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Alexander Lytchak, Viktor Schroeder

Institutions: University of Bonn, University of Zurich

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Lytchak, A ; Schroeder, Viktor

Abstract: Affine functions on CAT(kappa) spaces

AFFINE FUNCTIONS ON $CAT(\kappa)$-SPACES

ALEXANDER LYTCHAK & VIKTOR SCHROEDER

1. INTRODUCTION

This paper is devoted to the structure of singular metric spaces admitting affine functions. Since we are dealing with quite general spaces it is reasonable to require the maps to be Lipschitz.

Definition 1.1. *A Lipschitz map $f : X \to Y$ between geodesic metric spaces is called affine, if it maps each geodesic to a geodesic parametrized proportional to the arclength. In the case $Y = \mathbb{R}$ we call f an affine function.*

The easiest example of an affine map occurs in the situation that X splits as $X' \times Y$ and f is the projection $p : X' \times Y \to Y$. In the case $Y = \mathbb{R}$ we get affine functions. In [AB] situations are studied, in which the existence of an affine function $f : X \to \mathbb{R}$ already implies the existence of a Euclidean de Rham factor. To obtain these results one has to assume that the space is geodesically complete. Without this assumption one cannot expect the existence of a splitting. The best one can hope for is the existence of an isometric embedding of X into a product with a flat factor. Indeed our main result is

Theorem 1.1. *Let X be a $CAT(\kappa)$ space. Then there is a canonically defined isometric embedding $i : X \to Y \times H$, where Y is a geodesic metric space and H is a Hilbert space. Every affine function $f : X \to \mathbb{R}$ factors as $f = \hat{f} \circ p_H \circ i$ where p_H is the projection onto H and $\hat{f} : H \to \mathbb{R}$ is an affine function. Moreover each isometry of X determines an isometry of Y and of H. Finally the completion of Y is $CAT(0)$ if X is $CAT(0)$.*

Remark 1.1. In the case that X is a Hadamard space and the affine function is a Busemann function this result was shown in [AdB]. This was the motivation for our work.

Remark 1.2. If we assume (instead of the $CAT(\kappa)$ condition) that X is an Alexandrov space with lower curvature bound and finite dimension (maybe with nonempty boundary) a corresponding theorem can be proved in essentially the same way. This generalizes results of Alexander and Bishop [AB].

Without a curvature assumption a corresponding result is no longer true:

Example 1.3. Let X and Y be geodesic spaces, $|| \cdot ||$ a strongly convex norm on a two dimensional vector space. Let $Z = X \times_{||\cdot||} Y$ be the nonstandard metric product in the sense of [BFS]. Then the projections $p : Z \to X$ and $Z \to Y$ are affine. In particular if Y is a strongly convex Banach space one gets many affine functions on Z. Moreover convex subsets of non-standard products admit affine functions. Such a space does not necessarily admit a non-trivial isometric embedding into a space with a direct Euclidean factor.

The next example describes a more complicated space with a non-trivial affine function which doesnot admit an embedding even into a nonstandard product.

Example 1.4. Let B_1 and B_2 be two Banach spaces with strongly convex and smooth norms. Let v_i be a unit vector in B_i. Denote by γ_i the line $\gamma_i(t) = tv_i$ and by $f_i : B_i \to \mathbb{R}$ the Busemann function of γ_i. By identifying γ_1 and γ_2 we glue B_1 and B_2 to a space X. Observe now that the function $f : X \to \mathbb{R}$ that arises from f_1 and f_2 is affine.

For general metric spaces it is not clear which implications the existence of an affine function has. Under the additional assumption that the affine functions separate the points in X one can prove that X is isometric to a convex subset of a Banach space.

All the proofs in [In],[Ma1],[Ma2] and [AB] have in common that the non-Euclidean factor can be recognized as a convex subset of X. Our proof is quite different and the outline of the argument is as follows: Let $\mathcal{A}$ be the space of affine functions on X modulo the constant functions. If X is $CAT(\kappa)$, then $\mathcal{A}$ and its dual space $H := \mathcal{A}^*$ are Hilbert spaces (section 4). There is a naturally defined *evaluation map* $F : X \to H$. In section 5 we prove that the function $\tilde{d} : X \times X \to [0, \infty)$, $\tilde{d}(y, z) = \sqrt{d(y, z)^2 - ||F(y) - F(z)||^2}$ defines a pseudometric on X. Let $Y = X/\tilde{d}$ be the corresponding metric space. We finally show that $i : X \to Y \times H$, $x \mapsto ([x], F(x))$ satisfies the properties of Theorem 1.1.

Remark 1.5. We note that in general the factor Y cannot be embedded isometrically into X. This makes it difficult to obtain geometric properties of Y. We do not know, if the $CAT(\kappa)$ property of X implies $CAT(\kappa)$ for Y. In the special case $\kappa = 0$ we can however prove this.

2. Preliminaries

By d we will denote the distance in metric spaces without an extra reference to the space. A *pseudo metric* is a metric for which the distance between different points may be zero. It defines a unique metric space.

A *geodesic* in a metric space is a length minimizing curve parametrized propotionally to arclength. A metric space is *geodesic* if all pair of points are connected by a geodesic. A subspace of a geodesic space is *convex* if it is geodesic with respect to the induced metric. A $CAT(\kappa)$ space is a complete geodesic metric space in which triangles are not thicker than in the space of constant curvature κ. We refer to [BH] for more detailed discussion of these spaces.

A map $f : X \to Y$ is called L-*Lipschitz* if $d(f(x), f(z)) \leq Ld(x, z)$. The smallest L is called the optimal Lipschitz constant. For a Lipschitz function $f : X \to R$ we denote by $|\nabla_x f|$ the *absolut gradient* at x which is given by $\sup\{0, \limsup_{z \to x} \frac{f(z)-f(x)}{d(x,z)}\}$. If the space X is geodesic, the optimal Lipschitz constant is the supremum of all absolut gradients.

Remark that a Lipschitz function $f : X \to \mathbb{R}$ is affine iff it is convex and concave, i.e. if its restriction to each geodesic is convex and concave. For a convex (in particular for an affine) function f the absolut gradient $|\nabla_x f|$ is semi-continuous in x (compare [P]).

3. Affine functions on general spaces

Let X be an arbitrary geodesic metric space. The set of all affine functions on X is a vector space and will be denoted by $\tilde{\mathcal{A}}(X)$. It always contains the one-dimensional subspace $Const(X)$ of constant functions. For each point $x \in X$ the space $\mathcal{A}_x$ of all affine functions vanishing at x is a complement of $Const$ in $\tilde{\mathcal{A}}(X)$. By $\mathcal{A}(X)$ or simply $\mathcal{A}$ we will denote the quotient vector space $\tilde{\mathcal{A}}(X)/Const(X)$. For an affine function $f : X \to \mathbb{R}$ we denote with $[f] \in \mathcal{A}$ the corresponding element of $\mathcal{A}$. The best Lipschitz constant defines a norm on the space $\mathcal{A}$. Equipped with this norm $\mathcal{A}$ is a normed vector space. It is complete (even if X is not complete), hence it is a Banach space.

Consider the evaluation map $E : X \times X \to \mathcal{A}^*$ from the product $X \times X$ to the dual space of $\mathcal{A}$ given by $E(x, y)([f]) = f(y) - f(x)$. We have

$$E(x, z)([f]) - E(\bar{x}, \bar{z})([f]) \leq ||f|| \; (\; d(x, \bar{x}) + d(z, \bar{z}) \;).$$

Moreover the map E is strongly affine in the sense that it maps geodesics to affine lines of the Banach space $\mathcal{A}^*$. Observe that $E(x, z) = 0$ iff the points x and z cannot be separated by an affine map on X.

By $E_x : X \to \mathcal{A}^*$ we denote the restriction $E_x(z)([f]) = f(z) - f(x)$. We have $E_y = E_x + E(x, y)$.

4. Affine functions on $CAT(\kappa)$ spaces

Let X be a $CAT(\kappa)$ space and $f : X \to \mathbb{R}$ affine. For $x \in X$ let $C_x = CS_x$ be the tangent cone at the point $x \in X$ which is the cone over the space of directions S_x. Then f induces a homogeneous affine function (the *directional derivative*) $D_x f : C_x \to \mathbb{R}$ (compare [K]). The absolute gradient $\mid \nabla_x f \mid$ is equal to $\sup\{D_x f(v) \mid v \in C_x, d(0, v) = 1\}$. The function $D_x f$ inherits the Lipschitz constant from f.

The following splitting result is basic:

Lemma 4.1. *Let X be a $CAT(0)$ space. Let $f : X \to \mathbb{R}$ be an affine function. Assume that for some line γ in X we have $(f \circ \gamma)' = ||f|| > 0$. Then X splits as $X = Z \times \mathbb{R}$ and f is given by $f(z, t) = ||f||t$.*

Proof. We may assume that $||f|| = 1$ and $f(\gamma(0)) = 0$. Let $x \in X$ be arbitrary. For the rays γ_x^+ and γ_x^- starting at x and asymptotic to γ^+ resp. γ^- we immediatly obtain $(f \circ \gamma_x^+)' = 1$ and $(f \circ \gamma_x^-)' = -1$. Therefore $|f(\gamma_x^+(1)) - f(\gamma_x^-(1))| = 2$. Since f is 1-Lipschitz we deduce that $d(\gamma_x^+(1), \gamma_x^-(1)) = 2$ and hence the concatenation of γ_x^+ and γ_x^- is a line γ_x which is parallel to γ. Therefore through each point $x \in X$ there is a line paralell to γ and we may apply the well known splitting theorem ([BH]). Now the last statement is clear too. $\square$

Proposition 4.2. *Let X be a $CAT(\kappa)$ space and $f : X \to \mathbb{R}$ an affine function. Assume that y is an inner point of a geodesic starting at x. Then $|\nabla_y f| \geq |\nabla_x f|$.*

Proof. Let X be a $CAT(\kappa)$ space. We may assume that $d(x, y) < \frac{\pi}{3\sqrt{\kappa}}$ and that for some point z we have $d(z, y) = d(x, y) = \frac{1}{2}d(x, z)$. Moreover we may assume $f(x) = 0$. Let $f(y) = r$. Let η be a geodesic starting at x with $a = (f \circ \eta)' > 0$. Consider the midpoint m_t of the geodesic between z and $\eta(t)$ for small t. We have $f(m_t) = \frac{2r + at}{2}$. On the other hand the $CAT(\kappa)$ assumption implies $d(y, m_t) \leq \frac{t}{2} + At^2$ for some fixed $A \geq 0$ depending only on κ. This implies

$$|\nabla_y f| \geq \frac{f(m_t) - f(y)}{d(m_t, y)} \geq \frac{\frac{at}{2}}{\frac{t}{2} + At^2} = \frac{a}{1 + 2At}$$

For $t \to 0$ we obtain $|\nabla_y f| \geq a$. Since η is arbitrary we have $|\nabla_y f| \geq |\nabla_x f|$. $\square$

We see that for each affine function f on X, the set X_ϵ of all points $x \in X$ such that $|\nabla_x f| > ||f|| - \epsilon$ is open, dense and convex in X. From the theorem of Baer we obtain:

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Corollary 4.3. *Let f_k be a sequence of affine function. Then the set X^0 of points x such that $|\nabla_x(-f_j)| = |\nabla_x f_j| = ||f_j||$ for all j is convex and dense in X.*

Now we can deduce

Corollary 4.4. *Let X be a $CAT(\kappa)$ space. Then $\mathcal{A}$ is a Hilbert space.*

Proof. Let f, g be two affine functions. We have to prove $||f + g||^2 + ||f - g||^2 = 2(||f||^2 + ||g||^2)$. By Corollary 4.3 there exists $x \in X$ sucht that $|\nabla_x h| = ||h||$ for $h = \pm f, \pm g, f + g, f - g$. For simplicity let $h' := D_x h : C_x \to \mathbb{R}$. Then the functions h' are homogeneous, affine and $||h'|| = ||h||$. Let $0 \in C_x$ be the cone point of $C_x = CS_x$, where S_x is the space of directions in x. Since $|| \pm f'|| = ||f||$ we have $|\nabla_0 f'| = ||f||$ and $|\nabla_0(-f')| = ||f||$. Thus there are $v^+, v^- \in S_x$ such that $f'(v^+) = -f'(v^-) = ||f'||$, hence $|f'(v^+) - f'(v^-)| = 2||f'||$. This implies $d(v^+, v^-) = 2$, where this distance is measured in the cone CS_x. Thus the concatenation of the two rays $\gamma^+(s) = sv^+$ and $\gamma^-(s) = sv^-$ for $s \in [0, \infty)$ is a line in the cone C_x, and $(f' \circ \gamma)' = ||f'||$ along this line. By Lemma 4.1 the $CAT(0)$ space C_x splits as $Z \times \mathbb{R}$ and $f'(z, t) = ||f'|| \cdot t$. In the same way C_x can be decomposed as $Z' \times \mathbb{R}$ such that $g(z', s) = ||g'|| \cdot s$. By the properties of the Euclidean de Rham factor of a $CAT(0)$ space (compare [BH] p. 235), C_x splits as $Z'' \times E$, where E is a one or twodimensional Euclidean space and $f' = \hat{f}' \circ p_E$, $g' = \hat{g}' \circ p_E$, where p_E is the projection onto E and $\hat{f}', \hat{g}'$ are affine functions on the Euclidean space. Thus the equation $||f' + g'||^2 + ||f' - g'||^2 = 2(||f'||^2 + ||g'||^2)$ and hence the corresponding equation for f, g holds. $\qquad\square$

We come back to the affine maps $E_x : X \to \mathcal{A}^*$ defined in section 3. In the case that $\mathcal{A}^*$ is a Hilbert space, these maps are normalized in the following sense.

Definition 4.5. *Let X be a geodesic metric space, H be a Hilbert space and $F : X \to H$ an affine map. We call F normalized, if F is 1-Lipschitz and for each unit vector $v \in H$ the affine function $F^v : X \to \mathbb{R}$ given by $F^v(x) = \langle F(x), v \rangle$ satisfies $||F^v|| = 1$.*

Example 4.1. Let $H_0 \subset H$ be a Hilbert subspace. Then the orthogonal projection $p : H \to H_0$ is normalized . If $F : X \to H$ is normalized, then so is the composition $p \circ F$.

Observe that if $F : X \to H$ is normalized, then the linear hull of the convex set $C = F(X)$ is dense in H. By the very definition the canonical evaluation maps $E_x : X \to \mathcal{A}^*$ are normalized.

Definition 4.6. *Let X be a $CAT(\kappa)$ space, H a Hilbert space and $F : X \to H$ a normalized affine map. We call a point $x \in X$ regular if C_x has the splitting $C_x = C'_x \times H_x$, with a Hilbert space H_x, such that $D_x F$ is the projection onto H_x.*

Corollary 4.7. *If H is separable, then the set of regular points is convex and dense in X.*

Proof. Let e_i, $i \in \mathbb{N}$ be a dense subset of the unit vectors in H and let $F_i = F^{e_i}$ be the corresponding affine functions. By Corollary 4.3 the set $W \subset X$ of points x such that $|\nabla_x(F_i)| = |\nabla_x(-F_i)| = ||F_i||$ for all $i \in \mathbb{N}$ is convex and dense. For $x \in W$ let C_x be the tangent cone which is a $CAT(0)$ space and splits an Euclidean de Rham factor $C_x = C''_x \times H'_x$. By the proof of Corollary 4.4 the homogeneous affine function $D_x F_i : C_x \to \mathbb{R}$ has the form $D_x F_i(z'', h') = \langle v_i, h' \rangle$, where $v_i \in H'_x$ is a unit vector. Let $H_x \subset H'_x$ be the closure of the span of the v_i and $C_x = C'_x \times H_x$ be the corresponding splitting where $C'_x = C''_x \times H_x^{\perp}$. By construction $D_x F$ is the projection onto H_x. $\square$

5. PROOF OF THEOREM 1.1

The proof of the main theorem is based on the following fact

Theorem 5.1. *Let X be a $CAT(\kappa)$ space, H be a Hilbert space and $F : X \to H$ be a normalized affine map. Then $\tilde{d} : X \times X \to [0, \infty)$ given by $\tilde{d}(y, z) = \sqrt{d(y, z)^2 - ||F(y) - F(z)||^2}$ defines a pseudo metric on X.*

Proof. By definition $\tilde{d}$ is symmetric and since F is 1-Lipschitz, $\tilde{d}$ is nonnegative.

Since F is affine, we have for each point m on a geodesic yz that

$$(5.1) \qquad \tilde{d}(y, z) = \tilde{d}(y, m) + \tilde{d}(m, z).$$

We will prove that $\tilde{d}$ satisfies the triangle inequality and therefore defines a pseudometric. We will first show that the triangle inequality is satisfied in the neighborhood of every point. Consider therefore three points $x, y, z \in X$ with pairwise distance $< \frac{\pi}{2\sqrt{\kappa}}$ and assume that $\tilde{d}(x, z) > \tilde{d}(x, y) + \tilde{d}(y, z)$. We may assume that $F(x) = 0$. Denote by H_0 the linear hull of $F(y)$ and $F(z)$ in H. Replacing F by the composition $p \circ F$, where $p : H \to H_0$ is the orthogonal projection, we may assume that $H = H_0$ and the Hilbert space is at most 2-dimensional. In particular the set of regular points is dense in X by Corollary 4.3. Hence we can assume that x, y and z are regular points.

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In particular $C_y = C'_y \times H_y$ where H_y is a Euclidean space of dimension ≤ 2 such that $D_y F$ is the projection onto H_y.

Assume for a moment that all $\tilde{z}$ near z satisfy the equality $\tilde{d}(y, \tilde{z}) = 0$. Then $||F(\tilde{z}) - F(y)|| = d(\tilde{z}, y)$ and since F is 1-Lipschitz this implies that all the initial vector of the geodesic $y\tilde{z}$ lie in the H_y factor of C_y. It follows that C'_y is trivial and F is an isometric embedding. We are done in this case.

Hence replacing z by a nearby point we may assume that $\tilde{d}(y, z) > 0$. Set $\rho = \frac{d(y,z)}{\tilde{d}(y,z)}$.

Let $\gamma : [0, d(y, z)] \to X$ be a unit speed geodesic between y and z. Consider the function h given by $h(t) = \tilde{d}(x, y) + \tilde{d}(y, \gamma(t)) - \tilde{d}(x, \gamma(t))$. We have $h(0) = 0$, $h(d(y, z)) < 0$ and by equation (5.1) for $s > t$ the equality

$$h(s) - h(t) = \tilde{d}(x, \gamma(t)) + \tilde{d}(\gamma(t), \gamma(s)) - \tilde{d}(x, \gamma(s))$$

holds.

The function h is Lipschitz (hence differentiable almost everywhere) and satisfies $h(t_0) < 0$ for some t_0, hence we find some $\varepsilon > 0$, $t \in [0, d(y, z)]$ such that $h'(t) = -3\varepsilon$. Replacing y by $\gamma(t)$ we may assume $t = 0$. For a very small number $r << \epsilon$ set $z = \gamma(r)$. We then have $h(r) \leq -2\varepsilon r$. Because of equation (5.1) we still have $\rho = \frac{d(y,z)}{\tilde{d}(y,z)}$ and ρ does not depend on ε and r.

Let $\eta_0 : [0, s_0] \to X$ resp. $\eta_1 : [0, s_1] \to X$ be geodesics from x to y resp. to z. For $0 < t \leq 1$ set $y_t = \eta_0(ts_0)$ and $z_t = \eta_1(ts_1)$.

We have $\tilde{d}(x, z_t) = t\tilde{d}(x, z)$; $\tilde{d}(x, y_t) = t\tilde{d}(x, y)$. Moreover $||F(x) - F(z_t)|| = t||F(z)||$; $||F(x) - F(y_t)|| = t||F(y)||$ and $||F(z_t) - F(y_t)|| = t||F(z) - F(y)||$.

Since X is a $CAT(\kappa)$ space and the pairwise distances of the points x, y, z are by assumption $< \frac{\pi}{2\sqrt{\kappa}}$, there exists $A \geq 0$ depending only on κ such that

$$d(y_t, z_t) \leq t(d(y, z) + A \, d(y, z)^2)$$

We compute

$$\begin{aligned}
\tilde{d}(y_t, z_t) &= \sqrt{d(y_t, z_t)^2 - t^2||F(y) - F(z)||^2} \\
&\leq \sqrt{t^2(d(y, z) + A \, d(y, z)^2)^2 - t^2||F(y) - F(z)||^2} \\
&\leq t\sqrt{\tilde{d}(y, z)^2 + Bd(y, z)^3} \\
&= t\tilde{d}(y, z)\sqrt{(1 + B\rho^2 d(y, z))} \\
&\leq t\tilde{d}(y, z)(1 + Cd(y, z))
\end{aligned}$$

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for some constant B depending only on κ and some constant C depending only on ρ and the curvature bound κ. If $r = d(y, z)$ has been choosen small enough we thus obtain

$$(5.2) \qquad \tilde{d}(y_t, z_t) \leq t\tilde{d}(y, z) + t\varepsilon\tilde{d}(y, z)$$

Since $h(r) \leq -2\varepsilon r$, $r = d(y, z)$ we obtain

$$(5.3) \qquad \tilde{d}(x, y) + \tilde{d}(y, z) - \tilde{d}(x, z) \leq -2\varepsilon d(y, z) \leq -2\varepsilon\tilde{d}(y, z)$$

It follows that

$$\tilde{d}(x, z_t) = t\tilde{d}(x, z)$$
$$\geq t(\tilde{d}(x, y) + \tilde{d}(y, z) + 2\varepsilon\tilde{d}(y, z))$$
$$\geq \tilde{d}(x, y_t) + \tilde{d}(y_t, z_t) + t\varepsilon\tilde{d}(y, z)$$

where we used equation (5.3) for the first and equation (5.2) for the second inequality. Going to the limit $t \to 0$ we see that for the affine function $D_x F : C_x \to H_x$ the correponding function $\tilde{d}_x : C_x \times C_x \to \mathbb{R}$, $\tilde{d}_x(v, w) = \sqrt{d_x^2(v, w) - ||D_x F(v) - D_x F(w)||^2}$ is not a pseudo metric. But $D_x F : C_x \to H_x$ is just the projection onto the Euclidean factor of $C_x = C'_x \times H_x$ (since x is regular). Hence $\tilde{d}_x$ is just the metric on C'_x. Contradiction.

This contradiction shows that $\tilde{d}$ satisfies the triangle inequality in the neighborhood of each point. Using the $CAT(\kappa)$ property of X and equation (5.1) it is not difficult to prove that the triangle inequality holds for all triples of points.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$
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Let $F : X \to H$ as in the assumption of Theorem 5.1, then $\tilde{d}$ defines a pseudometric on X. Let $Y = X/\tilde{d}$ be the induced metric space. A point in Y is an equivalence class $[x]$ where $x \sim x'$ iff $\tilde{d}(x, x') = 0$. Theorem 5.1 implies immediately that the map $X \to Y \times H$, $x \mapsto ([x], F(x))$ is an isometric embedding.

For the proof of Theorem 1.1 we use the affine map $F = E_o : X \to \mathcal{A}^*$, where E_o is the evaluation map for some basepoint $o \in X$. By the discussion of section 3 and section 4, the assumptions of Theorem 5.1 are satisfied. Hence $i : X \to Y \times \mathcal{A}^*$, $x \mapsto ([x], E_o(x))$ is an isometric embedding.

If $f \in \tilde{\mathcal{A}}(X)$ is an affine function on X, then define $\hat{f} : \mathcal{A}^* \to \mathbb{R}$ by

$$\hat{f}(\xi) := \xi([f]) + f(o)$$

Then $\hat{f}$ is an affine function on $\mathcal{A}^*$ and

$$\hat{f}(E_o(x)) = E_o(x)([f]) + f(o) = f(x) - f(o) + f(o) = f(x)$$

hence $\hat{f} \circ p_{\mathcal{A}^*} \circ i = f$ as required.

We show now that Y is a geodesic metric space. Indeed if $[y], [z] \in Y$ and $\gamma : [0, 1] \to X$ is a geodesic from x to y, then $t \mapsto [\gamma(t)]$ is a geodesic in Y due to equation (5.1).

We finally prove that the completion of Y is $CAT(0)$ if X is $CAT(0)$. Let therefore $[x], [y], [z] \in Y$ be arbitrary and let $[m] = [\gamma(\frac{1}{2})]$ be a midpoint of $[y]$ and $[z]$. We have to prove the Bruhat-Tits $CAT(0)$ inequality (see e.g. [BH] p.163):

$$(5.4) \qquad \tilde{d}^2([x], [m]) \leq \frac{1}{2}\tilde{d}^2([x], [y]) + \frac{1}{2}\tilde{d}^2([x], [z]) - \frac{1}{4}\tilde{d}^2([y], [z])$$

Since X is $CAT(0)$ we have

$$d^2(x, m) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z)$$

and since F is affine we see

$$||F(x) - F(m)||^2 = \frac{1}{2}||F(x) - F(y)||^2 + \frac{1}{2}||F(x) - F(z)||^2 - \frac{1}{4}||F(y) - F(z)||^2.$$

Subtracting the two formulas we obtain inequality (5.4).

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Alexander Lytchak,
Mathematisches Institut,
Universität Bonn,
Beringstrasse 1,
D-53115 Bonn, Germany
lytschak@math.uni-bonn.de

Viktor Schroeder,
Institut für Mathematik,
Universität Zürich,
Winterthurer Strasse 190,
CH-8057 Zürich, Switzerland
vschroed@math.unizh.ch