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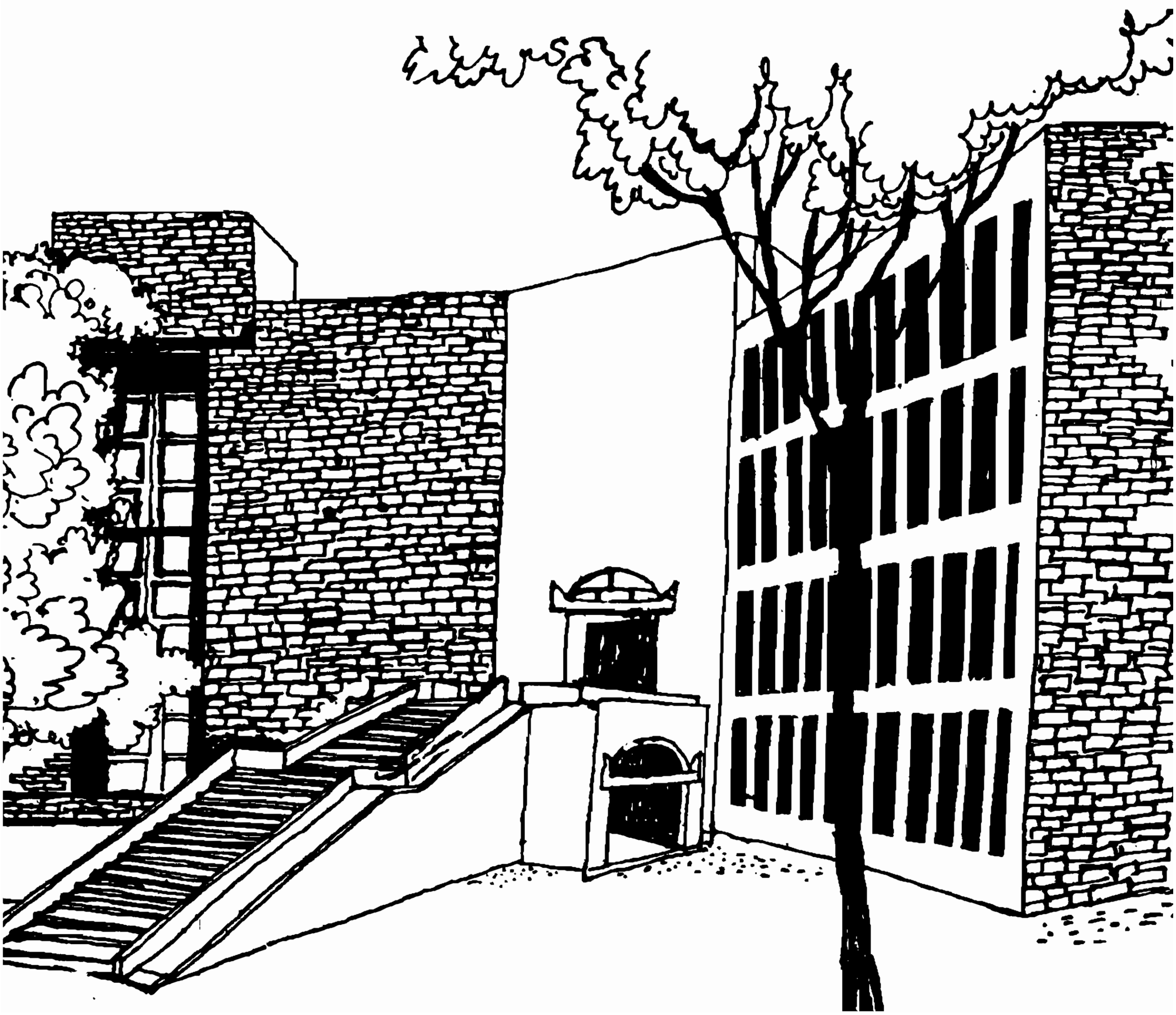

AFFINE COST SHARE EQUILIBRIA FOR ECONOMIES WITH PUBLIC GOODS

By

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Abstract

In this paper we propose the concept of an affine cost share equilibrium and show that under natural assumptions any optimal allocation corresponds to an affine cost share equilibrium. With linear cost functions and under mild regularity assumptions we show that an optimal allocation is a ratio equilibrium with redistribution. In an appendix to the paper we propose a new proof of the existence of a ratio equilibrium when the preferences of the agents are representable by strictly quasi-concave. continuous and strictly monotonically increasing utility functions.

1. Introduction :- We consider an economy in which a public good is produced by a regulated monopoly and is consumed by a finite number of consumers. Since production takes place in a regulated monopoly, the assumption of profit maximization by the firm is infructuous. The appropriate behavioral hypothesis is that of a firm attempting to cover its costs of production, which might include a profit margin as a component.

The first solution concept in the Lindahlian tradition, which was provided for such an economy is due to Kaneko (1977), called the ratio equilibrium. Subsequently Mas-Colell and Silvestre (1989) proposed the concept of cost-share equilibria for such economies. The optimality and core compatibility of cost-share equilibria was established in the latter contribution. We find the cost-share equilibria as a useful paradigm in modelling decentralized resource allocation in economies with public goods.

In this paper, we first show that if the cost function of the firms is convex and preferences of the consumers are representable by semi-strictly quasi-concave utility functions then essentially all optimal allocations can be obtained as "affine cost-share equilibria". An affine cost-share equilibria is similar in spirit to a two-part tariff which may differ across individuals. There is a lump-sum contribution on all consumption of the public good and a contribution proportional to costs. Then we show that if the cost function is linear, the equilibrium is in fact a ratio equilibrium with redistribution, provided a mild regularity condition is satisfied.

In an appendix to the paper, we propose a new simple proof of the existence of a ratio equilibrium.

Our framework of analysis throughout the paper is that which has been developed in Moulin (1990) i.e. a model of a two good economy with one private and one public good.

2. The Model :- Following Moulin (1990) we study an economy with one public good $y \geq 0$ producible from a single private good (money)

$x \geq 0$. Taking the latter as numeraire a technology is given by a cost function $c: \mathbb{R}_+ \to \mathbb{R}_+$. We assume throughout that $c(0)=0$ and that c is strictly increasing, unbounded above, and continuous.

There are N consumers, each endowed with a strictly positive amount of numeraire: let $w_i > 0$ be agent i's endowment of the private good. The preferences of consumer i are described by a utility function $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ which is assumed to be continuous, strictly increasing and satisfies semi-strict quasi-concavity: $\forall (y^0, x^0_i), (y^1, x^1_i) \in \mathbb{R}^2_+$ with $u_i(y^0, x^0_i) \neq u_i(y^1, x^1_i)$ and $t \in (0,1), u_i\big((1-t)(y^0, x^0_i) + t(y^1, x^1_i)\big) > \min(u_i(y^0, x^0_i), u_i(y^1, x^1_i))$.

A <u>state</u> of the economy is a vector $(y,x) \in \mathbb{R}_+ \times \mathbb{R}^N_+$. A state (y,x) is <u>feasible</u> if $c(y) \leq \Sigma_i (w_i - x_i)$. A feasible state (y,x) is <u>optimal</u> if there is no other feasible state (y^1, x^1) such that $u_i(y', x'_i) \geq u_i(y, x_i)$ for all i, with strict inequality for at least one i.

Following Mas-Colell and Silvestre (1989), we define a <u>cost share system</u> to be a family $g = \langle g_1, \ldots, g_N \rangle$ of N functions $g_i : \mathbb{R}_+ \to \mathbb{R}$ such that $g_i(0)=0$ and $\Sigma_i g_i(y) = c(y)$ for all $y > 0$.

A <u>cost share equilibrium</u> is a pair $\{(\bar{y}, \bar{x}), g\}$ where $(\bar{y}, \bar{x})$ is a feasible state and g is a cost share system with the property that for every i, $\bar{x}_i = w_i - g_i(\bar{y})$ and $u_i(\bar{y}, \bar{x}_i) \geq u_i(y, w_i - g_i(y))$ for all $y \geq 0$, and $w_i - g_i(y) \geq 0$.

A cost share system $g = \langle g_1, \ldots, g_N \rangle$ is said to be <u>affine</u> if $g_i(y) = a_i + b_i . c(y)$, where $b_i \geq 0 \; \forall$ i and $b_i > 0$ for some i. An <u>affine cost share equilibria</u> is a cost share equilibrium with an affine cost share system.

An affine cost share equilibrium is said to be a <u>ratio equilibrium with redistribution</u> if $\Sigma_i a_i = 0$. It is said to be a <u>ratio equilibrium</u> if $a_i = 0 \; \forall$ i (see Kaneko (1977)).

We close this section with a result due to Mas-Collel and Silvestre (1989):

Proposition 1 :- Any Cost Share Equilibrium yields an optimal state.

It is easy to show, as has been asserted by Mas-Colell and

Silvestre (1989), that provided $g_i(y) \geq 0\ \forall y \geq 0$ and $i \in \{1,...,N\}$, a cost share equilibrium yields an allocation in the core of the economy.

3. Optimality as Affine Cost Share Equilibrium :- In this section we show that essentially any optimal state can be supported as an affine cost-share equilibrium.

Proposition 2 :- Assume that c is convex and $(\bar{y},\bar{x})$ is an optimal state with $\bar{y}>0$. Then $(\bar{y},\bar{x})$ is an affine cost share equilibrium allocation.

Proof :- First observe that $c:\mathbb{R}_+ \to \mathbb{R}_+$ is convex, continuous strictly increasing and unbounded above with $c(0)=0$ implies that $c:\mathbb{R}_+ \to \mathbb{R}_+$ is well defined $c^{-1}(0)=0$, c^{-1} is strictly increasing, unbounded above and concave.

Let $U = \{(x,\alpha_1,...,\alpha_N) \in \mathbb{R}_+ \times \mathbb{R}^N_+ / \exists\ x_i \in \mathbb{R}_+ \forall\ i \in \{1,...,N\}$ and $y \geq 0$

such that $\Sigma^N_{i=1} x_i = x$ and $u_i(c^{-1}(\alpha_i), x_i) > u_i(\bar{y},\bar{x}_i)\ \forall\ i\}$.

It is easily verified that U is convex, nonempty and $(\Sigma^N_{i=1}$

$\bar{x}_i, c(\bar{y}),...,c(\bar{y})) \notin U$.

Thus by the separating hyperplane theorem there exists $(\Gamma,\mu_1,...,\mu_N) \in \mathbb{R}^{N+1}_+ \setminus \{0\}$ such that

$$\Gamma x + \Sigma^N_{i=1}\alpha_i \mu_i > \Gamma\bar{x} + (\Sigma^N_{i=1} \alpha_i)c(\bar{y})\ \forall (x,\alpha_1,...,\alpha_N) \in U.$$

By the strict monotonicity of the u_i's it follows that $\Gamma>0$ and $\mu_i > 0\ \forall\ i \in \{1,...,N\}$.

Let $b_i = \mu_i / \Gamma > 0$, $i \in \{1,...,N\}$.

$$\therefore\ x + \Sigma^N_{i=1} b_i \alpha_i > \bar{x} + (\Sigma^N_{i=1} b_i)c(\bar{y})\ \forall (x,\alpha_1,...,\alpha_N) \in U.$$

Suppose $u_i(x_i, y) > u_i(\bar{x}_i, \bar{y})$ and $x_i + b_i c(y) < \bar{x}_i + b_i c(\bar{y})$.

Let $x_j = x_j + \frac{1}{N-1}[x_i + b_i c(\bar{y}) - x_i - b_i c(y)]$.

Let $\alpha_i = c(y)$ and $\alpha_j = c(\bar{y}) \forall j \neq i$.

$\therefore u_j(x_j, c^{-1}(\alpha_j)) > u_j(\bar{x}_j, \bar{y}) \forall j \neq i$.

$u_i(x_i, c^{-1}(\alpha^i)) > u_i(\bar{x}_i, \bar{y})$.

Further,

$\Sigma^N_{j=1} x_j + \Sigma^N_{j=1} \alpha_j b_j = x_i + b_i c(y) + \Sigma_{j \neq i} x_j$
$+ (\Sigma_{j \neq i} b_j) c(\bar{y})$

$= x_i + b_i c(y) + \Sigma_{j \neq i} \bar{x}_j + x_i$
$+ b_i c(\bar{y})$

$- x_i - b_i c(y)$
$+ (\Sigma_{j \neq i} b_j) c(\bar{y})$
$= \Sigma^N_{j=1} \bar{x}_j + (\Sigma^N_{j=1} b_j) c(\bar{y})$,

which is a contradiction.

Thus $u_i(x_i, y) > u_i(\bar{x}_i, \bar{y}) \Rightarrow x_i + b_i c(y) \geq x_i + b_i c(\bar{y})$.

Suppose $x_i + b_i c(y) = \bar{x}_i + b_i c(\bar{y})$. By continuity and strict monotonicity of u, since $\bar{y} > 0$ there exists $\epsilon > 0$ such that either $u_i(x_i - \epsilon, y) > u_i(\bar{x}_i, \bar{y})$ or $u_i(x_i, y - \epsilon) > u_i(\bar{x}_i, \bar{y})$; however $(x_i - \epsilon) + b_i c(y) < x_i + b_i c(y) = \bar{x}_i + b_i c(\bar{y})$ and $x_i + b_i c(y - \epsilon) < \bar{x}_i + b_i c(\bar{y}) = x_i + b_i c(\bar{y})$ contradicting what we obtained above. Thus

$u_i(x_i, y) > u_i(\bar{x}_i, \bar{y}) \Rightarrow x_i + b_i c(y) > \bar{x}_i + b_i c(\bar{y})$.

Now let $a_i = w_i - \bar{x}_i - b_i c(\bar{y})$, and define

$g_i(y) = a_i + b_i c(y)$.

Then $\max\limits_{\substack{x_i \in \mathbb{R}_+ \\ y \in \mathbb{R}_+}} u_i(x_i, y)$

s.t. $g_i(y) + x_i \leq w_i$

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$$\Leftrightarrow \max_{\substack{x_i \in \mathbb{R}_+ \\ y \in \mathbb{R}_+}} u_i(x_i, y)$$

$$\text{s.t. } [w_i - \bar{x}_i - b_i c(\bar{y})] + b_i c(y) + x_i \leq w_i$$

$$\Leftrightarrow \max_{\substack{x_i \in \mathbb{R}_+ \\ y \in \mathbb{R}_+}} u_i(x_i, y)$$

$$\text{s.t. } x_i + b_i c(y) \leq \bar{x}_i + b_i c(\bar{y})$$

which completes the proof.

Q.E.D.

4. Linear Cost Functions and Ratio Equilibria With Redistribution:-

In this section we obtain a result stronger than proposition 2, when the cost function is of the form $c(y)=c.y \; \forall y \geq 0$, and a mild regularity condition is satisfied.

Proposition 3 :- Assume that $c(y)=c.y \; \forall y \geq 0$ where c is a constant greater than zero. Suppose $(b_1, \ldots, b_N)$ is uniquely defined in proposition 2, where $(\bar{y}, \bar{x})$ is an optimal state. Then $(\bar{y}, \bar{x})$ is a ratio equilibrium allocation with redistribution.

Proof :- Let $x + cy \leq \Sigma^N_{i=1} \bar{x}_i + c\bar{y}$ and $x + (\Sigma^N_{i=1} b_i) cy > \Sigma^N_{i=1} \bar{x}_i + (\Sigma^N_{i=1} b_i) c\bar{y}$.

Then since U is strictly convex and $(b_1, \ldots, b_N)$ as obtained in proposition 2 is unique, there exists $t \in (0,1)$ such that $(tx + (1-t)\Sigma^N_{i=1} \bar{x}_i , tcy + (1-t)c\bar{y}, \ldots, tcy + (1-t)c\bar{y}) \in U$. However, $[tx + (1-t)\Sigma^N_{i=1} \bar{x}_i] + c[ty + (1-t)\bar{y}] < \Sigma^N_{i=1} \bar{x}_i + c\bar{y}$, contradicting the optimality of $(\bar{y}, \bar{x})$. Thus $x + (\Sigma^N_{i=1} b_i) cy \leq \Sigma^N_{i=1} \bar{x}_i + (\Sigma^N_{i=1} b_i) c\bar{y}$. This establishes that $\Sigma^N_{i=1} b_i = 1$ and the proposition.

Q.E.D.

In an appendix to this paper, we provide a new proof of the existence of a ratio equilibrium when for simplicity, the utility functions are assumed to be strictly quasi-concave. The proof of the existence of a ratio equilibrium when utility functions are semi-strictly quasi-concave is only a slight modification of the one that we have provided.

Appendix : On the Existence of a ratio equilibrium

Assumption 1 :- $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ is strictly increasing, continuous and strictly quasi-concave $\forall i \in \{1,...,N\}$.

Assumption 2 :- $w_i > 0$ $\forall i \in \{1,...,N\}$.

Assumption 3 :- $c : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing and convex.

For $b_i \geq 0$, define $f_i(b_i)$ as follows:

$f_i(b_i)$ solves $\max u_i(y, w_i - b_i c(y)$

s.t. $\Sigma^N_{i=1} w_i + 1 \geq c(y) \geq 0$

$w_i - b_i c(y \geq 0$

Under our assumptions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function for each $i \in \{1,...,N\}$.

<u>Claim 1</u> :- If $c(f_i(b_i)) \leq \Sigma^N_{i=1} w_i$, then $f_i(b_i)$ solves

$\max u_i(y, w_i - b_i c(y))$

s.t. $y \geq 0$

$w_i - b_i c(y) \geq 0$

<u>Proof</u> :- Let $w_i - b_i c(f_i(b_i)) = x_i$ and suppose

$\arg\max_{\substack{y \geq 0 \\ w_i - b_i c(y) \geq 0}} u_i(y, w_i - b_i c(y)) = y' \neq f_i(b_i)$. Let $x'_i = w_i - b_i c(y')$

$\therefore c(y') > \Sigma^N_{i=1} w_i + 1)$.

Observe, $x_i + b_i c(f_i(b_i)) = w_i$ and $x'_i + b_i c(y')$. Thus $w_i = [tx_i + (1-t)x'_i] + b_i[tc_i(f_i(b_i) + (1-t)c(y')] \geq [tx_i + (1-t)x'_i] + b_i c(tf_i(b_i) + (1-t)y')$ $\forall t \in (0,1)$ by the convexity of c and the non-negativity of b_i. Further, $u_i(tx_i + (1-t)x'_i, tf_i(b_i)$

$+(1-t)y'))>u_i(f_i(b_i),x_i)\ \forall\ t\in(0,1)$ by the strict quasi-concavity of u_i and there exists $t\in(0,1)$ such that $c(tf_i(b_i)+(1-t)y')\leq\Sigma^N_{i=1} w_i+1$ by the continuity of c, and thus contradicting the definition of $f_i(b_i)$.

Q.E.D.

Let $\Delta=\{(b_1,\ldots,b_N)\in\mathbb{R}^N_+/\Sigma^N_{i=1} b_i=1\}$ and define the function $\bar{f}:\Delta\to\mathbb{R}$ as follows:

$$\bar{f}(b_1,\ldots,b_N)=\Sigma^N_{i=1} b_i f_i(b_i).$$

Observe that $\bar{f}$ is a continuous function.

Now define N functions $(\hat{h}_1,\ldots,\hat{h}_N)\equiv\hat{h},\ \hat{h}_i:\Delta\to\mathbb{R}\ \forall i\in\{1,\ldots,N\}$ as follows:

$$\hat{h}_i(b)=\frac{b_i+\max(0,f_i(b_i)-\bar{f}(b))}{1+\Sigma^N_{j=1}\max(0,f_i(b_i)-\bar{f}(b))}$$

Observe that $\hat{h}(b)\in\Delta\ \forall\ b\in\Delta$ and $h:\Delta\to\Delta$ is continuous. Thus by Brouwer's Fixed Point Theorem, there exists $\bar{b}\in\Delta$ such that $\hat{h}(\bar{b})=\bar{b}$.

Thus,

$$\bar{b}_i=\frac{\bar{b}_i+\max(0,f_i(\bar{b}_i)-\bar{f}(\bar{b}))}{1+\Sigma^N_{j=1}\max(0,f_j(\bar{b}_j)-\bar{f}(\bar{b}))}$$

$$\therefore \bar{b}_i\ \Sigma^N_{j=1}\max(0,f_j(\bar{b}_j)-\bar{f}(\bar{b}))=\max(0,f_i(\bar{b}_i)-\bar{f}(\bar{b}))\quad \forall i\in\{1,\ldots,N\}.$$

$$\therefore \bar{b}_i(f_i(\bar{b}_i)-\bar{f}(\bar{b}))\Sigma^N_{j=1}\max(0,f_j(\bar{b}_j)-\bar{f}(\bar{b}))=(f_i(\bar{b}_i)-\bar{f}(\bar{b}))\max(0,f_i(\bar{b}_i)-\bar{f}(\bar{b}))$$

Summing over $i\in\{1,\ldots,N\}$ we get

$$0=\Sigma^N_{i=1}(f_i(\bar{b}_i)-\bar{f}(\bar{b}))\max(0,f_i(\bar{b}_i)-\bar{f}(\bar{b}))$$

$$\therefore f_i(\bar{b}_i)\leq\bar{f}(\bar{b})\quad\forall i\in\{1,\ldots,N\}.$$

Since $\bar{f}(\bar{b})=\Sigma^N_{i=1}\bar{b}_i f_i(\bar{b}_i)$ with $\bar{b}\in\Delta$, we get

$$f_i(\bar{b}_i)=\bar{f}(\bar{b})\quad\forall i\in\{1,\ldots,N\}.$$

Since $b_i\, c(\bar{f}(\bar{b}) = b_i\, c(f_i\ (\bar{b}_i\ )) \leqq w_i\ \forall i \in \{1,\ldots,N\}$, we get $c(\bar{f}(\bar{b})) = c(f_i\ (\bar{b}_i\ )) \leqq \Sigma^N_{i=1}\ w_i$.

Thus by claim 1, we have the following theorem:

<u>Theorem 1</u> :- Under the above assumptions a ratio equilibrium exists for the public good economy.

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