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AFFINE GEOMETRIC PROOFS OF THE BANACH STONE THEOREMS OF KADISON AND KAUP

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1. Introduction and preliminaries. In 1951, Kadison [**14**] proved the following non-commutative extension of the Banach Stone Theorem, thereby showing that the geometry of a C^*-algebra determines some aspects of its algebraic structure.

THEOREM A. *Let T be a surjective linear isometry of a unital C^*-algebra A onto a unital C^*-algebra B. Then there is a unitary element u in B and a Jordan $*$-isomorphism ρ of A onto B such that*

$$Tx = u\rho(x), \qquad x \in A. \tag{1.1}$$

Recall that the proof of the Banach Stone theorem, i.e., the special case of Theorem A in which A and B are abelian, say $A = C(X)$ and $B = C(Y)$, uses duality and the intimate relation between the topological space X (respectively Y) and the algebra $C(X)$ (respectively $C(Y)$).

This approach was abandoned by Kadison because "the sparseness of knowledge concerning the pure states of an operator algebra makes this procedure seem difficult" [**14**, p. 326]. Instead he gives an intrinsic proof, depending mainly on spectral theory and the geometry of the underlying Hilbert spaces on which A and B act. Kadison points out that ρ preserves the quantum mechanical structure of the C^*-algebras, i.e., the linear structure and the power structure of self-adjoint elements. It follows, and this is significant for the viewpoint expressed in this paper, that T, given by (1.1), preserves powers of the form $aa^*a, aa^*aa^*a, \ldots$, and hence, by polarization, that T preserves the triple product $ab^*c + cb^*a$.

By using some basic results on operator algebras which appeared in the decade after Kadison's theorem, we present here a proof of Theorem A which is similar in spirit to the approach of Banach and Stone. Namely, we use affine geometric properties of the convex set of states, i.e., instead of pure states or extreme points, we consider faces, or extremal subsets of the state space.

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This approach has the advantage of being independent of the order structure. As such, it can be applied to a category of algebraic objects (the JB^*-triples) which includes C^*-algebras and Jordan C^*-algebras, is determined by geometric and holomorphic properties, and usually does not have a global order structure. In this way we are able to also give an elementary proof of Kaup's generalization to JB^*-triples of Kadison's theorem. Before introducing the category of JB^*-triples we recall two other analogues of Theorem A.

A significant generalization of Theorem A was obtained in 1973 by Harris [**12**], by the use of holomorphic methods. A norm closed subspace of $B(H,K)$ which is closed under the triple product $(x,y,z) \to xy^*z + zy^*x$ is called a J^*-*algebra*. This is now recognized as a misnomer since J^*-algebras are not in general closed under a binary product or an involution (J^*-algebras are now referred to as JC^*-triples). Equivalently, J^*-algebras are precisely the norm closed subspaces of C^*-algebras which are closed for the operation $x \to xx^*x$.

By introducing the analog of Möbius transformations, first studied by Potapov [**17**], Harris [**12**] showed that the open unit ball of a J^*-algebra is a bounded symmetric domain. As an application of his techniques, he obtained the following generalization of Theorem A.

THEOREM B. *Let T be a surjective linear isometry of a J^*-algebra A onto a J^*-algebra B. Then T is a J^*-isomorphism, i.e.,*

$$T(xy^*z + zy^*x) = Tx(Ty)^*Tz + Tz(Ty)^*Tx, \qquad x, y, z \in A. \tag{1.2}$$

The study of Jordan operator algebras, pioneered by Størmer [**20, 21**], and Topping [**23**], reached a certain level of maturity with the Gelfand-Naimark Theorem of Alfsen-Schultz-Størmer [**1**], cf [**11**].

The conclusion of Theorem A suggested a Banach Stone Theorem for Jordan C^*-algebras (also called JB^*-algebras), and such a Theorem was proved in 1978 by Wright and Youngson [**26**]:

THEOREM C. *Let T be a unital surjective linear isometry of a unital Jordan C^*-algebra A onto a unital Jordan C^*-algebra B. Then T is a Jordan $*$-isomorphism, i.e.,*

$$T(a \circ b) = Ta \circ Tb, \qquad a, b \in A,$$

and

$$T(a^*) = (Ta)^*, \qquad a \in A.$$

The proof of Theorem C leans heavily on the extensive analysis of JB-algebras (= the self adjoint parts of Jordan C^*-algebras) in [**1**] (cf. [**11**]), and the assumption $T1 = 1$, which allows a reduction to JB-algebras. The key step is to show that (orthogonal) projections in A are mapped by T into (orthogonal) projections in B. This idea will play a similar role in our proofs of Theorems D and E below, with projection replaced by tripotent and partial isometry, respectively.

A common generalization of each of the algebraic-topological structures in Theorems A, B, and C is a JB^*-triple. This class arises naturally in the study of bounded symmetric domains in complex Banach spaces (Kaup [**16**]) and appears as the range of an arbitrary contractive projection on a C^*-algebra (Friedman-Russo [**8**]). Thus Theorem D below, due to Kaup [**16**], includes the previous three theorems as particular cases. Kaup's proof of Theorem D uses the deep connection between JB^*-triples and bounded symmetric domains and therefore depends strongly on the elaborate machinery of infinite dimensional holomorphy, cf. [**15, 24**].

Let's recall that a JB^*-*triple* is a complex Banach space U endowed with a continuous sesqui-linear map $D : U \times U \to B(U)$ such that, for $x \in U, D(x,x)$ is Hermitian positive, $||D(x,x)|| = ||x||^2$, and, setting $\{xyz\} := D(x,y)z$, one has

$$\{xyz\} = \{zyx\}$$

and

$$\{xy\{uvz\}\} + \{u\{yxv\}z\} = \{\{xyu\}vz\} + \{uv\{xyz\}\}.$$

For example, a C^*-algebra is a JB^*-triple with $\{xyz\} = 1/2(xy^*z + zy^*x)$, and a Jordan C^*-algebra is a JB^*-triple with $\{xyz\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (z \circ x) \circ y^*$.

THEOREM D. *Let T be a surjective linear isometry of a JB^*-triple U onto a JB^*-triple V. Then T is a JB^*-triple isomorphism, i.e.,*

$$T\{xyz\} = \{Tx, Ty, Tz\}, \qquad \text{for } x, y, z \in U. \tag{1.3}$$

The purpose of this note is to give a proof of Theorem D which is elementary in the sense that it uses only simple affine geometric properties of the dual unit ball of a JB^*-triple, together with analogs of standard operator algebraic theoretical tools (spectral, polar, and Jordan decompositions; biduals, and a Theorem of Effros). As a consequence we also obtain new proofs of Theorems A, B and C.

In order to make our exposition as simple and self contained as possible, and to foster a better understanding between operator algebraists and complex analysts concerning the interplay between their respective fields, we will first prove (in §2) the version of Theorem D in which U and V are C^*-algebras (see Theorem E below). At the end of this paper (in §3), we shall indicate, mostly by reference to the recent literature on JB^*-triples, how the proof given here for C^*-algebras can be modified to obtain a proof of Kaup's Theorem D.

Thus our first goal is to give a geometric proof of

THEOREM E. *Let T be a surjective linear isometry of a C^*-algebra A onto a C^*-algebra B. Then*

$$T(xy^*z + zy^*x) = Tx(Ty)^*Tz + Tz(Ty)^*Tx, \qquad x, y, z \in A. \tag{1.4}$$

A sketch of our proof will emphasize its geometric character and simplicity. Since T is an isometry, its adjoint T^* is an affine isometry of the unit ball of the dual of B onto the unit ball of the dual of A. As such, T^* maps faces (= extremal convex subsets) into faces. Since orthogonality of functionals can be expressed in terms of a norm condition, T^* preserves orthogonality of faces as well.

Motivated by the one to one correspondence between projections in a von Neumann algebra and certain faces in its normal state space (cf. [**5, 18**]), we show that there is a similar correspondence between partial isometries in a von Neumann algebra and certain faces in the unit ball of its predual. The fact that a von Neumann algebra is generated by its partial isometries (via the polar and spectral decompositions) then shows that T has the required algebraic property (1.4).

Returning to the general case, any JB^*-triple satisfies the "C^*-norm condition" $||\{xxx\}|| = ||x||^3$. It follows easily from this that the converse of Theorem D holds. Therefore Theorem D has the following consequence.

Corollary. *Let δ be a bounded linear operator on a JB^*-triple. Then δ is a triple derivation if and only if δ is skew-hermitian.*

By a *triple derivation* on a JB^*-triple U we mean a linear mapping $\delta : U \to U$ satisfying

$$\delta\{abc\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\},$$

for $a, b, c \in U$. An operator H on any Banach space X is *skew-hermitian* if $\exp tH$ is an isometry of X for all real t. The corollary follows since, by a standard argument, δ is a bounded triple derivation if and only if $\exp t\delta$ is an automorphism of the triple structure, for all real t.

The importance of this corollary stems from the following quantum mechanical considerations. The time evolution operator of a quantum mechanical system is given by a one parameter group of isometries of the state space of the system. The adjoint of this flow, acting on the space of observables, is therefore infinitesimally generated, in general, by a skew-hermitian operator. If the space of observables is represented by the self-adjoint elements of a C^*-algebra (cf. Bratteli-Robinson [**3**]), or even by the elements of a JB-algebra (cf. Segal [**19**]), it has been customary to consider only the special case in which the above generator is a derivation of the *binary* structure, which is not the most general skew-hermitian operator.

On the other hand, as the corollary shows, one need consider only (triple) derivations as infinitesimal generators of the time evolution of a quantum mechanical system, provided that the space of observables is taken to be a JB^*-triple. Justification for the use of a JB^*-triple as a space of observables can be found in the Gelfand-Naimark Theorem for JB^*-triples of Friedman-Russo [**9**].

The following five theorems, which are standard facts in operator algebra theory, will be used in our proof of Theorem E (cf. [**22**, Chapter III]).

Theorem 1. (Bidual) *The bidual of a C^*-algebra A is a von Neumann algebra which contains A as a C^*-subalgebra via the canonical embedding.*

Theorem 2. (Spectral and polar decomposition of operators) *Let x be any element of a von Neumann algebra M.*

(a) *There is a unique partial isometry u in M with the properties $x = u|x|$ (where $|x| = (x^*x)^{1/2}$), and uu^* = the projection on the closure of the range of x.*

(b) *There is a partial isometry valued spectral measure $\sigma \to v(\sigma)$ on the Borel subsets of $[0, ||x||]$ such that*

$$x = \int_0^{||x||} \lambda dv_\lambda.$$

Theorem 3. (Polar decomposition of functionals) *Let f be a normal functional on a von Neumann algebra M. Then there is a positive normal functional φ and a partial isometry u with $||\varphi|| = ||f||, f(x) = \varphi(ux)$, for $x \in M$, and uu^* is the support projection of φ.*

Theorem 4. (Jordan decomposition)

(a) *Let f be a normal hermitian functional on a von Neumann algebra M. Then there exist normal positive functionals g and h with*

$$f = g - h, \quad ||f|| = ||g|| + ||h||.$$

(b) *Normal positive functionals g and h have orthogonal support projections if and only if $||g - h|| = ||g|| + ||h||$.*

Theorem 5. (Neutrality) *Let f be a normal functional on a von Neumann algebra M and let e be any projection in M. The following are equivalent*:

(a) $f = f \cdot e$, *where $f \cdot e(x) = f(xe)$ for $x \in M$;*

(b) $||f|| = ||f \cdot e||$.

Theorem 5 is due to Effros [**5**] and has a physical interpretation: the state f is unchanged by the filter determined by e if its intensity is unchanged by the filter.

Note that the Theorems A–E show that the geometry of the unit ball of the spaces involved is rich enough to determine the non-associative algebraic structure. In Theorem C one needs an assumption of order preserving. By contrast, in order to capture the *associative* structure (in the case of a C^*-algebra) one needs to assume, as shown by M.-D. Choi, that the isometry is unital and a matricial order isomorphism.

2. Isometries of C^*-algebras. In this section we shall give a proof of Theorem E.

Let A and B be C^*-algebras and let $T : A \to B$ be a surjective linear isometry. Then $T^{**} : A^{**} \to B^{**}$ is a surjective linear isometry of the von Neumann algebra A^{**} onto the von Neumann algebra B^{**} (by Theorem 1). We shall show that, with $\{abc\} := 1/2(ab^*c + cb^*a)$,

$$T^{**}\{abc\} = \{T^{**}a, T^{**}b, T^{**}c\}, \quad \text{for } a, b, c \in A^{**}. \tag{2.1}$$

Since A is a C^*-subalgebra of A^{**} and T^{**} extends T, we will have

$$T\{xyz\} = \{Tx, Ty, Tz\}, \quad \text{for } x, y, z \in A, \tag{2.2}$$

which is (1.4).

We have thus reduced the proof of (2.2) to the case where A and B are von Neumann algebras and T is weak $*$-continuous. Moreover, by the standard polarization formula

$$\{abc\} = 1/8 \sum_{\substack{\alpha^4=1 \\ \beta^2=1}} \alpha\beta(a + \alpha b + \beta c)^{(3)},$$

it suffices to prove (2.2) with $x = y = z$, i.e.,

$$T(x^{(3)}) = (Tx)^{(3)}, \quad \text{for } x \in A, \tag{2.3}$$

where we have written $x^{(3)}$ for $\{xxx\} = xx^*x$.

The proof of (2.3) depends on the following two properties of T acting on partial isometries.

Proposition 1. *Let T be a weak*-continuous surjective linear isometry of a von Neumann algebra A onto a von Neumann algebra B.*

(a) *If u is a partial isometry in A then Tu is a partial isometry in B.*

(b) *If u and v are orthogonal partial isometries in A, then Tu and Tv are orthogonal partial isometries in B.*

Partial isometries u, v are *orthogonal* if their left and right support projections are orthogonal, i.e., $uu^*vv^* = 0$ and $u^*uv^*v = 0$.

Assume Proposition 1 and let $x \in A$. For any $\varepsilon > 0$ there are (by Theorem 2) orthogonal partial isometries $u_1, u_2, \ldots, u_n$ and positive scalars $\lambda_1, \ldots, \lambda_n$ such that $||x - y|| < \varepsilon$ and $||y|| \leq ||x||$, where $y = \sum_{i=1}^n \lambda_i u_i$.

It follows that $||Tx - \Sigma\lambda_i Tu_i|| < \varepsilon$ and, by Proposition 1, that $||x^{(3)} - \Sigma\lambda_i^3 u_i|| < 3\varepsilon||x||^2$, $||(Tx)^{(3)} - \Sigma\lambda_i^3 Tu_i|| < 3\varepsilon||x||^2$. Therefore $||T(x^{(3)}) - (Tx)^{(3)}|| < 6\varepsilon||x||^2$, and since ε is arbitrary, (2.3) follows.

Before proving Proposition 1, and hence Theorem E, we show how to obtain Theorem A from Theorem E. Assume (1.4). Since $T1 = T(11^*1) = T1(T1)^*T1, T1$ is a partial isometry. Moreover, $2Tx = T(11^*x + x1^*1) = \ell(Tx) + (Tx)r$ where $\ell = T1(T1)^*$ and $r = (T1)^*T1$. Since T is onto, $2z = \ell z + zr$ for all z in B. This implies $\ell = r = 1$, so $T1$ is unitary.

Now set $\tilde{T}x = (T1)^*Tx$ for $x \in A$. Then $\tilde{T}$ is a unital surjective isometry and so $\tilde{T}(xy + yx) = \tilde{T}(x1^*y + y1^*x) = \tilde{T}x\tilde{T}y + \tilde{T}y\tilde{T}x$, and $\tilde{T}(x^*) = \tilde{T}(1x^*1) = \tilde{T}1(\tilde{T}x)^*\tilde{T}1 = (\tilde{T}x)^*$. Since $Tx = T1\tilde{T}x, T$ satisfies (1.1).

In order to prove Proposition 1, we prepare five lemmas, some of which are of independent interest.

Let v be partial isometry in a C^*-algebra A. Setting $\ell = vv^*$ and $r = v^*v$, the projections $P_j(v), j = 0, 1, 2$, on A are defined by

$$P_2(v)x = \ell xr, \qquad P_1(v)x = (1-\ell)xr + \ell x(1-r)$$
$$P_0(v)x = (1-\ell)x(1-r), \qquad x \in A.$$

The decomposition $x = x_2 + x_1 + x_0$, where $x_j = P_j(v)x$, is called the *Pierce decomposition* of x relative to v. Note that $P_j(v)A$ is the j-eigenspace of the map $x \to vv^*x + xv^*v, j = 0, 1, 2$. We have

$$(2.3') \quad \begin{aligned} ||P_2(v)x + P_0(v)x|| &= \max(||P_2(v)x||,\ ||P_0(v)x||), \qquad x \in A, \\ ||P_2(v)^*g + P_0(v)^*h|| &= ||P_2(v)^*g|| + ||P_0(v)^*h||, \qquad g, h \in A^*. \end{aligned}$$

LEMMA 1. *Let v be a partial isometry in a C^*-algebra A.*

(a) *$A_v := v^*Ar$, with $r = v^*v$, is a C^*-subalgebra of A, with unit r. If A is a von Neumann algebra, so is A_v.*

(b) *The map $x \to vx$ is a linear isometric bijection of A_v onto $P_2(v)A$ with inverse $a \to v^*a$. Thus $P_2(v)A$ becomes a C^*-algebra with unit v and operations*

$$a \cdot b := av^*b, \qquad a^{\#} := va^*v.$$

(c) *The map $f \to f|P_2(v)A$ is an affine isometry of $\{f \in A^* : f(v) = \|f||\}$ onto $(P_2(v)A)^*_+$. If A is a von Neumann algebra, this map restricts to an affine isometry of $\{f \in A_* : f(v) = ||f||\}$ onto $(P_2(v)A)_{*,+}$.*

PROOF. Verifications of (a), (b), (c) are straightforward calculations which the reader is encouraged to carry out. Theorem 5 is needed in the proof of (c). □

The Pierce 2 subspace $P_2(v)A$ will occur frequently in the sequel. It will be denoted by $A_2(v)$. If A is a von Neumann algebra, then by Lemma 1(c), the normal state space of $A_2(v)$ is affinely isometric to the norm exposed face F_v defined by $F_v = \{f \in A_* : f(v) = ||f|| = 1\}$. Norm exposed faces of $A_{*,1}$ will be studied below in Lemmas 3, 4, and 5.

Orthogonal partial isometries were considered in Proposition 1. More generally, elements x, y in a C^*-algebra are *orthogonal* if $xy^* = 0 = y^*x$. This is equivalent to $D(x, y) = 0$, where $D(x, y)$ is the operator $z \to 1/2(xy^*z + zy^*x)$ on A. If u is a partial isometry in A and $x \in A$, then x and u are orthogonal if and only if $x \in P_0(u)A$.

LEMMA 2. *Let f and g be normal functionals on a von Neumann algebra A and let u and v be the partial isometries occurring in their polar decompositions, respectively. Then u and v are orthogonal if and only if*

$$||f + g|| = ||f - g|| = ||f|| + ||g||. \tag{2.4}$$

PROOF. Suppose u and v are orthogonal, and set $w_\pm = u \pm v$. Then $||w_\pm|| = 1$ and $(f \pm g)(w_\pm) = f(u) + g(v) = \|f|| + ||g||$. Therefore (2.4) holds.

For the converse let w^* be the partial isometry occurring in the polar decomposition of $f+g$. Then, by (2.4), $||f|| + ||g|| = ||f+g|| = (f+g)(w) = f(w) + g(w) \leq |f(w)| + |g(w)| \leq ||f|| + ||g||$. Thus $f(w) = \|f||, g(w) = ||g||$, so, by Lemma 1(c), the restrictions $\varphi = f|A_2(w)$ and $\psi = g|A_2(w)$ are normal positive functionals on $A_2(w)$ for which $||\varphi - \psi|| = ||\varphi|| + ||\psi||$. If p and q denote the support projections in $A_2(w)$ of φ and ψ respectively, then, by Theorem 4(b), p and q are orthogonal projections in $A_2(w)$, i.e.,

$$
\begin{aligned}
&p = p \cdot p = pw^*p, \qquad q = q \cdot q = qw^*q,\\
&p = p^{\#} = wp^*w, \qquad q = q^{\#} = wq^*w,\\
&p \cdot q = pw^*q = 0.
\end{aligned}
\tag{2.5}
$$

From (2.5) we have $p = p \cdot p^{\#} \cdot p = pw^*(wp^*w)w^*p = p(ww^*pw^*w)^*p = pp^*p$ and $pq^* = p(q^{\#})^* = p(wq^*w)^* = pw^*qw^* = 0$. Therefore p and q are orthogonal partial isometries in A.

For $x \in A_2(w), \varphi(x) = \varphi(p \cdot x \cdot p)$. But $p \cdot x \cdot p = pw^*xw^*p = (pw^*p)w^*xw^*(pw^*p) = p(wp^*w)^*x(wp^*w)^*p = pp^*xp^*p$. Moreover, since $f(w) = \|f||, f$ vanishes off $A_2(w)$ (Theorem 5). Thus for $x \in A, f(x) = f(pp^*xp^*p)$. Since uu^* and u^*u are the left and right supports of f on A, we have $uu^* \leq p^*p$ and $u^*u \leq pp^*$.

Similarly $vv^* \leq q^*q$ and $v^*v \leq qq^*$. Since p and q are orthogonal partial isometries in A, so are u and v. □

The next lemma examines the relation between partial isometries in a von Neumann algebra A, and norm exposed faces in the unit ball $A_{*,1}$ of its predual.

Recall that by a *norm exposed face* of $A_{*,1}$ we mean a non-empty subset F_x of $A_{*,1}$ of the form

$$F_x = \{f \in A_* : ||f|| = f(x) = 1\},$$

where $x \in A, ||x|| = 1$. Note that if u is a non-zero partial isometry in A, then, by Lemma 1(c), $F_u \neq \phi$.

LEMMA 3. *For each x in a von Neumann algebra A with $||x|| = 1$ and $F_x \neq \phi$, there is a partial isometry w in A with $F_x = F_w$. Moreover $x = y + w$ with y orthogonal to w.*

PROOF. Let $x = \int_0^1 \lambda dv_\lambda$ be the polar spectral decomposition of x. We shall show that $F_x = F_w$, where $w = \int_{\{1\}} \lambda dv_\lambda$.

For $\varepsilon > 0$, write

$$x = \int_{[0,1-\varepsilon)} \lambda dv_\lambda + \int_{[1-\varepsilon,1]} \lambda dv_\lambda = x_1(\varepsilon) + x_2(\varepsilon), \text{ say.}$$

Let v_i be the partial isometry occurring in the polar decomposition of x_i. Then v_1 and v_2 are orthogonal and $x_i = P_2(v_i)x_i$. If $f \in F_x$,

$$\begin{aligned} 1 = \langle f, x\rangle = \langle f, x_1 + x_2\rangle &= \langle (P_2(v_1)^* + P_2(v_2)^*)f, x_1 + x_2\rangle \\ &\le |\langle P_2(v_1)^* f, x_1\rangle| + |\langle P_2(v_2)^* f, x_2\rangle| \\ &\le ||P_2(v_1)^* f||(1-\varepsilon) + ||P_2(v_2)^* f||. \end{aligned}$$

By (2.3′), $1 + \varepsilon||P_2(v_1)^* f|| \le ||P_2(v_1)^* f|| + ||P_2(v_2)^* f|| = ||(P_2(v_1)^* + P_2(v_2)^*)f|| \le ||f|| = 1$. Thus $P_2(v_1)^* f = 0$, and $f(x_1) = \langle f, P_2(v_1)x_1\rangle = \langle P_2(v_1)^* f, x_1\rangle = 0$. Therefore, for all $\varepsilon > 0, f(x_2) = 1$. Now write $x_2 = \int_{[1-\varepsilon,1)} \lambda dv_\lambda + w$, and note that, by Lebesgue bounded convergence, $w^* - \lim_{\varepsilon\to 0} x_2 = w$. Thus $f(w) = \lim_{\varepsilon\to 0} f(x_2) = 1$. We have proved that $F_x \subset F_w$.

Conversely, if $f \in F_w$, then, by Theorem 5, $f = P_2(w)^* f$ and $f(x) = f(P_2(w)(y + w)) = f(w) = 1$. Thus $F_w \subset F_x$ and the proof is complete. □

Lemma 3 says that the map $u \to F_u$ from the set of partial isometries in a von Neumann algebra A to the set of norm exposed faces in the unit ball A_{*1} of the predual A_* is onto. In fact, this map is also one to one. Indeed, by Theorem 4(a) and Lemma 1(c), for any partial isometry $u, P_2(u)^* A_* = \operatorname{sp} F_u$. Also $u \in P_2(u)A$. Therefore u is determined by its values on $\operatorname{sp} F_u$. It follows that if $F_u = F_w$, then $u = w$.

The following lemma strengthens Lemma 2. To facilitate its statement we define $f \diamond g$ to mean that the normal functionals f and g satisfy (2.4). Motivated by Theorem 4(b) and Lemma 2 we say f and g are *orthogonal* if $f \diamond g$.

LEMMA 4. *Let u and v be partial isometries in a von Neumann algebra A. Then u and v are orthogonal if and only if $f \diamond g$ for all pairs $(f, g) \in F_u \times F_v$.*

PROOF. If u and v are orthogonal, then $u \in P_0(v)A, v \in P_0(u)A$, and $||u \pm v|| = 1$. If $(f, g) \in F_u \times F_v$, then, by Theorem 5, $f = P_2(u)^* f, g = P_2(v^*)g$, so that $f(v) = f(P_2(u)P_0(u)v) = 0$ and similarly $g(u) = 0$. Hence $||f|| + ||g|| = f(u) + g(v) = (f \pm g)(u \pm v) \leq \|f + g|| \leq ||f|| + ||g||$, so that $f \diamond g$.

For the converse, we use the fact that, in any von Neumann algebra, the identity element is the supremum, hence the weak*-limit of a sequence of support projections of normal states. By Lemma 1, u is the identity of $A_2(u)$ and so $u = w^* - \lim p_n$ where p_n is the support projection of the normal state $f_n|A_2(u)$ for some $f_n \in F_u$. It is clear that p_n is the partial isometry occurring in the polar decomposition of f_n. Similarly $v = w^* - \lim q_n$ and q_n is the partial isometry occurring in the polar decomposition of some $g_n \in F_v$. By Lemma 2 and our hypothesis, p_n and q_m are orthogonal in A. Therefore $uv^* = \lim_n(\lim_m p_n q_m^*) = 0$, and similarly $v^*u = 0$. □

Our final lemma in this section gives a geometric characterization of a partial isometry.

For any set S of normal functionals, let $S^\diamond = \{f \in A_* : f \diamond g$ for all g in $S\}$.

LEMMA 5. *Let x be an element of a von Neumann algebra A. Then x is a non-zero partial isometry if and only if $||x|| = 1, F_x \neq 0$, and $f(x) = 0$ for all f in $F_x^\diamond$.*

PROOF. If $x = u$ is a non-zero partial isometry then $||u|| = 1$ and we have already noted that $F_u \neq \phi$. Let $f \in F_u^\diamond$ and let v be the partial isometry occurring in the polar decomposition of f. By Lemma 4, u and v are orthogonal, i.e., $u = P_0(v)u$. Also $f = P_2(v)^* f$. Therefore $f(u) = (P_2(v)^* f)(u) = f(P_2(v)P_0(v)u) = 0$.

Conversely, let x satisfy the conditions stated in the lemma. By Lemma 3, $F_x = F_w$ for a partial isometry w and $y := x - w$ is orthogonal to w. We show that $g(y) = 0$ for all $g \in A_*$, forcing $y = 0$.

Since $y = P_0(w)y$, we may assume $g = P_0(w)^* g$. Then if v is the partial isometry occurring in the polar decomposition of g, we have $v \in P_0(w)A$, i.e., v and w are orthogonal. Then, by Lemma 4, $g \in F_w^\diamond = F_x^\diamond$, so that $g(x) = 0$ by assumption. Since also $g(w) = P_0(w)^* g(P_2(w)w) = g(P_0(w)P_2(w)w) = 0$ we have $g(y) = g(x - w) = 0$. □

We are now ready to prove Proposition 1. We note first that, since $T : A \to B$ is weak*-continuous, $S := T^*|B_*$ is a linear isometry of B_* onto A_*. Therefore, by direct verification,

$$S(F_x) = F_{T^{-1}x} \quad \text{for } x \in B. \tag{2.6}$$

Proof of Proposition 1.

(a) If u is a partial isometry in A, then $||u|| = 1, F_u \neq \phi$, and $f(u) = 0$ for all f in $F_u^\diamond$.

Since T and S are surjective isometries, it follows that $||Tu|| = 1, F_{Tu} = S^{-1}F_u \neq \phi$ and $h(Tu) = 0$ for all $h \in F_{Tu}^\diamond$. By Lemma 5, Tu is a partial isometry in B.

(b) If u and v are orthogonal partial isometries in A, then, by Lemma 4 and (2.6), $||f \pm g|| = ||f|| + ||g||$ for all $(f, g) \in F_u \times F_v$, and $||h \pm k|| = ||h|| + ||k||$ for $(h, k) \in S^{-1}F_u \times S^{-1}F_v = F_{Tu} \times F_{Tv}$. By Lemma 4 again, Tu and Tv are orthogonal. □

The proof of Theorem E is now complete.

3. Isometries of JB^*-triples. In this section we shall outline a proof of Theorem D which parallels the proof of Theorem E given in §2.

This proof is affine geometric in character and avoids nearly all of the extensive machinery of infinite dimensional holomorphy required in Kaup's original proof.

We begin by stating the analogs for JB^*-triples of Theorems 1 to 5 of §1. These results were obtained by several authors and most have appeared since 1983.

It is natural to define a *JBW^*-triple* to be a JB^*-triple which is the normed dual of some Banach space with respect to which the triple product is separately weak*-continuous. Dineen, in [**4**], proved that the bidual of a JB^*-triple is a JB^*-triple, and Barton-Timoney [**2**] proved that a dual JB^*-triple is a JBW^*-triple. Hence we have

THEOREM I. *The bidual of a JB^*-triple U is a JBW^*-triple which contains U as a JB^*-subtriple via the canonical embedding.*

The analog of Theorem 2 involves the functional calculus in a JB^*-triple. For purposes of motivation, we first take a close look, from a different viewpoint, at the functional calculus in a C^*-algebra.

Let A be any C^*-algebra and let x be a self adjoint element of A. The C^*-subalgebra A_x of A generated by x is isomorphic to $C_0(\sigma(x) \cup \{0\})$, so that, for any continuous function f on $\sigma(x) \cup \{0\}$ with $f(0) = 0, f(x)$ is an element of A.

Now let U_x be the smallest norm closed subspace of A closed under the triple product $y \to yy^*y$. It is easy to see that $U_x \subset A_x$ corresponds to the functions h in $C_0(\sigma(x) \cup \{0\})$ satisfying $h(-\lambda) = -h(\lambda)$ whenever λ and $-\lambda$ both belong to $\sigma(x)$. This analysis leads easily to the fact that U_x is (triple) isomorphic to the commutative C^*-algebra $C_0(S)$, where $S = (\sigma(x) \cup (-\sigma(x)) \cup \{0\}) \cap [0, \infty)$, and the triple product on $C_0(S)$ is given by $\{fgh\} = f\overline{g}h$.

The following theorem, due to Kaup [**15, 16**], (cf. Friedman-Russo [**6**, Theorem 2; **7**, Remark 1.9]) implies that the preceding discussion is valid for arbitrary elements x in a C^*-algebra.

THEOREM II.

(a) *Let U be a JB^*-triple and let $x \in U$. Then the JB^*-subtriple of U generated by x (i.e., the smallest norm closed subtriple of U containing x) is isomorphic to a commutative C^*-algebra $C_0(S)$, where $S \subset [0, \infty)$.*

(b) *If U is a JBW^*-triple, then the JBW^*-subtriple of U generated by x, namely $\overline{U}_x^{w^*}$, is isomorphic to a commutative von Neumann algebra.*

(c) *For each $x \in U, U$ a JBW^*-triple, there is a tripotent-valued spectral measure $\sigma \to v(\sigma)$ on the Borel subsets of $[0, ||x||]$ such that*

$$x = \int_0^{||x||} \lambda dv_\lambda.$$

An element e in a JB^*-triple is a *tripotent* if $\{eee\} = e$.

In order to state the analogs of Theorems 3 and 5 for JB^*-triples, we need the following notions (cf. Friedman-Russo [**7**, §1]). Each tripotent e gives rise to a Pierce decomposition (generalizing the Pierce decomposition described in §2) and Pierce projections as follows. Let $Q(x)y := \{xyx\}$, for $x, y \in U$ and define

$$P_2(e) = Q(e)^2, \qquad P_1(e) = 2(D(e,e) - Q(e)^2),$$
$$P_0(e) = I - 2D(e,e) + Q(e)^2.$$

We let $U_k(e)$ denote the range of $P_k(e)$, for $k = 0, 1, 2$. Note that $U_k(e)$ is the $k/2$-eigenspace of $D(e,e)$. Then we have

$$U = U_2(e) \oplus U_1(e) \oplus U_0(e), \tag{3.1}$$

$$\{U_i(e), U_j(e), U_k(e)\} \subset U_{i-j+k}(e) \tag{3.2}$$

(where $U_\ell(e) := \{0\}$ if $\ell \notin \{0,1,2\}$),

$$\{U_2(e), U_0(e), U\} = \{U_0(e), U_2(e), U\} = \{0\}. \tag{3.3}$$

Moreover, $U_2(e)$ is a complex Jordan algebra, with unit e and operations

$$x \circ y := \{xey\}, \qquad z^\# = \{eze\}. \tag{3.4}$$

It is convenient at this point to state the JB^*-triple analogs of Theorem 5 (which is proved in Friedman-Russo [**7**, Proposition 1]) and Lemma 1.

Theorem V. *Let U be a JB^*-triple, let f be a bounded linear functional on U and let e be any tripotent of U.*

(a) *If $||P_2(e)^* f|| = ||f||$, then $P_2(e)^* f = f$;*

(b) *If $||P_0(e)^* f|| = ||f||$, then $P_0(e)^* f = f$.*

Lemma I. *Let e be a tripotent in a JB^*-triple U.*

(a) *$U_2(e)$ is a JB^*-algebra with unit e and operations given by* (3.4). *If U is a JBW^*-triple, then $U_2(e)$ is a JBW^*-algebra.*

(b) *The map $f \to f|U_2(e)$ is an affine isometry of $\{f \in U^* : f(e) = ||f||\}$ onto $(U_2(e))^*_+$. If U is a JBW^*-triple* (*with predual U_**), *this map restricts to an affine isometry of $\{f \in U_* : f(e) = ||f||\}$ onto $(U_2(e))_{*,+}$.*

The proof of Lemma I(a) is indicated in Friedmann-Russo [**7**, p. 70], and the proof of Lemma I(b) follows from Theorem V (cf. Friedman-Russo [**10**, Lemma 1.1]).

We can now state the analog for JB^*-triples of Theorem 3. The proof is in Friedman-Russo [**7**, Proposition 2].

THEOREM III. *Let U be a JBW^*-triple and let f be a normal functional on U, i.e., $f \in U_*$. Then there is a unique tripotent e in U, denoted by $e(f)$, such that $f = P_2(e)^* f$, and $f|U_2(e)$ is a faithful normal positive functional on the JBW^*-algebra $U_2(e)$.*

Before stating the analog of Theorem 4 we recall that JB-algebras are precisely the self-adjoint parts of JB^*-algebras (Wright [**25**]). The same result holds for JBW-algebras and JBW^*-algebras. The proof of Theorem IV can therefore be found in Iochum [**13**] [cf. also [**11**]).

THEOREM IV.

(a) *Let f be any normal functional on a JBW-algebra (equivalently, f is any normal hermitian functional on a JBW^*-algebra). Then $f = g - h$ for some positive normal functionals g, h with $||g - h|| = ||g|| + ||h||$.*

(b) *Positive normal functionals g and h on a JBW^*-algebra have orthogonal support projections if and only if $||g - h|| = ||g|| + ||h||$.*

The argument given at the beginning of §2 to prove (2.3) now applies verbatim with C^*-algebra replaced by JB^*-triple and partial isometry replaced by tripotent. In particular, this reduces the proof of Theorem D to the proof of the following analog of Proposition 1.

PROPOSITION I. *Let T be a weak $*$-continuous surjective linear isometry of a JBW^*-triple U onto a JBW^*-triple V.*

(a) *If e is a tripotent in U, then Te is a tripotent in V;*

(b) *If e_1 and e_2 are orthogonal tripotents in U, then Te_1 and Te_2 are orthogonal tripotents in V.*

We recall that tripotents e_1, e_2 are said to be *orthogonal* if $e_2 \in U_0(e_1)$.

As in §2, in order to prove Proposition I we need to prepare four more lemmas.

LEMMA II. *Let f and g be normal functionals on a JBW^*-triple U and let $e(f), e(g)$ be the tripotents given by Theorem* III. *Then $e(f), e(g)$ are orthogonal if and only if*

$$||f+g|| = ||f-g|| = ||f|| + ||g||. \tag{3.5}$$

PROOF. If $e(f), e(g)$ are orthogonal, then (3.5) follows by the same argument as in the proof of Lemma 2.

Conversely, if (3.5) holds, then, with $w := e(f+g)$, the argument in Lemma 2 yields $f(w) = \|f||, g(w) = ||g||$, so that, by Theorem V, $f = P_2(w)^* f$ and $g = P_2(w)^* g$ and, by Friedman-Russo [**7**, Proposition 3], $ef \in U_2(w)$ and $e(g) \in U_2(w)$.

Setting $\varphi = f|U_2(w), \psi = g|U_2(w)$, we still have $||\varphi+\psi|| = ||\varphi-\psi|| = ||\varphi|| + ||\psi||$, so, by Theorem IV, φ and ψ have orthogonal support projections in $U_2(w)$. These support projections coincide with $e(\varphi), e(\psi)$ because φ and ψ are positive functionals on $U_2(w)$. Moreover, by Friedman-Russo [**10**, Proposition 2.2] $e(\varphi) = P_2(w)e(f) = e(f)$ since $e(f) \in U_2(w)$. Similarly $e(\psi) = e(g)$. Since $e(\varphi)$ and $e(\psi)$ are orthogonal in $U_2(w)$, it follows that $e(f)$ and $e(g)$ are orthogonal in U. □

The next lemma is proved in Friedman-Russo [**7**, Proposition 8].

LEMMA III. *For each x in a JBW^*-triple U, with $||x|| = 1$ and $F_x \neq \phi$ (where F_x is the norm exposed face of $U_{*,1}$ given by $F_x = \{f \in U_{*,1} : f(x) = \|f|| = 1\}$), there is a tripotent w in U with $F_x = F_w$. Moreover $x = y + w$ with y orthogonal to w.*

The orthogonality of y and w is expressed by $y \in U_0(w)$ or $D(w, y) = 0$.

As in §2 we define $f \diamond g$ to mean that (3.5) holds for the normal functionals f, g.

LEMMA IV. *Let e_1 and e_2 be tripotents in a JBW^*-triple U. Then e_1 and e_2 are orthogonal if and only if $F_{e_1} \diamond F_{e_2}$, i.e., $f \diamond g$ for all pairs $(f, g) \in F_{e_1} \times F_{e_2}$.*

PROOF. The proof of the only if part is the same as the corresponding proof of Lemma 4.

For the converse, we use the fact that, in any JBW^*-algebra, the unit element is a weak $*$-limit of a sequence of support projections of normal states. Thus, by Lemma I, $e_1 = w^* - \lim p_n$, where p_n is the support projection of the normal state $f_n|U_2(e_1)$ for some $f_n \in F_{e_1}$. It is clear that $p_n = e(f_n)$. Similarly $e_2 = w^* - \lim q_n$ with $q_n = e(g_n), g_n \in F_{e_2}$. By Lemma II and our hypothesis, p_n and q_n are orthogonal in U. Therefore, for any $z \in U, D(e_1, e_2)z = \lim_n(\lim_m\{p_n q_m z\}) = 0$, i.e., e_1 and e_2 are orthogonal. □

We can now give a geometric characterization of tripotents.

LEMMA V. *Let x be an element of a JBW^*-triple U. Then x is a non-zero tripotent if and only if $||x|| = 1, F_x \neq \phi$, and $f(x) = 0$ for all $f \in F_x^\diamond$ where, for $S \subset U_*$, $S^\diamond = \{f \in U_* : f \diamond g$ for all $g \in S\})$.*

PROOF. The proof is identical to the proof of Lemma 5. To show that $e(g) \in U_0(w)$, use Friedman-Russo [**7**, Proposition 3]. □

The proof of Proposition I now follows the exact same lines as the proof of Proposition 1 in §2. Thus Theorem D has been proved.

As noted earlier, Theorem D includes as special cases each of Theorems A, B, C (and E). Theorem D, specialized to JB^*-algebras does not require the assumption that $T1 = 1$. Since multiplication by a unitary element is not in general an isometry, one cannot reduce the case $T1 \neq 1$ to the case $T1 = 1$ by staying in the category of JB^*-algebras. On the other hand, since $\{11x\} = x$ for all x in the unital JB^*-algebra A, it follows from Theorem E, with $u = T1$, that $\{uuy\} = y$ for all y in the JB^*-algebra B, i.e., $B = B_2(u)$. Thus the isometry T is a unital Jordan C^*-isomorphism from A to the JB^*-algebra $B_2(u)$ with structure given by (3.4).

The ideas expressed in this paper suggest an affine geometric approach to the study of JB^*-triples. The second and third authors, motivated by the geometry imposed by measuring processes on the set of observables of a quantum mechanical system, have introduced and studied the class of "facially symmetric spaces" for this purpose. (Cf. *A geometric spectral theorem,* Quart. J. Math., Oxford Ser. (2), **37** (1986), 263–277, and a preprint: *Affine structure of facially symmetric spaces.*)

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