AFFINE APPROACH TO QUANTUM SCHUBERT CALCULUS

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Abstract

This paper presents a formula for products of Schubert classes in the quantum cohomology ring of the Grassmannian. We introduce a generalization of Schur symmetric polynomials for shapes that are naturally embedded in a torus. Then we show that the coefficients in the expansion of these toric Schur polynomials, in terms of the regular Schur polynomials, are exactly the 3-point Gromov-Witten invariants, which are the structure constants of the quantum cohomology ring. This construction implies three symmetries of the Gromov-Witten invariants of the Grassmannian with respect to the groups S_3, $(\mathbb{Z}/n\mathbb{Z})^2$, and $\mathbb{Z}/2\mathbb{Z}$. The last symmetry is a certain curious duality *of the quantum cohomology which inverts the quantum parameter q. Our construction gives a solution to a problem posed by Fulton and Woodward about the characterization of the powers of the quantum parameter q which occur with nonzero coefficients in the quantum product of two Schubert classes. The curious duality switches the smallest such power of q with the highest power. We also discuss the affine nil-Temperley-Lieb algebra that gives a model for the quantum cohomology.*

Contents

DUKE MATHEMATICAL JOURNAL
Vol. 128, No. 3, © 2005
Received 21 July 2003. Revision received 10 August 2004.
2000 *Mathematics Subject Classification*. Primary 05E05; Secondary 14M15, 14N35.
Author's work supported in part by National Science Foundation grant number DMS-0201494.

1. Introduction

It is well known that the Schubert calculus is related to the theory of symmetric functions. The cohomology ring of the Grassmannian is a certain quotient of the ring of symmetric functions. Schubert classes form a linear basis in the cohomology and correspond to the Schur symmetric polynomials. There is a more general class of symmetric polynomials known as the skew Schur polynomials. The problem of multiplying two Schubert classes is equivalent to the problem of expanding a given skew Schur polynomial in the basis of ordinary Schur polynomials. The coefficients that appear in this expansion are explicitly computed using the Littlewood-Richardson rule.

Recently, in a series of papers by various authors, attention has been drawn to the small quantum cohomology ring of the Grassmannian. This ring is a certain deformation of the usual cohomology. Its structure constants are the 3-point Gromov-Witten invariants, which count the numbers of certain rational curves of fixed degree.

In this paper we present a quantum cohomology analogue of skew Schur polynomials. These are certain symmetric polynomials labeled by shapes that are embedded in a torus. We show that the Gromov-Witten invariants are the expansion coefficients of these toric Schur polynomials in the basis of ordinary Schur polynomials. The toric Schur polynomials are defined as sums over certain cylindric semistandard tableaux. Note that these tableaux already appeared (under different names) in [GK] and [BCF].

This construction implies several nontrivial results. For example, it reproduces the known result that the Gromov-Witten invariants are symmetric with respect to the action of the product of two cyclic groups. Also, it gives a certain curious duality of the Gromov-Witten invariants which exchanges the quantum parameter q and its inverse.* Geometrically, this duality implies that the number of rational curves of small degree equals the corresponding number of rational curves of high degree. Another corollary of our construction is a complete characterization of all powers of q with nonzero coefficient which appear in the expansion of the quantum product of two Schubert classes. This problem was posed in a recent paper by Fulton and Woodward [FW], in which the lowest power of q was calculated. By virtue of the curious duality, the problem of computing the highest power of q is equivalent to finding the lowest power.

The general outline of the paper follows. In Section 2 we review main definitions and results related to symmetric functions and to classical and quantum cohomologies of the Grassmannian. In Section 3 we introduce our main tool—toric shapes and toric tableaux. In Section 4 we discuss the quantum Pieri formula and quantum Kostka numbers. In Section 5 we define toric Schur polynomials and prove our main result on their Schur expansion. In Section 6 we discuss the cyclic symmetry and the curious

*After the original version of this paper appeared in the e-print arXiv, Hengelbrock informed us that he independently found this duality of the quantum cohomology, for $q = 1$ (see [H]).

duality of the Gromov-Witten invariants. In Section 7 we describe all powers of the quantum parameter which appear in the quantum product. In Section 8 we discuss the action of the affine nil-Temperley-Lieb algebra on quantum cohomology. In Section 9 we give final remarks, open questions, and conjectures.

2. Preliminaries

We note some definitions and results related to symmetric functions and to classical and quantum cohomology rings of the Grassmannian (see [M], [F] and [A], [AW], [B], [Bu], [BCF], [FW] for the quantum part of the story).

2.1. Symmetric functions

Let $\Lambda_k = \mathbb{Z}[x_1, \dots, x_k]^{S_k}$ be the ring of *symmetric polynomials* in $x_1, \dots, x_k$. The ring Λ of *symmetric functions* in the infinite set of variables $x_1, x_2, \dots$ is defined as the inverse limit $\Lambda = \varprojlim \Lambda_k$ in the category of graded rings. In other words, the elements of the ring Λ are formal power series (with bounded degrees) in the variables $x_1, x_2, \dots$ which are invariant under any finite permutation of the variables. The ring Λ is freely generated by the *elementary symmetric functions* $e_i = \sum_{a_1 < \cdots < a_i} x_{a_1} \cdots x_{a_i}$ and, alternatively, by the *complete homogeneous symmetric functions* $h_j = \sum_{b_1 \le \cdots \le b_j} x_{b_1} \cdots x_{b_j}$:

$$\Lambda = \mathbb{Z}[e_1, e_2, e_3, \dots] = \mathbb{Z}[h_1, h_2, h_3, \dots].$$

The two sets of generators can be recursively expressed from each other using the identity $(1 + t\, e_1 + t^2 e_2 + \cdots) \cdot (1 - t\, h_1 + t^2 h_2 - \cdots) = 1$.

For a partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_l \ge 0)$, the *Young diagram* of shape λ is the set $\{(i, j) \in \mathbb{Z}^2 \mid 1 \le j \le \lambda_i\}$. It is usually represented as a collection of $|\lambda| = \lambda_1 + \cdots + \lambda_l$ boxes arranged on the plane as one would arrange elements of a matrix (see Fig. 1). For a pair of partitions λ and μ such that $\lambda_i \ge \mu_i$, for all i, the *skew Young diagram* of shape λ/μ is the set-theoretic difference of the two Young diagrams of shapes λ and μ. A *semistandard Young tableau* of *shape* λ/μ and *weight* $\beta = (\beta_1, \dots, \beta_r)$ is a way to fill the boxes of the skew Young diagram with numbers $1, \dots, r$ such that β_i is the number of i's, for $i = 1, \dots, r$, and the entries in the tableau are weakly increasing in the rows and strictly increasing in the columns of the Young diagram. For a tableau T of weight β, let $\mathbf{x}^T = \mathbf{x}^\beta = x_1^{\beta_1} \cdots x_r^{\beta_r}$.

The *skew Schur function* $s_{\lambda/\mu}$ is defined as the sum

$$s_{\lambda/\mu} = s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \text{ of shape } \lambda/\mu} \mathbf{x}^T$$

over all semistandard Young tableaux T of shape λ/μ. It is a homogeneous element in the ring Λ of symmetric functions of degree $|\lambda/\mu| = |\lambda| - |\mu|$. By our convention,

$s_{\lambda/\mu} = 0$ if some of the conditions $\lambda_i \geq \mu_i$ fail. If $\mu = \emptyset$ is the empty partition, then we obtain the usual *Schur function* $s_\lambda = s_{\lambda/\emptyset}$. The set of Schur functions s_λ, for all partitions λ, forms a $\mathbb{Z}$-basis of Λ.

The specializations $s_\lambda(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k, 0, 0, \ldots) \in \Lambda_k$ of the Schur functions are called the *Schur polynomials.* The set of Schur polynomials, where λ ranges over partitions with at most k parts, forms a $\mathbb{Z}$-basis of the ring of symmetric polynomials $\Lambda_k = \mathbb{Z}[e_1, \ldots, e_k]$.

The *Jacobi-Trudy formula* expresses the Schur functions s_λ in terms of the elementary or complete homogeneous symmetric functions:

$$s_\lambda = \det(h_{\lambda_i+j-i})_{1\leq i,\, j\leq l} = \det(e_{\lambda'_i+j-i})_{1\leq i,\, j\leq s}, \tag{1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition and $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is its *conjugate partition,* whose Young diagram is obtained by transposition of the Young diagram of λ (see Fig. 1). Here we assume that $e_0 = h_0 = 1$ and $e_i = h_j = 0$ for $i, j < 0$.

The *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$ are defined as the structure constants of the ring of symmetric functions Λ in the basis of Schur functions:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu,$$

where the sum is over partitions ν such that $|\nu| = |\lambda| + |\mu|$. The coefficients $c^\nu_{\lambda\mu}$ are nonnegative integers. The famous *Littlewood-Richardson rule* gives an explicit combinatorial formula for these numbers.

The Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$ can also be expressed using skew Schur functions, as follows. Let $\langle \cdot\,, \cdot\rangle$ be the inner product in the space of symmetric functions Λ such that the usual Schur functions s_λ form an orthogonal basis. Then we have $\langle s_\lambda,\, s_\mu \cdot s_\nu\rangle = \langle s_{\lambda/\mu},\, s_\nu\rangle$ (see [M]). Thus the coefficients of expansion of a skew Schur function in the basis of the usual Schur functions are exactly the Littlewood-Richardson coefficients:

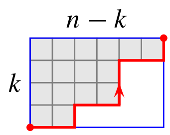
$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu}\, s_\nu. \tag{2}$$

2.2. Cohomology of Grassmannians

Let Gr_{kn} be the variety of k-dimensional subspaces in $\mathbb{C}^n$. It is a complex projective variety called the *Grassmann variety* or the *Grassmannian*. There is a cellular decomposition of the Grassmannian Gr_{kn} into Schubert cells Ω°_λ. These cells are indexed by partitions λ whose Young diagrams fit inside the $(k \times (n-k))$-rectangle. Let

$$P_{kn} = \left\{\lambda = (\lambda_1, \ldots, \lambda_k) \,\middle|\, n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0\right\}$$

be the set of such partitions. The boundary of the Young diagram of a partition $\lambda \in P_{kn}$ corresponds to a lattice path in the $(k \times (n-k))$-rectangle from the lower-left corner to the upper-right corner. Such a path can be encoded as a sequence $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$ of 0's and 1's with $\omega_1 + \cdots + \omega_n = k$, where 0's correspond to the right steps and 1's correspond to the upward steps in the path (see Fig. 1). We say that $\omega(\lambda)$ is the *01-word* of a partition $\lambda \in P_{kn}$. The 01-words are naturally associated with cosets of the symmetric group S_n modulo the maximal parabolic subgroup $S_k \times S_{n-k}$.



$k = 4,\ n = 10,$
$\lambda = (6, 4, 4, 2),\ |\lambda| = 16,$
$\omega(\lambda) = (0, 0, 1, 0, 0, 1, 1, 0, 0, 1),$
$\lambda^\vee = (4, 2, 2, 0),\ \lambda' = (4, 4, 3, 3, 1, 1)$

Figure 1. A partition in P_{kn}

Fix a standard flag of coordinate subspaces $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$. For $\lambda \in P_{kn}$ with $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, the *Schubert cell* Ω°_λ consists of all k-dimensional subspaces $V \subset \mathbb{C}^n$ with prescribed dimensions of intersections with the elements of the coordinate flag: $\dim(V \cap \mathbb{C}^i) = \omega_n + \omega_{n-1} + \cdots + \omega_{n-i+1}$ for $i = 1, \ldots, k$. The closures $\Omega_\lambda = \bar{\Omega}^\circ_\lambda$ of Schubert cells are called the *Schubert varieties*. Their fundamental cohomology classes $\sigma_\lambda = [\Omega_\lambda]$, $\lambda \in P_{kn}$, called the *Schubert classes*, form a $\mathbb{Z}$-basis of the *cohomology ring* $\mathrm{H}^*(\mathrm{Gr}_{kn})$ of the Grassmannian. Thus $\dim \mathrm{H}^*(\mathrm{Gr}_{kn}) = |P_{kn}| = \binom{n}{k}$. We have $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{Gr}_{kn})$.

The cohomology ring of the Grassmannian is generated by either of the following two families of *special Schubert classes:* $\sigma_{(1^i)} = c_i(\mathcal{V}^*)$, $i = 1, \ldots, k$, and $\sigma_{(j)} = c_j(\mathbb{C}^n/\mathcal{V})$, $j = 1, \ldots, n-k$, where $\mathcal{V}$ is the universal subbundle on Gr_{kn} and c_i denotes the ith Chern class. Here $(1^i) = (1, \ldots, 1)$ is the partition with i parts equal to 1, and (j) is the partition with one part j.

The cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{kn})$ is canonically isomorphic to the quotient of the polynomial ring $\mathbb{Z}[e_1, \ldots, e_k, h_1, \ldots, h_{n-k}]$ modulo the ideal generated by the coefficients in the t-expansion of $(1 + t\,e_1 + \cdots + t^k e_k)(1 - t\,h_1 + \cdots + (-t)^{n-k} h_{n-k}) - 1$. This isomorphism is given by $\sigma_{(1^i)} \mapsto e_i$, $i = 1, \ldots, k$, and $\sigma_{(j)} \mapsto h_j$, $j = 1, \ldots, n-k$. Equivalently, we can present the cohomology $\mathrm{H}^*(\mathrm{Gr}_{kn})$ as the quotient

$$\mathrm{H}^*(\mathrm{Gr}_{kn}) \simeq \Lambda_k / \langle h_{n-k+1}, \ldots, h_n \rangle = \Lambda / \langle e_i, h_j \mid i > k,\ j > n-k \rangle. \quad (3)$$

The ideal in Λ in the last expression is spanned by the Schur functions s_λ, whose shapes do not fit inside the $(k \times (n-k))$-rectangle. In this isomorphism, the Schubert classes σ_λ, for $\lambda \in P_{kn}$, map to (the cosets of) the Schur functions s_λ.

This isomorphism implies that the structure constants of the cohomology ring

$H^*(\mathrm{Gr}_{kn})$ in the basis of Schubert classes are the Littlewood-Richardson coefficients:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu \in P_{kn}} c_{\lambda\mu}^{\nu}\, \sigma_\nu \quad \text{for } \lambda, \mu \in P_{kn}.$$

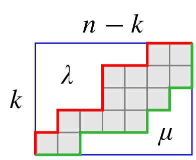
In particular, the structure constants in $H^*(\mathrm{Gr}_{kn})$ do not depend on k and n.

Let $c_{\lambda\mu\nu} = \int_{Gr_{kn}} \sigma_\lambda \cdot \sigma_\mu \cdot \sigma_\nu$ be the intersection numbers of the three Schubert varieties Ω_λ, Ω_μ, and Ω_ν. Let $\nu^\vee = (n-k-\nu_k, \ldots, n-k-\nu_1)$ denote the *complement partition* to $\nu \in P_{kn}$; that is, $\nu^\vee$ is obtained from ν by taking the complement to its Young diagram in the $(k \times (n-k))$-rectangle and then rotating it by 180° degrees (see Fig. 1). Then $c_{\lambda\mu\nu} = c_{\lambda\mu}^{\nu^\nu}$. This equality of the structure constants and the intersection numbers follows from the fact that the basis of Schubert classes σ_λ is self-dual with respect to the Poincaré pairing: $\int_{Gr_{kn}} \sigma_\lambda \cdot \sigma_{\mu^\vee} = \delta_{\lambda\mu}$ (Kronecker's delta). This provides a geometric explanation for the nonnegativity of the Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu^\vee} = c_{\lambda\mu\nu}$ and implies their S_3-symmetry with respect to permutations of λ, μ, and ν.

Expression (2) for the Littlewood-Richardson coefficients in terms of the skew Schur functions can be written equivalently as

$$s_{\mu^\vee/\lambda} = \sum_{\nu \in P_{kn}} c_{\lambda\mu}^{\nu}\, s_{\nu^\vee} \quad \text{for } \lambda, \mu \in P_{kn}. \tag{4}$$

Here we use S_3-symmetry of the Littlewood-Richardson coefficients. Note that expression (4) depends on particular values of k and n.



$k = 5,\ n = 12,$
$\lambda = (5, 3, 3, 1, 0),\ \mu = (5, 2, 1, 0, 0)$

Figure 2. Skew shape associated with $\sigma_\lambda \cdot \sigma_\mu$

Formula (4) says that the coefficient $c_{\lambda\mu}^{\nu}$ of σ_ν in the expansion of the product $\sigma_\lambda \cdot \sigma_\mu \in H^*(\mathrm{Gr}_{kn})$ is equal to the coefficient of $s_{\nu^\vee}$ in the expansion of the skew Schur function $s_{\mu^\vee/\lambda} \in \Lambda$. In other words, the Poincaré dual of the product $\sigma_\lambda \cdot \sigma_\mu$ in $H^*(\mathrm{Gr}_{kn})$ corresponds to the skew Schur function $s_{\mu^\vee/\lambda}$ under isomorphism (3). Note that the shape $\mu^\vee/\lambda$ is obtained from the $(k \times (n-k))$-rectangle by removing the shape λ in the northwest corner and removing the shape μ (rotated by 180°) in the southeast corner (see Fig. 2). In particular, $\sigma_\lambda \cdot \sigma_\mu \neq 0$ if and only if $\mu^\vee/\lambda$ is a valid skew shape; that is, the removed shapes do not overlap in the $(k \times (n-k))$-rectangle. In this paper we present analogues of formulas (2) and (4) for the quantum cohomology ring of Gr_{kn}.

2.3. Quantum cohomology of Grassmannians

The (small) *quantum cohomology ring* $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ of the Grassmannian is an algebra over $\mathbb{Z}[q]$, where q is a variable of degree n. As a linear space, the quantum cohomology is equal to the tensor product $\mathrm{H}^*(\mathrm{Gr}_{kn}) \otimes \mathbb{Z}[q]$. Thus Schubert classes σ_λ, $\lambda \in P_{kn}$, form a $\mathbb{Z}[q]$-linear basis of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$.

The product in $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ is a certain q-deformation of the product in $\mathrm{H}^*(\mathrm{Gr}_{kn})$. It is defined using the (3-point) Gromov-Witten invariants. The *Gromov-Witten invariant* $C^d_{\lambda\mu\nu}$, usually denoted $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu\rangle_d$, counts the number of rational curves of degree d in Gr_{kn} which meet generic translates of the Schubert varieties Ω_λ, Ω_μ, and Ω_ν, provided that this number is finite. The last condition implies that the Gromov-Witten invariant $C^d_{\lambda\mu\nu}$ is defined if $|\lambda| + |\mu| + |\nu| = nd + k(n-k)$. (Otherwise, we set $C^d_{\lambda\mu\nu} = 0$.) If $d = 0$, then a degree zero curve is just a point in Gr_{kn} and $C^0_{\lambda\mu\nu} = c_{\lambda\mu\nu}$ are the usual intersection numbers. In general, the geometric definition of the Gromov-Witten invariants $C^d_{\lambda\mu\nu}$ implies that they are nonnegative integer numbers. We use the notation $\sigma * \rho$ for the "quantum product" of two classes σ and ρ, that is, their product in the ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. This product is a $\mathbb{Z}[q]$-linear operation. Thus it is enough to specify the quantum product of any two Schubert classes. It is defined as

$$\sigma_\lambda * \sigma_\mu = \sum_{d,\nu} q^d\, C^{\nu,d}_{\lambda\mu}\, \sigma_\nu, \tag{5}$$

where the sum is over nonnegative integers d and partitions $\nu \in P_{kn}$ such that $|\nu| = |\lambda| + |\mu| - d\,n$ and the structure constants are the Gromov-Witten invariants $C^{\nu,d}_{\lambda\mu} = C^d_{\lambda\mu\nu^\vee}$. Properties of the Gromov-Witten invariants imply that the quantum product is a commutative and associative operation. In the "classical limit" $q \to 0$, the quantum cohomology ring becomes the usual cohomology.

Unlike the usual Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$, the Gromov-Witten invariants $C^{\nu,d}_{\lambda\mu}$ depend not only on three partitions λ, μ, and ν but also on the numbers k and n. If $n > |\lambda| + |\mu|$, then $C^{\nu,d}_{\lambda\mu} = \delta_{d\,0} \cdot c^\nu_{\lambda\mu}$. Thus all "quantum effects" vanish in the limit $n \to \infty$.

The quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ is canonically isomorphic to the quotient

$$\mathrm{QH}^*(\mathrm{Gr}_{kn}) \simeq \mathbb{Z}[q, e_1, \dots, e_k, h_1, \dots, h_{n-k}]/I^q_{kn} = (\Lambda_k \otimes \mathbb{Z}[q])/J^q_{kn}, \tag{6}$$

where the ideal I^q_{kn} is generated by the coefficients in the t-expansion of the polynomial $(1 + t\,e_1 + \cdots + t^k e_k)(1 - t\,h_1 + \cdots + (-t)^{n-k} h_{n-k}) - 1 - (-1)^{n-k} q\, t^n$, and $J^q_{kn} = \langle h_{n-k+1}, \dots, h_{n-1}, h_n + (-1)^k q\rangle$. As in the classical case, isomorphism (6) is given by $\sigma_{1^i} \mapsto e_i$, $i = 1, \dots, k$, and $\sigma_j \mapsto h_j$, $j = 1, \dots, n-k$. For $\lambda \in P_{kn}$, the class σ_λ maps to (the coset of) the Schur polynomial $s_\lambda(x_1, \dots, x_k) \in \Lambda_k$. Note, however, that the Schur polynomials $s_\lambda(x_1, \dots, x_k)$, for $\lambda \notin P_{kn}$, may represent nonzero elements in $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. In what follows, by a slight abuse of notation, we denote the special Schubert classes in (quantum) cohomology by $e_i = \sigma_{(1^i)}$ and $h_j = \sigma_{(j)}$, for $i = 1, \dots, k$, $j = 1, \dots, n-k$.

Bertram, Ciocan-Fontanine, and Fulton [BCF] expressed the Gromov-Witten invariants as alternating sums of the Littlewood-Richardson coefficients by showing how to reduce a Schur function $s_\lambda \in \Lambda_k$ modulo the ideal J^q_{kn} in (6).

The Jacobi-Trudy formula in (1) specializes to the expression for a Schubert class in terms of the special Schubert classes, known as the *Giambelli formula.* Bertram's *quantum Giambelli formula* in [B] claims that the same expression remains valid in the quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$:

$$\sigma_\lambda = \det(h_{\lambda_i+j-i})_{1\le i,\, j\le k} = \det(e_{\lambda'_i+j-i})_{1\le i,\, j\le n-k}, \tag{7}$$

where $\lambda = (\lambda_1, \dots, \lambda_k) \in P_{kn}$, $\lambda' = (\lambda'_1, \dots, \lambda'_{n-k}) \in P_{n-k\,n}$ is its conjugate partition, and we assume that $e_0 = h_0 = 1$ and $e_i = h_j = 0$ unless $0 \le i \le k$ and $0 \le j \le n-k$.

Let us also mention the *duality isomorphism* of the quantum cohomology rings

$$\mathrm{QH}^*(\mathrm{Gr}_{kn}) \simeq \mathrm{QH}^*(\mathrm{Gr}_{n-k\,n}). \tag{8}$$

In this isomorphism, a Schubert class σ_λ in $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ maps to the Schubert class $\sigma_{\lambda'}$ in $\mathrm{QH}^*(\mathrm{Gr}_{n-k\,n})$. In particular, the generators e_i of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ map to the generators h_j of $\mathrm{QH}^*(\mathrm{Gr}_{n-k\,n})$, and vice versa.

3. Cylindric and toric tableaux

Let us fix two positive integer numbers k and n such that $n > k \ge 1$, and let us define the *cylinder* $\mathscr{C}_{kn}$ as the quotient

$$\mathscr{C}_{kn} = \mathbb{Z}^2/(-k, n-k)\,\mathbb{Z}.$$

In other words, $\mathscr{C}_{kn}$ is the quotient of the integer lattice $\mathbb{Z}^2$ modulo the action of the *shift operator* $\mathrm{Shift}_{kn} : (i, j) \mapsto (i-k, j+n-k)$. For $(i, j) \in \mathbb{Z}^2$, let $\langle i, j\rangle = (i, j) + (-k, n-k)\,\mathbb{Z}$ be the corresponding element of the cylinder $\mathscr{C}_{kn}$.

For a partition $\lambda \in P_{kn}$ and an integer r, let $\lambda[r] = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots)$ be the integer sequence, infinite in both directions, such that

(a) $\alpha_{i+r} = \lambda_i + r$ for $i = 1, \dots, k$, and

(b) $\alpha_i = \alpha_{i+k} + (n-k)$ for any $i \in \mathbb{Z}$.

These are exactly all weakly decreasing sequences satisfying condition (b).

The coordinatewise partial order on $\mathbb{Z}^2$ induces the partial order structure $\preceq$ on the cylinder $\mathscr{C}_{kn}$. A subset in a partially ordered set is called an *order ideal* if whenever it contains an element a it also contains all elements that are less than a. All order ideals in $\mathscr{C}_{kn}$ are of the form $\{\langle i, j\rangle \in \mathscr{C}_{kn} \mid (i, j) \in \mathbb{Z}^2,\ j \le \lambda[r]_i\}$ for $\lambda \in P_{kn}$, $r \in \mathbb{Z}$. We call sequences of the form $\lambda[r]$ *cylindric loops* because the boundary of the corresponding order ideal forms a closed loop on the cylinder $\mathscr{C}_{kn}$. We can think of cylindric loops as infinite Shift_{kn}-invariant lattice paths on the plane. The cylindric loop $\lambda[r]$ is obtained by shifting the loop $\lambda[0]$ by r steps in the southeast direction, that is, by shifting it by the vector (r, r) (see Fig. 3). (As usual, we arrange pairs (i, j) on the plane as one would arrange matrix elements.)

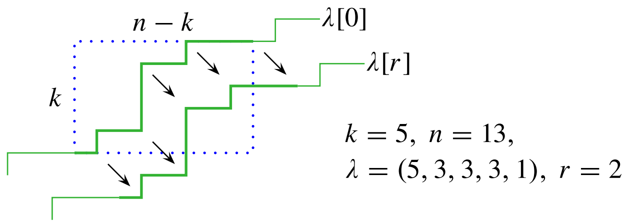


Figure 3. A cylindric loop $\lambda[r]$

For two cylindric loops $\lambda[r]$ and $\mu[s]$ such that $\lambda[r]_i \geq \mu[s]_i$ for any i, we define the *cylindric Young diagram* of *type* (k, n) and *shape* $\lambda[r]/\mu[s]$ as the set-theoretic difference of the corresponding order ideals in the cylinder $\mathscr{C}_{kn}$:

$$\left\{\langle i, j\rangle \in \mathscr{C}_{kn} \,\middle|\, (i, j) \in \mathbb{Z}^2,\ \lambda[r]_i \geq j > \mu[s]_i\right\}.$$

This diagram consists of the elements of the cylinder $\mathscr{C}_{kn}$ (represented by boxes) located between the two cylindric loops. Cylindric Young diagrams are exactly all finite subsets in $\mathscr{C}_{kn}$ closed with respect to the operation of taking intervals. Let $|\lambda[r]/\mu[s]|$ denote the number of boxes in the cylindric Young diagram.

For two partitions $\lambda, \mu \in P_{kn}$ and a nonnegative integer d, let $\lambda/d/\mu$ be shorthand for the cylindric shape $\lambda[d]/\mu[0]$. Each cylindric Young diagram $\lambda[r]/\mu[s]$ is obtained by the shift of the diagram of shape $\lambda/d/\mu$, $d = r - s$, by s steps in the southeast direction. We often use the more compact notation $\lambda/d/\mu$ for cylindric shapes. Each skew Young diagram of shape λ/μ, with $\lambda, \mu \in P_{kn}$, that fits inside the $(k \times (n - k))$-rectangle gives rise to the cylindric Young diagram of shape $\lambda/0/\mu$. In this sense we regard skew Young diagrams as a special case of cylindric Young diagrams.

Define the *pth row* of the cylinder $\mathscr{C}_{kn}$ as the set $\{\langle i, j\rangle \mid i = p\}$, the *qth column* as the set $\{\langle i, j\rangle \mid j = q\}$, and the rth *diagonal* as the set $\{\langle i, j\rangle \mid j - i = r\}$. They depend only on the residues $p \pmod k$, $q \pmod{n-k}$, and $r \pmod n$. Thus the cylinder $\mathscr{C}_{kn}$ has exactly k rows, $n - k$ columns, and n diagonals. The restriction of the partial order $\preceq$ on $\mathscr{C}_{kn}$ to a row, column, or diagonal induces a linear order on it. Thus the intersection of a cylindric Young diagram with a row, column, or diagonal of $\mathscr{C}_{kn}$ contains at most one linearly ordered interval, called row, column, or diagonal, respectively, of the cylindric diagram. Notice that the cylindric Young diagram of shape $\lambda/d/\mu$ has exactly d elements in the $(-k)$th diagonal.

Definition 3.1
A *semistandard cylindric tableau* of shape $\lambda[r]/\mu[s]$ and weight $\beta = (\beta_1, \ldots, \beta_l)$ is a function $T : D \mapsto \{1, \ldots, l\}$ on the cylindric Young diagram D of shape $\lambda[r]/\mu[s]$ such that $\beta_i = \#\{a \in D \mid T(a) = i\}$, for $i = 1, \ldots, l$, and the function T weakly increases in the rows and strictly increases in the columns of the diagram.

The semistandard cylindric tableaux are equivalent to (0, 1)-*cylindric partitions* introduced by Gessel and Krattenthaler [GK] and to *proper tableaux* of Bertram, Ciocan-Fontanine, and Fulton [BCF] (though notation of [GK] and [BCF] is different from ours).

Figure 4 gives an example of a cylindric tableau for $k = 3$ and $n = 8$. It has shape $\lambda[r]/\mu[s] = (5, 2, 1)[3]/(4, 1, 1)[1]$ and weight $\beta = (4, 4, 4, 4, 2)$. Here we present the tableau as a Shift_{kn}-symmetric function defined on an infinite subset in $\mathbb{Z}^2$. Representatives of Shift_{kn}-equivalence classes of entries are displayed in bold font. We also indicate the (i, j)-coordinate system in $\mathbb{Z}^2$, the shift operator Shift_{kn}, and the $(n - k)$th and $(-k)$th diagonal.

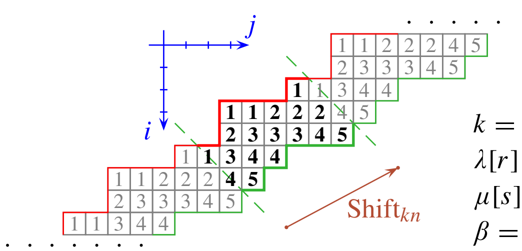


Figure 4. A semistandard cylindric tableau

Let $\mathscr{T}_{kn} = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n-k)\mathbb{Z}$ be the integer $(k \times (n-k))$-*torus*. The torus $\mathscr{T}_{kn}$ is the quotient of the cylinder

$$\mathscr{T}_{kn} = \mathscr{C}_{kn}/(k, 0)\mathbb{Z} = \mathscr{C}_{kn}/(0, n-k)\mathbb{Z}. \tag{9}$$

Elements in rows, columns, and diagonals of the torus $\mathscr{T}_{kn}$, which are defined as images of rows, columns, and diagonals of the cylinder, are cyclically ordered.

Definition 3.2
A cylindric shape $\lambda[r]/\mu[s]$ is called a *toric shape* if the restriction of the natural projection $p : \mathscr{C}_{kn} \to \mathscr{T}_{kn}$ to the cylindric Young diagram D of shape $\lambda[r]/\mu[s]$ is

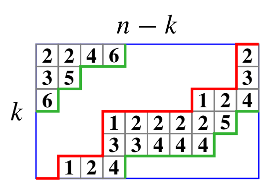
an injective embedding $D \hookrightarrow \mathcal{T}_{kn}$. A *semistandard toric tableau* is a semistandard cylindric tableau of a toric shape.

LEMMA 3.3
A cylindric shape is toric if and only if all columns of its diagram contain at most k elements. Also, a cylindric shape is toric if and only if all rows of its diagram contain at most $n-k$ elements.

Proof
Both statements immediately follow from (9). □

A cylindric loop $\lambda[r]$ can also be regarded as a closed loop on the torus $\mathcal{T}_{kn}$. The Young diagram of a toric shape $\lambda[r]/\mu[s]$ is formed by the elements of the torus $\mathcal{T}_{kn}$ between two nonintersecting loops $\lambda[r]$ and $\mu[s]$.

The tableau given in Figure 4 is *not* a toric tableau. It has two columns with more than three elements and two rows with more than five elements. Figure 5 gives an example of a toric tableau drawn inside the torus $\mathcal{T}_{kn}$ for $k = 6$ and $n = 16$. It has shape $\lambda/d/\mu = (9, 7, 6, 2, 2, 0)/2/(9, 9, 7, 3, 3, 1)$ and weight $\beta = (3, 9, 4, 6, 2, 2)$.



$k = 6,\ n = 16,$
$\lambda[d] = (9, 7, 6, 2, 2, 0)[2],$
$\mu = (9, 9, 7, 3, 3, 1),$
$\beta = (3, 9, 4, 6, 2, 2)$

Figure 5. A semistandard toric tableau of shape $\lambda/d/\mu$

Note that two different cylindric loops related by the shift by k steps in the south direction, that is, by the vector $(k, 0)$, represent the same loop on the torus $\mathcal{T}_{kn}$. For a partition $\lambda \in P_{kn}$ with $\omega(\lambda) = (\omega_1, \dots, \omega_n)$, let $\mathrm{diag}_0(\lambda) = \omega_{k+1} + \cdots + \omega_n$ be the number of elements in the 0th diagonal of its Young diagram. The number $\mathrm{diag}_0(\lambda)$ is also equal to the size of the *Durfee square*—the maximal square inside the Young diagram. For a cylindric loop $\lambda[r]$, let $\lambda^{\downarrow}[r^{\downarrow}]$ be the cylindric loop such that $r^{\downarrow} = r + \mathrm{diag}_0(\lambda)$ and $\lambda^{\downarrow} \in P_{kn}$ is the partition whose 01-word is equal to $\omega(\lambda^{\downarrow}) = (\omega_{k+1}, \dots, \omega_n, \omega_1, \dots, \omega_k)$. Then the cylindric loop $\lambda^{\downarrow}[r^{\downarrow}]$ is the shift of $\lambda[r]$ by the vector $(k, 0)$. This implies the following claim.

LEMMA 3.4
For any $\lambda \in P_{kn}$ and integer r, the two cylindric loops $\lambda[r]$ and $\lambda^{\downarrow}[r^{\downarrow}]$ represent the same loop on the torus $\mathcal{T}_{kn}$. Any two cylindric loops that are equivalent on the torus

can be related by one or several transformations $\lambda[r] \mapsto \lambda^{\downarrow}[r^{\downarrow}]$.

4. Quantum Pieri formula and quantum Kostka numbers

Bertram's quantum Pieri formula in [B] gives a rule for the quantum product of the Schubert classes with the generators $e_1, \dots, e_k$ and $h_1, \dots, h_{n-k}$ of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. Thus this formula determines the multiplicative structure of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. We can formulate this formula using our notation, as follows.

Let us say that a cylindric shape $\lambda[r]/\mu[s]$ is a *horizontal* i*-strip* (resp., *vertical* i*-strip*) if $|\lambda[r]/\mu[s]| = i$ and each column (resp., row) of its diagram contains at most one element.

PROPOSITION 4.1 (Quantum Pieri formula)
For any $\mu \in P_{kn}$, $i = 1, \dots, k$, $j = 1, \dots, n-k$, *the quantum products* $e_i * \sigma_\mu$ *and* $h_j * \sigma_\mu$ *are given by the sums*

$$e_i * \sigma_\mu = \sum_{\substack{\lambda/d/\mu \text{ is} \\ \text{a vertical } i\text{-strip}}} q^d \, \sigma_\lambda \qquad \text{and} \qquad h_j * \sigma_\mu = \sum_{\substack{\lambda/d/\mu \text{ is} \\ \text{a horizontal } j\text{-strip}}} q^d \, \sigma_\lambda$$

over $d \in \{0, 1\}$ *and* $\lambda \in P_{kn}$ *satisfying the stated conditions.*

Note that, for any vertical or horizontal strip $\lambda/d/\mu$, we have $d = 0$ or 1. Bertram proved this formula using quot schemes. Buch [Bu] gave a simple proof of the quantum Pieri formula using only the definition of Gromov-Witten invariants. For the sake of completeness, we give here a short combinatorial proof of Proposition 4.1 using the Jacobi-Trudy formula (1).

Proof (cf. [BCF])
Let us first prove the formula for $e_i * \sigma_\mu$. In order to find the quantum product $e_i * \sigma_\mu$, we need to express the product $e_i \cdot s_\mu \in \Lambda_k$ of the elementary symmetric polynomial with the Schur polynomial as a linear combination of s_λ, for $\lambda \in P_{kn}$, modulo the ideal J^q_{nk} (see (6)). The classical Pieri formula says that the product $e_i \cdot s_\mu \in \Lambda_k$ equals the sum of Schur functions $e_i \cdot s_\mu = \sum s_\tau$ over all partitions τ with at most k rows such that τ/μ is a (classical) vertical i-strip.

If $\tau \in P_{kn}$, then we recover all terms with $d = 0$. Suppose that $\tau \notin P_{kn}$. Then $\tau_1 = n - k + 1$. The top row in the Jacobi-Trudy determinant for s_τ is

$$(h_{n-k+1}, h_{n-k+1}, \dots, h_n) \equiv \big(0, \dots, 0, (-1)^{k-1} q\big) \pmod{J^q_{kn}}$$

(see (1)). This determinant is equivalent to q times its $((k-1) \times (k-1))$-minor obtained by removing the first row and the last column. If $\tau_k \geq 1$, we get $s_\tau \equiv q \cdot s_\lambda$ (mod J^q_{kn}), where $\lambda = (\tau_2 - 1, \dots, \tau_k - 1, 0) \in P_{kn}$; otherwise, $s_\tau \in J^k_{kn}$. In our notation this means that the *cylindric* shape $\lambda/1/\mu$ is a vertical i-strip. This gives all terms with $d = 1$.

The second formula for $h_j * \sigma_\mu$ follows from the first formula and the duality isomorphism (8) between $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ and $\mathrm{QH}^*(\mathrm{Gr}_{n-k\,n})$, which switches the e_i with the h_j and vertical strips with horizontal strips. □

Define the *quantum Kostka number* $K^{\beta}_{\lambda/d/\mu}$ as the number of semistandard cylindric tableaux of shape $\lambda/d/\mu$ and weight β. These tableaux are in one-to-one correspondence with chains of cylindric loops $\lambda^{(0)}[d_0] = \mu[0]$, $\lambda^{(1)}[d_1], \ldots, \lambda^{(l)}[d_l] = \lambda[d]$ such that $\lambda^{(i)}[d_i]/\lambda^{(i-1)}[d_{i-1}]$ is a horizontal β_i-strip for $i = 1, \ldots, l$. Applying the quantum Pieri formula repeatedly, we immediately recover the following result.

COROLLARY 4.2 (see [BCF, Sec. 3])
For a partition $\mu \in P_{kn}$ and an integer vector $\beta = (\beta_1, \ldots, \beta_l)$ with $0 \le \beta_i \le n - k$, we have $\sigma_\mu * h_{\beta_1} * \cdots * h_{\beta_l} = \sum_{d,\lambda} q^d K^{\beta}_{\lambda/d/\mu} \sigma_\lambda$ in $\mathrm{QH}^*(\mathrm{Gr}_{kn})$, where the sum is over nonnegative integers d and partitions $\lambda \in P_{kn}$.

Corollary 4.2 and the commutativity of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ imply the following claim.

COROLLARY 4.3
The quantum Kostka numbers $K^{\beta}_{\lambda/d/\mu}$ are invariant under permuting elements β_i of the vector β.

It is not hard to give a direct combinatorial proof of this statement by showing that the operators of adding horizontal (vertical) r-strips to cylindric shapes commute pairwise. This argument is almost the same as in the classical case.

5. Toric Schur polynomials

In this section we define toric Schur polynomials. Then we prove our main result.

For a cylindric shape $\lambda/d/\mu$, with $\lambda, \mu \in P_{kn}$ and $d \in \mathbb{Z}_{\ge 0}$, we define the *cylindric Schur function* $s_{\lambda/d/\mu}(\mathbf{x})$ as the formal series in the infinite set of variables $x_1, x_2, \ldots$ given by

$$s_{\lambda/d/\mu}(\mathbf{x}) = \sum_T \mathbf{x}^T = \sum_\beta K^{\beta}_{\lambda/d/\mu}\, \mathbf{x}^\beta,$$

where the first sum is over all semistandard cylindric tableaux T of shape $\lambda/d/\mu$, the second sum is over all possible monomials $\mathbf{x}^\beta$, and $\mathbf{x}^T = \mathbf{x}^\beta = x_1^{\beta_1} \cdots x_l^{\beta_l}$ for a cylindric tableau T of weight $\beta = (\beta_1, \ldots, \beta_l)$.

Recall that the diagrams of shape $\lambda/0/\mu$ are exactly the cylindric diagrams associated with a skew shape λ/μ. Thus

$$s_{\lambda/0/\mu}(\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x})$$

is the usual skew Schur function.

Corollary 4.3 implies the following claim.

PROPOSITION 5.1
The cylindric Schur function $s_{\lambda/d/\mu}(\mathbf{x})$ belongs to the ring Λ of symmetric functions.

Let us define the *toric Schur polynomial* as the specialization

$$s_{\lambda/d/\mu}(x_1, \dots, x_k) = s_{\lambda/d/\mu}(x_1, \dots, x_k, 0, 0, \dots)$$

of the cylindric Schur function $s_{\lambda/d/\mu}(\mathbf{x})$. Here, as before, k is the number of rows in the torus $\mathcal{T}_{kn}$. Proposition 5.1 implies that $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ belongs to the ring Λ_k of symmetric polynomials in $x_1, \dots, x_k$. The name "toric" is justified by the following lemma.

LEMMA 5.2
The toric Schur polynomial $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ is nonzero if and only if the shape $\lambda/d/\mu$ is toric.

Proof
Let us use Lemma 3.3. If the shape $\lambda/d/\mu$ is not toric, then it contains a column with greater than k elements. Thus there are no cylindric tableaux of shape $\lambda/d/\mu$ and weight $\beta = (\beta_1, \dots, \beta_k)$ (given by a k-vector). This implies that $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ is zero. If $\lambda/d/\mu$ is toric, then all columns have at most k elements. There are cylindric tableaux of this shape and some weight $\beta = (\beta_1, \dots, \beta_k)$. For example, we can put the consecutive numbers $1, 2, \dots$ in each column starting from the top. This implies that $s_{\lambda/d/\mu}(x_1, \dots, x_k) \neq 0$. □

We are now ready to formulate and prove our main result. Each toric Schur polynomial $s_{\lambda/d/\mu}(x_1, \dots, x_n)$ can be uniquely expressed in the basis of the usual Schur polynomials $s_\nu(x_1, \dots, x_k)$. The next theorem links this expression to the Gromov-Witten invariants $C^{\lambda,d}_{\mu\nu}$ which give the quantum product (5) of Schubert classes.

THEOREM 5.3
For two partitions $\lambda, \mu \in P_{kn}$ and a nonnegative integer d, we have

$$s_{\lambda/d/\mu}(x_1, \dots, x_k) = \sum_{\nu \in P_{kn}} C^{\lambda,d}_{\mu\nu}\, s_\nu(x_1, \dots, x_k).$$

Proof
By the quantum Giambelli formula (7), we have

$$\sigma_\mu * \sigma_\nu = \sum_{w \in S_k} (-1)^{\mathrm{sign}(w)} \, \sigma_\mu * h_{\nu_1+w_1-1} * h_{\nu_2+w_2-2} * \cdots * h_{\nu_k+w_k-k},$$

where the sum is over all permutations $w = (w_1, \dots, w_k)$ in S_k. Each of the summands in the right-hand side is given by Corollary 4.2. Extracting the coefficients of $q^d \sigma_\lambda$ from both sides, we get

$$C^{\lambda,d}_{\mu\nu} = \sum_{w \in S_k} (-1)^{\mathrm{sign}(w)} \, K^{\nu+w(\rho)-\rho}_{\lambda/d/\mu},$$

where $\nu + w(\rho) - \rho = (\nu_1 + w_1 - 1, \dots, \nu_k + w_k - k)$. Let us define the operator A_ν which acts on polynomials $f \in \mathbb{Z}[x_1, \dots, x_k]$ as

$$A_\nu(f) = \sum_{w \in S_k} (-1)^{\mathrm{sign}(w)} \, [\text{coefficient of } \mathbf{x}^{\nu+w(\rho)-\rho}](f).$$

Then the previous expression can be written as

$$C^{\lambda,d}_{\mu\nu} = A_\nu\big(s_{\lambda/d/\mu}(x_1, \dots, x_k)\big). \tag{10}$$

We claim that $A_\nu(s_\lambda(x_1, \dots, x_k)) = \delta_{\lambda\nu}$. Of course, this is a well-known identity. This is also a special case of (10) for $\mu = \emptyset$ and $d = 0$. Indeed, $C^{\lambda,0}_{\emptyset\,\nu} = c^{\lambda}_{\emptyset\,\nu} = \delta_{\lambda\nu}$ because the Schubert class $\sigma_\emptyset$ is the identity element in $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. Thus $A_\nu(f)$ is the coefficient of s_ν in the expansion of f in the basis of Schur polynomials. According to (10), the Gromov-Witten invariant $C^{\lambda,d}_{\mu\nu}$ is the coefficient of s_ν in the expansion of $s_{\lambda/d/\mu}$, as needed. □

Let us reformulate our main theorem as follows.

COROLLARY 5.4
For two partitions $\lambda, \mu \in P_{kn}$ and a nonnegative integer d, we have

$$s_{\mu^\vee/d/\lambda}(x_1, \dots, x_k) = \sum_{\nu \in P_{kn}} C^{\nu,d}_{\lambda\mu} \, s_{\nu^\vee}(x_1, \dots, x_k).$$

In other words, the coefficient of $q^d \, \sigma_\nu$ in the expansion of the quantum product $\sigma_\lambda * \sigma_\mu$ is exactly the same as the coefficient of $s_{\nu^\vee}$ in the Schur expansion of the toric Schur polynomial $s_{\mu^\vee/d/\lambda}$. In particular, $\sigma_\lambda * \sigma_\mu$ contains nonzero terms of the form $q^d \sigma_\nu$ if and only if the toric Schur polynomial $s_{\mu^\vee/d/\lambda}$ is nonzero, that is, if and only if $\mu^\vee/d/\lambda$ forms a valid toric shape.

Proof
The first claim is equivalent to Theorem 5.3. Indeed, using the S_3-symmetry of the Gromov-Witten invariants, we obtain $C_{\lambda\mu}^{\nu,d} = C_{\lambda\mu\nu^\vee}^{d} = C_{\lambda\nu^\vee}^{\mu^\vee,d}$. The second claim follows from Lemma 5.2. □

This statement means that the image of the toric Schur polynomial $s_{\mu^\vee/d/\lambda}$ in the cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{kn})$ under the natural projection (see (3)) is equal to the Poincaré dual of the coefficient of q^d in the quantum product $\sigma_\lambda * \sigma_\mu$ of two Schubert classes. In other words, the coefficient of q^d in $\sigma_\lambda * \sigma_\mu$ is associated with the toric shape $\mu^\vee/d/\lambda$ in the same sense that the usual product $\sigma_\lambda \cdot \sigma_\mu$ is associated with the skew shape $\mu^\vee/\lambda$, (see (4)).

Theorem 5.3 implies that all toric Schur polynomials $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ are *Schur-positive*; that is, they are positive linear combinations of the usual Schur polynomials. Indeed, the coefficients are the Gromov-Witten invariants, which are positive according to their geometric definition. Note, however, that cylindric Schur functions $s_{\lambda/d/\mu}(\mathbf{x})$ (in infinitely many variables) may not be Schur-positive. For example, for $k = 1$ and $n = 3$, we have

$$s_{\emptyset/1/\emptyset}(\mathbf{x}) = \sum_{a\le b\le c,\, a<c} x_a x_b x_c = s_{21}(\mathbf{x}) - s_{1^3}(\mathbf{x}).$$

Krattenthaler remarked that [GK, Prop. 1] implies the following *dual Jacobi-Trudy formula* for the cylindric Schur functions:

$$s_{\lambda/d/\mu}(\mathbf{x}) = \sum_{l_1+\cdots+l_{n-k}=0} \det_{1\le i,\, j\le n-k} \left(e_{\lambda[d]'_i - i - \mu'_j + j + n\, l_i}(\mathbf{x})\right), \tag{11}$$

where $e_m(\mathbf{x})$ denotes the mth elementary symmetric function, and the sequence $\lambda[d]' = (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots)$ is given by
(a) $\alpha_{i+d} = \lambda'_i + d$ for $i = 1, \dots, n-k$, and
(b) $\alpha_i = \alpha_{i+n-k} + k$ for any $i \in \mathbb{Z}$ (cf. Sec. 3).
This formula is based on an interpretation of cylindric tableaux in terms of families of nonintersecting lattice paths (see [GK]). For example, for $k = 1$ and $n = 3$, we have

$$s_{\emptyset/1/\emptyset}(\mathbf{x}) = \sum_{l_1+l_2=0} \begin{Vmatrix} e_{2+3l_1} & e_{3l_2} \\ e_{3+3l_1} & e_{1+3l_2} \end{Vmatrix} = e_2\, e_1 - e_0\, e_3 - e_3\, e_0.$$

6. Symmetries of Gromov-Witten invariants

In this section we show that the Gromov-Witten invariants are symmetric with respect to certain actions of the groups S_3, $(\mathbb{Z}/n\mathbb{Z})^2$, and $\mathbb{Z}/2\mathbb{Z}$ on triples (λ, μ, ν). While the S_3-symmetry is trivial and the cyclic symmetry has already appeared in several papers, the $(\mathbb{Z}/2\mathbb{Z})$-symmetry seems to be the most intriguing. We call it the *curious duality*.

In this section it is convenient to use the following notation for the Gromov-Witten invariants:

$$C_{\lambda\mu\nu}(q) := q^d\, C^d_{\lambda\mu\nu} = q^d\, C^{\nu^\vee,d}_{\lambda\mu}.$$

Recall that d can be expressed as $d = (|\lambda| + |\mu| + |\nu| - k(n-k))/n$. Also, let

$$\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn}) = \mathrm{QH}^*(\mathrm{Gr}_{kn}) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}].$$

6.1. S_3-symmetry

The invariants $C_{\lambda\mu\nu}(q)$ are symmetric with respect to the six permutations of λ, μ, and ν. This is immediately clear from their geometric definition. We have already mentioned and used this symmetry on several occasions.

6.2. Cyclic "hidden" symmetry

Let us define the *cyclic shift* operation S on the set P_{kn} of partitions, as follows. Let $\lambda \in P_{kn}$ be a partition with the 01-word $\omega(\lambda) = (\omega_1, \dots, \omega_n)$ (see Sec. 2). Its cyclic shift $S(\lambda)$ is the partition $\tilde{\lambda} \in P_{kn}$ whose 01-word $\omega(\tilde{\lambda})$ is equal to $(\omega_2, \omega_3, \dots, \omega_n, \omega_1)$. Also, for the same λ, let $\phi_i = \phi_i(\lambda)$, $i \in \mathbb{Z}$, be the sequence such that $\phi_i = \omega_1 + \cdots + \omega_i$ for $i = 1, \dots, n$, and $\phi_{i+n} = \phi_i + k$ for any $i \in \mathbb{Z}$.

PROPOSITION 6.1

For three partitions $\lambda, \mu, \nu \in P_{kn}$ and three integers a, b, c with $a + b + c = 0$, we have

$$C_{S^a(\lambda)\, S^b(\mu)\, S^c(\nu)}(q) = q^{\phi_a(\lambda)+\phi_b(\mu)+\phi_c(\nu)}\, C_{\lambda\mu\nu}(q).$$

This symmetry was noticed by several people. The first place where it appeared in print is Seidel's paper [S]. Agnihotri and Woodward in [AW, Prop. 7.2] explained the symmetry using the Verlinde algebra. In [P2] we gave a similar property of the quantum cohomology of the complete flag manifold. We call this property the *hidden symmetry* because it cannot be detected in full generality on the level of the classical cohomology. It comes from symmetries of the extended Dynkin diagram of type A_{n-1}, which is an n-circle. This symmetry is especially transparent in the language of toric shapes.

Proof

It is clear from the definition that toric shapes possess cyclic symmetry. More precisely, for a shape $\kappa = \lambda/d/\mu$, the shape $S(\kappa) = S(\lambda)/\tilde{d}/S(\mu)$, where $\tilde{d} - d = \omega_1(\mu) - \omega_1(\lambda)$, is obtained by rotation of κ. Thus their toric Schur polynomials are the same: $s_\kappa = s_{S(\kappa)}$. This fact, empowered by Theorem 5.3, proves the proposition for $(a, b, c) = (0, 1, -1)$. The general case follows by induction from this claim and the S_3-symmetry. $\square$

COROLLARY 6.2
For any $\lambda, \mu \in P_{kn}$, and $a \in \mathbb{Z}$, we have

$$\sigma_{S^a(\lambda)} * \sigma_{S^{-a}(\mu)} = q^{\phi_a(\lambda)+\phi_{-a}(\mu)}\, \sigma_\lambda * \sigma_\mu$$

in the ring $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$.

The cyclic shift operation $\lambda \mapsto S(\lambda)$ can be described in terms of the action of the following two Schubert classes: $E = e_k = \sigma_{1^k}$ and $H = h_{n-k} = \sigma_{n-k}$. The following claim easily follows from the quantum Pieri formula (Prop. 4.1).

PROPOSITION 6.3
For $\lambda \in P_{kn}$, we have

$$E * \sigma_\lambda = q^{\omega_n(\lambda)}\sigma_{S^{-1}(\lambda)} \qquad \textit{and} \qquad H * \sigma_\lambda = q^{1-\omega_1(\lambda)}\sigma_{S(\lambda)}$$

in the quantum cohomology ring. In particular, we have $E^n = q^k$, $H^n = q^{n-k}$, $E * H = q$ in the quantum cohomology. The class $E^{n-k} = H^k = \sigma_{(n-k)^k}$ is the fundamental class of a point.

The powers of E and H involve all Schubert classes σ_λ with rectangular shapes λ that have k rows or $n-k$ columns. We have $E^j = \sigma_{(j)^k}$ for $j = 0, 1, \ldots, n-k$, and we have $E^{n-k+i} = q^i\, \sigma_{(n-k)^{k-i}}$ for $i = 0, 1, \ldots, k$. Also, $H^i = \sigma_{(n-k)^i}$ for $i = 0, 1, \ldots, k$, and $H^{k+j} = q^j\, \sigma_{(n-k-j)^k}$ for $j = 0, 1, \ldots, n-k$.

6.3. Curious duality
The quantum product has the following symmetry related to the Poincaré duality: $\sigma_\lambda \mapsto \sigma_{\lambda^\vee}$.

THEOREM 6.4
For three partitions $\lambda, \mu, \nu \in P_{kn}$ and three integers a, b, c with $a + b + c = n - k$, we have

$$C_{\lambda^\vee\, \mu^\vee\, \nu^\vee}(q) = q^{\phi_a(\lambda)+\phi_b(\mu)+\phi_c(\nu)}\, C_{S^a(\lambda)\, S^b(\mu)\, S^c(\nu)}(q^{-1}).$$

Before we prove this theorem, let us reformulate it in algebraic terms. Let D be the $\mathbb{Z}$-linear involution on the space $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$ given by

$$D : q^d\, \sigma_\lambda \longmapsto q^{-d}\, \sigma_{\lambda^\vee}.$$

Notice that $D(1) = \sigma_{(n-k)^k}$ is the fundamental class of a point. It is an invertible element in the ring $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$. By Proposition 6.3, we have $D(1) = H^k$ and

$$\big(D(1)\big)^{-1} = q^{k-n} H^{n-k}.$$

Let us define another map $\tilde{D} : \mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn}) \to \mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$ as the normalization of D given by

$$\tilde{D}(f) = D(f) * \big(D(1)\big)^{-1}.$$

According to Proposition 6.3, the map $\tilde{D}$ is explicitly given by

$$\tilde{D} : q^d\, \sigma_\lambda \longmapsto q^{-d-\mathrm{diag}_0(\lambda)}\, \sigma_{S^{n-k}(\lambda^\vee)},$$

where $\mathrm{diag}_0(\lambda) = \phi_{n-k}(\lambda^\vee) = k - \phi_k(\lambda)$ is the size of the 0th diagonal of the Young diagram of λ.

THEOREM 6.5
The map $\tilde{D}$ is a homomorphism of the ring $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$. The map $\tilde{D}$ is also an involution. It inverts the quantum parameter: $\tilde{D}(q) = q^{-1}$.

For $q = 1$, the involution $\tilde{D}$ was independently discovered from a different point of view by Hengelbrock [H]. He showed that it comes from complex conjugation of the points in Spec R, where $R = \mathrm{QH}^*(\mathrm{Gr}_{kn})/\langle q-1\rangle$.

The claim that $\tilde{D}$ is an involution of $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$ implies that the map $\lambda \mapsto \tilde{\lambda} = S^{n-k}(\lambda^\vee)$ is an involution on partitions in P_{kn} and $\mathrm{diag}_0(\lambda) = \mathrm{diag}_0(\tilde{\lambda})$. It is easy to see this combinatorially. Indeed, if the 01-word of λ is $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, then the 01-word of $\tilde{\lambda}$ is $\omega(\tilde{\lambda}) = (\omega_k, \omega_{k-1}, \ldots, \omega_1, \omega_n, \omega_{n-1}, \ldots, \omega_{k+1})$ and $\mathrm{diag}_0(\lambda) = \mathrm{diag}_0(\tilde{\lambda}) = \omega_{k+1} + \cdots + \omega_n$.

Theorem 6.5 is equivalent to the following property of the involution D.

PROPOSITION 6.6
We have the identity

$$D(f * g) * D(h) = D(f) * D(g * h)$$

for any $f, g, h \in \mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$.

We need the following lemma.

LEMMA 6.7
For any $f \in \mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$ and any $i = 0, \ldots, k$, we have

$$D(f * e_i) = q^{-1}\, D(f) * h_{n-k} * e_{k-i}. \tag{12}$$

Here we assume that $e_0 = 1$.

Proof

Since $D(q^k f) = q^{-k} D(f)$, it is enough to prove the lemma for a Schubert class $f = \sigma_\lambda$. According to the quantum Pieri formula (Prop. 4.1), $\sigma_\lambda * e_i$ is given by the sum over all possible ways to add a vertical i-strip to the cylindric loop $\lambda[0]$. Thus $D(\sigma_\lambda * e_i)$ is given by the sum over all possible ways to *remove* a vertical i-strip from $\lambda^\vee[0]$. In other words, we have

$$D(\sigma_\lambda * e_i) = \sum q^{-d}\, \sigma_\mu,$$

where the sum is over $\mu \in P_{kn}$ and d such that $\lambda^\vee/d/\mu$ is a vertical i-strip. By Proposition 6.3, the right-hand side of (12) is equal to

$$q^{-1}\, \sigma_{\lambda^\vee} * h_{n-k} * e_{k-i} = q^{-\omega_1(\lambda^\vee)}\, \sigma_{S(\lambda^\vee)} * e_{k-i}.$$

We obtain exactly the same expressions in both cases. Indeed, *removing* a vertical i-strip from a cylindric shape means exactly the same as *cyclically shifting* the shape and then *adding* a vertical $(k-i)$-strip. By looking at the formula for a minute, we also see that the powers of q in both cases are equal to each other. □

Proof of Proposition 6.6

Again, since multiplying g by a power of q does not change the formula, it is enough to verify the statement when g belongs to some set that spans the algebra $\mathrm{QH}^*_{\langle q\rangle}(\mathrm{Gr}_{kn})$ over $\mathbb{Z}[q, q^{-1}]$. Let us prove the statement when $g = e_{i_1} * e_{i_2} * \cdots * e_{i_l}$. If $l = 1$, then by Lemma 6.7, we have

$$D(f * e_i) * D(h) = q^{-1}\, h_{n-k} * e_{k-1} * D(f) * D(h) = D(f) * D(e_i * h).$$

The general case follows from this case. We just need to move the l factors $e_{i_1}, \ldots, e_{i_l}$ one by one from the first D to the second D. □

Proof of Theorem 6.5

Proposition 6.6 with $h = 1$ says that $D(f * g) * D(1) = D(f) * D(g)$. It is equivalent to saying that the normalization $\tilde{D}$ is a homomorphism. We already proved combinatorially that $\tilde{D}$ is an involution. Let us also deduce this fact algebraically from Proposition 6.6:

$$\tilde{D}\big(\tilde{D}(f)\big) = \frac{D(D(f)/D(1)) * D(D(1))}{D(1)} = \frac{D(D(f)) * D(D(1)/D(1))}{D(1)} = f.$$

The fact that $\tilde{D}(q) = q^{-1}$ is clear from the definition. □

COROLLARY 6.8
The coefficient of $q^d\,\sigma_{\nu^\vee}$ in the quantum product $\sigma_{\lambda^\vee} * \sigma_{\mu^\vee}$ is equal to the coefficient of $q^{\mathrm{diag}_0(\nu)-d}\,\sigma_{S^k(\nu)}$ in the quantum product $\sigma_\lambda * \sigma_\mu$.

Proof
By setting $f = \sigma_{\lambda^\vee}$, $g = \sigma_{\mu^\vee}$, and $h = 1$ in Proposition 6.6, we obtain

$$D(\sigma_{\lambda^\vee} * \sigma_{\mu^\vee}) * D(1) = \sigma_\lambda * \sigma_\mu.$$

Since $D(1) = \sigma_{(n-k)^k}$ is the fundamental class of a point, we get, by Proposition 6.3,

$$D(q^d * \sigma_{\nu^\vee}) * D(1) = q^{-d} * \sigma_\nu * \sigma_{(n-k)^k} = q^{-d}\,H^k * \sigma_\nu = q^{\mathrm{diag}_0(\nu)-d}\,\sigma_{S^k(\nu)}.$$

Here we used the fact that $\mathrm{diag}_0(\nu) = k - \phi_k(\nu)$. □

We can now prove the first claim of this subsection.

Proof of Theorem 6.4
Corollary 6.8 is equivalent to the special case of Theorem 6.4 for $a = b = 0$ and $c = n - k$. The general case follows by Proposition 6.1. □

The statement of Corollary 6.8 means that the terms in the expansion of the quantum product $\sigma_\lambda * \sigma_\mu$ are in one-to-one correspondence with the terms in the expansion of the quantum product $\sigma_{\lambda^\vee} * \sigma_{\mu^\vee}$ so that the coefficients of corresponding terms are equal to each other. Notice that terms with low powers of q correspond to terms with high powers of q and vice versa. This property seems mysterious from the point of view of quantum cohomology. Why should the number of some rational curves of high degree be equal to the number of some rational curves of low degree? This curious duality is also "hidden" on the classical level. For example, if $|\lambda| + |\mu|$ is sufficiently small, then the product $\sigma_\lambda \cdot \sigma_\mu$ is always nonzero and the product $\sigma_{\lambda^\vee} \cdot \sigma_{\mu^\vee}$ always vanishes in the classical cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{kn})$.

Let us reformulate this duality in terms of toric Schur polynomials. For a toric shape $\kappa = \lambda[r]/\mu[s]$, let us define the *complement toric shape* as

$$\kappa^\vee = \mu^\downarrow[s^\downarrow]/\lambda[r],$$

where the transformation $\mu[s] \mapsto \mu^\downarrow[s^\downarrow]$ is the same as in Lemma 3.4.

This definition has the following simple geometric meaning. The image of the diagram of shape $\kappa^\vee$ in the torus $\mathcal{T}_{kn}$ is the complement to the image of the diagram of shape κ (see Fig. 6). If $\kappa = \lambda/d/\mu$, then $\kappa^\vee$ is obtained by a shift of the toric shape $\mu^\downarrow/d'/\lambda$, where $\mu^\downarrow = S^k(\mu)$ and $d' = \mathrm{diag}_0(\mu) - d = \phi_{n-k}(\mu^\vee) - d$. Thus the toric Schur polynomial $s_{(\lambda/d/\mu)^\vee}$ is equal to $s_{\mu^\downarrow/d'/\lambda}$.

COROLLARY 6.9

For any toric shape κ, the coefficients in the Schur expansion of the toric Schur polynomial s_κ correspond to the coefficients in the Schur expansion of the toric Schur polynomial $s_{\kappa^\vee}$ as follows:

$$s_\kappa = \sum_{\nu \in P_{kn}} a_\nu \, s_\nu \ \text{ has the same coefficients } a_\nu \text{ as in } s_{\kappa^\vee} = \sum_{\nu \in P_{kn}} a_\nu \, s_{\nu^\vee}.$$

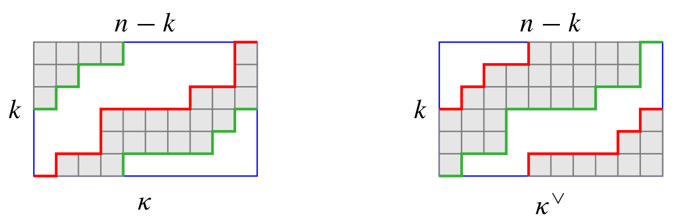


Figure 6. The complement toric shape

Proof

Suppose that $\kappa = \lambda/d/\mu$. By Theorem 5.3, the coefficient of s_ν in the Schur expansion of s_κ is equal to $C_{\mu\nu}^{\lambda,d} = C^d_{\mu\nu\lambda^\vee}$. On the other hand, the coefficient of $s_{\nu^\vee}$ in the Schur expansion of $s_{\kappa^\vee} = s_{\mu^\downarrow/d'/\lambda}$ is equal to

$$C_{\lambda\,\nu^\vee}^{\mu^\downarrow,d'} = C^{d'}_{\lambda\,\nu^\vee S^{n-k}(\mu^\vee)}.$$

The equality of these two coefficients is a special case of Theorem 6.4. □

6.4. Essential interval

In many cases the hidden symmetry and the curious duality imply that a Gromov-Witten invariant vanishes. In some cases these symmetries allow us to reduce a Gromov-Witten invariant to a certain Littlewood-Richardson coefficient. For three partitions $\lambda, \mu, \nu \in P_{kn}$, let us define three numbers

$$d_{\min}(\lambda,\mu,\nu) = -\min_{a+b+c=0} \big(\phi_a(\lambda)+\phi_b(\mu)+\phi_c(\nu)\big),$$
$$d_{\max}(\lambda,\mu,\nu) = -\max_{a+b+c=k-n} \big(\phi_a(\lambda)+\phi_b(\mu)+\phi_c(\nu)\big),$$
$$d(\lambda,\mu,\nu) = \frac{|\lambda|+|\mu|+|\nu|-k(n-k)}{n},$$

where in the first and second cases the maximum and minimum are taken over all triples of integers a, b, and c which satisfy the given condition.

Let us say that the integer interval $[d_{\min}(\lambda, \mu, \nu), d_{\max}(\lambda, \mu, \nu)] \subset \mathbb{Z}$ is the *essential interval* for the triple of partitions $\lambda, \mu, \nu \in P_{kn}$.

PROPOSITION 6.10
Let $\lambda, \mu, \nu \in P_{kn}$ be three partitions, and let $d_{\min} = d_{\min}(\lambda, \mu, \nu)$, $d_{\max} = d_{\max}(\lambda, \mu, \nu)$. Then the Gromov-Witten invariant $C^d_{\lambda\mu\nu}$ is equal to zero unless $d = d(\lambda, \mu, \nu)$ and $d_{\min} \le d \le d_{\max}$. If $d = d_{\min}$ and (a, b, c) is a triple such that $a + b + c = 0$ and $d = -(\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu))$, then

$$C^{d_{\min}}_{\lambda\mu\nu} = c_{S^a(\lambda)S^b(\mu)S^c(\nu)}.$$

Similarly, if $d = d_{\max}$ and (a, b, c) is a triple such that $a + b + c = k - n$ and $d = -(\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu))$, then

$$C^{d_{\max}}_{\lambda\mu\nu} = c_{S^{-a}(\lambda^\vee)S^{-b}(\mu^\vee)S^{-c}(\nu^\vee)}.$$

Proof
The claim that $C^d_{\lambda\mu\nu} = 0$ unless $d = d(\lambda, \mu, \nu)$ follows directly from the definition of the Gromov-Witten invariants. Proposition 6.1 says that $C^d_{\lambda\mu\nu} = C^{\tilde{d}}_{S^a(\lambda)S^b(\mu)S^c(\nu)}$, where $\tilde{d} = d + \phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu)$. The Gromov-Witten invariant in the right-hand side vanishes if $\tilde{d} < 0$, and it is a Littlewood-Richardson coefficient if $\tilde{d} = 0$. This proves that $C^d_{\lambda\mu\nu} = 0$ for $d < d_{\min}$ and that $C^{d_{\min}}_{\lambda\mu\nu}$ is a Littlewood-Richardson coefficient. Similarly, the statement that $C^d_{\lambda\mu\nu} = 0$ for $d > d_{\max}$ and $C^{d_{\max}}_{\lambda\mu\nu}$ is a Littlewood-Richardson coefficient is a consequence of Theorem 6.4. □

7. Powers of q in the quantum product of Schubert classes

In this section we discuss the following problem: What is the set of all powers q^d which appear with nonzero coefficients in the Schubert-expansion of a given quantum product $\sigma_\lambda * \sigma_\mu$? The lowest such power of q was established in [FW]. Some bounds for the highest power of q were found in [Y]. In this section we present a simple answer to this problem. We thank here Anders Buch, who remarked that our main theorem resolves this problem and made several helpful suggestions.

We have already formulated the answer to this problem in Corollary 5.4. The quantum product $\sigma_\lambda * \sigma_\mu$ contains nonzero terms with given power q^d if and only if $\mu^\vee/d/\lambda$ forms a valid toric shape.

Let $D_{\min}$ be the minimal d such that $\mu^\vee/d/\lambda$ forms a valid toric shape, and let $D_{\max}$ be the maximal such d. Graphically, this means that the loop $\mu^\vee[D_{\min}]$ drawn on the torus $\mathcal{T}_{kn}$ touches (but does not cross) the *southeast* side of the toric loop $\lambda[0]$. Similarly, the toric loop $\mu^\vee[D_{\max}]$ touches the *northwest* side of the toric loop $\lambda[0]$. Figure 7 gives an example for $k = 6$, $n = 16$, $\lambda = (9, 6, 6, 4, 3, 0)$, and $\mu^\vee = (6, 4, 3, 2, 2, 1)$. We have $D_{\min} = 2$ and $D_{\max} = 3$.

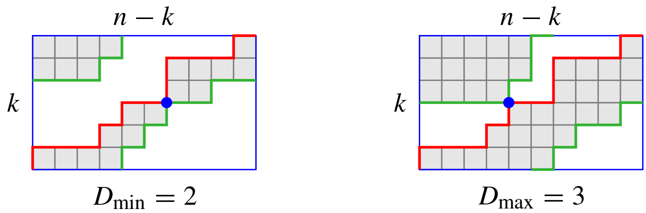


Figure 7. The lowest power $D_{\min}$ and the highest power $D_{\max}$

Let us give an explicit expression for the numbers $D_{\min}$ and $D_{\max}$. Recall that, for $\lambda \in P_{kn}$ with 01-word $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, the sequence $\phi_i(\lambda)$, $i \in \mathbb{Z}$, is defined by $\phi_i(\lambda) = \omega_1 + \cdots + \omega_i$ for $i = 1, \ldots, n$ and $\phi_{n+i}(\lambda) = \phi_i(\lambda) + k$ for any $i \in \mathbb{Z}$ (see Sec. 6). For any $\lambda, \mu \in P_{kn}$, define two integers $D_{\min}(\lambda, \mu)$ and $D_{\max}(\lambda, \mu)$ by

$$D_{\min}(\lambda, \mu) = -\min_{i+j=0} \big(\phi_i(\lambda) + \phi_j(\mu)\big),$$
$$D_{\max}(\lambda, \mu) = -\max_{i+j=k-n} \big(\phi_i(\lambda) + \phi_j(\mu)\big),$$

where in both cases the maximum or minimum is taken over all integers i and j which satisfy the given condition (cf. Sec. 6.4).

THEOREM 7.1
For any pair $\lambda, \mu \in P_{kn}$, we have $D_{\min}(\lambda, \mu) \le D_{\max}(\lambda, \mu)$, and the set of all d's such that the power q^d appears in $\sigma_\lambda * \sigma_\mu$ with nonzero coefficient is exactly the integer interval $D_{\min}(\lambda, \mu) \le d \le D_{\max}(\lambda, \mu)$. In particular, the quantum product $\sigma_\lambda * \sigma_\mu$ is always nonzero.

The claim about the lowest power of q with nonzero coefficient is due to Fulton and Woodward [FW]. Some bounds for the highest power of q were given by Yong in [Y]. He also formulated a conjecture that the powers of q which appear in the expansion of the quantum product $\sigma_\lambda * \sigma_\mu$ form an interval of consecutive integers.

Proof
Let us first verify that $D_{\min}(\lambda, \mu) \le D_{\max}(\lambda, \mu)$. We need to check that, for any integers i and j, we have $-\phi_i(\lambda) - \phi_{-i}(\mu) \le -\phi_j(\lambda) - \phi_{-j+k-n}(\mu)$ or, equivalently, $\phi_j(\lambda) - \phi_i(\lambda) \le \phi_{-i}(\mu) - \phi_{-j+k-n}(\mu) = \phi_{-i}(\mu) - \phi_{-j+k}(\mu) + k$. We may assume that $j \in [i, i+n[$ because the function ϕ_j satisfies the condition $\phi_{j+n} = \phi_j + k$. Then we have $\phi_j(\lambda) - \phi_i(\lambda) \le \min(j-i, k)$. Indeed, $\phi_j(\lambda) - \phi_i(\lambda) \le \phi_{i+n} - \phi_i = k$ and $\phi_j(\lambda) - \phi_i(\lambda) \le j - i$ because $\phi_{s+1} - \phi_s \in \{0, 1\}$ for any s. On the other hand, we have $\phi_{-i}(\mu) - \phi_{-j+k}(\mu) + k \ge \min(j-i, k)$ or, equivalently, $\phi_{k-j}(\mu) - \phi_{-i}(\mu) \le$

$\max(i+k-j,0)$. Indeed, if $k-j\le -i$, then the left-hand side is nonpositive and the right-hand side is zero; otherwise, $\phi_{k-j}(\mu)-\phi_{-i}(\mu)\le(k-j)-(-i)=i+k-j$. This proves the required inequality.

Let us now show that the values of d for which q^d occurs with nonzero coefficient in $\sigma_\lambda*\sigma_\mu$ form the interval $[D_{\min}(\lambda,\mu),D_{\max}(\lambda,\mu)]$. According to Corollary 5.4, the power q^d appears in the quantum product $\sigma_\lambda*\sigma_\mu$ whenever $\mu^\vee/d/\lambda$ is a valid toric shape. This is true if and only if the following two conditions are satisfied:

(a) $\mu^\vee[d]\ge\lambda[0]$; that is, $\mu^\vee[d]_i\ge\lambda[0]_i$ for all i; and

(b) $\lambda^\downarrow[0^\downarrow]\ge\mu^\vee[d]$, where $\lambda^\downarrow[0^\downarrow]=S^k(\lambda)[\mathrm{diag}_0(\lambda)]$ (cf. Lem. 3.4).

The first condition (a) can be written as $\phi_i(\lambda)-\phi_i(\mu^\vee)+d=\phi_i(\lambda)+\phi_{-i}(\mu)+d\ge 0$ for all i. It is equivalent to the inequality $d\ge D_{\min}(\lambda,\mu)$. The second condition (b) can be written as $\phi_i(\mu^\vee)-\phi_i(\lambda^\downarrow)+0^\downarrow-d=-\phi_{-i}(\mu)-(\phi_{i+k}(\lambda)-\phi_k(\lambda))+(k-\phi_k(\lambda))-d=-\phi_{i+k-n}(\lambda)-\phi_{-i}(\mu)-d\ge 0$ for all i. It is equivalent to the inequality $d\le D_{\max}(\lambda,\mu)$. □

The number $D_{\min}$ was defined in [FW] in terms of overlapping diagonals in two Young diagrams. We have $D_{\min}(\lambda,\mu)=\max_{i=-k,\dots,n-k}(\mathrm{diag}_i(\lambda)-\mathrm{diag}_i(\mu^\vee))$, where $\mathrm{diag}_i(\lambda)$ is the number of elements in the ith diagonal of shape λ. We also have $D_{\max}(\lambda,\mu)=\mathrm{diag}_0(\lambda)-\max_{i=-k,\dots,n-k}\big(\mathrm{diag}_i(\mu^\vee)-\mathrm{diag}_i(S^k(\lambda))\big)$. These expressions are equivalent to the definition of $D_{\min}$ and $D_{\max}$ in terms of the function ϕ_i, due to the following identities, which we leave as an exercise for the reader: $\mathrm{diag}_{i-k}(\lambda)-\mathrm{diag}_{i-k}(\mu^\vee)=\phi_i(\mu^\vee)-\phi_i(\lambda)$, $\mathrm{diag}_0(\lambda)=k-\phi_k(\lambda)$, $\phi_i(\mu^\vee)=-\phi_{-i}(\mu)$, and $\phi_i(S^k(\lambda))=\phi_{i+k}(\lambda)-\phi_k(\lambda)$.

Recall that in Section 6.4, for a triple of partitions $\lambda,\mu,\nu\in P_{kn}$, we defined the essential interval $[d_{\min},d_{\max}]$.

COROLLARY 7.2

For a pair of partitions $\lambda,\mu\in P_{kn}$, we have

$$[D_{\min}(\lambda,\mu),D_{\max}(\lambda,\mu)]=\bigcup_{\nu\in P_{kn}}[d_{\min}(\lambda,\mu,\nu),d_{\max}(\lambda,\mu,\nu)].$$

Proof

It is clear from the definitions that, for any λ,μ,ν,

$$[d_{\min}(\lambda,\mu,\nu),d_{\max}(\lambda,\mu,\nu)]\subseteq[D_{\min}(\lambda,\mu),D_{\max}(\lambda,\mu)].$$

Thus the right-hand side of the formula in Corollary 7.2 is contained in the left-hand side. On the other hand, by Proposition 6.10, the right-hand side contains the set of all d's such that q^d appears in $\sigma_\lambda*\sigma_\mu$, which is equal to the left-hand side of the expression, by Theorem 7.1. □

Let us show that the curious duality flips the interval $[D_{\min}, D_{\max}]$. Indeed, it follows from Theorem 6.4 that

$$C_{\lambda\mu}^{\nu,d} = C_{S^{n-k}(\lambda^\vee)\,\mu^\vee}^{\nu^\vee,\,\mathrm{diag}_0(\lambda)-d}.$$

In other words, the coefficient of $q^d\,\sigma_\nu$ in the quantum product $\sigma_\lambda * \sigma_\mu$ is exactly the same as the coefficient of $q^{\mathrm{diag}_0(\lambda)-d}\,\sigma_{\nu^\vee}$ in the quantum product $\sigma_{S^{n-k}(\lambda^\vee)} * \sigma_{\mu^\vee}$. This means that the set of all powers of q which occur in $\sigma_\lambda * \sigma_\mu$ is obtained from the set of all powers of q which occur in $\sigma_{S^{n-k}(\lambda^\vee)} * \sigma_{\mu^\vee}$ by the transformation $d \mapsto \mathrm{diag}_0(\lambda) - d$. In particular, we obtain the following statement.

COROLLARY 7.3
For any $\lambda, \mu \in P_{kn}$, we have

$$D_{\min}(\lambda, \mu) = \mathrm{diag}_0(\lambda) - D_{\max}\big(S^{n-k}(\lambda^\vee), \mu^\vee\big),$$
$$D_{\max}(\lambda, \mu) = \mathrm{diag}_0(\lambda) - D_{\min}\big(S^{n-k}(\lambda^\vee), \mu^\vee\big).$$

Recall that the map $\lambda \mapsto \tilde{\lambda} = S^{n-k}(\lambda^\vee)$ is an involution on P_{kn} such that $\mathrm{diag}_0(\lambda) = \mathrm{diag}_0(\tilde{\lambda})$ (see the second paragraph after Th. 6.5).

The lowest $D_{\min}$ and the highest $D_{\max}$ powers of q in the quantum product $\sigma_\lambda * \sigma_\mu$ can be easily recovered from the hidden symmetry and the curious duality of the Gromov-Witten invariants. Moreover, the Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$ in the case when $d = D_{\min}$ or $d = D_{\max}$ are equal to certain Littlewood-Richardson coefficients.

COROLLARY 7.4
Let $\lambda, \mu, \nu \in P_{kn}$. Let $D_{\min} = D_{\min}(\lambda, \mu)$ and $D_{\max} = D_{\max}(\lambda, \mu)$. By the definition, there are integers a and b such that $D_{\min} + \phi_a(\lambda) + \phi_{-a}(\mu) = 0$ and $D_{\max} + \phi_{-b}(\lambda) + \phi_{b+k-n}(\mu) = 0$. For such a and b, we have

$$C_{\lambda\mu}^{\nu,\,D_{\min}} = c_{S^a(\lambda)\,S^{-a}(\mu)}^{\nu} \qquad \text{and} \qquad C_{\lambda\mu}^{\nu,\,D_{\max}} = c_{S^b(\lambda^\vee)\,S^{n-k-b}(\mu^\vee)}^{\nu^\vee}.$$

Proof
If $D_{\min}(\lambda, \mu) = d_{\min}(\lambda, \mu, \nu)$, then the statement about $C_{\lambda\mu}^{\nu,\,D_{\min}}$ is a special case of Proposition 6.10. If $D_{\min}(\lambda, \mu) < d_{\min}(\lambda, \mu, \nu)$, then, by the same proposition, both sides are equal to zero. Similarly, the statement about $C_{\lambda\mu}^{\nu,\,D_{\max}}$ follows from Proposition 6.10. □

This statement means that, for a *toric* shape $\kappa = \mu^\vee/d/\lambda$ with $d = D_{\min}$, there always exists a cyclic shift $S^a(\kappa)$ which is equal to the *skew* shape $S^a(\kappa) = S^a(\mu^\vee)/0/S^a(\lambda)$ (cf. Fig. 7). If $d = D_{\max}$, then the same is true for the complement toric shape $\kappa^\vee$.

8. Affine nil-Temperley-Lieb algebra

In this section we discuss the affine nil-Temperley-Lieb algebra and its action on the quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$. This section justifies the word "affine" that appears in the title of this paper. The affine nil-Temperley-Lieb algebra presents a model for the quantum cohomology of the Grassmannian.

For $n \geq 2$, let us define the *affine nil-Temperley-Lieb algebra* AnTL_n as the associative algebra with 1 over $\mathbb{Z}$ with generators a_i, $i \in \mathbb{Z}/n\mathbb{Z}$, and the following defining relations:

$$a_i\,a_i = a_i\,a_{i+1}\,a_i = a_{i+1}\,a_i\,a_{i+1} = 0, \qquad a_i\,a_j = a_j\,a_i \quad \text{if } i - j \not\equiv \pm 1. \tag{13}$$

The subalgebra of AnTL_n generated by $a_1, \dots, a_{n-1}$ is called the *nil-Temperley-Lieb algebra*. Its dimension is equal to the nth Catalan number. According to Fomin and Green [FG], this algebra can also be defined as the algebra of operators acting on the space of formal combinations of Young diagrams by adding boxes to diagonals. In the next paragraph we extend this action to the affine nil-Temperley-Lieb algebra.

Recall that $\omega(\lambda) = (\omega_1, \dots, \omega_n)$ denotes the 01-word of a partition $\lambda \in P_{kn}$ (see Sec. 2). Let us define $\lambda(\omega) \in P_{kn}$ as the partition with $\omega(\lambda) = \omega$. Let ϵ_i be the ith coordinate n-vector, and let $\epsilon_{ij} = \epsilon_i - \epsilon_j$. For $i, j \in \{1, \dots, n\}$, we define the $\mathbb{Z}[q]$-linear operator E_{ij} on the space $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ given in the basis of Schubert cells by

$$E_{ij} : \sigma_{\lambda(\omega)} \longmapsto \begin{cases} \sigma_{\lambda(\omega-\epsilon_{ij})} & \text{if } \omega - \epsilon_{ij} \text{ is a 01-word,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega - \epsilon_{ij}$ means the coordinatewise difference of two n-vectors. We define the action of the generators $a_1, \dots, a_n$ of the affine nil-Temperley-Lieb algebra AnTL_n on the quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ using operators E_{ij} as follows:

$$a_i = E_{i\,i+1} \quad \text{for } i = 1, \dots, n-1, \qquad \text{and} \qquad a_n = q \cdot E_{n1}.$$

It is an easy exercise to check that these operators satisfy relations (13).

This action can also be interpreted in terms of Young diagrams that fit inside the $(k \times (n-k))$-rectangle. For $i = 1, \dots, n-1$, we have $a_i(\sigma_\lambda) = \sigma_\mu$ if the shape μ is obtained by adding a box to the $(i-k)$th diagonal of the shape λ, or $a_i(\sigma_\lambda) = 0$ if it is not possible to add such a box. Also, $a_n(\sigma_\lambda) = q \cdot \sigma_\mu$ if the shape μ is obtained from the shape λ by removing a rim hook of size $n-1$, or $a_n(\sigma_\lambda) = 0$ if it is not possible to remove such a rim hook. Notice that the partition $\mu \in P_{kn}$ is obtained from $\lambda \in P_{kn}$ by removing a rim hook of size $n-1$ if and only if the order ideal $D_{\mu[r+1]}$ in the cylinder $\mathscr{C}_{kn}$ is obtained from $D_{\lambda[r]}$ by adding a box to the $(n-k)$th diagonal. Thus the generators a_i, $i = 1, \dots, n$, of the affine nil-Temperley-Lieb algebra naturally act on order ideals in $\mathscr{C}_{kn}$ by adding boxes to $(i-k)$th diagonals.

Let us say a few words about a relation between the affine nil-Temperley-Lieb algebra AnTL_n and the *affine Lie algebra* $\widehat{\mathfrak{sl}}_n$ (without central extension). The vector space $\mathrm{H}^*(\mathrm{Gr}_{kn}) \otimes \mathbb{C}$ can be regarded as the kth *fundamental representation* Φ_k of the Lie algebra $\mathfrak{sl}_n$. A Schubert class $\sigma_{\lambda(\omega)}$ corresponds to the weight vector of weight ω. These are exactly the weights obtained by conjugations of the kth fundamental weight. The generator e_i of $\mathfrak{sl}_n$ acts on $\mathrm{H}^*(\mathrm{Gr}_{kn})$ as the operator a_i above by adding a box to the $(i-k)$th diagonal of the shape λ. (The generators e_i of $\mathfrak{sl}_n$ should not be confused with elementary symmetric functions.) The generator f_i acts as the adjoint to the operator e_i by removing a box from the $(i-k)$th diagonal. Recall that every representation Γ of $\mathfrak{sl}_n$ gives rise to the *evaluation module* $\Gamma(q)$, which is a representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ (see [K]). Then the space $\mathrm{QH}^*(\mathrm{Gr}_{kn}) \otimes \mathbb{C}[q, q^{-1}]$ can be regarded as the evaluation module $\Phi_k(q)$ of the k-fundamental representation:

$$\mathrm{QH}^*(\mathrm{Gr}_{kn}) \otimes \mathbb{C}[q, q^{-1}] \simeq \Phi_k(q).$$

This equality is just a formal identification of two linear spaces over $\mathbb{C}[q, q^{-1}]$ given by mapping a Schubert class to the corresponding weight vector in $\Phi_k(q)$. This $\mathbb{C}[q, q^{-1}]$-linear action of $\widehat{\mathfrak{sl}}_n$ on $\mathrm{QH}^*(\mathrm{Gr}_{kn}) \otimes \mathbb{C}[q, q^{-1}]$ is explicitly given by

$$\begin{aligned}
&e_i = E_{i\,i+1},\ i = 1, \ldots, n-1, \qquad \text{and} \qquad e_n = q \cdot E_{n1},\\
&f_i = E_{i+1\,i},\ i = 1, \ldots, n-1, \qquad \text{and} \qquad f_n = q^{-1} \cdot E_{1n},\\
&h_i : \sigma_\lambda \mapsto \big(\omega_i(\lambda) - \omega_{i+1}(\lambda)\big)\sigma_\lambda \quad \text{for } i = 1, \ldots, n,
\end{aligned}$$

where we assume that $\omega_{n+1}(\lambda) = \omega_1(\lambda)$.

Let $\mathfrak{n}$ be the subalgebra of the affine algebra $\widehat{\mathfrak{sl}}_n$ generated by $e_1, \ldots, e_n$. The affine nil-Temperley-Lieb algebra (with complex coefficients) is exactly the following quotient of the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$:

$$\mathrm{AnTL}_n \otimes \mathbb{C} \simeq U(\mathfrak{n}) / \langle (e_i)^2 \mid i = 1, \ldots, n \rangle.$$

Indeed, Serre's relations modulo the ideal $\langle (e_i)^2 \rangle$ degenerate to the defining relations (13) of AnTL_n. Notice that the squares of the generators $(e_i)^2$ and $(f_i)^2$ vanish in all fundamental representations Φ_k and in their evaluation modules $\Phi_k(q)$. The action of the affine nil-Temperley-Lieb algebra AnTL_n on $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ described above in this section is exactly the action deduced from the evaluation module $\Phi_k(q)$.

Let us show how the affine nil-Temperley-Lieb algebra AnTL_n is related to cylindric shapes. Let κ be a cylindric shape of type (k, n) for *some* k. Let us pick any cylindric tableau T of shape κ and *standard* weight $\beta = (1, \ldots, 1)$. For $i = 1, \ldots, |\kappa|$, let d_i be k plus the index of the diagonal that contains the entry i in the tableau T. Let us define $\mathbf{a}_\kappa = a_{d_1} \cdots a_{d_{|\kappa|}}$. The monomials for different tableaux of the same shape can be related by the commuting relations $a_i \cdot a_j = a_j \cdot a_i$. Thus the monomial $\mathbf{a}_\kappa$ does not depend on the choice of tableau. For two cylindric shapes κ and $\tilde{\kappa}$ of types (k, n) and $(\tilde{k}, n)$, let us write $\kappa \sim \tilde{\kappa}$ whenever $\mathbf{a}_\kappa = \mathbf{a}_{\tilde{\kappa}}$. Clearly, $\mathbf{a}_\kappa$ does not change if we shift the shape κ in the southeast direction. Thus $\kappa \sim \tilde{\kappa}$ for any $\tilde{\kappa}$ obtained from κ by such a shift. Moreover, if the diagram D_κ of κ has several connected components, then we can shift each connected component independently. These shifts of connected components generate the equivalence relation "$\sim$". Any nonvanishing monomial in AnTL_n is equal to $\mathbf{a}_\kappa$ for some κ. Thus the map $\kappa \mapsto \mathbf{a}_\kappa$ gives a one-to-one correspondence between cylindric shapes (modulo the "$\sim$"-equivalence) and nonvanishing monomials in the algebra AnTL_n.

For any $\mu \in P_{kn}$ and a cylindric shape κ, there is at most one cylindric loop $\lambda[d]$ of type (k, n) such that $\lambda/d/\mu \sim \kappa$. The action of a monomial $\mathbf{a}_\kappa$ on $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ is given by

$$\mathbf{a}_\kappa : \sigma_\mu \longmapsto \begin{cases} q^d\, \sigma_\lambda & \text{if } \lambda/d/\mu \sim \kappa, \\ 0 & \text{if there are no such } \lambda \text{ and } d. \end{cases} \tag{14}$$

So far in this section we have treated the quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ as a linear space. Let us now show that the action of the affine nil-Temperley-Lieb algebra AnTL_n is helpful for describing the multiplicative structure of $\mathrm{QH}^*(\mathrm{Gr}_{kn})$.

Let us define the elements $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ and $\mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$ in the algebra AnTL_n, as follows. For a proper subset I in $\mathbb{Z}/n\mathbb{Z}$, let $\prod_{i\in I}^{\circlearrowright} a_i \in \mathrm{AnTL}_n$ be the product of a_i, $i \in I$, taken in an order such that if $i, i+1 \in I$, then a_{i+1} goes before a_i. This product is well defined because all such orderings of a_i, $i \in I$, are obtained from each other by switching commuting generators. Also, let $\prod_{i\in I}^{\circlearrowleft} a_i \in \mathrm{AnTL}_n$ be the element obtained by reversing the "cyclic order" of a_i's in $\prod_{i\in I}^{\circlearrowright} a_i$. For $r = 1, \ldots, n-1$, define

$$\mathbf{e}_r = \sum_{|I|=r} \prod_{i\in I}^{\circlearrowright} a_i \qquad \text{and} \qquad \mathbf{h}_r = \sum_{|I|=r} \prod_{i\in I}^{\circlearrowleft} a_i,$$

where the sum is over all r-element subsets I in $\mathbb{Z}/n\mathbb{Z}$. For example,

$$\mathbf{e}_1 = \mathbf{h}_1 = a_1 + \cdots + a_n,$$

$$\mathbf{e}_2 = a_2\, a_1 + a_3\, a_2 + \cdots + a_n\, a_{n-1} + a_1\, a_n + \sum^{\mathrm{c}} a_i\, a_j,$$

$$\mathbf{h}_2 = a_1\, a_2 + a_2\, a_3 + \cdots + a_{n-1}\, a_n + a_n\, a_1 + \sum^{\mathrm{c}} a_i\, a_j,$$

where $\sum^{\mathrm{c}} a_i\, a_j$ is the sum of products of (unordered) pairs of commuting a_i and a_j, that is, where i and j are not adjacent elements in $\mathbb{Z}/n\mathbb{Z}$. In the spirit of [FG], we can say that the $\mathbf{e}_r$ are elementary symmetric polynomials and the $\mathbf{h}_r$ are the complete homogeneous symmetric polynomials in *noncommutative* variables $a_1, \ldots, a_n$. Notice that the element $\mathbf{e}_r$ (resp., $\mathbf{h}_r$) in AnTL_n is the sum of monomials $\mathbf{a}_\kappa$ for all non-"$\sim$"-equivalent cylindric vertical (resp., horizontal) r-strips κ.

LEMMA 8.1
The elements $\mathbf{e}_i$ *and* $\mathbf{h}_j$ *in the algebra* AnTL_n *commute pairwise. For* $i + j > n$*, we have* $\mathbf{e}_i \cdot \mathbf{h}_j = 0$*. These elements are related by the equation*

$$\Big(1 + \sum_{i=1}^{n-1} \mathbf{e}_i\, t^i\Big) \cdot \Big(1 + \sum_{j=1}^{n-1} \mathbf{h}_j\, (-t)^j\Big) = 1 + \Big(\sum_{k=1}^{n-1} (-1)^{n-k}\, \mathbf{e}_k \cdot \mathbf{h}_{n-k}\Big) t^n. \qquad (15)$$

Proof
Let us first show that $\mathbf{e}_i \cdot \mathbf{h}_j = 0$ for $i + j > n$. Indeed, by the pigeonhole principle, every monomial in the expansion of $\mathbf{e}_i \cdot \mathbf{h}_j$ contains two copies of some generator a_s. If there is such a monomial that does not vanish in AnTL_n, then it is of the form $\mathbf{a}_\kappa$ and the shape κ contains at least two elements in the $(s - k)$th diagonal. Thus κ should contain a (2×2)-rectangle. But it is impossible to cover a (2×2)-rectangle by a horizontal and a vertical strip.

Two elements $\mathbf{h}_i$ and $\mathbf{h}_j$ commute because the coefficient of a monomial $\mathbf{a}_\kappa$ in $\mathbf{h}_i \cdot \mathbf{h}_j$ is equal to the number of cylindric tableaux of shape κ and weight (i, j), which is the same as the number of tableaux of weight (j, i), by Corollary 4.3.

Let us check that the coefficient of t^l in the left-hand side of (15) is zero, for $0 < l < n$. Indeed, any monomial that occurs in the expansion of $\mathbf{e}_i \cdot \mathbf{h}_j$, $i + j \le n - 1$, avoids at least one variable a_r. Assume, without loss of generality, that $r = n$. If we remove all terms containing a_n from the left-hand side of (15), we obtain the expression $((1 + t\, a_{n-1}) \cdots (1 + t\, a_1)) \cdot ((1 - t\, a_1) \cdots (1 - t\, a_{n-1}))$. This equals 1 because $(1 + t\, a_s)(1 - t\, a_s) = 1$ in AnTL_n.

Finally, the relation (15) allows one to express the elements $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ in terms of $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$, which shows that the elements $\mathbf{e}_i$ commute with each other and with the elements $\mathbf{h}_j$. □

Recall that the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ is the quotient (6) of the polynomial ring over $\mathbb{Z}[q]$ in the variables $e_1, \dots, e_k, h_1, \dots, h_{n-k}$. These generators are the special Schubert classes $e_i = \sigma_{(1^i)}$ and $h_j = \sigma_{(j)}$. We can reformulate Bertram's quantum Pieri formula (see Prop. 4.1) as follows.

COROLLARY 8.2 (Quantum Pieri formula: AnTL_n-version)
For any $\lambda \in P_{kn}$*, the products of the Schubert class* σ_λ *in the quantum cohomology ring* $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ *with the generators* e_i *and* h_j *are given by*

$$e_i * \sigma_\lambda = \mathbf{e}_i(\sigma_\lambda) \qquad \text{and} \qquad h_j * \sigma_\lambda = \mathbf{h}_j(\sigma_\lambda),$$

where $i = 1, \dots, k$ *and* $j = 1, \dots, n - k$*.*

Indeed, by (14) the operators $\mathbf{e}_i$ and $\mathbf{h}_j$ act on $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ by adding cylindric vertical i-strips and horizontal j-strips, respectively.

The quantum Giambelli formula (7) implies the following statement.

COROLLARY 8.3
For any $\lambda \in P_{kn}$, the element

$$\mathbf{s}_\lambda = \det(\mathbf{h}_{\lambda_i+j-i})_{1\le i,\, j\le k} = \det(\mathbf{e}_{\lambda'_i+j-i})_{1\le i,\, j\le n-k} \in \mathrm{AnTL}_n$$

acts on the quantum cohomology $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ *as the operator of quantum multiplication by the Schubert class σ_λ.*

According to this claim and (14), for $\kappa = \nu/d/\mu$, the coefficient of $\mathbf{a}_\kappa$ in $\mathbf{s}_\lambda$ is the Gromov-Witten invariant $C^{\nu,d}_{\lambda\mu}$. Thus, even though the expansion of the determinant contains negative signs, all negative terms cancel, and $\mathbf{s}_\lambda$ always reduces to a *positive* expression in AnTL_n.

The algebra AnTL_n acts on $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ for all values of k. In order to single out one particular k, we need to describe certain $n-1$ central elements in the algebra AnTL_n. We say that a cylindric shape κ of type (k,n) is a *circular ribbon* if the diagram of κ contains no (2×2)-rectangle and $|\kappa| = n$. Up to the "$\sim$"-equivalence, there are exactly $\binom{n}{k}$ circular ribbons of type (k,n). Let us define the elements $\mathbf{z}_1,\dots,\mathbf{z}_{n-1}$ in AnTL_n as the sums $\mathbf{z}_k = \sum_\kappa \mathbf{a}_\kappa$ over all $\binom{n}{k}$ non-"$\sim$"-equivalent circular ribbons κ of type (k,n). These elements are also given by

$$\mathbf{z}_k = \mathbf{e}_k \cdot \mathbf{h}_{n-k}.$$

Indeed, a nonvanishing monomial in $\mathbf{e}_k \cdot \mathbf{h}_{n-k}$ should be of the type $\mathbf{a}_\kappa$, where κ contains no (2×2)-rectangle (cf. proof of Lemma 8.1). Since $|\kappa| = k+(n-k) = n$, the cylindric shape κ should be a circular ribbon. Then each circular ribbon of type (k,n) uniquely decomposes into a product of two monomials corresponding to a vertical k-strip and a horizontal $(n-k)$-strip.

LEMMA 8.4
The elements $\mathbf{z}_1,\dots,\mathbf{z}_{n-1}$ are central elements in the algebra AnTL_n*. For $k \ne l$, we have $\mathbf{z}_k \cdot \mathbf{z}_l = 0$.*

Proof
For any i, both elements $\mathbf{z}_k \cdot a_i$ and $a_i \cdot \mathbf{z}_k$ are given by the sum of monomials $\mathbf{a}_\kappa$ over all cylindric shapes κ, $|\kappa| = n+1$, that have exactly one (2×2)-rectangle centered in the $(i-k)$th diagonal. Thus $\mathbf{z}_k \cdot a_i = a_i \cdot \mathbf{z}_k$ for any i, which implies that $\mathbf{z}_k$ is a central element in AnTL_n. The second claim follows from (14). □

Let us define the algebra AnTL_{kn} as

$$\mathrm{AnTL}_{kn} = \mathrm{AnTL}_n \otimes \mathbb{Z}[q, q^{-1}] / \langle \mathbf{z}_1, \dots, \mathbf{z}_{k-1}, \mathbf{z}_k - q, \mathbf{z}_{k+1}, \dots, \mathbf{z}_{n-1} \rangle .$$

PROPOSITION 8.5
The ring $\mathrm{QH}^*_{\langle q \rangle}(\mathrm{Gr}_{kn}) = \mathrm{QH}^*(\mathrm{Gr}_{kn}) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}]$ *is isomorphic to the subalgebra of* AnTL_{kn} *generated by the elements* $\mathbf{e}_i$ *and/or* $\mathbf{h}_j$. *This isomorphism is given by the* $\mathbb{Z}[q, q^{-1}]$*-linear map that sends the generators* e_i *and* h_j *of* $\mathrm{QH}^*_{\langle q \rangle}$ *to the elements* $\mathbf{e}_i$ *and* $\mathbf{h}_j$ *in* AnTL_{kn}*, respectively.*

Proof
By Corollary 8.2, the algebra AnTL_{kn} acts faithfully on $\mathrm{QH}^*_{\langle q \rangle}(\mathrm{Gr}_{kn})$. The only thing that we need to check is that the elements $\mathbf{e}_i$ and $\mathbf{h}_j$ in AnTL_{kn} satisfy the same relations as the elements e_i and h_j in the quantum cohomology do (cf. (6)). The right-hand side of equation (15) becomes $1 + (-1)^{n-k} q\, t^n$ in the algebra AnTL_{kn}. It remains to show that $\mathbf{e}_i = \mathbf{h}_j = 0$ in AnTL_{kn} whenever $i > k$ and $j > n - k$. By Lemma 8.1, we have $\mathbf{e}_i \cdot \mathbf{h}_{n-k} = \mathbf{h}_j \cdot \mathbf{e}_k = 0$ for $i > k$ and $j > n - k$. Since $\mathbf{z}_k = \mathbf{e}_k \cdot \mathbf{h}_{n-k} = q$, both elements $\mathbf{e}_k$ and $\mathbf{h}_{n-k}$ are invertible in AnTL_{kn}. Thus $\mathbf{e}_i = \mathbf{h}_j = 0$, as needed. □

Remark 8.6
Fomin and Kirillov [FK] defined a certain quadratic algebra and a set of its pairwise commuting elements, called *Dunkl elements*. According to the *quantum Monk's formula* from [FGP], the multiplication in the quantum cohomology ring $\mathrm{QH}^*(Fl_n)$ of the *complete flag manifold* Fl_n can be written in terms of the Dunkl elements. A conjecture from [FK], which was proved in [P1], says that these elements generate a subalgebra isomorphic to $\mathrm{QH}^*(\mathrm{Fl}_n)$. This section shows that the affine nil-Temperley-Lieb algebra AnTL_n is, in a sense, a Grassmannian analogue of Fomin-Kirillov's quadratic algebra. The pairwise commuting elements $\mathbf{e}_i$ and $\mathbf{h}_j$ are analogues of the Dunkl elements. It would be interesting to extend these two opposite cases to the quantum cohomology of an arbitrary partial flag manifold.

9. Open questions, conjectures, and final remarks

9.1. Quantum Littlewood-Richardson rule
The problem that still remains open is to give a generalization of the Littlewood-Richardson rule to the quantum cohomology ring of the Grassmannian. As we have already mentioned, it is possible to use the quantum Giambelli formula in order to derive a rule for the Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$ which involves an *alternating* sum (e.g., see [BCF] or Cor. 8.3 in the present paper). The problem is to present a

subtraction-free rule for the Gromov-Witten invariants. In other words, one would like to get a combinatorial or algebraic construction for the Gromov-Witten invariants which would imply their nonnegativity. There are several possible approaches to this problem. Buch, Kresch, and Tamvakis [BKT] showed that the Gromov-Witten invariants of Grassmannians are equal to some intersection numbers for two-step flag manifolds, and they conjectured a rule for the latter numbers.

In the next subsection we propose an algebraic approach to this problem via representations of symmetric groups.

9.2. Toric Specht modules

For any toric shape $\kappa = \lambda/d/\mu$, let us define a representation S^κ of the symmetric group S_N, where $N = |\kappa|$, as follows. Let us fix a labeling of the boxes of κ by numbers $1, \ldots, N$. Recall that every toric shape has rows and columns (see Sec. 3). The rows (columns) of κ give a decomposition of $\{1, \ldots, N\}$ into a union of disjoint subsets. Let $R_\kappa \subset S_N$ and $C_\kappa \subset S_N$ be the *row stabilizer* and the *column stabilizer*, correspondingly. Let $\mathbb{C}[S_N]$ denote the group algebra of the symmetric group S_N. The *toric Specht module* S^κ is defined as the subspace of $\mathbb{C}[S_N]$ given by

$$S^\kappa = \Big(\sum_{u \in R_\kappa} u\Big)\Big(\sum_{v \in C_\kappa} (-1)^{\mathrm{sign}(v)} v\Big)\mathbb{C}[S_N].$$

It is equipped with the action of S_N by left multiplications.

If κ is a usual shape λ, then S^λ is known to be an irreducible representation of S_N. The following conjecture proposes how the S_N-module S^κ decomposes into irreducible representations, for an arbitrary toric shape κ.

CONJECTURE 9.1

For a toric shape $\kappa = \lambda/d/\mu$, the coefficients of irreducible components in the toric Specht module $S^{\lambda/d/\mu}$ are the Gromov-Witten invariants:

$$S^{\lambda/d/\mu} = \bigoplus_{\nu \in P_{kn}} C_{\mu\nu}^{\lambda,d}\, S^\nu.$$

Equivalently, the toric Specht module $S^{\lambda/d/\mu}$ is expressed in terms of the irreducible modules S^ν in exactly the same way that the toric Schur polynomial $s_{\lambda/d/\mu}$ is expressed in terms of the usual Schur polynomials s_ν.

This conjecture is true (and well known) for skew shapes (see [JP]). We have verified this conjecture for several toric shapes. For example, it is easy to prove the conjecture for $k \leq 2$. If the conjecture is true in general, it would provide an algebraic explanation of nonnegativity of the Gromov-Witten invariants.

Note that Reiner and Shimozono [RS] have investigated Specht modules for some class of shapes, called *percent-avoiding*, which is more general than skew shapes. However, toric shapes are not percent-avoiding, except for some degenerate cases.

9.3. Representations of GL(k) *and crystal bases*

According to Theorem 5.3, each toric Schur polynomial $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ is Schur-positive. The usual Schur polynomials in k variables are the characters of irreducible representations of the general linear group $\mathrm{GL}(k)$. Thus we obtain the following statement.

COROLLARY 9.2

For any toric shape $\lambda/d/\mu$, there exists a representation $V_{\lambda/d/\mu}$ of $\mathrm{GL}(k)$ *such that $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ is the character of $V_{\lambda/d/\mu}$.*

It would be extremely interesting to present a more explicit construction for this representation $V_{\lambda/d/\mu}$.

Recall that with every representation of $\mathrm{GL}(k)$ it is possible to associate its *crystal*, which is a certain directed graph with labeled edges (e.g., see [KN]). This graph encodes the corresponding representation of $U_q(\mathfrak{gl}_k)$ modulo $\langle q\rangle$. Its vertices correspond to the elements of a certain *crystal basis*, and the edges describe the action of generators on the basis elements. It is well known (see [KN]) that crystals are intimately related to the Littlewood-Richardson rule.

The vertices of the crystal for $V_{\lambda/d/\mu}$ should correspond to the toric tableaux of shape $\lambda/d/\mu$. Its edges should connect the vertices in a certain prescribed manner. In a recent paper [St], Stembridge described simple local conditions that would ensure that a given graph is a crystal of some representation. Thus, in order to find the crystal for $V_{\lambda/d/\mu}$, it would be enough to present a graph on the set of toric tableaux which complies with Stembridge's conditions.

Actually, an explicit construction of the crystal for $V_{\lambda/d/\mu}$ would immediately produce the following subtraction-free combinatorial rule for the Gromov-Witten invariants: The Gromov-Witten invariant $C_{\mu\nu}^{\lambda,d}$ is equal to the number of toric tableaux T of shape $\lambda/d/\mu$ and weight ν such that there are no directed edges in the crystal with initial vertex T. The last condition means that the element in the crystal basis given by T is annihilated by the operators $\tilde{e}_i$.

Note that all of the numerous (re)formulations of the Littlewood-Richardson rule and all explicit constructions of crystals for representations of $\mathrm{GL}(k)$ use some kind of ordering of elements in shapes. The main difficulty with toric shapes is that they are cyclically ordered and there is no natural way to select a linear order on a cycle.

9.4. Verlinde algebra and fusion product

Several people have observed that the specialization of the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ at $q = 1$ is isomorphic to the *Verlinde algebra* (also known as the *fusion ring*) of $\mathrm{U}(k)$ at level $n - k$ (see Witten [W] for a physical proof and Agnihotri [A] for a mathematical proof). This ring is the Grothendieck ring of representations of $\mathrm{U}(k)$ modulo some identifications. A Schubert class σ_λ corresponds to the irreducible representation V_λ with highest weight given by the partition λ.

All constructions of this paper for the quantum product make perfect sense for the Verlinde algebra and its product, called the *fusion product.* Our curious duality might have a natural explanation in terms of the Verlinde algebra.

9.5. Geometrical interpretation

The relevance of skew Young diagrams to the product of Schubert classes in the cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{kn})$ has a geometric explanation (see [F]). It is possible to see that the intersection of two Schubert varieties $\Omega_\lambda \cap \tilde{\Omega}_\mu$ (where $\tilde{\Omega}_\mu$ is taken in the opposite Schubert decomposition) is empty unless $\mu/\lambda^\vee$ is a valid skew shape. A natural question to ask is, How does one extend this construction to the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}_{kn})$ and toric shapes? It would be interesting to obtain a "geometric" proof of our result on toric shapes (Cor. 5.4), and also to present a geometric explanation of the curious duality (Th. 6.4).

9.6. Generalized flag manifolds

The main theorem of [FW] is given in a uniform setup of the generalized flag manifold G/P, where G is a complex semisimple Lie group and P is its parabolic subgroup. It describes the minimal monomials q^d in the quantum parameters q_i which occur in the quantum product of two Schubert classes. It would be interesting to describe all monomials q^d which occur with nonzero coefficients in the quantum product.

In [P3] we proved several results for G/B, where B is a Borel subgroup. We showed that there is a unique minimal monomial q^d which occurs in the quantum product of two Schubert classes. This monomial has a simple interpretation in terms of directed paths in the quantum Bruhat graph from [BFP]. For the flag manifold $\mathrm{SL}(n)/B$, we gave a complete characterization of all monomials q^d which occur in the quantum product. In order to do this, we defined path Schubert polynomials in terms of paths in the quantum Bruhat graph, and we showed that their expansion coefficients in the basis of usual Schubert polynomials are the Gromov-Witten invariants for the flag manifold.

In forthcoming publications we will address the question of extending the constructions of [P3] and the present paper to the general case G/P.

Acknowledgments. I thank Sergey Fomin, Christian Krattenthaler, Victor Ostrik, Josh Scott, Mark Shimozono, and Chris Woodward for interesting discussions and helpful correspondence. I thank Michael Entov, whose question on quantum cohomology gave the first impulse for writing this paper. I am grateful to Anders Buch, who pointed out the problem of Fulton and Woodward and made several helpful suggestions. I thank William Fulton for several helpful comments, corrections, and suggestions for improvement of the exposition. I also thank the anonymous referees for thoughtful comments.

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