	20%	8%	72%	0.15
Ex Ante Efficient	0.9%	18.40%	80.70%	0.51

5.5 Sensitivity check

In order to check how robust the calculation is with respect to changes in parameters, I calculate welfare gain from implementing the ex ante efficient allocation over the economy

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[30]Gong and Webb (2006) and Brown (2003) study the redistributional effects of mandatory annuitization and find that education and racial groups with lower mortality (generally college-educated whites) unambiguously gain from mandatory annuitization. My finding here is that whether the lowest mortality group gains depends on how much mandatory annuitization there is.

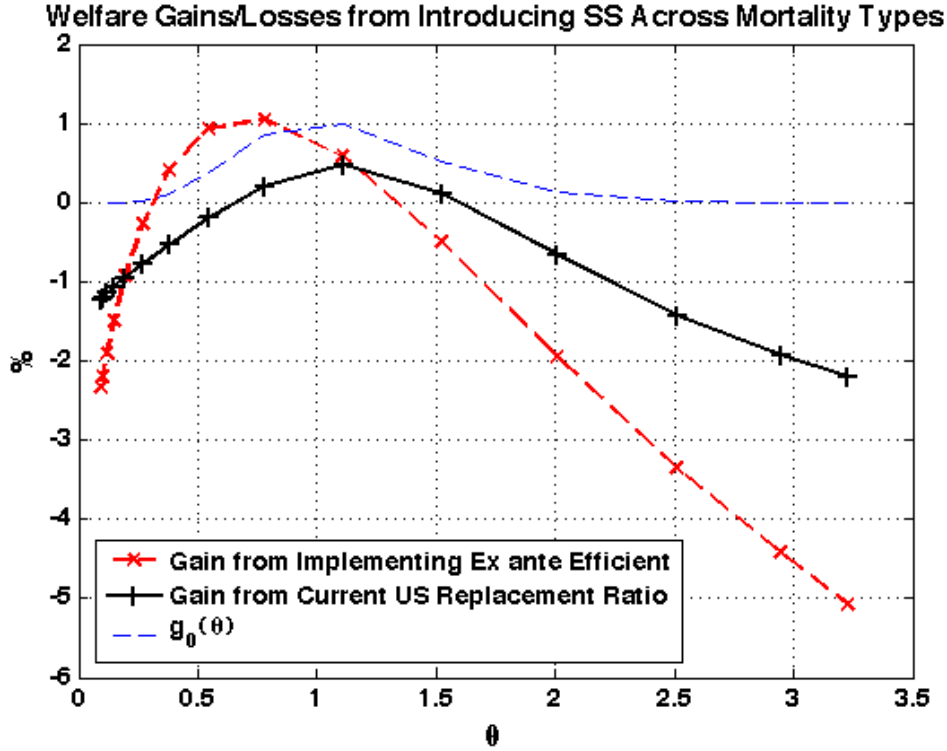## Welfare Gains/Losses from Introducing SS Across Mortality Types

FIGURE 10: Welfare gain by individual mortality types. The plain dashed line is the initial distribution of types. The solid line with + symbols is the welfare gain of having the current U.S. replacement ratio over no social security at all. The dashed line with × is the welfare gain from implementing ex ante efficient allocation over no social security.

with no social security for different levels of parameters. The results are summarized in Figure 11. Panel (a) is the result of welfare calculation for various levels of risk aversion while holding the bequest parameter, ξ, at its benchmark value ($\xi = 0.52$). As I argue in the appendix C, using the two-period example, increasing the risk aversion coefficient does not lead to high welfare gains. The reason is that for high levels of risk aversions, high mortality types have a greater demand for insurance. Therefore, they buy more at any price. On the other hand, low mortality types (who have an upward-sloping consumption profile) prefer a smoother and flatter profile of consumption. Therefore, they demand less annuity and increase their consumption at younger ages. These two effects result in a flatter profile of annuity purchase. That in turn implies that the price of annuity in the market is closer to the overall actuarially fair price. Therefore, for high levels of risk aversion, the quality of risk sharing in the market is better and there are fewer gains from implementing the ex ante efficient allocation, even though the social value of insurance is increased.

Panel (b) in Figure 11 shows the calculated welfare gain for various levels of bequest parameter, holding the coefficient of risk aversion at its benchmark level ($\gamma = 1.09$). The
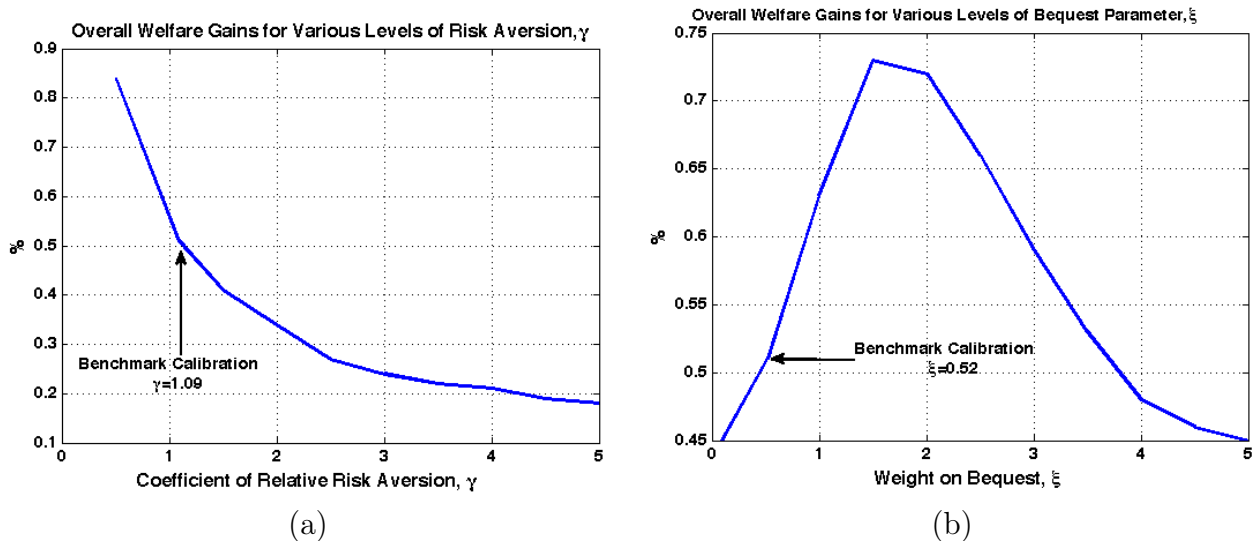
FIGURE 11: Panel (a) shows the overall welfare gain from implementing the ex ante efficient allocation over an economy with no social security for various levels of risk aversion. Panel (b) shows the same calculation for various levels of bequest parameter.

results of comparative static for the two-period example suggest that when starting from low levels of ξ and increasing it, the price in the annuity will rise. [31] The reason is that increasing weight on bequest reduces the value of annuitization (since it increases the value of death at the margin). This affects the higher mortality types more, i.e., everything else equal, their demand is reduced more relative to low mortality types for higher levels of ξ. As a result, the profile of annuity purchase across types becomes steeper and the price of annuity rises further away from the overall actuarially fair price. However, the price of annuity in the market is bounded above (by the fair price of the lowest mortality types). As the bequest parameter increases, eventually the price hits the upper bound and does not increase further. However, the value of annuity insurance in the economy reduces as we increase the weight on bequest. Therefore, even though the market outcome does not provide good risk sharing, the welfare gain eventually decreases for the high bequest parameter because the annuity insurance has little value.

Figure 11 shows that there are combinations of parameters that give rise to sizable welfare gains (for example, coefficient of risk aversion is less than 0.5 and bequest parameter of 0.52). However, at these values for parameters the extent of annuitization is much less than what it is in data. To check whether it is possible to produce large welfare gains while being consistent with 10 percent annuitized wealth at age 70, I calculate welfare gain for various levels of risk aversion and at the same time adjust the bequest parameter so that in the

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[31]Appendix C

economy with social security, the fraction of wealth annuitized through pension stays at 10 percent.[32] The results are reported in Table 4. Welfare gain is no larger than half a percentage in consumption.

TABLE 4: Welfare Gain for Various Parameters while keeping annuitized wealth fixed

γ	ξ (Chosen to match the fraction of pension wealth at 70 equal to 11%)	Welfare Gain
1.09	0.52	0.51%
1.5	0.63	0.41%
2	0.8	0.34%
2.5	0.9	0.3%
3	1.38	0.27%
4	2.8	0.18%

6 Conclusion

Friedman (1962) argues that governments ought to give individuals the freedom of choice to "purchase their annuities from private concerns." However, if individuals are heterogeneous in their mortality and have private information about it, freedom of choice does not lead to the best outcome. Given choices, individuals with high mortality (who have lower risk of survival to old ages) do not participate in the annuity market and limit the risk sharing available to individuals with low mortality (who have a higher risk of survival). Mandatory annuitization can improve welfare by forcing everyone to pool their longevity risk and therefore can improve ex ante welfare. In this paper I investigated the quantitative importance of this welfare improvement and found that it is small. The welfare cost to high mortality types from losing their freedom of choice over the desired level of annuitization almost offsets the improved longevity insurance to low mortality types in a mandatory annuitization scheme. This results in a small welfare gain.

The environment studied in this paper abstracts from any heterogeneity other than mortality types (e.g., in tastes, bequest motives, abilities, income shocks, etc.). It also abstracts from any distortionary effects of policy on labor supply and retirement decisions. An obvious first step to extend the results is to include these features in the environment. In particular, the negative correlation between mortality and measures of individual ability (e.g., educa-

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[32]However, except for the benchmark parameters, the fraction of wealth annuitized through social security will be off.

tion) is well documented.[33] However, the implication of these empirical results for the design of an optimal retirement plan is unknown. A challenge in incorporating these heterogeneities in the model is that the ex ante efficient allocation no longer has the simple type-independent form.[34] In such an environment mandatory annuitization, not only has small value but may also have negative welfare effects.[35] Another extension would be to include endogenous labor supply and/or the retirement decision.

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[33]See, for example Deaton and Paxson (1999).

[34]In fact, the two-dimensionality of private information makes the characterization of the constraint efficient allocation very difficult.

[35]Einav et al. (2007) provide an instructive example of the negative welfare effects of mandatory annuitization when individuals are heterogeneous in mortality risk as well as their preference for bequest.

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Appendix

A Proof of Proposition 1

Proof. Let's write the first-order conditions for type θ

$$q^a c_0^{-\gamma} \geq P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma (s + a + z)^{-\gamma} \text{ "="} \text{ if } a > 0 \tag{19}$$

$$q^l c_0^{-\gamma} \geq (1 - P(\theta))(l + s)^{-\gamma} \text{ "="} \text{ if } l > 0 \tag{20}$$

$$c_0^{-\gamma} \geq P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma (s + a + z)^{-\gamma} + (1 - P(\theta))(l + s)^{-\gamma} \text{ "="} \text{ if } l > 0. \tag{21}$$

The steps of the proof are the following. Step 1, I consider three different cases for prices and characterize the demand for insurance for each case. Step 2, I show that only one of the cases can be an equilibrium.

Step 1

Case I : $q^a + q^l < 1$

Checking the first-order conditions, it is immediate that $s = 0$. We can solve for annuity and life insurance demand:

$$a(q^a, q^l; P(\theta)) = max\left[\left(\frac{P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma}{q^a}\right)^{\frac{1}{\gamma}} c_0(q^a, q^l; \theta) - z, 0\right]$$

$$l(q^a, q^l; P(\theta)) = \left(\frac{1 - P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} c_0(q^a, q^l; \theta).$$

Replacing these equations into the budget constraint, we get

$$c_0(q^a, q^l; P(\theta)) = \frac{w(1-\tau) + q^a z}{1 + q^l \left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a}\right)^{\frac{1}{\gamma}}}$$

or

$$c_0(q^a, q^l; P(\theta)) = \frac{w(1-\tau)}{1 + \left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}}}$$

if $a(q^a, q^l; P(\theta)) = 0$. Demand for annuity and life insurance is

$$a(q^a, q^l; P(\theta)) = max \left[\frac{\left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a}\right)^{\frac{1}{\gamma}} (w(1-\tau) + q^a z)}{1 + q^l \left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a}\right)^{\frac{1}{\gamma}}} - z, 0 \right]$$

and

$$l(q^a, q^l; P(\theta)) = \frac{\left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} (w(1-\tau) + q^a z)}{1 + q^l \left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a}\right)^{\frac{1}{\gamma}}}.$$

Note that an important property of these demand functions is that they are monotone. In particular, at any given set of prices, $l(q^a, q^l; P(\theta))$ is strictly decreasing and $a(q^a, q^l; P(\theta))$ is strictly increasing (whenever non-zero) in $P(\theta)$.[36]

Case II : $q^a + q^l > 1$

This is the case in which any given type θ holds, at most, one type of survival-contingent contract. No individual holds both types of contract, or otherwise the first order condition for liquid assets, s, is violated. The intuitive reason is that, in this case, purchasing both insurance contracts is more expensive than holding liquid assets. Suppose for some type θ, demand for life insurance is positive. That means $a(q^a, q^l; P(\theta)) = 0$ and

$$l(q^a, q^l; P(\theta)) = max \left[\frac{\left[\left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} - \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{1-q^l}\right)^{\frac{1}{\gamma}} \right] (w(1-\tau) + (1-q^l)z)}{1 + q^l \left(\frac{1-P(\theta)}{q^l}\right)^{\frac{1}{\gamma}} + (1-q^l) \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{1-q^l}\right)^{\frac{1}{\gamma}}} + z, 0 \right].$$

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[36]Remember that θ indexed mortality. Survival probability, $P(\theta)$, is a strictly decreasing function of θ.

Note that at any set of prices, $l(q^a, q^l; P(\theta))$ is a decreasing function of $P(\theta)$. Also, there exists a cutoff $\theta_l(q^a, q^l)$ such that $l(q^a, q^l; P(\theta)) > 0$ if and only if $\theta > \theta_l(q^a, q^l)$. Now suppose demand for annuity is positive. Then $l(q^a, q^l; P(\theta)) = 0$ and

$$a(q^a, q^l; P(\theta)) = max \left[\frac{\left[\left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a} \right)^{\frac{1}{\gamma}} - \left(\frac{1-P(\theta)}{1-q^a} \right)^{\frac{1}{\gamma}} \right] (w(1-\tau) + q^a z)}{1 + (1-q^a)\left(\frac{1-P(\theta)}{1-q^a} \right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a} \right)^{\frac{1}{\gamma}}} - z, 0 \right] \quad (22)$$

and similarly, $a(q^a, q^l; P(\theta))$ is a strictly decreasing function of θ for each given set of prices. There is a cutoff $\theta_a(q^a, q^l)$ such that $a(q^a, q^l; P(\theta)) > 0$ if and only if $\theta > \theta_a(q^a, q^l)$.

Case III : $q^a + q^l = 1$

Under these prices, every individual's portfolio choice is indeterminate. However, we can show that

$$a(q^a, q^l; P(\theta)) - l(q^a, q^l; P(\theta)) = \frac{\left[\left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a} \right)^{\frac{1}{\gamma}} - \left(\frac{1-P(\theta)}{q^l} \right)^{\frac{1}{\gamma}} \right] (w(1-\tau) + q^a z)}{1 + q^l \left(\frac{1-P(\theta)}{q^l} \right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a} \right)^{\frac{1}{\gamma}}} - z.$$

In the case in which $a(q^a, q^l; P(\theta)) > 0$ or

$$l(q^a, q^l; P(\theta)) = \frac{\left(\frac{1-P(\theta)}{q^l} \right)^{\frac{1}{\gamma}} (w(1-\tau) + q^a z)}{1 + q^l \left(\frac{1-P(\theta)}{q^l} \right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1+\xi^{\frac{1}{\gamma}})\gamma}{q^a} \right)^{\frac{1}{\gamma}}}$$

if $a(q^a, q^l; P(\theta)) = 0$.

So far I have characterized some key properties of individuals' demand for insurance. In particular, I showed that they are monotone in individual mortality type. At any given price, individuals with higher risk purchase more insurance. In the next proposition I use this property to identify the restriction that equilibrium prices have to satisfy.

Step 2

Start with Case I and II in which $q^a + q^l < 1$ or $q^a + q^l > 1$. In this case, I showed that demand for annuity and life insurance is determinant and monotone in θ (for any fixed set of prices). Consider annuity pricing equation (equation (9))

$$
\begin{aligned}
q^a \int a(q^a, q^l; P(\theta)) dG(\theta) &= \int P(\theta) a(q^a, q^l; P(\theta)) dG(\theta) \\
&= \left(\int P(\theta) dG(\theta) \right) \left(\int a(q^a, q^l; P(\theta)) dG(\theta) \right) \\
&\quad - Cov(P(\theta), a(q^a, q^l; P(\theta))) \\
&> \left(\int P(\theta) dG(\theta) \right) \left(\int a(q^a, q^l; P(\theta)) dG(\theta) \right).
\end{aligned}
$$

The last inequality follows from the fact that $a(q^a, q^l; P(\theta))$ is a monotone increasing function of $P(\theta)$. Now, if $\int a(q^a, q^l; P(\theta)) dG(\theta) > 0$, we get

$$
q^a > \int P(\theta) dG(\theta).
$$

For the case that $\int a(q^a, q^l; P(\theta)) dG(\theta) = 0$ equilibrium belief about payouts, $f^a(\theta)$ puts all weights on the lowest θ (which is zero). Therefore, $q^a = P(0)$ and the above inequality are satisfied. Similarly, we can show that monotone decreasing property of $l(q^a, q^l; P(\theta))$ implies

$$
q^l > \int (1 - P(\theta)) dG(\theta),
$$

therefore

$$
q^l + q^a > 1.
$$

It remains to be shown that the same is true for the demand profiles derived under assumption $q^a + q^l = 1$. In that case we know that $a(q^a, q^l; P(\theta)) - l(q^a, q^l; P(\theta))$ is monotone decreasing in θ. If all demands for life insurance and annuity are zero, then prices cannot be such that $q^a + q^l = 1$ (because of the extreme belief in the equilibrium). Let's rewrite the zero profit conditions for both type of insurers (equations (9) and (10)):

$$
\begin{aligned}
q^a \int a(q^a, q^l; P(\theta)) dG(\theta) &= \int P(\theta) a(q^a, q^l; P(\theta)) dG(\theta) \\
q^l \int l(q^a, q^l; P(\theta)) dG(\theta) &= \int (1 - P(\theta)) l(q^a, q^l; P(\theta)) dG(\theta).
\end{aligned}
$$

Add these equation and replace $q^l = 1 - q^a$ (for simplicity, I suppress the dependence on

prices):

$$q^a \int (a(P(\theta)) - l(P(\theta)))dG(\theta) = \int P(\theta)(a(P(\theta)) - l(P(\theta)))dG(\theta)$$
$$= \left(\int P(\theta)dG(\theta) \right) \left(\int (a(P(\theta)) - l(P(\theta)))dG(\theta) \right)$$
$$- Cov(P(\theta), (a(P(\theta)) - l(P(\theta)))\theta))$$
$$> \left(\int P(\theta)dG(\theta) \right) \left(\int (a(P(\theta)) - l(P(\theta)))dG(\theta) \right).$$

Therefore, $\int (a(P(\theta)) - l(P(\theta)))dG(\theta) \neq 0$ and we get

$$q^q > \int P(\theta)dG(\theta).$$

Similarly, we can show that $q^l > \int (1 - P(\theta))dG(\theta)$. Contradiction.
Therefore, equilibrium prices must be such that $q^l + q^a > 1$. ∎

B   Proof of Theorem 1

Proof. Consider demand for annuity by type θ (equation (22)) and assume it is positive at sum set of prices. I rewrite it as

$$a(q^a, q^l; P(\theta)) = \phi(q^a, P(\theta))(e(1 - \tau) + q^a z) - z,$$

in which

$$\phi(q^a, P(\theta)) = \frac{\left[\left(\frac{P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma}{q^a} \right)^{\frac{1}{\gamma}} - \left(\frac{1 - P(\theta)}{1 - q^a} \right)^{\frac{1}{\gamma}} \right](w(1 - \tau) + q^a z)}{1 + (1 - q^a) \left(\frac{1 - P(\theta)}{1 - q^a} \right)^{\frac{1}{\gamma}} + q^a \left(\frac{P(\theta)(1 + \xi^{\frac{1}{\gamma}})^\gamma}{q^a} \right)^{\frac{1}{\gamma}}}.$$

Notice that

$$\frac{\partial \phi(q^a, P(\theta))}{\partial P(\theta)} > 0.$$

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Also, we know that the social security budget is balanced. Therefore, $z = \frac{\tau w}{\mathrm{E}[P(\theta)]}$ and

$$
\begin{aligned}
\frac{\partial a}{\partial \tau} &= \phi(q^a, P(\theta))(-w + \frac{q^a \tau w}{\mathrm{E}[P(\theta)]}) - \frac{\tau w}{\mathrm{E}[P(\theta)]} \\
&= -\phi(q^a, P(\theta))w - \frac{\tau w}{\mathrm{E}[P(\theta)]}(1 - q^a \phi(q^a, P(\theta))) < 0.
\end{aligned}
\tag{23}
$$

It can be also checked that if $q^a > \mathrm{E}[P(\theta)]$ (which we know is the case in an equilibrium),

$$
\frac{\partial^2 a}{\partial \tau \partial P(\theta)} = \frac{\partial \phi}{\partial P(\theta)}(\frac{q^a \tau w}{\mathrm{E}[P(\theta)]} - w) > 0.
\tag{24}
$$

Now let's write the pricing equation (equation (9)) in expectation terms:

$$
q^a \mathrm{E}[a(q^a, q^l; P(\theta))] = \mathrm{E}[P(\theta)a(q^a, q^l; P(\theta))].
$$

Taking derivatives with respect to τ, we get

$$
\frac{\partial q^a}{\partial \tau}\mathrm{E}[a(q^a, q^l; P(\theta))] + q^a \mathrm{E}[\frac{\partial a}{\partial \tau}] = \mathrm{E}[P(\theta)\frac{\partial a}{\partial \tau}],
$$

therefore,

$$
\begin{aligned}
\frac{\partial q^a}{\partial \tau}\mathrm{E}[a(q^a, q^l; P(\theta))] &= \mathrm{E}[(P(\theta) - q^a)\frac{\partial a}{\partial \tau}] \\
&= \mathrm{E}[(P(\theta) - q^a)]\mathrm{E}[\frac{\partial a}{\partial \tau}] + \mathrm{Cov}[P(\theta) - q^a, \frac{\partial a}{\partial \tau}] \\
&> 0.
\end{aligned}
$$

The last inequality follows from (23)(24) and the fact that in equilibrium, $q^a > \mathrm{E}[P(\theta)]$. ∎

C   Comparative Statics Examples

Risk Aversion

Suppose there is no demand for bequest ($\xi = 0$) and there is no social security. Rewrite the first-order conditions for individual θ.[37]

$$
q^a c_0^{-\gamma} = P(\theta)a^{-\gamma}
$$

---

[37]Since annuity makes survival contingent payment it dominates the liquid assets, and therefore I omit the first-order condition with respect to s. For more detailed analysis of this, see Brown et al. (2005).

and the budget constraint

$$c_0 + q^a a = w.$$

Therefore,

$$a(q^a; P(\theta)) = \frac{w}{q^a + \left(\frac{q^a}{P(\theta)}\right)^{\frac{1}{\gamma}}}.$$

Fixing price q^a and taking a derivative with respect to γ,

$$\frac{\partial a(q^a; P(\theta))}{\partial \gamma} = \frac{w \left(\frac{q^a}{P(\theta)}\right)^{\frac{1}{\gamma}} \log\left(\frac{q^a}{P(\theta)}\right)}{\gamma^2 \left(q^a + \left(\frac{q^a}{P(\theta)}\right)^{\frac{1}{\gamma}}\right)^2}.$$

Notice that for individuals with $P(\theta) < q^a$, increasing γ leads to more demand for insurance. This is because more risk-averse individuals demand more insurance. However, notice that if $P(\theta) > q^a$, then the effect is reversed. For these individuals, the elasticity of substitution effect dominates. They favor a smoother consumption path across time, and therefore their demand for annuity reduces (instead they consume more in the first period). Overall, higher demand by high mortality types ($P(\theta) < q^a$) and lower demand by low mortality types ($P(\theta) > q^a$) means that the overall risk in the insurers' pool is lower. That is because there is more demand by good risk types (who are less likely to claim their coverage) and less demand by high risk types (who are more likely to claim their coverage). This leads to a reduction in the equilibrium price.

Lemma 1 *Equilibrium annuity price is a decreasing function of coefficient of risk aversion.*

Proof.

Consider the zero profit condition of insurers:

$$\int (q^a - P(\theta)) a(q^a; P(\theta)) dG(\theta) = 0.$$

Take the derivative with respect to γ:

$$\int \frac{\partial q^a}{\partial \gamma} a(q^a; P(\theta)) dG(\theta) + \int (q^a - P(\theta)) [\frac{\partial q^a}{\partial \gamma} \frac{\partial a(q^a; P(\theta))}{\partial q^a} + \frac{\partial a(q^a; P(\theta))}{\partial \gamma}] dG(\theta) = 0.$$

Collecting terms:

$$\frac{\partial q^a}{\partial \gamma} = \frac{-\int (q^a - P(\theta)) \frac{\partial a(q^a; \theta)}{\partial \gamma} dG(\theta)}{\int [a(q^a; \theta) + (q^a - P(\theta)) \frac{\partial a(q^a; \theta)}{\partial q^a}] dG(\theta)}.$$

The denominator is always positive since $\frac{\partial a(q^a;\theta)}{\partial q^a}$ and $P(\theta)$ are both decreasing functions of θ. The numerator is always negative since $\frac{\partial a(q^a;\theta)}{\partial \gamma} > 0$ whenever $P(\theta) < q^a$ and vice versa. Therefore, it follows that

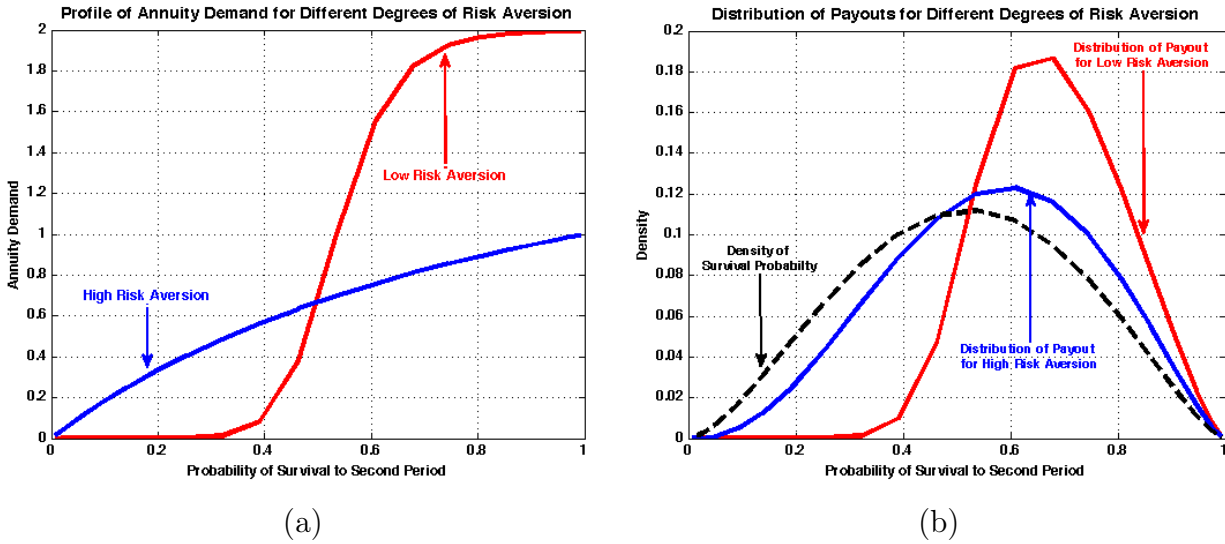$$\frac{\partial q^a}{\partial \gamma} < 0.$$

∎

FIGURE 12: Panel (a) shows the profile of annuity purchase across survival types for a given price and for different degrees of risk aversion. Panel (b) shows the anticipated distribution of payouts for various degrees of risk aversion.

Figure 12 is a graphical illustration of this result. Panel (a) shows the profile of annuity purchase as a function of probability of survival, $P(\theta)$, for a given price. We see that at a higher level of risk aversion, the profile of annuity purchase is less steep. This leads to a less skewed anticipated distribution for payouts (panel (b)) and, in turn, the prices are going to be lower. Figure 13 illustrates how the price changes with degree of risk aversion. Also, panel (b) shows how the welfare difference between equilibrium allocations and ex ante efficient allocations changes with various levels of risk aversion. At very low risk aversion, insurance is not valued and therefore the welfare difference is small. At high levels if risk aversion, the social value of insurance is higher. However, since the equilibrium price is lower, the better insurance is available in the market and the overall welfare difference is reduced.
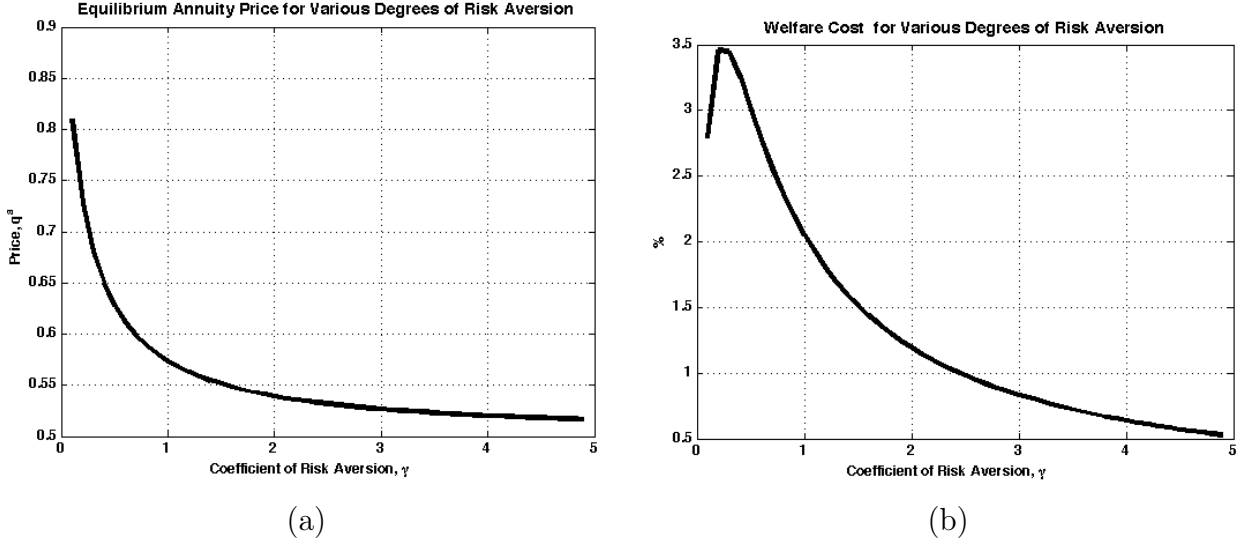
FIGURE 13: Panel (a) shows how the equilibrium price changes with the degree of risk aversion. Panel (b) shows the welfare cost as function of risk aversion.

Bequest

Suppose that the utility function for both consumption and bequest is logarithmic and the weight on bequest is ξ. The demand for annuity (when it is positive) is

$$a(q^a; P(\theta)) = \frac{(1+\xi)\frac{P(\theta)}{q} - \xi\frac{(1-P(\theta))}{1-q}}{1 + P(\theta) + \xi}w.$$

Once again I hold the price fixed and take the derivative with respect to ξ

$$\frac{\partial a(q^a; P(\theta))}{\partial \xi} = \frac{wP(\theta)(P(\theta) - q)}{q(1 - q)(1 + P(\theta) + \xi)^2}.$$

At any given price, individuals with $P(\theta) > q$ have more demand for annuity as the weight on bequest increases. On the other hand, individuals with $P(\theta) < q$ have less demand as bequest parameters increase and their demand eventually drops to zero. Figure 14 illustrates this effect for levels of ξ at a given price.

Lemma 2 *Equilibrium annuity price is an increasing function of bequest parameter, ξ.*

Proof.

Given the expression for $\frac{\partial a(q^a; \theta)}{\partial \xi}$, the proof works exactly as the proof for the previous lemma. ∎
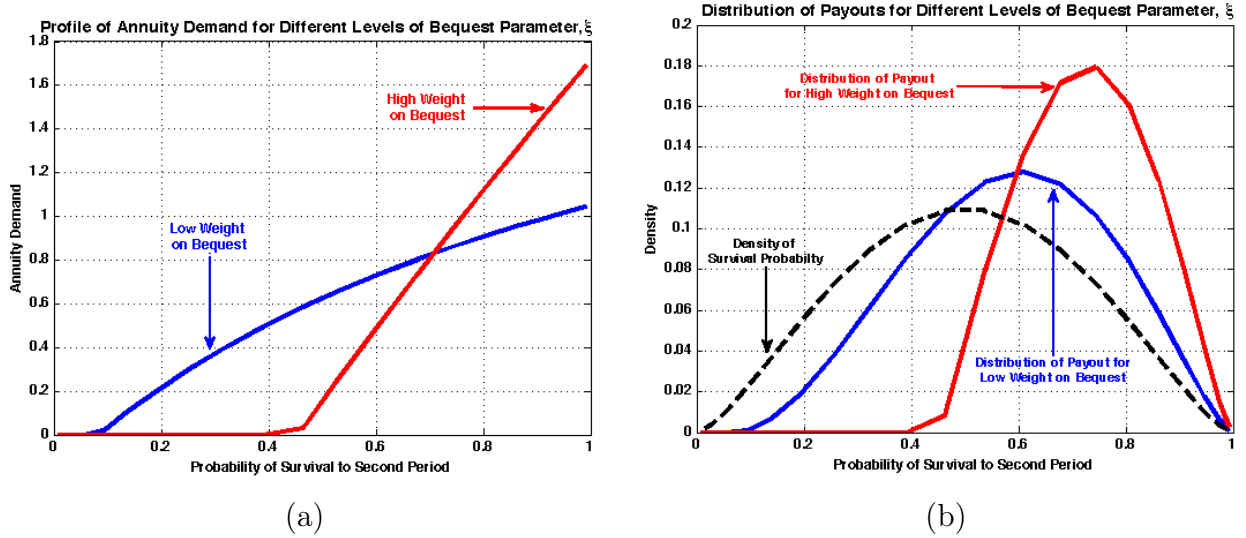
47

FIGURE 14: Panel (a) shows that higher weight on bequest makes the profile of annuity purchase more steep. This, in turn, leads to more skewed anticipated distribution of payouts (panel (b)) and higher annuity prices.

Although this lemma shows that increasing the weight on bequest leads to an increase in prices, the effect on welfare is unclear. Panel (b) in Figure 15 shows how the welfare changes with ξ. When ξ is very low, a small an increase leads to increase in the welfare cost. This is the case when individuals of various risk types purchase positive annuity coverage. However, because the annuity profile is steeper when ξ increases, the price increases and the welfare cost goes up. But when ξ is high enough so that the price is at its highest possible value, a further increase in ξ reduces the welfare cost. It is true that at this high price, many individuals do not purchase annuity or purchase very little. But at the same time, the value of dying with positive assets is higher (because ξ is higher). Since the price cannot be above one, a further increase in ξ cannot increase the price and cannot make the market outcome worse. On the other hand, it increases the value of dying with positive liquid assets and overall welfare becomes higher. As a result, the distance between market allocation and ex ante efficient allocation goes down.

D   Estimation of Mortality Heterogeneity

In this appendix I describe the procedure for estimating the distribution of mortality heterogeneity using the HRS data on subjective survival probabilities. The data that is used is the answer to the following question:

> Using any number from 0 to 10 where 0 equals *absolutely no chance* and 10 equals
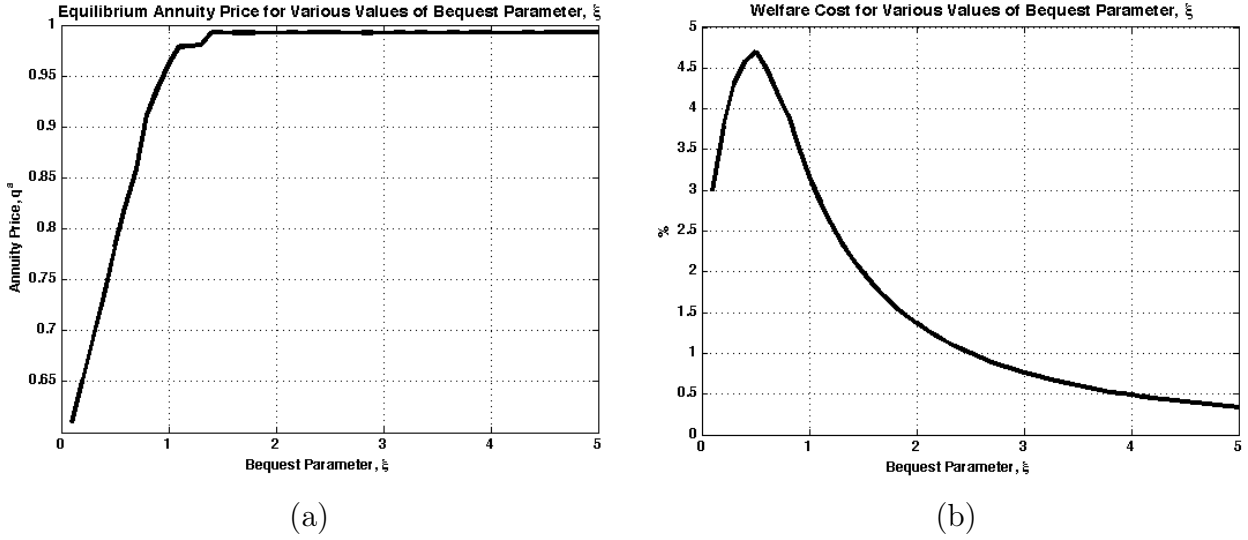
FIGURE 15: Equilibrium annuity prices (panel (a)) and welfare (panel (b)) as function of weight on bequest.

> *absolutely certain*, what do you think are the chances you will live to be 75 and more?

It is documented by Hurd and McGarry (1995, 2002) that the responses aggregated quite closely to the predictions of life tables and varies appropriately with known risk factors and determinants of mortality. Therefore, these responses can potentially be used as true probabilities. However, at the individual level there are some difficulties. A large fraction of responses are either zero or one. Gan et al. (2003) documented that 30% of respondents in wave 1 (survey year 1992) and 19% of respondents in wave 2 (survey year 1994) responded zero or one and argue that these responses cannot represent true probabilities since the distribution of true probabilities should be continuous, and moreover, true probabilities cannot be equal zero or one. They propose a Bayesian update model to recover the true subjective survival curves for each respondent. Here I follow their procedure to estimate the heterogeneity in mortality using the data on male respondents who answered the question about survival probability in wave 1 (year 1992).

Notations and assumptions on mortality model

Before I describe the estimation procedure, recall the notations and assumptions of the mortality model:

- θ: frailty index

- $H_t(\theta)$: cumulative mortality hazard for individual of type θ. I assume that

$$H_t(\theta) = \theta H_t,$$

  in which H_t is independent of θ.

- $P_t(\theta)$: probability that an individual with type θ survive to age t from birth. Note that by definition we have

$$P_t(\theta) = \exp(-H_t(\theta)) = \exp(-\theta H_t).$$

- $g_0(\theta)$: density of type distribution at birth. I assume that θ has gamma distribution with mean one and variance $\sigma_\theta^2 = \frac{1}{k}$.

$$g_0(\theta) = k^k \theta^{k-1} \frac{\exp(-k\theta)}{\Gamma(k)}.$$

- $\bar{P}_t$: average (life table) probability of survival to age t from birth.

$$\bar{P}_t = \int_0^\infty P_t(\theta) g_0(\theta) d\theta. \tag{25}$$

  Also let $\bar{H}_t = -\log(\bar{P}_t)$ denote the life table cumulative mortality hazard.

- $g_t(\theta)$: density of types who survive to age t.

$$g_t(\theta) = \frac{g_0(\theta) P_t(\theta)}{\bar{P}_t}.$$

It is shown in the text (equation (18)) that frailty types who survive to any age t also have gamma distribution

$$g_t(\theta) = \mathcal{G}\left(\frac{1}{k + H_t}, k\right) = (H_t + k)^k \theta^{k-1} \frac{\exp(-\theta(H_t + k))}{\Gamma(k)},$$

and the restriction (25) implies

$$H_t = \frac{\exp(\sigma_\theta^2 \bar{H}_t) - 1}{\sigma_\theta^2}. \tag{26}$$

Therefore, if σ_θ^2 is known, H_t can be computed from $\bar{H}_t$ (which is known from life table).

Suppose there are n respondent. The data available for each respondent $i = 1, \ldots, n$ is

- r_t^i: the number that is reported to the survival probability question (normalized to be between zero and one). This is what respondent i reported at age t about his probability of survival to age 75.

- T^i: the age at which respondent i left the sample. It is also known whether the respondent was alive or dead at age T^i

Suppose the respondent i has frailty type θ^i. Then, the probability that this individual survives to age 75 is

$$\frac{P_{75}(\theta^i)}{P_t(\theta^i)} = \exp(-\theta^i(H_{75} - H_t)).$$

However because of error in report, this individual instead make a report $r_t^i \neq \frac{P_{75}(\theta^i)}{P_t(\theta^i)}$. To model this error in reporting we assume that self-reported survival probability r_t^i has a density $f(r_t^i | \frac{P_{75}(\theta^i)}{P_t(\theta^i)})$. The difference between the actual report and the true survival probability is measured as error in report. I assume that conditional on $\frac{P_{75}(\theta^i)}{P_t(\theta^i)}$, r_t^i has a censored normal distribution, i.e., there is a μ_t^i and σ_f^2 such that

$$f\left(r | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}\right) = \phi\left(\frac{r - \mu_t^i}{\sigma_f}\right) \text{ for } 0 < r < 1,$$

and

$$\Pr\left(r = 0 | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}\right) = 1 - \Phi\left(\frac{\mu_t^i}{\sigma_f}\right),$$

$$\Pr\left(r = 1 | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}\right) = 1 - \Phi\left(\frac{1 - \mu_t^i}{\sigma_f}\right),$$

in which $\phi(\cdot)$ is the standard normal p.d.f and $\Phi(\cdot)$ is standard normal c.d.f. Furthermore, I assume that each individual makes no error on average ($\mathrm{E}[r | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}] = \frac{P_{75}(\theta^i)}{P_t(\theta^i)}$). Therefore, for each $\frac{P_{75}(\theta^i)}{P_t(\theta^i)}$ the following restriction must hold

$$\frac{P_{75}(\theta^i)}{P_t(\theta^i)} = \Pr\left(r = 1 | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}\right) + \int_0^1 r f\left(r | \frac{P_{75}(\theta^i)}{P_t(\theta^i)}\right) dr. \tag{27}$$

Note that this implies that the (uncensored) normal distribution has the same variance for all θ types and all ages. However, the mean depends on type and also on the age at which the report is being made (hence μ_t^i is indexed by both i and t).

Estimation procedure

The prior on the density of types alive at age t is given by $g_t(\theta)$. Once a report r_t^i is observed, we can form a posterior about the respondent i's type, given his report. I denote this posterior belief by $\hat{g}_t(\theta|r_t^i)$ and

$$\hat{g}_t(\theta|r_t^i) = \frac{g_t(\theta)f(r_t^i|\frac{P_{75}(\theta)}{P_t(\theta)})}{\int_0^\infty g_t(\eta)f(r_t^i|\frac{P_{75}(\eta)}{P_t(\eta)})d\eta}.$$

Furthermore, we can use this posterior to form expectation about respondent i's frailty, given the report r_t^i. Let $\hat{\theta}(r_t^i) = \int_0^\infty \theta\hat{g}_t(\theta|r_t^i)d\theta$ be the conditional expectation of frailty type. Using $\hat{\theta}(r_t^i)$ and H_τ we can find estimate of true probability of survival to any age τ for individual i (conditional on being alive at t)

$$\hat{P}_{t.\tau}^i = \exp(-\hat{\theta}(r_t^i)(H_\tau - H_t)).$$

Now we can apply these estimate of survival probabilities to the observed death and survival records to estimate model's parameter values. In other words, suppose the individual i is at age t at the time of report and suppose he stays in the sample up to age T^i. The log-likelihood function is

$$\log L = \sum_{i \text{ is alive at } T^i} \log(\hat{P}_{t.T^i}^i) + \sum_{i \text{ is dead at } T^i} (\log(\hat{P}_{t.T^i-1}^i) + \log(1 - \hat{P}_{T^i-1,T^i}^i)).$$

The parameters of $g_0(\cdot)$ and $f(\cdot|\cdot)$ are estimated by maximizing the above log-likelihood function. There are two parameters that we need to estimate. One is the variance of initial type distribution, σ_θ^2 (the mean is normalized to one). The other one is the variance of the (uncensored) normal distribution that is used to construct $f(\cdot|\cdot)$ (given this variance, the mean can be computed for each θ). Therefore, the likelihood function is a function of two variables, σ_θ^2 and σ_f^2. To evaluate the likelihood function we use the following procedure:

- Given a guess of σ_θ^2 and σ_f^2, and using $\bar{H}_t$ (computed form life table) we can compute H_t.

- For each respondent i we should find the posterior density $\hat{g}_t(\theta|r_t^i)$. This is then used to evaluate the integral $\int_0^\infty \theta\hat{g}_t(\theta|r_t^i)d\theta$ numerically to find estimate of frailty for respondent i ($\hat{\theta}(r_t^i)$). Therefore, we need to know the value of $\hat{g}_t(\theta|r_t^i)$ on a finite number of points (on a grid of θ). For each of these points, we can find μ_t^i from equation (27). Once that is known, $f(\cdot|\frac{P_{75}(\theta)}{P_t(\theta)})$ can be evaluated (for each point of the grid of θ). We

can now compute $\hat{\theta}(r_t^i)$.

- Given $\hat{\theta}(r_t^i)$ and H_t the probability of survival to any age $(\hat{P}_{t.\tau}^i)$ can be computed for respondent i.

- Likelihood of the respondent i's survival to age T^i,conditional on being alive at age t, is $\hat{P}_{t.T^i}^i$. Likelihood of respondent i's survival to age T^i-1 and die at age T^i ,conditional on being alive at age t, is $\hat{P}_{t.T^i-1}^i(1 - \hat{P}_{T^i-1,T^i}^i)$.

Once the log-likelihood is evaluated we can use standard procedure for numerical optimization to find its maximum.

<div align="center">

TABLE 5:   Estimation Results

Variance of initial frailty distribution, σ_θ^2	Variance of reporting error, σ_f^2
0.1256	0.0348
(0.07)	(0.0224)

</div>

Table 5 shows the result of estimation. The number is parenthesis are standard errors. We can see from the left column that the estimated variance for frailty heterogeneity, σ_θ^2, is very close to the number that Gan et al. (2003) report as "hazard scaling"in their model.