Affine Hecke algebras and raising operators for Macdonald polynomials

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Introduction.

In this paper we introduce certain raising operators and lowering operators for Macdonald polynomials (of type A_{n-1}) by means of the Dunkl operators due to I. Cherednik. The raising operators we discuss below are a natural q-analogue of the raising operators for Jack polynomials introduced by L. Lapointe and L. Vinet [LV1, LV2]. As an application of our raising operators, we will prove the integrality of double Kostka coefficients which had been conjectured by I.G. Macdonald [Ma1] (apart from the positivity conjecture). We will also include some application to a double analogue of the multinomial coefficients.

Let $\mathbb{K} = \mathbb{Q}(q,t)$ be the field of rational functions in two indeterminates (q,t) and $\mathbb{K}[x]^W$ the algebra of symmetric polynomials in n variables $x = (x_1, \cdots, x_n)$ over $\mathbb{K}$, W being the symmetric group $\mathfrak{S}_n$ of degree n. The *Macdonald polynomials* $P_\lambda(x) = P_\lambda(x;q,t)$ (or *symmetric functions with two parameters*, in the terminology of I.G. Macdonald [Ma1]), are a family of symmetric polynomials parametrized by partitions, and they form a $\mathbb{K}$-basis of $\mathbb{K}[x]^W$. One way to characterize these polynomials is, among others, to consider the joint eigenfunctions in $\mathbb{K}[x]^W$ for the commuting family of q-difference operators

$$(1)\qquad D_x^{(r)} = t^{\binom{r}{2}} \sum_{\substack{I\subset[1,n]\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i} \quad (r = 0,1,\cdots,n).$$

The Macdonald polynomial $P_\lambda(x)$ is characterized as the joint eigenfunction of $D_x^{(r)}$ $(r = 0,1,\cdots,n)$ that has the leading term $m_\lambda(x)$ under the dominance order of partitions when it is expressed as a linear combination of monomial symmetric functions $m_\mu(x)$. As in [Ma1] (VI.8.3), we also use another normalization $J_\lambda(x) = c_\lambda P_\lambda(x)$, called the "integral form" of $P_\lambda(x)$.

We define the operators B_m and A_m $(m = 0,1,\cdots,n)$, involving q-shift operators and permutations, as

$$(2)\qquad B_m = \sum_{k_1<k_2<\cdots<k_m} x_{k_1}\cdots x_{k_m}(1-t^mY_{k_1})(1-t^{m-1}Y_{k_2})\cdots(1-tY_{k_m}),$$

$$A_m = \sum_{k_1<k_2<\cdots<k_m} \frac{1}{x_{k_1}\cdots x_{k_m}}(1-Y_{k_1}^*)(1-tY_{k_2}^*)\cdots(1-t^{m-1}Y_{k_m}^*),$$

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by means of the *Dunkl operators* Y_k and their dual version Y_k^* $(k = 1, \cdots, n)$. The Dunkl operators for Macdonald polynomials, defined in terms of a representation of the (extended) affine Hecke algebra, are due to Cherednik [C]. (For our normalization of Y_k and Y_k^*, see Section 2.) The main result of this paper is:

Theorem 1. *The operators B_m and A_m are raising and lowering operators, respectively, in the sense that*

$$B_m J_\lambda(x) = J_{\lambda+(1^m)}(x), \quad \text{and} \quad A_m J_\lambda(x) = a_\lambda J_{\lambda-(1^m)}(x), \tag{3}$$

for each partition λ with $\ell(\lambda) \leq m$, with some $a_\lambda \in \mathbb{Z}[q,t]$.

(See Theorem 3.1 in Section 3.)

Theorem 1 implies that, for any partition $\lambda = (\lambda_1, \cdots, \lambda_n)$, the Macdonald polynomial $J_\lambda(x)$ is obtained by a successive application of the operators B_m starting from $J_0(x) = 1$:

$$J_\lambda(x) = (B_n)^{\lambda_n}(B_{n-1})^{\lambda_{n-1}-\lambda_n} \cdots (B_1)^{\lambda_1-\lambda_2}(1). \tag{4}$$

We can make use of this expression to study transition coefficients between Macdonald polynomials $J_\lambda(x) = J_\lambda(x;q,t)$ and other symmetric functions. In particular we have

Theorem 2. *For any partition λ and μ, the double Kostka coefficient $K_{\lambda,\mu}(q,t)$ is a polynomial in q and t with integral coefficients.*

(See Theorem 3.2). Let us recall ([Ma1], (VI.8.11)) that the *double Kostka coefficients* (or (q,t)-*Kostka coefficients*) $K_{\lambda,\mu}(q,t)$ are defined via decomposition

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t) S_\lambda(x;t), \tag{5}$$

where $S_\lambda(x;t)$ are the so-called *big Schur functions* ([Ma1], (III.4.5)). Theorem 2 gives a partial affirmative answer to the conjecture of Macdonald proposed in [Ma1], (VI.8.18?).

It also turns out that the raising operators B_m and the lowering operators A_m preserve the ring of symmetric functions $\mathbb{K}[x]^W$. On $\mathbb{K}[x]^W$, the action of these operators is described by the following W-invariant q-difference operators

$$(B_m)_{\text{sym}} = \sum_{r=0}^{m} (-1)^r t^{\binom{r}{2}+(m-n+1)r} \sum_{\substack{I\subset[1,n]\\ |I|=r}} x_I e_{m-r}(x_{[1,n]\setminus I}) a_I(x) T_{q,x}^I, \tag{6}$$

$$(A_m)_{\text{sym}} = \sum_{r=0}^{m} (-1)^r t^{\binom{r}{2}} \sum_{\substack{I\subset[1,n]\\ |I|=r}} x_I^{-1} e_{m-r}(x_{[1,n]\setminus I}^{-1}) a_I(x) T_{q,x}^I,$$

respectively (Theorem 7.1). Here we used the abbreviated notations

$$x_I = \prod_{i\in I} x_i, \quad T_{q,x}^I = \prod_{i\in I} T_{q,x_i}, \quad a_I(x) = \prod_{i\in I} \frac{tx_i - x_j}{x_i - x_j}, \tag{7}$$

and, for a subset $J \subset [1,n]$, $e_s(x_J)$ denotes the elementary symmetric function of degree s in the variables $(x_j)_{j\in J}$.

After some preliminaries on Macdonald's q-difference operators and the Dunkl operators (Sections 1 and 2), we formulate our main results in Section 3. Theorem 1 will be proved in Sections 5 and 6 by means of the *Mimachi basis* (defined in Section 4), which is a family of rational functions in two sets of variables that realizes some representation of the Hecke algebra. Explicit formulas (6) for the raising and the lowering q-difference operators will be determined in Section 7.

In the last section, we will give an application of our results to some combinatorial problem. We will introduce a double analogue of the multinomial coefficients in terms of the so-called modified Macdonald polynomials. The *modified Macdonald polynomials* $\widetilde{J}_\lambda(x;q,t)$ are defined using the λ-ring notations as

$$\widetilde{J}_\lambda(x;q,t) = J_\lambda(\frac{x}{1-t};q,t). \tag{8}$$

It is well-known (see e.g. [GH]) that the double Kostka coefficients are characterized also as the transition coefficients between the modified Macdonald polynomials and the Schur functions:

$$\widetilde{J}_\mu(x;q,t) = \sum_\mu K_{\lambda,\mu}(q,t)\, s_\lambda(x). \tag{9}$$

Let us introduce a family of polynomials $B_{\lambda,\mu}(q,t)$ via decomposition

$$\widetilde{J}_\lambda(x;q,t) = \sum_\mu B_{\lambda,\mu}(q,t)\, m_\mu(x) \tag{10}$$

in terms of the monomial symmetric functions. The polynomiality of these coefficients follows from our Theorem 2. Note also that $B_{\lambda,\mu}(q,t) = 0$ unless $|\lambda| = |\mu|$.

Theorem 3. *For any partitions λ and μ with $|\lambda| = |\mu|$, we have*

(1) $B_{\lambda,\mu}(q,t) \in \mathbb{Z}[q,t]$,

(2) $B_{\lambda,\mu}(1,1) = \binom{|\mu|}{\mu_1,\mu_2,\dots}$,

(3) $B_{(\ell),\mu}(q,t) = q^{n(\mu')} \begin{bmatrix} |\mu| \\ \mu_1,\mu_2,\dots \end{bmatrix}_q$ *if* $\ell = |\mu|$,

(4) $B_{\lambda',\mu}(q,t) = q^{n(\lambda')} t^{n(\lambda)} B_{\lambda,\mu}(t^{-1},q^{-1})$.

It follows from Macdonald's conjecture [Ma1], (VI.8.18?) that

Conjecture 4?. $B_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ *for any partitions λ and μ.*

Hence, one can consider the polynomials $B_{\lambda,\mu}(q,t)$ as a natural two-parameter deformation of the classical multinomial coefficients.

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Notes: After we completed this paper, we found a direct proof for the fact that the q-difference operators (6) are raising operators for Macdonald polynomials. This method, without Dunkl operators, also provides an elementary proof of the integrality of double Kostka coefficients. For this direct approach, see our forthcoming paper [KN].

Contents.

§1: Macdonald's q-difference operators.

In this section, we will make a brief review of some basic properties of the Macdonald polynomials (associated with the root system of type A_{n-1}, or the symmetric functions with two parameters) and the commuting family of q-difference operators which have Macdonald polynomials as joint eigenfunctions. For details, see Macdonald's book [Ma1].

Let $\mathbb{K} = \mathbb{Q}(q,t)$ be the field of rational functions in two indeterminates q, t and consider the ring $\mathbb{K}[x] = \mathbb{K}[x_1, \cdots, x_n]$ of polynomials in n variables $x = (x_1, \cdots, x_n)$ with coefficients in $\mathbb{K}$. Under the natural action of the symmetric group $W = \mathfrak{S}_n$ of degree n, the subring of all symmetric polynomials will be denoted by $\mathbb{K}[x]^W$.

The *Macdonald polynomials* $P_\lambda(x) = P_\lambda(x;q,t)$ (associated with the root system of type A_{n-1}) are symmetric polynomials parametrized by the *partitions* $\lambda = (\lambda_1, \cdots, \lambda_n)$ ($\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$). They form a $\mathbb{K}$-basis of the invariant ring $\mathbb{K}[x]^W$ and are characterized as the joint eigenfunctions of a commuting family of q-difference operators $\{D_x^{(r)}\}_{r=0}^n$. For each $r = 0, 1, \cdots, n$, the q-difference operator $D_x^{(r)}$ is defined by

$$(1.1) \qquad D_x^{(r)} = \sum_{\substack{I \subset [1,n] \\ |I|=r}} t^{\binom{r}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i},$$

where T_{q,x_i} stands for the q-shift operator in the variable x_i : $(T_{q,x_i} f)(x_1, \cdots, x_n) = f(x_1, \cdots, qx_i, \cdots, x_n)$. The summation in (1.1) is taken over all subsets I of the interval $[1,n] = \{1, 2, \cdots, n\}$ consisting of r elements. Note that $D_x^{(0)} = 1$ and $D_x^{(n)} = t^{\binom{n}{2}} T_{q,x_1} \cdots T_{q,x_n}$. Introducing a parameter u, we will use the generating function

$$(1.2) \qquad D_x(u) = \sum_{r=0}^n (-u)^r D_x^{(r)}$$

of these operators $\{D_x^{(r)}\}_{r=0}^n$. Note that the operator $D_x(u)$ has the determinantal expression

$$(1.3) \qquad D_x(u) = \frac{1}{\Delta(x)} \det(x_j^{n-i}(1 - ut^{n-i} T_{q,x_j}); 1 \leq i,j \leq n),$$

where $\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the difference product of $x_1, \cdots, x_n$. It is well known that the q-difference operators $D_x^{(r)}$ $(0 \leq r \leq n)$ commute with each other, or equivalently, $[D_x(u), D_x(v)] = 0$. Furthermore the Macdonald polynomial $P_\lambda(x)$ satisfies the q-difference equation

$$(1.4) \qquad D_x(u) P_\lambda(x) = c_\lambda^n(u) P_\lambda(x), \quad \text{with} \quad c_\lambda^n(u) = \prod_{i=1}^n (1 - ut^{n-i} q^{\lambda_i}),$$

for each partition $\lambda = (\lambda_1, \cdots, \lambda_n)$. Recall that each $P_\lambda(x)$ can be written in the form

$$(1.5) \qquad P_\lambda(x) = m_\lambda(x) + \sum u_{\lambda\mu} m_\mu(x) \quad (u_{\lambda\mu} \in \mathbb{K}),$$

where, for each partition μ, $m_\mu(x)$ stands for the monomial symmetric function of monomial type μ, and $\leq$ is the dominance order of partitions. The Macdonald polynomials $P_\lambda(x)$ are determined uniquely by the conditions (1.4) and (1.5).

We recall here on the "reproducing kernel" of the Macdonald polynomials. Consider another set of variables $y = (y_1, \cdots, y_m)$ and assume that $m \leq n$. We define the function $\Pi(x, y) = \Pi(x, y; q, t)$ by

$$\Pi(x, y) = \prod_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty}, \tag{1.6}$$

where $(x; q)_\infty = \prod_{k=0}^{\infty}(1 - xq^k)$. The convergence of the infinite product above may be understood in the sense of formal power series (or of absolute convergence assuming that q is a complex variable with $|q| < 1$). It is known that the function $\Pi(x, y)$ has the expression

$$\Pi(x, y) = \sum_{\ell(\lambda)\leq m} b_\lambda P_\lambda(x)P_\lambda(y) \quad (b_\lambda \in \mathbb{K}), \tag{1.7}$$

where the summation is taken over all partitions λ with length $\leq m$, and each partition $\lambda = (\lambda_1, \cdots, \lambda_m, 0, \cdots, 0)$ with $\ell(\lambda) \leq m$ is identified with the truncation $(\lambda_1, \cdots, \lambda_m)$ when it is used as the suffix for $P_\lambda(y)$. The coefficients $b_\lambda = b_\lambda(q, t)$ in (1.7) are determined as

$$b_\lambda = \prod_{s\in\lambda} \frac{1 - t^{\ell(s)+1}q^{a(s)}}{1 - t^{\ell(s)}q^{a(s)+1}}, \tag{1.8}$$

in terms of the leg-length $\ell(s) = \lambda'_j - i$ and the arm-length $a(s) = \lambda_i - j$ for a box $s = (i, j)$ in the Young diagram representing the partition λ.

We remark that, by (1.4), expression (1.7) is equivalent to the formula

$$D_x(u)\Pi(x, y) = (u; t)_{n-m}D_y(ut^{n-m})\Pi(x, y). \tag{1.9}$$

Since

$$D_x(u) = \sum_{I\subset[1,n]} (-u)^{|I|}t^{\binom{|I|}{2}} \prod_{\substack{i\in I \\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \tag{1.10}$$

We have

$$D_x(u)\Pi(x, y) = \Pi(x, y)F(u; x, y), \tag{1.11}$$

where

$$F(u; x, y) = \sum_{I\subset[1,n]} (-u)^{|I|}t^{\binom{|I|}{2}} \prod_{\substack{i\in I \\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{\substack{i\in I \\ 1\leq k\leq m}} \frac{1 - x_iy_k}{1 - tx_iy_k}. \tag{1.12}$$

Hence formula (1.9) is also equivalent to

$$F(u; x, y) = (u; t)_{n-m}F(ut^{n-m}; y, x). \tag{1.13}$$

(See also [MN1].)

§2: Affine Hecke algebras and the Dunkl operators.

By the work of I. Cherednik [C], it is known that Macdonald's q-difference operators $\{D_x^{(r)}\}$ are reconstructed from the structure of affine Hecke algebras. We recall here how the Dunkl operators for Macdonald polynomials are defined, and how Macdonald's commuting family of q-difference operators are recovered from the Dunkl operators. (See also I.G. Macdonald [Ma2], and A.A. Kirillov Jr. [KJr].) Our convention of the affine Hecke algebra and the definition of Dunkl operators are slightly different from the ones in the references cited above.

We denote by $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ the free $\mathbb{Z}$-module of rank n with basis $\{\epsilon_i\}_{i=1}^n$ and take the canonical symmetric bilinear form $\langle\ ,\ \rangle : P \times P \to \mathbb{Z}$ such that $\langle \epsilon_i, \epsilon_j\rangle = \delta_{ij}$ for each $1 \le i, j \le n$. The elements of P will be called (integral) *weights*, and a weight $\lambda = \sum_{i=1}^n \lambda_i\epsilon_i \in P$ will be identified freely with the n-tuple of integers $(\lambda_1, \cdots, \lambda_n)$. The action of the Weyl group $W = \mathfrak{S}_n$ on P will be fixed so that $w(\epsilon_i) = \epsilon_{w(i)}$ for $i = 1, \cdots, n$, namely, $w(\lambda)_i = \lambda_{w^{-1}(i)}$ for each $\lambda \in P$ and $i = 1, \cdots, n$. We will take the *simple roots* $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and the *simple transpositions* $s_i = (i, i+1)$ for $i = 1, \cdots, n-1$, as usual.

From this section on, we set $\tau_i = \tau_{x_i} = T_{q,x_i}$ $(i = 1, \cdots, n)$ to avoid the conflict with the notation of Hecke algebras. We use the notation of multi-indices both for the multiplication operators and for the q-shift operators:

$$x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \quad \text{and} \quad \tau^\mu = \tau_1^{\mu_1} \cdots \tau_n^{\mu_n} \tag{2.1}$$

for each $\lambda = (\lambda_1, \cdots, \lambda_n), \mu = (\mu_1, \cdots, \mu_n) \in P$. We denote by $\mathcal{D}_{q,x} = \mathbb{K}(x)[\tau^{\pm 1}]$ the $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}(\mathbb{K}(x))$ generated by the multiplication by elements of $\mathbb{K}(x)$ and the q-shift operators τ^μ $(\mu \in P)$:

$$\mathcal{D}_{q,x} = \mathbb{K}(x)[\tau^{\pm 1}] = \bigoplus_{\mu \in P} \mathbb{K}(x)\tau^\mu. \tag{2.2}$$

Note that the q-shift operators τ^μ act on $\mathbb{K}(x)$ as $\mathbb{K}$-algebra automorphisms and that the commutation relations between multiplication operators and q-shift operators are determined accordingly. In particular we have

$$\tau^\mu x^\lambda = q^{\langle \mu, \lambda\rangle} x^\lambda \tau^\mu \quad (\lambda, \mu \in P). \tag{2.3}$$

Each element w of the Weyl group $W = \mathfrak{S}_n$ acts on $\mathbb{K}(x)$ as the $\mathbb{K}$-algebra automorphism of $\mathbb{K}(x)$ such that $w(x_i) = x_{w(i)}$ for $i = 1, \cdots, n$. We denote by $\mathcal{D}_{q,x}[W]$ the $\mathbb{K}$-algebra of q-difference operators involving permutations:

$$\mathcal{D}_{q,x}[W] = \bigoplus_{w \in W} \mathcal{D}_{q,x} w = \bigoplus_{\mu \in P, w \in W} \mathbb{K}(x)\tau^\mu w. \tag{2.4}$$

Note that we have the commutation relations

$$w\, x^\lambda = x^{w(\lambda)} w \quad \text{and} \quad w\, \tau^\mu = \tau^{w(\mu)} w \tag{2.5}$$

for all $\lambda, \mu \in P$ and $w \in W$.

The $\mathbb{K}$-subalgebra $\mathbb{K}[\tau^{\pm 1}; W]$ of $\mathcal{D}_{q,x}[W]$, generated by the q-shift operators and the Weyl group, is isomorphic to the group ring $\mathbb{K}[\widetilde{W}]$ of the extended affine Weyl group $\widetilde{W} = P \rtimes W$. Let us describe the commutation relations of this algebra in terms of generators. Firstly, the Weyl group W is generated by the simple transpositions s_i $(i = 1, \cdots, n-1)$. We define the element s_0 (corresponding to the affine simple root $\delta - \epsilon_1 + \epsilon_n$) by

$$s_0 = s_{\epsilon_1 - \epsilon_n} \tau_1 \tau_n^{-1}. \tag{2.6}$$

These elements $s_0, s_1, \cdots, s_{n-1}$ generate the affine Weyl group $W^{\rm aff} = Q^\vee \rtimes W$, $Q^\vee$ being the coroot lattice. Note that $W^{\rm aff} = Q^\vee \rtimes W$ is a subgroup of our $\widetilde{W} = P \rtimes W$. The fundamental relations among $s_0, s_1, \cdots, s_{n-1}$ are given by

$$\begin{array}{lll} \text{(i)} & s_i^2 = 1 & (i = 0, 1, \cdots, n-1), \\ \text{(ii)} & s_i s_j = s_j s_i & (|i-j| \geq 2), \\ \text{(iii)} & s_i s_j s_i = s_j s_i s_j & (|i-j| = 1), \end{array} \tag{2.7}$$

if $n \geq 3$. Here the suffices for $s_0, s_1, \cdots, s_{n-1}$ are understood as elements of $\mathbb{Z}/n\mathbb{Z}$, and $|a|$ stands for the representative r of the class $a + n\mathbb{Z}$ such that $0 \leq r < n$. If $n = 2$, the fundamental relations are simply given by (2.7.i). In order to obtain the whole extended affine Weyl group $\widetilde{W} = P \rtimes W$, we need to adjoin an element, denoted by ω below, corresponding to the rotation of the Coxeter diagram. We set

$$\omega = s_{n-1} s_{n-2} \cdots s_1 \tau_1 = \tau_n s_{n-1} s_{n-2} \cdots s_1. \tag{2.8}$$

As to this element ω, we have the commutation relations

$$\text{(iv)} \quad \omega\, s_i = s_{i-1}\, \omega \quad (i = 0, 1, \cdots, n-1). \tag{2.9}$$

We remark that ω has the infinite order in our $\widetilde{W}$, and that ω^n coincides with the Euler operator $\tau_1 \cdots \tau_n$. Summarizing, the extended affine Weyl group $\widetilde{W} = P \rtimes W$ is generated by $s_0, s_1, \cdots, s_{n-1}$ and ω, and their fundamental relations are given by (i) – (iv) in (2.7) and (2.9). Note also that the q-shift operators $\tau_1, \cdots, \tau_n$ are recovered by the formula

$$\tau_i = s_i s_{i+1} \cdots s_{n-1}\, \omega\, s_1 \cdots s_{i-1} \tag{2.10}$$

for $i = 1, \cdots, n-1$.

One important fact is that the Hecke algebra $H(\widetilde{W})$ of the extended affine Weyl group $\widetilde{W} = P \rtimes W$ can be realized in the algebra $\mathcal{D}_{q,x}[W]$ of q-difference operators with permutations. We define the elements T_i $(i = 0, 1, \cdots n-1)$ in $\mathcal{D}_{q,x}[W]$ by

$$T_i = t + \frac{1 - t x_i / x_{i+1}}{1 - x_i / x_{i+1}} (s_i - 1) \tag{2.11}$$

for $i = 1, \cdots, n-1$ and

$$T_0 = t + \frac{1 - t q x_n / x_1}{1 - q x_n / x_1} (s_0 - 1). \tag{2.12}$$

Then the following relations are verified in $\mathcal{D}_{q,x}[W]$:

$$(2.13)\qquad \begin{array}{rll} \text{(i)} & (T_i - t)(T_i + 1) = 0 & (i = 0, 1, \cdots, n-1), \\ \text{(ii)} & T_i T_j = T_j T_i & (|i-j| \geq 2), \\ \text{(iii)} & T_i T_j T_i = T_j T_i T_j & (|i-j| = 1), \\ \text{(iv)} & \omega\, T_i = T_{i-1}\, \omega & (i = 0, 1, \cdots, n-1), \end{array}$$

with indices understood as elements of $\mathbb{Z}/n\mathbb{Z}$. We denote by $H(\widetilde{W})$ the subalgebra of $\mathcal{D}_{q,x}[W]$ generated by $T_0, T_1, \cdots, T_{n-1}$ and $\omega^{\pm 1}$. One can also show that (2.13) gives a complete list of the fundamental relations among the generators $T_0, T_1, \cdots, T_{n-1}$ and $\omega^{\pm 1}$. This is a realization of the extended affine Hecke algebra (defined by the generators $T_0, \cdots, T_{n-1}, \omega^{\pm 1}$ and the relations (i) – (iv) above) in the algebra of q-difference operators with permutations. Note also that the $\mathbb{K}$-subalgebra of $\mathcal{D}_{q,x}[W]$ generated by $T_1, \cdots, T_{n-1}$ is isomorphic to the Hecke algebra $H(\mathfrak{S}_n)$ of the symmetric group. In what follows, we will also use the operators $\overline{T}_i = t^{-1} T_i$ $(i = 0, 1, \cdots, n-1)$, which satisfy the quadratic relations $(\overline{T}_i - 1)(\overline{T}_i + t^{-1}) = 0$.

We now define the *Dunkl operators* $Y_1, \cdots, Y_n$ and the *dual Dunkl operators* $Y_1^*, \cdots, Y_n^*$ in the extended affine Hecke algebra $H(\widetilde{W})$. For each $i = 1, \cdots, n$, we set

$$(2.14)\qquad \begin{aligned} Y_i &= \overline{T}_i \overline{T}_{i+1} \cdots \overline{T}_{n-1}\, \omega\, \overline{T}_1^{-1} \cdots \overline{T}_{i-1}^{-1} \\ &= t^{-n+2i-1}\, T_i T_{i+1} \cdots T_{n-1}\, \omega\, T_1^{-1} \cdots T_{i-1}^{-1} \end{aligned}$$

and

$$(2.15)\qquad \begin{aligned} Y_i^* &= \overline{T}_i^{-1} \overline{T}_{i+1}^{-1} \cdots \overline{T}_{n-1}^{-1}\, \omega\, \overline{T}_1 \cdots \overline{T}_{i-1} \\ &= t^{n-2i+1} T_i^{-1} T_{i+1}^{-1} \cdots T_{n-1}^{-1}\, \omega\, T_1 \cdots T_{i-1}. \end{aligned}$$

Note that $Y_1 = \overline{T}_1 \overline{T}_2 \cdots \overline{T}_{n-1}\, \omega$ and $Y_n = \omega\, \overline{T}_1^{-1} \cdots \overline{T}_{n-1}^{-1}$. Comparing these formulas with (2.10), one sees that both Y_i and Y_i^* reduce to τ_i when $t \to 1$. By (2.13), one can show that the Dunkl operators $Y_1, \cdots, Y_n$ commute with each other, and that

$$(2.16)\qquad H(\widetilde{W}) = \bigoplus_{w \in W} \mathbb{K}[Y^{\pm 1}] T_w = \bigoplus_{\mu \in P,\, w \in W} \mathbb{K}\, Y^\mu T_w,$$

where $Y^\mu = Y_1^{\mu_1} \cdots Y_n^{\mu_n}$. For each $w \in W$, T_w and $\overline{T}_w$ is defined by

$$(2.17)\qquad T_w = T_{i_1} \cdots T_{i_p}, \quad \overline{T}_w = \overline{T}_{i_1} \cdots \overline{T}_{i_p} = t^{-\ell(w)} T_w$$

by taking any reduced decomposition $w = s_{i_1} \cdots s_{i_p}$ $(1 \leq i_1, \cdots, i_p \leq n-1)$; these elements do not depend on the choice of reduced decompositions. The Dunkl operators satisfy the following commutation relations with $\overline{T}_1, \cdots, \overline{T}_{n-1}$:

$$(2.18)\qquad \begin{aligned} &\overline{T}_i Y_{i+1} \overline{T}_i = Y_i, \quad \overline{T}_i Y_j = Y_j \overline{T}_i \qquad (j \neq i, i+1), \\ &\overline{T}_i Y_i^* \overline{T}_i = Y_{i+1}^*, \quad \overline{T}_i Y_j^* = Y_j^* \overline{T}_i \qquad (j \neq i, i+1) \end{aligned}$$

for $i = 1, \cdots, n-1$.

Let us define a $\mathbb{K}$-algebra homomorphism $\eta : \mathbb{K}[\tau^{\pm 1}] \to \mathbb{K}[Y^{\pm 1}]$ by the substitution $\tau_i \mapsto t^{n-i} Y_i$ for $i = 1, \cdots, n$. Then it is known that η induces the isomorphism

$$\eta : \mathbb{K}[\tau^{\pm 1}]^W \xrightarrow{\sim} \mathcal{Z}H(\widetilde{W}) \tag{2.19}$$

from the invariant ring $\mathbb{K}[\tau^{\pm 1}]^W$ onto the center of $H(\widetilde{W})$ (Bernstein's theorem). This theorem also implies that, for any $f = f(\tau) \in \mathbb{K}[\tau^{\pm 1}]^W$, the operator $\eta(f) = f(t^{n-1}Y_1, t^{n-2}Y_2, \cdots, Y_n) \in \mathcal{D}_{q,x}[W]$ is W-invariant, hence preserves the $\mathbb{K}$-algebra $\mathbb{K}[x]^W$ of symmetric polynomials. In this way, the center $\mathcal{Z}H(\widetilde{W})$ of the extended affine Hecke algebra provides a commuting family of q-difference operators acting on $\mathbb{K}[x]^W$. It turns out also that this family of operators is diagonalized simultaneously by the Macdonald polynomials $P_\lambda(x)$. In fact we have

$$f(t^{n-1}Y_1, t^{n-2}Y_2, \cdots, Y_n) P_\lambda(x) = f(t^{n-1}q^{\lambda_1}, t^{n-2}q^{\lambda_2}, \cdots, q^{\lambda_n}) P_\lambda(x), \tag{2.20}$$

for any partition $\lambda = (\lambda_1, \cdots, \lambda_n)$. In particular, the q-difference operator $D_x(u)$ of Macdonald can be recovered as the restriction of an operator in $\mathcal{Z}H(\widetilde{W})$. Namely we have

$$(1 - ut^{n-1}Y_1)(1 - ut^{n-2}Y_2) \cdots (1 - uY_n)|_{\mathbb{K}[x]^W} = D_x(u). \tag{2.21}$$

We remark that the dual Dunkl operators $Y_1^*, \cdots, Y_n^*$ defined by (2.15) also have properties similar to $Y_1, \cdots, Y_n$ in relation to Macdonald polynomials. They commute with each other, and are related with Macdonald's q-difference operator through the formula

$$(1 - uY_1^*)(1 - utY_2^*) \cdots (1 - ut^{n-1}Y_n^*)|_{\mathbb{K}[x]^W} = D_x(u). \tag{2.22}$$

(See Section 4.)

§3: Raising operators and transition coefficients.

In this section we introduce certain raising and lowering operators for Macdonald polynomials. After stating our main result (Theorem 3.1), we discuss some of its consequences, including the polynomiality and the integrality of transition coefficients related to Macdonald polynomials. At the end of this section, we formulate the key lemma for the proof of Theorem 3.1, and give a direct proof for a special case, to show some of the ideas in our proof of the general case in Sections 4–6.

By means of the Dunkl and the dual Dunkl operators, we can construct certain raising and lowering operators for Macdonald polynomials. Using the Dunkl operators $Y_1, \cdots, Y_n$ of (2.14), we define the operator $B_m^x \in \mathcal{D}_{q,x}[W]$ by

$$B_m^x = \sum_{1 \le k_1 < \cdots < k_m \le n} x_{k_1} \cdots x_{k_m} (1 - t^m Y_{k_1})(1 - t^{m-1} Y_{k_2}) \cdots (1 - tY_{k_m}). \tag{3.1}$$

for each $m = 1, 2, \cdots, n$. Similarly we define the operator $A_m^x \in \mathcal{D}_{q,x}[W]$ by

$$A_m^x = \sum_{1 \le k_1 < \cdots < k_m \le n} \frac{1}{x_{k_1} \cdots x_{k_m}} (1 - Y_{k_1}^*)(1 - tY_{k_2}^*) \cdots (1 - t^{m-1} Y_{k_m}^*), \tag{3.2}$$

where the $Y_1^*, \cdots, Y_n^*$ are the dual Dunkl operators defined in (2.15).

Theorem 3.1. (1) *The operators* $B^x_m \in \mathcal{D}_{q,x}[W]$ $(m = 1, \cdots, n)$ *are raising operators for the Macdonald polynomials such that*

$$B^x_m\, P_\lambda(x) = \prod_{i=1}^{m} (1 - t^{m-i+1} q^{\lambda_i}) P_{\lambda+(1^m)}(x) \tag{3.3}$$

for all partition λ *with* $\ell(\lambda) \le m$, *where* $(1^m) = \epsilon_1 + \cdots + \epsilon_m$.
(2) *The operators* $A^x_m \in \mathcal{D}_{q,x}[W]$ $(m = 1, \cdots, n)$ *are lowering operators for the Macdonald polynomials such that*

$$A^x_m\, P_\lambda(x) = \prod_{i=1}^{m} \frac{(1 - t^{m-i} q^{\lambda_i})(1 - t^{n-i+1} q^{\lambda_i - 1})}{(1 - t^{m-i+1} q^{\lambda_i - 1})} P_{\lambda-(1^m)}(x) \tag{3.4}$$

if $\ell(\lambda) \le m$. *In particular, one has* $A^x_m\, P_\lambda(x) = 0$ *if* $\ell(\lambda) < m$.

The proof of Theorem 3.1 will be given later in Sections 4, 5 and 6. On the $\mathbb{K}$-algebra $\mathbb{K}[x]^W$ of symmetric polynomials, the operators B^x_m and A^x_m act as q-difference operators. Explicit formulas for these q-difference operators will be given in Section 7. In this section, we will discuss some of the consequences of Theorem 3.1.

From Theorem 3.1, it follows that the Macdonald polynomial $P_\lambda(x)$ for any partition λ can be obtained from the constant function $P_0(x) = 1$ by an iterated application of the raising operators B^x_m. Namely we have

$$P_\lambda(x) = \text{const.}\, (B^x_n)^{\lambda_n} (B^x_{n-1})^{\lambda_{n-1} - \lambda_n} \cdots (B^x_1)^{\lambda_1 - \lambda_2}(1). \tag{3.5}$$

In other words,

$$P_\lambda(x) = \text{const.}\, B^x_{\mu_1} B^x_{\mu_2} \cdots B^x_{\mu_s}(1), \tag{3.6}$$

where $\mu = (\mu_1, \cdots, \mu_s)$ $(\mu_1 \ge \cdots \ge \mu_s > 0)$ is the conjugate partition λ' of λ. If we take the normalization

$$J_\lambda(x) = c_\lambda P_\lambda(x), \quad c_\lambda = \prod_{s \in \lambda} (1 - t^{\ell(s)+1} q^{a(s)}), \tag{3.7}$$

of the Macdonald polynomials ([Ma1]), the action of the operator B^x_m and A^x_m are given as follows:

$$\begin{aligned} & B^x_m J_\lambda(x) = J_{\lambda+(1^m)}(x), \\ & A^x_m J_\lambda(x) = \prod_{i=1}^{m} (1 - t^{m-i} q^{\lambda_i})(1 - t^{n-i+1} q^{\lambda_i - 1})\, J_{\lambda-(1^m)}(x) \end{aligned} \tag{3.8}$$

for $\ell(\lambda) \le m$. In particular we have

$$J_\lambda(x) = (B^x_n)^{\lambda_n} (B^x_{n-1})^{\lambda_{n-1} - \lambda_n} \cdots (B^x_1)^{\lambda_1 - \lambda_2}(1). \tag{3.9}$$

Namely, the "const." in (3.5) and (3.6) is precisely the reciprocal of the c_λ mentioned above.

We can make use of expression (3.9) to study transition coefficients for Macdonald polynomials $J_\lambda(x) = J_\lambda(x;q,t)$. By (1.5) and (3.7), the "integral form" $J_\lambda(x;q,t)$ has an expression

$$J_\lambda(x;q,t) = c_\lambda(q,t) \sum_{\mu \le \lambda} u_{\lambda\mu}(q,t)\, m_\mu(x), \tag{3.10}$$

where $c_\lambda(q,t) = c_\lambda \in \mathbb{Z}[q,t]$ as in (3.7) and $u_{\lambda\mu}(q,t) \in \mathbb{K} = \mathbb{Q}(q,t)$ $(u_{\lambda\lambda}(q,t) = 1)$. Following Macdonald [Ma1], Chapter VI, let us define the *double Kostka coefficients* $K_{\lambda,\mu}(q,t)$ via decomposition

$$J_\mu(x;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t) S_\lambda(x;t), \tag{3.11}$$

in terms of the *big Schur functions* $S_\lambda(x;t)$ (by taking the stable limit as $n \to \infty$, to be more precise). For the definition of $S_\lambda(x;t)$, we refer to [Ma1], (III.4.5). By means of expression (3.9), we can prove that $c_\lambda(q,t)u_{\lambda\mu}(q,t)$ and $K_{\lambda\mu}(q,t)$ are in the polynomial ring $\mathbb{Z}[q,t]$ with integral coefficients, for any partitions λ and μ.

Theorem 3.2. (1) *For each partition λ, the Macdonald polynomial $J_\lambda(x;q,t)$ is expressed as a linear combination of monomial symmetric functions with coefficients in the ring $\mathbb{Z}[q,t]$.*
(2) *For any partitions λ and μ, the double Kostka coefficient $K_{\lambda,\mu}(q,t)$ is a polynomial in q and t with integral coefficients.*

Proof. Since the divided difference operators $\frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(s_i - 1)$ $(i = 1, \cdots, n-1)$ preserve the ring $\mathbb{Z}[t,t^{-1}][x]$, so do the operators T_i as well as T_i^{-1}. This implies that the Dunkl operators $Y_1, \cdots, Y_n$, and accordingly the raising operators B_m^x preserve the ring $\mathbb{Z}[q;t,t^{-1}][x]^W$. Hence we have $J_\lambda(x;q,t) \in \mathbb{Z}[q;t,t^{-1}][x]^W$. Note that this statement is valid for any n. Hence, also in infinite variables, $J_\lambda(x_1, x_2, \cdots; q,t)$ is a linear combination of monomial symmetric functions with coefficients in $\mathbb{Z}[q;t,t^{-1}]$. We can now apply the duality of $J_\lambda(x_1,x_2,\cdots;q,t)$ and $J_{\lambda'}(x_1,x_2,\cdots;t,q)$ with respect to q and t in [Ma1], (VI.8.6), and conclude that $J_\lambda(x_1,x_2,\cdots;q,t)$ is a linear combination of monomial symmetric functions with coefficients regular at $t = 0$, since the involution $\omega_{q,t}$ does not give rise to any singularity at $t = 0$ or $q = 0$. Hence the coefficients must belong eventually to $\mathbb{Z}[q,t]$. This also implies $J_\lambda(x;q,t) \in \mathbb{Z}[q,t][x]^W$, which proves Statement (1).
As in [Ma1], p.241, the transition coefficients between monomial symmetric functions and big Schur functions have the form $p(t)/q(t)$, where $p(t), q(t) \in \mathbb{Z}[t]$ and $q(0) = 1$. Note in particular that they belong to the ring $\mathbb{Q}[t]_{(t)}$ of rational functions in t, regular at $t = 0$. Combining this fact with Statement (1) of Theorem, we see that each double Kostka coefficients can be written as a finite sum of the form

$$K_{\lambda,\mu}(q,t) = \sum_{k \ge 0} \frac{p_{\lambda\mu}^{(k)}(t)}{q_{\lambda\mu}^{(k)}(t)} q^k, \tag{3.12}$$

where all $p_{\lambda\mu}^{(k)}(t)$ and $q_{\lambda\mu}^{(k)}(t)$ belong to $\mathbb{Z}[t]$ and $q_{\lambda\mu}^{(k)}(0) = 1$. In particular we have $K_{\lambda,\mu}(q,t) \in \mathbb{Q}[t]_{(t)}[q]$. We can now apply the duality with respect to q and t again, between $J_\lambda(x;q,t)$ and $J_{\lambda'}(x;t,q)$ and between $S_\lambda(x;t)$ and $S_{\lambda'}(x;q)$, to conclude $K_{\lambda,\mu}(q,t) = K_{\lambda',\mu'}(t,q)$ ([Ma1],(VI.8.15)). Hence, we have $K_{\lambda,\mu}(q,t) \in \mathbb{Q}[t]_{(t)}[q] \cap \mathbb{Q}[q]_{(q)}[t]$. By using Taylor expansions at $t = q = 0$, one can easily see that the intersection of the two subalgebras $\mathbb{Q}[t]_{(t)}[q]$ and $\mathbb{Q}[q]_{(q)}[t]$ coincides precisely with $\mathbb{Q}[q,t]$. Hence we have $K_{\lambda,\mu}(q,t) \in \mathbb{Q}[q,t]$. In expression (3.12), it means that, for any k, $p^{(k)}_{\lambda\mu}(t)/q^{(k)}_{\lambda\mu}(t) \in \mathbb{Q}[t]$, i.e., $q^{(k)}_{\lambda\mu}(t)$ divides $p^{(k)}_{\lambda\mu}(t)$. Since $q^{(k)}_{\lambda\mu}(0) = 1$, it follows that $p^{(k)}_{\lambda\mu}(t)/q^{(k)}_{\lambda\mu}(t) \in \mathbb{Z}[t]$ for all k. Namely we have $K_{\lambda,\mu}(q,t) \in \mathbb{Z}[q,t]$. □

Our raising operators B^x_m $(m = 0, 1, \cdots, n)$ can be regarded as a natural q-analogue of those introduced by Lapointe-Vinet [LV1, LV2] for Jack polynomials, although the limits of our B^x_m does not give exactly their operators. For the comparison with the operators of Lapointe-Vinet, we look at the "quasi-classical limits" of our operators as $q \to 1$. Introducing the parameter β, let us take the limit as $q \to 1$ with rescaling $t = q^\beta$. Then our Dunkl operators Y_k $(k = 1, \cdots, n)$ have the following quasi-classical limits:

$$\mathcal{D}_k = \lim_{q\to 1} \frac{1 - Y_k}{1 - q} = x_k \frac{\partial}{\partial x_k} + \beta \sum_{\alpha>0} \frac{\langle \alpha, \epsilon_k \rangle}{1 - x^\alpha}(s_\alpha - 1), \tag{3.13}$$

summed over all positive roots $\alpha = \epsilon_i - \epsilon_j$ $(i < j)$. Note that these Dunkl operators commute with each other, i.e., $[\mathcal{D}_i, \mathcal{D}_j] = 0$. With these $\mathcal{D}_k$, the quasi-classical limit $\lim_{q\to 1}(B^x_m/(1-q)^m)$ of the raising operator B^x_m is given by

$$\sum_{k_1<\cdots<k_m} x_{k_1} x_{k_2} \cdots x_{k_m} (\mathcal{D}_{k_1} + m\beta)(\mathcal{D}_{k_2} + (m-1)\beta) \cdots (\mathcal{D}_{k_m} + \beta). \tag{3.14}$$

Similarly, the quasi-classical limits of the lowering operators A^x_m are given by

$$\sum_{k_1<\cdots<k_m} \frac{1}{x_{k_1} x_{k_2} \cdots x_{k_m}} (\mathcal{D}^*_{k_1})(\mathcal{D}^*_{k_2} + \beta) \cdots (\mathcal{D}^*_{k_m} + (m-1)\beta), \tag{3.15}$$

where

$$\mathcal{D}^*_k = \lim_{q\to 1} \frac{1 - Y^*_k}{1 - q} = x_k \frac{\partial}{\partial x_k} - \beta \sum_{\alpha>0} \frac{\langle \alpha, \epsilon_k \rangle}{1 - x^{-\alpha}}(s_\alpha - 1), \tag{3.16}$$

Theorem 3.1 will be proved by investigating the action of the operators B^x_m and A^x_m $(m \le n)$ on the function $\Pi(x,y)$ with the auxiliary variables $y = (y_1, \cdots, y_m)$. In fact we will prove the following key lemma in Sections 5 and 6.

Lemma 3.3. *The operators B^x_m and A^x_m act on $\Pi(x,y)$ for $y = (y_1, \cdots, y_m)$ as follows:*

$$B^x_m \, \Pi(x,y) = \frac{1}{y_1 \cdots y_m} D_y(1) \, \Pi(x,y), \tag{3.17}$$

$$A^x_m \, \Pi(x,y) = y_1 \cdots y_m D_y(t^{n-m+1}) \, \Pi(x,y). \tag{3.18}$$

It is easy to see that Lemma 3.3 implies Theorem 3.1. In fact, we have

$$\begin{aligned} y_1 \cdots y_m \, B^x_m \, \Pi(x,y) &= \sum_{\ell(\lambda)\le m} b_\lambda B^x_m(P_\lambda(x)) \, y_1 \cdots y_m \, P_\lambda(y) \\ &= \sum b_\lambda B^x_m(P_\lambda(x)) \, P_{\lambda+(1^m)}(y). \end{aligned} \tag{3.19}$$

As to the action of $D_y(1)$, we have

$$D_y(1)\,\Pi(x,y) = \sum_{\ell(\lambda)\le m} b_\lambda\, P_\lambda(x)\, c_\lambda^m(1)\, P_\lambda(y) \tag{3.20}$$
$$= \sum_{\ell(\lambda)\le m} c^m_{\lambda+(1^m)}(1)\, b_{\lambda+(1^m)}\, P_{\lambda+(1^m)}(x)\, P_{\lambda+(1^m)}(y),$$

since $c^m_\lambda(1) = (1-t^{m-1}q^{\lambda_1})\cdots(1-tq^{\lambda_{m-1}})(1-q^{\lambda_m}) = 0$ if $\lambda_m = 0$. Since (3.19) and (3.20) are equal to each other by (3.17), we obtain

$$b_\lambda\, B^x_m(P_\lambda(x)) = c^m_{\lambda+(1^m)}(1)\, b_{\lambda+(1^m)}\, P_{\lambda+(1^m)}(x), \tag{3.21}$$

by comparing the coefficients of $P_{\lambda+(1^m)}(y)$. This proves statement (1) of Theorem 3.1. A similar computation based on (3.18) shows that

$$b_\lambda\, A^x_m(P_\lambda(x)) = c^m_{\lambda-(1^m)}(t^{n-m+1})\, b_{\lambda-(1^m)}\, P_{\lambda-(1^m)}(x), \tag{3.22}$$

which proves statement (2) of Theorem 3.1.

Remark 3.4. It follows from Theorem 3.1 that, if λ is a partition such that $\ell(\lambda) \le m \le n$, then

$$[D_x(u), B_m]J_\lambda(x) = (c^n_{\lambda+(1^m)}(u) - c^n_\lambda(u))B_m J_\lambda(x). \tag{3.23}$$

In the case of Jack polynomials, the last formula corresponds to Proposition 4.1 of Lapointe-Vinet [LV2] and appears to be the main step of their proof of a Rodrigues type formula for Jack polynomials [LV1]. It would be interesting to find a direct proof of formula (3.23).

Before the proof of Lemma 3.3, we will give a proof of formula (3.17) for the case $m = 1$, to show some of the ideas of our proof.

We have to compute

$$B^x_1\,\Pi(x,y) = \sum_{k=1}^n x_k(1-tY_k)\prod_{i=1}^n \frac{(tx_iy;q)_\infty}{(x_iy;q)_\infty}. \tag{3.24}$$

Since $\Pi(x,y)$ is symmetric in $x = (x_1,\cdots,x_n)$, we have

$$Y_k\,\Pi(x,y) = \overline{T}_k\,\overline{T}_{k+1}\cdots\overline{T}_{n-1}\tau_n\Pi(x,y) \tag{3.25}$$
$$= \Pi(x,y)\,\overline{T}_k\,\overline{T}_{k+1}\cdots\overline{T}_{n-1}\left(\frac{1-x_ny}{1-tx_ny}\right)$$
$$= \Pi(x,y)\,\overline{T}_k\,\overline{T}_{k+1}\cdots\overline{T}_{n-1}t^{-1}\left(1+\frac{t-1}{1-tx_ny}\right).$$

Hence

$$(1-tY_k)\,\Pi(x,y) = \Pi(x,y)\,\overline{T}_k\,\overline{T}_{k+1}\cdots\overline{T}_{n-1}\left(\frac{1-t}{1-tx_ny}\right). \tag{3.26}$$

The key identity in our computation is

$$\overline{T}_k\left(\frac{1}{1-tx_{k+1}y}\right)=\frac{1}{1-tx_ky}\frac{1-x_{k+1}y}{1-tx_{k+1}y}\quad(k=1,\cdots,n-1). \tag{3.27}$$

Using this identity repeatedly, we get

$$\overline{T}_k\overline{T}_{k+1}\cdots\overline{T}_{n-1}\left(\frac{1}{1-tx_ny}\right)=\frac{1}{1-tx_ky}\prod_{k<i\le n}\frac{1-x_iy}{1-tx_iy}. \tag{3.28}$$

This implies

$$(1-tY_k)\,\Pi(x,y)=\Pi(x,y)\left(\frac{1-t}{1-tx_ky}\prod_{k<i\le n}\frac{1-x_iy}{1-tx_iy}\right). \tag{3.29}$$

We now use the formula

$$\frac{(1-t)x_ky}{1-tx_ky}=1-\frac{1-x_ky}{1-tx_ky} \tag{3.30}$$

to compute the summation

(3.31)

$$\begin{aligned}\sum_{k=1}^{n}\frac{(1-t)x_ky}{1-tx_ky}\prod_{k<i\le n}\frac{1-x_iy}{1-tx_iy}&=\sum_{k=1}^{n}\left(\prod_{k<i\le n}\frac{1-x_iy}{1-tx_iy}-\prod_{k\le i\le n}\frac{1-x_iy}{1-tx_iy}\right)\\&=1-\prod_{1\le i\le n}\frac{1-x_iy}{1-tx_iy}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}y\sum_{k=1}^{n}x_k(1-tY_k)\,\Pi(x,y)&=\Pi(x,y)\left(1-\prod_{1\le i\le n}\frac{1-x_iy}{1-tx_iy}\right)\\&=(1-\tau_y)\Pi(x,y).\end{aligned} \tag{3.32}$$

This gives formula (3.17) of Lemma 3.3 for $m=1$. A similar computation can be done to prove (3.18) for $m=1$.

For the proof of Lemma 3.3, we will make use of a certain class of rational functions in (x,y) which is related to a systematic generalization of formula (3.28) above. Such rational functions, which we call *Mimachi basis* below, were proposed by [Mi] in the study of integral representations of q-KZ equations. It actually gives a realization of some representations of the Hecke algebra $H(W)$ as is clarified in [MN2].

The proof of Lemma 3.3 will be divided into two parts. After reformulating the Mimachi basis in Section 4 so as to fit for our purpose, we will describe in Section 5 the action of the operator $D_y(u)$ on $\Pi(x,y)$, in terms of the Mimachi basis. On the other hand, we will analyze in Section 6 the action of the operators B^x_m and A^x_m on $\Pi(x,y)$ by computing the action of Dunkl operators explicitly to establish the formulas (3.17) and (3.18).

§4: Mimachi basis and a representation of the Hecke algebra.

In this section, we will formulate the Mimachi basis in our context. Keeping the notations of the previous section, we work with the field $\mathbb{K}(x,y)$ of rational functions in $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_m)$. For the moment, we do *not* assume that $m\le n$.

Note first that the field $\mathbb{K}(x,y)$ has a natural structure of left module over the Hecke algebra $H(W)=H(\mathfrak{S}_n^x)$ defined through the operators

$$T_i^x = t + \frac{1-tx_i/x_{i+1}}{1-x_i/x_{i+1}}(s_i-1) \quad (i=1,\cdots,n-1) \tag{4.1}$$

as in Section 2. We use the superscript x to remind that these operators are realized as operators with respect to the variables $x=(x_1,\cdots,x_n)$. As to this $H(\mathfrak{S}_n^x)$-module structure of $\mathbb{K}(x,y)$, we construct some explicit finite dimensional subrepresentations of $\mathbb{K}(x,y)$.

Let us introducing some notations. Let $I\subset[1,n]$ and $K\subset[1,m]$ be subsets of indices for the variables x and y, respectively, and assume that $|I|=|K|=r$ ($0\le r\le m\wedge n$). By writing these sets as $I=\{i_1<\cdots<i_r\}$ and $K=\{k_1<\cdots<k_r\}$, we define a rational function $h_K^I=h_K^I(x,y)\in\mathbb{K}(x,y)$ as follows:

(4.2)
$$h_K^I(x,y) = \sum_{\sigma\in\mathfrak{S}_r}\prod_{1\le\mu\le r}\left(\frac{t-1}{1-tx_{i_\mu}y_{k_{\sigma(\mu)}}}\prod_{i_\mu<j\le n}\frac{t(1-x_jy_{k_{\sigma(\mu)}})}{1-tx_jy_{k_{\sigma(\mu)}}}\right)\prod_{1\le\mu<\nu\le r}\frac{ty_{k_{\sigma(\mu)}}-y_{k_{\sigma(\nu)}}}{y_{k_{\sigma(\mu)}}-y_{k_{\sigma(\nu)}}}.$$

Note that each $h_K^I(x,y)$ is a symmetric function in the variables $(y_{k_1},\cdots,y_{k_r})$, while, as a function of x, it strongly depends on the inclusion $I\subset[1,n]$.

For a fixed $K\subset[1,m]$ with $|K|=r$, let $V_{n,r;K}$ be the $\mathbb{K}$-vector subspace of $\mathbb{K}(x,y)$ spanned by the rational functions $h_K^I(x,y)$ ($I\subset[1,n]$, $|I|=r$):

$$V_{n,r;K} = \sum_{\substack{I\subset[1,n]\\|I|=r}}\mathbb{K}\,h_K^I(x,y)\subset\mathbb{K}(x,y). \tag{4.3}$$

Theorem 4.1. *For each $K\subset[1,m]$ with $|K|=r$ ($0\le r\le m\wedge n$), the vector subspace $V_{n,r;K}$ is an $H(\mathfrak{S}_n^x)$-submodule of $\mathbb{K}(x,y)$. The rational functions $\{h_K^I(x,y);\ I\subset[1,n],\ |I|=r\}$ form a $\mathbb{K}$-basis of $V_{n,r,K}$; hence $\dim_{\mathbb{K}}V_{n,r;K}=\binom{n}{r}$. Furthermore, for each $i=1,\cdots,n-1$, the action of the operator T_i^x on $h_K^I=h_K^I(x,y)$ is described as follows:*

$$\begin{aligned} &\text{(i)} && T_i^x\,h_K^I = t\,h_K^I && (i,i+1\in I \quad \text{or} \quad i,i+1\notin I),\\ &\text{(ii)} && T_i^x\,h_K^I = h_K^{s_i(I)} && (i\notin I, i+1\in I),\\ &\text{(iii)} && T_i^x\,h_K^I = t\,h_K^{s_i(I)}+(t-1)h_K^I && (i\in I, i+1\notin I). \end{aligned} \tag{4.4}$$

Note that formula (iii) of (4.4) is equivalent to $(T_i^x)^{-1}h_K^I = h_K^{s_i(I)}$. We will not prove this theorem here since it is a reformulation of a part of the results of [MN2].

From formula (4.4), it turns out that, for each K, the $H(\mathfrak{S}_n^x)$-module $V_{n,r;K}$ is isomorphic to the induced representation $\mathrm{Ind}^{H(\mathfrak{S}_n)}(\mathrm{triv}_{H(\mathfrak{S}_{n-r}\times\mathfrak{S}_r)})$ of the trivial representation of $H(\mathfrak{S}_{n-r}\times\mathfrak{S}_r)$.

For each subset $I \subset [1,n]$ with $|I| = r$, take the permutation $w_I \in \mathfrak{S}_r$ defined by

$$w_I = \begin{pmatrix} 1 & \cdots & n-r & n-r+1 & \cdots & n \\ j_1 & \cdots & j_{n-r} & i_1 & \cdots & i_r \end{pmatrix}, \tag{4.5}$$

where $I = \{i_1 < \cdots < i_r\}$ and $[1,n]\backslash I = \{j_1 < \cdots < j_{n-r}\}$. Note that a reduced decomposition of w_I is given by

$$w_I = s_{i_r}\cdots s_{n-1}s_{i_{r-1}}\cdots s_{n-2}\cdots s_{i_1}\cdots s_{n-r}, \tag{4.6}$$

and that the length of w_I is determined as

$$\ell(w_I) = \sum_{\nu=1}^{r}(n-r+\nu-i_\nu) = rn - \binom{r}{2} - \sum_{i\in I} i. \tag{4.7}$$

From (4.4) it follows that

$$h_K^I(x,y) = T_{w_I}^x\, h_K^{[n-r+1,n]}(x,y), \tag{4.8}$$

since $T_{w_I}^x$ has the expression

$$T_{w_I}^x = T_{i_r}\cdots T_{n-1}T_{i_{r-1}}\cdots T_{n-2}\cdots T_{i_1}\cdots T_{n-r}. \tag{4.9}$$

Recall that each permutation $w \in \mathfrak{S}_n$ can be uniquely written in the form $w = w_I w' w''$ with some $I \subset [1,n]$ $(|I| = r)$, $w' \in \mathfrak{S}_{n-r}$, $w'' \in \mathfrak{S}_r$ and that $\ell(w) = \ell(w_I) + \ell(w') + \ell(w'')$. Hence the elements T_{w_I} $(I \subset [1,n], |I| = r)$ form a free basis of $H(\mathfrak{S}_n)$ as a right $H(\mathfrak{S}_{n-r}\times\mathfrak{S}_r)$-module. This shows that the $H(\mathfrak{S}_n)$-module $V_{n,r;K}$ is isomorphic to the induced representation mentioned above. Note that the canonical generator of the induced representation corresponds to the rational function $h_K^{[n-r+1,n]}$. We will call the rational functions $\{h_K^I(x,y); I \subset [1,n],\ |I| = r\}$ the *Mimachi basis* of the representation $V_{n,r;K} = H(\mathfrak{S}_n^x)\, h_K^{[n-r+1,n]}$.

These properties of the Mimachi basis will be used in the Section 6.

For the proof of Lemma 3.3, we will need a *dual version* of the Mimachi basis as well. Let $\iota : \mathbb{K}(x,y) \to \mathbb{K}(x,y)$ be the involutive $\mathbb{Q}$-*algebra* automorphism determined by

$$\iota(q) = q^{-1},\ \iota(t) = t^{-1},\ \iota(x_i) = x_i^{-1}\ ,\ \iota(y_k) = y_k^{-1}, \tag{4.10}$$

for $i = 1,\cdots,n$ and $k = 1,\cdots,m$. Note that ι is *not* an anti-automorphism. It is easily checked that

$$\iota(wf) = w\iota(f) \quad (w \in W = \mathfrak{S}_n^x), \quad \iota(\tau_x^\mu f) = \tau_x^\mu \iota(f) \quad (\mu \in P), \tag{4.11}$$

for all $f \in \mathbb{K}(x,y)$. Hence the involution ι intertwines the action of the extended affine Weyl group $\widetilde{W} = P \rtimes W$. As to the action of the extended affine Hecke algebra $H(\widetilde{W})$, we have

$$\iota(T_i^x f) = (T_i^x)^{-1}\iota(f) \quad (i = 0,1,\ldots,n-1), \quad \iota(\omega f) = \omega\,\iota(f) \tag{4.12}$$

Namely, the action of $H(\widetilde{W})$ is reversed by the involution, denoted by the same symbol, such that

$$\iota(T_i) = (T_i)^{-1} \quad (i = 0, 1, \cdots, n-1), \quad \iota(\omega) = \omega. \tag{4.13}$$

Note also that $\iota(T_w) = (T_{w^{-1}})^{-1}$ for each $w \in W$. By this involution, the Dunkl operators and the dual Dunkl operators defined in (2.14), (2.15) are interchanged to each other:

$$\iota(Y_i) = Y_i^* \quad (i = 1, \cdots n). \tag{4.14}$$

Formula (2.22) for the dual Dunkl operators is also obtained from (2.21) by this dualizing procedure. (Note that $\iota(D_x(u)) = D_x(ut^{1-n})$.)

It is easily seen that, by the involution ι, the rational function $h_K^I = h_K^I(x, y)$ of (4.2) is transformed into

(4.15)

$$\sum_{\sigma \in \mathfrak{S}_r} \prod_{1 \le \mu \le r} \left(\frac{(t-1)x_{i_\mu} y_{k_{\sigma(\mu)}}}{1 - t x_{i_\mu} y_{k_{\sigma(\mu)}}} \prod_{i_\mu < j \le n} \frac{1 - x_j y_{k_{\sigma(\mu)}}}{1 - t x_j y_{k_{\sigma(\mu)}}} \right) \prod_{1 \le \mu < \nu \le r} \frac{t y_{k_{\sigma(\mu)}} - y_{k_{\sigma(\nu)}}}{t(y_{k_{\sigma(\mu)}} - y_{k_{\sigma(\nu)}})};$$

namely we have

$$\iota(h_K^I) = t^{-rn - \binom{r}{2} + \sum_{i \in I} i} \, x_I y_K \, h_K^I \quad (|I| = |K| = r), \tag{4.16}$$

where $x_I = \prod_{i \in I} x_i$, $y_K = \prod_{k \in K} y_k$. If we set $\ell(I) = \ell(w_I)$ with the notation of (4.7), this can be written as

$$\iota(h_K^I) = t^{-\ell(I) - 2\binom{r}{2}} \, x_I y_K \, h_K^I \quad (|I| = |K| = r). \tag{4.17}$$

By using the involution ι, we see that these $\iota(h_K^I)$ have similar properties as (4.4):

$$\begin{aligned} &\text{(i)} \quad T_i^x \, \iota(h_K^I) = t \, \iota(h_K^I) && (i, i+1 \in I \quad \text{or} \quad i, i+1 \notin I), \\ &\text{(ii)} \quad T_i^x \, \iota(h_K^I) = t \, \iota(h_K^{s_i(I)}) + (t-1)\iota(h_K^I) && (i \notin I, i+1 \in I), \\ &\text{(iii)} \quad T_i^x \, \iota(h_K^I) = \iota(h_K^{s_i(I)}) && (i \in I, i+1 \notin I). \end{aligned} \tag{4.18}$$

The vector subspace $\iota(V_{n,r;K}) \subset \mathbb{K}(x, y)$ is again an $H(\mathfrak{S}_n^x)$-module isomorphic to the induced representation $\mathrm{Ind}^{H(\mathfrak{S}_n)}(\mathrm{triv}_{H(\mathfrak{S}_r \times \mathfrak{S}_{n-r})})$; in this case the canonical generator corresponds to $\iota(h_K^{[1,r]})$.

§5: Action of $D_y(u)$ on $\Pi(x,y)$.

We will now describe the action of Macdonald's q-difference operator $D_y(u)$ on the kernel $\Pi(x,y)$ in terms of the Mimachi basis. Recall that

$$D_y(u)\Pi(x,y) = \Pi(x,y)F(u;y,x) \tag{5.1}$$

with the notation of (1.12), where

$$F(u;y,x) = \sum_{K\subset[1,m]} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{\substack{k\in K\\ \ell\notin K}} \frac{ty_k - y_\ell}{y_k - y_\ell} \prod_{\substack{1\le i\le n\\ k\in K}} \frac{1-x_iy_k}{1-tx_iy_k}. \tag{5.2}$$

We will give an explicit development of this $F(u;y,x)$ in terms of the Mimachi basis $h_K^I(x,y)$.

We start with a simple formula

$$t^m \prod_{1\le k\le m} \frac{1-xy_k}{1-txy_k} = 1 + \sum_{k=1}^m \frac{t-1}{1-txy_k} \prod_{\substack{1\le \ell\le m\\ \ell\ne k}} \frac{ty_k-y_\ell}{y_k-y_\ell}, \tag{5.3}$$

which can be proved by developing the left hand side into partial fractions. In order to generalize this formula, let us introduce the notation

$$a_{K|L}(y) = \prod_{\substack{k\in K\\ \ell\in L\setminus K}} \frac{ty_k-y_\ell}{y_k-y_\ell}, \quad a_K(y) = a_{K|[1,m]}(y), \tag{5.4}$$

$$b_K^I(x,y) = \prod_{\substack{i\in I\\ k\in K}} \frac{t(1-x_iy_k)}{1-tx_iy_k} = t^{|I||K|} \prod_{\substack{i\in I\\ k\in K}} \frac{1-x_iy_k}{1-tx_iy_k},$$

for $I\subset[1,n]$ and $K\subset L\subset[1,m]$. With this notation, one can use (5.3) to expand $b_{[1,m]}^{[1,n]} = b_{[1,m]}^{[1,n]}(x,y)$ as follows:

(5.5)

$$\begin{aligned}
b_{[1,m]}^{[1,n]} &= \prod_{k=1}^m \frac{t(1-x_1y_k)}{1-tx_1y_k}\, b_{[1,m]}^{[2,n]} \\
&= \left(1 + \sum_{k=1}^m \frac{t-1}{1-tx_1y_k} \prod_{\ell\ne k} \frac{ty_k-y_\ell}{y_k-y_\ell}\right) b_{[1,m]}^{[2,n]} \\
&= b_{[1,m]}^{[2,n]} \\
&\quad + \sum_{k=1}^m \left(\frac{t-1}{1-tx_1y_k} \prod_{1<i\le n} \frac{t(1-x_iy_k)}{1-tx_iy_k} \prod_{\ell\ne k} \frac{ty_k-y_\ell}{y_k-y_\ell}\right) b_{\{1\cdots\hat{k}\cdots m\}}^{[2,n]}
\end{aligned}$$

One can use this formula repeatedly to decompose $b_{[1,m]}^{[2,n]}$, $b_{\{1\cdots\hat{k}\cdots m\}}^{[2,n]}$, and so on. By tracing this procedure, we reach the Mimachi basis

Proposition 5.1. *With the notation of (5.4), one has*

$$b^{[1,n]}_{[1,m]}(x,y) = \sum_{\substack{K\subset[1,m],I\subset[1,n]\\ |K|=|I|}} a_K(y)h^I_K(x,y). \tag{5.6}$$

Furthermore, if $J=[n-s+1,n]$ for some $s=0,1,\cdots,n$, one has

$$b^J_L(x,y) = \sum_{\substack{I\subset J,K\subset L\\ |I|=|K|}} a_{K|L}(y)h^I_K(x,y), \tag{5.7}$$

for any subset $L\subset[1,m]$.

Note that formula (5.6) does not depend on the ordering of the variables y_ℓ. Although it depends on the ordering of x_j, formula (5.7) is obtained from (5.6) simply by renaming the variables $x_1,\cdots,x_n$ as $x_{n-s+1},\cdots,x_n$ with n replaced by $n-s$. For more details of the proof of this proposition, see [MN2].

We now try to express $F(u;y,x)$ in terms of the Mimachi basis. With the notation of (5.4), the function $F(u;y,x)$ can be written as

$$F(u;y,x) = \sum_{L\subset[1,m]} (-u)^{|L|}t^{\binom{|L|}{2}-|L|n}a_L(y)b^{[1,n]}_L(x,y). \tag{5.8}$$

By (5.7), we can rewrite the right-hand side into

$$\sum_{\substack{I\subset[1,n],K\subset[1,m]\\ |I|=|K|}} h^I_K(x,y) \sum_{K\subset L\subset[1,m]} (-u)^{|L|}t^{\binom{|L|}{2}-|L|n}a_L(y)a_{K|L}(y). \tag{5.9}$$

By the definition (5.4), it is easily seen that

$$a_L(y)a_{K|L}(y) = a_K(y)a_{L\backslash K|[1,m]\backslash K}(y). \tag{5.10}$$

Write $L=K\cup M$ with $M\subset[1,m]\backslash K$. Then the second summation of (5.9) takes the form

$$(-u)^{|K|}t^{\binom{|K|}{2}-|K|n}a_K(y)\sum_{M\subset[1,m]\backslash K} (-ut^{|K|-n})^{|M|}t^{\binom{|M|}{2}}a_{M|[1,m]\backslash K}(y). \tag{5.11}$$

Here we used the identity $\binom{a+b}{2}=\binom{a}{2}+\binom{b}{2}+ab$. This summation over $M\subset[1,m]\backslash K$ is nothing but the action of Macdonald's operator $D_z(ut^{|K|-n})$ for the variables $z=(y_\ell)_{\ell\in[1,m]\backslash K}$ on the constant function 1; hence it is equal to $(ut^{|K|-n};t)_{m-|K|}$. Therefore, (5.11) is equal to

$$(-u)^{|K|}(ut^{|K|-n};t)_{m-|K|}t^{\binom{|K|}{2}-|K|n}a_K(y). \tag{5.12}$$

Finally we get

$$F(u;y,x) = \sum_{|I|=|K|} (-u)^{|K|}(ut^{|K|-n};t)_{m-|K|}t^{\binom{|K|}{2}-|K|n}a_K(y)h^I_K(x,y), \tag{5.13}$$

where the summation is taken over all pairs of indexing sets $I\subset[1,n]$ and $K\subset[1,m]$ such that $|I|=|K|$. Since $(ut^{r-n};t)_{m-r}=(ut^{-n+m-1};t^{-1})_{m-r}$, formula (5.13) is simplified when the parameter u is specialized to t^{n-m+1}.

Proposition 5.2. *The function $F(u;y,x)$ is expressed in terms of the Mimachi basis as follows:*

(5.14)
$$F(u;y,x) = \sum_{r=0}^{m\wedge n} (-u)^r (ut^{r-n};t)_{m-r} t^{\binom{r}{2}-rn} \sum_{|I|=|K|=r} a_K(y) h_K^I(x,y),$$

where $I \subset [1,n]$ and $K \in [1,m]$. When $u = t^{n-m+1}$, this reduces to

$$F(t^{n-m+1};y,x) = (-1)^m t^{-\binom{m}{2}} \sum_{\substack{I\subset[1,n] \\ |I|=m}} h_{[1,m]}^I(x,y), \tag{5.15}$$

if $m \le n$, and $F(t^{n-m+1};y,x) = 0$ if $m > n$. Hence one has

$$D_y(t^{n-m+1})\Pi(x,y) = (-1)^m t^{-\binom{m}{2}} \Pi(x,y) \sum_{\substack{I\subset[1,n] \\ |I|=m}} h_{[1,m]}^I(x,y), \tag{5.16}$$

if $m \le n$.

We remark that formula (5.15) for $m = 1$ recovers the formula

$$1 - t^n \prod_{i=1}^n \frac{1-x_i y}{1-tx_i y} = \sum_{i=1}^n t^{n-i} \frac{1-t}{1-tx_i y} \prod_{i<j\le n} \frac{1-x_j y}{1-tx_j y}. \tag{5.17}$$

Note that (5.16) is also equal to $(t;t)_{n-m}^{-1} D_x(t)\Pi(x,y)$.

By the involution ι explained at the end of Section 4, we can easily obtain the dual version of the propositions above. One has only to notice that

$$\iota(a_{K|L}(y)) = t^{-|K|(|L|-|K|)} a_{K|L}(y), \quad \iota(b_K^I(x,y)) = t^{-|I||K|} b_K^I(x,y) \tag{5.18}$$

and

$$\iota(D_y(u)) = D_y(ut^{1-m}), \quad \iota(F(u;y,x)) = F(ut^{n-m+1};y,x). \tag{5.19}$$

Proposition 5.3. *With the notation of (5.4), one has*

$$t^{-mn} b_{[1,m]}^{[1,n]}(x,y) = \sum_{\substack{I\subset[1,n],K\subset[1,m] \\ |I|=|K|}} t^{-\ell(I)-|I|(m-1)} a_K(y)\, x_I y_K h_K^I(x,y). \tag{5.20}$$

Furthermore, if $J = [n-s+1,n]$ for some $s = 0,1,\cdots,n$, one has

$$t^{-|J||L|} b_L^J(x,y) = \sum_{\substack{I\subset J,K\subset L \\ |K|=|I|}} t^{-\ell(I)-|I|(|L|-1)} a_{K|L}(y)\, x_I y_K\, h_K^I(x,y), \tag{5.21}$$

for any subset $L \subset [1,m]$.

Here we used the notation $\ell(I) = \ell(x_I) = |I|n - \binom{|I|}{2} - \sum i$

Proposition 5.4. *The function $F(u;y,x)$ is expressed in terms of the Mimachi basis as follows:*

(5.22)
$$F(u;y,x)=\sum_{r=0}^{m\wedge n}(-u)^r(u;t)_{m-r}\sum_{|I|=|K|=r}t^{-\ell(I)-\binom{r}{2}}a_K(y)\,x_Iy_K\,h^I_K(x,y),$$

where $I\subset[1,m]$ and $K\subset[1,m]$. When $u=1$, this reduces to

$$F(1;y,x)=(-1)^mt^{-\binom{m}{2}}y_1\cdots y_m\sum_{\substack{I\subset[1,n]\\|I|=m}}t^{-\ell(I)}\,x_I\,h^I_{[1,m]}(x,y), \tag{5.23}$$

if $m\le n$, and $F(1;y,x)=0$ if $m>n$. Hence one has

$$D_y(1)\Pi(x,y)=(-1)^mt^{-\binom{m}{2}}y_1\cdots y_m\Pi(x,y)\sum_{\substack{I\subset[1,n]\\|I|=m}}t^{-\ell(I)}\,x_I\,h^I_{[1,m]}(x,y), \tag{5.24}$$

if $m\le n$.

We remark that formula (5.23) for $m=1$ recovers the formula

$$1-\prod_{i=1}^n\frac{1-x_iy}{1-tx_iy}=\sum_{i=1}^n\frac{(1-t)x_iy}{1-tx_iy}\prod_{i<j\le n}\frac{1-x_jy}{1-tx_jy}, \tag{5.25}$$

which we used in Section 3 to prove Lemma 3.3 for $m=1$. Note also that (5.24) is equal to $(t^{m-n};t)^{-1}_{n-m}D_x(t^{m-n})\Pi(x,y)$.

§6: Computation of the Dunkl operators acting on $\Pi(x,y)$.

In this section we will compute the action of the Dunkl operators and the operators

$$B^x_m=\sum_{1\le k_1<\cdots<k_m\le n}x_{k_1}\cdots x_{k_r}(1-t^mY_{k_1})\cdots(1-tY_{k_m})\quad\text{and} \tag{6.1}$$
$$A^x_m=\sum_{1\le k_1<\cdots<k_m\le n}\frac{1}{x_{k_1}\cdots x_{k_r}}(1-Y^*_{k_1})\cdots(1-t^{m-1}Y^*_{k_m})$$

on $\Pi(x,y)$ by means of the Mimachi basis, to complete the proof of Lemma 3.3. From now on, we assume that $m\le n$.

Recall first that the Dunkl operators satisfy the following commutation relations:

$$\overline{T}_iY_{i+1}\overline{T}_i=Y_i,\quad \overline{T_i}Y_j=Y_j\overline{T_i}\qquad(j\ne i,i+1), \tag{6.2}$$

for $i=1,\cdots,n-1$. If $\varphi(x)$ is a function symmetric in (x_i,x_{i+1}), i.e., $s_i\varphi(x)=\varphi(x)$, then $T_i\varphi(x)=t\varphi(x)$ and $\overline{T}_i\varphi(x)=\varphi(x)$; hence we have

$$\overline{T}_iY_{i+1}\varphi(x)=Y_i\varphi(x),\quad \overline{T}_iY_j\varphi(x)=Y_j\varphi(x)\quad(j\ne i,i+1) \tag{6.3}$$

In the following, we will use this property of the Dunkl operators extensively.

For the computation of the action of the operators B^x_m and A^x_m, let us introduce some abbreviated notations. For each subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[1, n]$, we set

$$x_I = x_{i_1} x_{i_2} \cdots x_{i_r}, \quad \tau_I = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_r} \quad \text{and} \quad Y_I = Y_{i_1} Y_{i_2} \cdots Y_{i_r}. \tag{6.4}$$

Note that the ordering of Y_i's does not matter since they are mutually commutative. Let $J = \{j_1 < j_2 < \cdots < j_m\}$ be a subset of $[1, n]$ and $\mathbf{u} = (u_1, u_2, \cdots, u_m)$ an m-tuple of parameters. We need to consider operators in $\mathcal{D}_{q,x}[W]$ of the form

$$\begin{aligned}(1 - u_1 Y_{j_1})(1 - u_2 Y_{j_2}) \cdots (1 - u_m Y_{j_m}) \\ = \sum_{r=0}^{m} (-1)^r \sum_{1 \le \nu_1 < \cdots < \nu_r \le m} u_{\nu_1} \cdots u_{\nu_r} Y_{j_{\nu_1}} \cdots Y_{j_{\nu_r}}.\end{aligned} \tag{6.5}$$

Such an expression will be abbreviated as follows:

$$(1 - \mathbf{u}\mathbf{Y})_J = \sum_{I \subset J} (-\mathbf{u})_{I|J} Y_I, \quad \text{with} \quad (-\mathbf{u})_{I|J} = \prod_{\substack{1 \le \nu \le m \\ j_\nu \in I}} (-u_\nu). \tag{6.6}$$

With this notation, one can write formulas like

$$B^x_m = \sum_{\substack{J \subset [1,n] \\ |J| = m}} x_J (1 - \mathbf{u}\mathbf{Y})_J \quad \text{with} \quad \mathbf{u} = (t^m, t^{m-1}, \cdots, t). \tag{6.7}$$

Since $\Pi(x, y)$ is a symmetric function in x and y, let us start with considering the action of Dunkl operators on *symmetric functions* in $x = (x_1, \cdots, x_n)$. Note that to consider the action of an operator $P \in \mathcal{D}_{q,x}[W]$ on the general symmetric function $f(x)$ is equivalent to working in the module $\mathcal{D}_{q,x}[W] / \sum_{w \in W} \mathcal{D}_{q,x}[W](w - 1)$ by identifying the symbol $f(x)$ with the modulo class of 1. For some time from now, $f(x)$ stands for the general symmetric function in x in this sense.

Proposition 6.1. *For the general symmetric function $f(x)$ in x, on has*

$$Y_I\, f(x) = \overline{T}_{w_I} \tau_{[n-r+1,n]} f(x) \tag{6.8}$$

for each $I \subset [1, n]$ with $|I| = r$, where $w_I \in W$ is the permutation of (4.6). Hence, for each $J \subset [1, n]$ with $|J| = m$, one has

$$\begin{aligned}(1 - \mathbf{u}\mathbf{Y})_J\, f(x) &= \sum_{I \subset J} (-\mathbf{u})_{I|J} Y_I\, f(x) \\ &= \sum_{r=0}^{m} \sum_{\substack{I \subset J \\ |I| = r}} (-\mathbf{u})_{I|J} \overline{T}_{w_I} \tau_{[n-r+1,n]} f(x),\end{aligned} \tag{6.9}$$

for any parameters $\mathbf{u} = (u_1, \cdots, u_m)$.

Proof. By using (6.2) repeatedly, one can show that, if $1 \le i < j \le n$, then

$$\begin{aligned}\overline{T}_i \cdots \overline{T}_{j-1} Y_j &= Y_i \overline{T}_i^{-1} \cdots \overline{T}_{j-1}^{-1}, \\ \overline{T}_i \cdots \overline{T}_{j-1} Y_k &= Y_k \overline{T}_i \cdots \overline{T}_{j-1} \quad (k < i \text{ or } k > j)\end{aligned} \tag{6.10}$$

Take the reduced decomposition of w_I as in (4.6). Then it follows from (6.10) that

$$\overline{T}_{w_I} Y_{n-r+1} \cdots Y_n = Y_{i_1} \cdots Y_{i_r} \iota(\overline{T}_{w_I}) = Y_{i_1} \cdots Y_{i_r} \overline{T}^{-1}_{w_I^{-1}}, \tag{6.11}$$

namely

$$\overline{T}_{w_I} Y_{[n-r+1,n]} \overline{T}_{w_I^{-1}} = Y_I. \tag{6.12}$$

This implies

$$Y_I\, f(x) = \overline{T}_{w_I} Y_{[n-r+1,n]} \overline{T}_{w_I^{-1}} f(x) = \overline{T}_{w_I} Y_{[n-r+1,n]} f(x), \tag{6.13}$$

since $f(x)$ is symmetric. It remains to show $Y_{[n-r+1,n]} f(x) = \tau_{[n-r+1,n]} f(x)$. Since $f(x)$ is symmetric, $Y_n f(x) = \omega f(x) = \tau_n f(x)$. One can show inductively that

$$Y_{n-r+1} \cdots Y_n f(x) = \tau_{n-r+1} \cdots \tau_n f(x), \tag{6.14}$$

noting that the function $\tau_{n-r+1} \cdots \tau_n f(x)$ is symmetric in $(x_1, \cdots, x_{n-r})$ and also in $(x_{n-r+1}, \cdots, x_n)$. The latter half of Proposition is clear. □

We now apply Lemma 6.2 to the kernel $\Pi(x, y)$ regarding y as parameters. Note for sure that the symbols τ_i, T_i, $\overline{T_i}$ and Y_i will be used below only for operators in the variables x. Since

$$\begin{aligned} \tau_{[n-r+1,n]} \Pi(x, y) &= \Pi(x, y) \prod_{\substack{i \in [n-r+1,n] \\ k \in [1,m]}} \frac{1 - x_i y_k}{1 - t x_i y_k} \\ &= \Pi(x, y)\, t^{-rm}\, b^{[n-r+1,n]}_{[1,m]}(x, y). \end{aligned} \tag{6.15}$$

we have

$$Y_I \Pi(x, y) = \Pi(x, y)\, t^{-rm}\, \overline{T}_{w_I}(b^{[n-r+1,n]}_{[1,m]}(x, y)), \tag{6.16}$$

for each $I \subset [1, n]$ with $|I| = r$. To compute the action of $\overline{T}_{w_I}$ on $b^{[n-r+1,n]}_{[1,n]}$, we use the expansion by the Mimachi basis:

$$b^{[n-r+1,n]}_{[1,m]}(x, y) = \sum_{\substack{H \subset [n-r+1,n], K \subset [1,m] \\ |H| = |K|}} a_K(y) h^H_K(x, y). \tag{6.17}$$

By using Theorem 4.1, one can describe the action $\overline{T}_{w_I} = t^{-\ell(I)} T_{w_I}$ on each $h^H_K(x, y)$. Another notation for use: for two disjoint subsets A, B of $[1, n]$, we define the *number of inversions* between A and B by

$$\ell(A; B) := \#\{(a, b) \in A \times B\, ;\, a > b\}. \tag{6.18}$$

Note that, with this notation, one has $\ell(I) = \ell(w_I) = \ell([1, n] \backslash I; I)$. The action of $\overline{T}_{w_I}$ on $h^H_K(x, y)$ is then given by

$$\overline{T}_{w_I}(h^H_K(x, y)) = t^{-\ell(I)} T_{w_I} h^H_K(x, y) = t^{-\ell([1,n]\backslash I; w_I(J))} h^{w_I(H)}_K(x, y) \tag{6.19}$$

The power of t in (6.16) is obtained by subtracting the contribution of the factors of T_{w_I} which acts trivially (i.e., by the multiplication by t). Hence we have

$$
\begin{aligned}
(6.20)\qquad &\overline{T}_{w_I}(b_{[1,m]}^{[n-r+1,n]}(x,y)) \\
&= \sum_{\substack{H\subset[n-r+1,n],K\subset[1,m]\\ |H|=|K|}} t^{-\ell([1,n]\setminus I;w_I(H))}a_K(y)h_K^{w_(H)}(x,y) \\
&= \sum_{\substack{H\subset I,K\subset[1,m]\\ |H|=|K|}} t^{-\ell([1,n]\setminus I;H)}a_K(y)h_K^{H}(x,y),
\end{aligned}
$$

where we renamed $w_I(H)$ as H. Summarizing, we get

Lemma 6.2. *For each subset $I\subset[1,n]$, one has*

$$
\begin{aligned}
(6.21)\qquad Y_I\,\Pi(x,y) &= \Pi(x,y)\,\overline{T}_{w_I}\prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}}\frac{1-x_iy_k}{1-tx_iy_k} \\
&= \Pi(x,y)\sum_{\substack{H\subset I,K\subset[1,m]\\ |H|=|K|}} t^{-\ell([1,n]\setminus I;H)-|I|m}a_K(y)h_K^{H}(x,y).
\end{aligned}
$$

Next we compute the action of the operator $(1-\mathbf{u}\mathbf{Y})_J=\prod_{\nu=1}^{m}(1-u_\nu Y_{j_\nu})$ on $\Pi(x,y)$ for each subset $J=\{j_1<\cdots<j_m\}$ of $[1,n]$ with $|J|=m$ and for general $\mathbf{u}=(u_1,\cdots,u_m)$. In view of (6.9) of Proposition 6.1, we have to compute

$$
(6.22)\qquad (1-\mathbf{u}\mathbf{Y})_J\,\Pi(x,y)=\Pi(x,y)\sum_{r=0}^{m}\sum_{\substack{I\subset J\\ |I|=r}}(-\mathbf{u})_{I|J}\overline{T}_{w_I}\prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}}\frac{1-x_iy_k}{1-tx_iy_k}.
$$

By Lemma 6.2, the summation

$$
(6.23)\qquad \sum_{r=0}^{m}\sum_{I\subset J:|I|=r}(-\mathbf{u})_{I|J}\overline{T}_{w_I}\prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}}\frac{1-x_iy_k}{1-tx_iy_k}.
$$

is now equal to

$$
\begin{aligned}
(6.24)\qquad &\sum_{I\subset J}(-\mathbf{u})_{I|J}\sum_{\substack{H\subset I,K\subset[1,m]\\ |H|=|K|}} t^{-\ell([1,n]\setminus I;H)-|I|m}a_K(y)h_K^{H}(x,y) \\
&= \sum_{H\subset J,K\subset[1,m]} a_K(y)h_K^{H}(x,y)\sum_{H\subset I\subset J}(-\mathbf{u})_{I|J}t^{-\ell([1,n]\setminus I;H)-|I|m}
\end{aligned}
$$

Write $I = H \cup A$ with $A \subset J \backslash H$. Then the summation over I above can be written in the form

(6.25)
$$\begin{aligned}&(-\mathbf{u})_{H|J} t^{-\ell([1,n]\backslash H;H)-|H|m} \sum_{A \subset J\backslash H} (-\mathbf{u})_{A|J} t^{\ell(A;H)-|A|m} \\ &= (-\mathbf{u})_{H|J} t^{-\ell(H)-|H|m} \sum_{A\subset J\backslash H} \prod_{\nu : j_\nu \in A} (-u_\nu t^{\ell(j_\nu;H)-m}) \\ &= (-\mathbf{u})_{H|J} t^{-\ell(H)-|H|m} \prod_{\nu : j_\nu \in J\backslash H} (1 - u_\nu t^{\ell(j_\nu;H)-m})\end{aligned}$$

If we set $u_\nu = ut^{m+1-\nu}$ $(\nu = 1, \cdots, m)$, the product at the end of (6.25) is simplified to $(u; t^{-1})_{m-|H|}$. This proves

Lemma 6.3. *If* $\mathbf{u} = (ut^m, ut^{m-1}, \cdots, ut)$, *then one has*

(6.26)
$$\begin{aligned}&\sum_{r=0}^{m} \sum_{I\subset J:|I|=r} (-\mathbf{u})_{I|J} \overline{T}_{w_I} \prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}} \frac{1-x_i y_k}{1-tx_i y_k}. \\ &= \sum_{\substack{I\subset J, K\subset[1,m]\\ |I|=|K|}} (-\mathbf{u})_{I|J} t^{-\ell(I)-|I|m} (u;t^{-1})_{m-|I|} a_K(y) h_K^I(x,y),\end{aligned}$$

for any subset $J \subset [1,n]$ *with* $|J| = m$. *Hence*

(6.27)
$$\begin{aligned}&(1-\mathbf{uY})_J\, \Pi(x,y) \\ &= \Pi(x,y) \sum_{\substack{I\subset J, K\subset[1,m]\\ |I|=|K|}} (-\mathbf{u})_{I|J} t^{-\ell(I)-|I|m} (u;t^{-1})_{m-|I|} a_K(y) h_K^I(x,y).\end{aligned}$$

If $\mathbf{u} = (t^m, t^{m-1}, \cdots, t)$ (i.e., $u = 1$), the right hand side of formula (6.26) reduces to the single term

(6.28)
$$(-1)^m t^{-\binom{m}{2}-\ell(J)} h_K^J(x,y).$$

Hence by Proposition 6.1, we obtain

Proposition 6.4. *Set* $\mathbf{u} = (t^m, t^{m-1}, \cdots, t)$. *Then for each subset* $J = \{j_1 < \cdots < j_m\}$ *of* $[1,n]$ *with* $|J| = m$, *one has*

(6.29)
$$\begin{aligned}(1-\mathbf{uY})_J\, \Pi(x,y) &= (1-t^m Y_{j_1})(1-t^{m-1}Y_{j_2}) \cdots (1-tY_{j_m})\, \Pi(x,y) \\ &= (-1)^m t^{-\binom{m}{2}} \Pi(x,y)\, t^{-\ell(J)} h_{[1,m]}^J(x,y).\end{aligned}$$

Furthermore the action of B_m^x *on* $\Pi(x,y)$ *is expressed as follows:*

(6.30)
$$\begin{aligned}B_m^x\, \Pi(x,y) &= \sum_{J\subset[1,n]:|J|=m} x_J (1-\mathbf{uY})_J\, \Pi(x,y) \\ &= (-1)^m t^{-\binom{m}{2}} \Pi(x,y) \sum_{\substack{J\subset[1,n]\\ |J|=m}} t^{-\ell(J)} x_J h_{[1,m]}^J(x,y)\end{aligned}$$

Comparing (6.30) with the expression (5.24) of Proposition 5.4 we obtain

Corollary. *If $m \le n$,*

$$B^x_m \Pi(x,y) = \frac{1}{y_1 \cdots y_m} D_y(1)\,\Pi(x,y). \tag{6.31}$$

This gives the proof of formula (3.17) of Lemma 3.3.

Remark. As to the case $|J| = s < m$, the same argument as above shows

$$\begin{aligned}(1-t^m Y_{j_1})\cdots(1-t^{m-s+1}Y_{j_s})\Pi(x,y)\\ = (-1)^s t^{-\binom{s}{2}-\ell(J)}\,\Pi(x,y) \sum_{\substack{K\subset[1,m]\\ |K|=s}} a_K(y) h^J_K(x,y).\end{aligned} \tag{6.32}$$

The computation of the action of A^x_m can be carried out similarly. In this case, we have

$$Y^*_I f(x) = \iota(\overline{T}_{w_I})\,\tau_{[n-r+1,n]} f(x) \quad (I \subset [1,n],\ |I| = r) \tag{6.33}$$

with the involution $\iota(\overline{T}_w) = (\overline{T}_{w^{-1}})^{-1}$ explained in Section 4, and

$$\begin{aligned}(1-\mathbf{v}\mathbf{Y}^*)_J f(x) &= (1-v_1 Y^*_{j_1})(1-v_2 Y^*_{j_2})(1-v_m Y^*_{j_m}) f(x)\\ &= \sum_{r=0}^{m} \sum_{\substack{I\subset J\\ |I|=r}} (-\mathbf{v})_{I|J}\,\iota(\overline{T}_{w_I})\,\tau_{[n-r+1,n]} f(x),\end{aligned} \tag{6.34}$$

for any symmetric function $f(x)$ in x. Hence we have

$$\begin{aligned}&(1-\mathbf{v}\mathbf{Y}^*)_J \Pi(x,y)\\ &= \Pi(x,y) \sum_{r=0}^{m} \sum_{\substack{I\subset J\\ |I|=r}} (-\mathbf{v})_{I|J}\,\iota(\overline{T}_{w_I}) \prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}} \frac{1-x_i y_k}{1-t x_i y_k}.\end{aligned} \tag{6.35}$$

To compute this formula, we have only to dualize Lemma 6.3 by the involution ι. From Lemma 6.3, we see that, if $\mathbf{u} = (ut^{-m}, \cdots, ut^{-1})$, we have

(6.36)

$$\begin{aligned}&\sum_{r=0}^{m} \sum_{I\subset J:|I|=r} (-\mathbf{u})_{I|J} t^{|I|m}\,\iota(\overline{T}_{w_I}) \prod_{\substack{i\in[n-r+1,n]\\ k\in[1,m]}} \frac{1-x_i y_k}{1-t x_i y_k}\\ &= \sum_{\substack{I\subset J, K\subset[1,m]\\ |I|=|K|}} (-\mathbf{u})_{I|J} (u;t)_{m-|I|} t^{|I|(m-|I|+1)} a_K(y) x_I y_K h^I_K(x,y).\end{aligned}$$

This implies that

(6.37)

$$\begin{aligned}&(1-t^m \mathbf{u}\mathbf{Y}^*)_J\,\Pi(x,y) = (1-uY^*_{j_1})\cdots(1-ut^{m-1}Y^*_{j_m})\Pi(x,y)\\ &= \Pi(x,y) \sum_{I\subset J, K\subset[1,m]} (-\mathbf{u})_{I|J} (u;t)_{m-|I|} t^{|I|(m-|I|+1)} a_K(y) x_I y_K h^I_K(x,y).\end{aligned}$$

If we set $u=1$, this formula reduces to

$$(1-t^m\mathbf{u}\mathbf{Y}^*)_J\Pi(x,y)=(-1)^m t^{-\binom{m}{2}}y_1\cdots y_m\Pi(x,y)\,x_J h^J_{[1,m]}(x,y). \tag{6.38}$$

Hence we have

$$\begin{aligned} A^x_m\Pi(x,y) &= \sum_{\substack{J\subset[1,n]\\ |J|=m}}\frac{1}{x_J}(1-t^m\mathbf{u}\mathbf{Y}^*)_J\,\Pi(x,y) \\ &= (-1)^m t^{-\binom{m}{2}}y_1\cdots y_m\Pi(x,y)\sum_{\substack{J\subset[1,n]\\ |J|=m}}h^J_{[1,m]}(x,y) \\ &= y_1\cdots y_m\,\Pi(x,y)\,F(t^{n-m+1};y,x)\quad\text{(by Proposition 5.2)} \\ &= y_1\cdots y_m\,D_y(t^{n-m+1})\Pi(x,y). \end{aligned} \tag{6.39}$$

This completes the proof of Lemma 3.3.

§7: q-Difference raising operators.

Let P be an operator in $\mathcal{D}_{q,x}[W]$, and express it in the form

$$P = \sum_{w \in W} P_w w \quad \text{with} \quad P_w \in \mathcal{D}_{q,x} \quad (w \in W). \tag{7.1}$$

We define the q-difference operator P_{sym} by

$$P_{\mathrm{sym}} = \sum_{w \in W} P_w \in \mathcal{D}_{q,x}. \tag{7.2}$$

Then, for any symmetric function $f(x)$, the operator P acts as the q-difference operator P_{sym}; namely, $Pf(x) = P_{\mathrm{sym}} f(x)$. By the method we used in the previous section, we can determine the q-difference operators on symmetric functions which arise from our raising and lowering operators.

Theorem 7.1. *The raising operator B_m and the lowering operator A_m preserve the ring $\mathbb{K}[x]^W$ of symmetric polynomials, for $m = 0, 1, \cdots, n$. Furthermore, these operators act on symmetric functions as the following q-difference operators :*

(7.3)

$$(B_m^x)_{\mathrm{sym}} = \sum_{r=0}^{m} (-1)^r t^{\binom{r}{2} + (m-n+1)r} \sum_{\substack{I \subset [1,n] \\ |I| = r}} x_I e_{m-r}(x_{[1,n]\setminus I}) a_I(x) \tau_I,$$

(7.4)

$$(A_m^x)_{\mathrm{sym}} = \sum_{r=0}^{m} (-1)^r t^{\binom{r}{2}} \sum_{\substack{I \subset [1,n] \\ |I| = r}} x_I^{-1} e_{m-r}(x_{[1,n]\setminus I}^{-1}) a_I(x) \tau_I.$$

Here $e_{m-r}(x_{[1,n]\setminus I})$ and $e_{m-r}(x_{[1,n]\setminus I}^{-1})$ are the elementary symmetric functions of degree $m-r$ in the $n-r$ variables $(x_j)_{j \in [1,n]\setminus I}$ and $(x_j^{-1})_{j \in [1,n]\setminus I}$, respectively, and $a_I(x) = \prod_{i \in I; j \in [1,n]\setminus I} (tx_i - x_j)/(x_i - x_j)$.

Before the proof of Theorem 7.1, we will prove that the q-difference operators $(B_m^x)_{\mathrm{sym}}$ and $(A_m^x)_{\mathrm{sym}}$ are W-invariant. Then the explicit formulas mentioned above will be determined only by the definition (6.1) and this W-invariance. In order to deal with B_m and A_m simultaneously, we consider the operator

$$\mathcal{P}_m(u) = \sum_{1 \le j_1 < \cdots < j_m \le n} x_{j_1} \cdots x_{j_m} (1 - ut^m Y_{j_1}) \cdots (1 - utY_{j_m}). \tag{7.5}$$

depending on a parameter u. With the notation of Section 6, we can also write

$$\mathcal{P}_m(u) = \sum_{\substack{J \subset [1,n] \\ |J| = m}} x_J (1 - \mathbf{u}\mathbf{Y})_J \quad \text{with} \quad \mathbf{u} = (ut^m, ut^{m-1}, \cdots, ut). \tag{7.6}$$

Note that the operators B_m and A_m are recovered from $\mathcal{P}_m(u)$ by

$$B_m = \mathcal{P}_m(1) \quad \text{and} \quad A_m = \iota(\mathcal{P}_m(t^{-m})), \tag{7.7}$$

where ι is the involution defined in Section 4. Then Theorem 7.1 is an immediate consequence of the following Propositions 7.2 and 7.3.

Proposition 7.2. *For each $m=0,1,\cdots,n$, the q-difference operator $\mathcal{P}_m(u)_{\rm sym}$ is W-invariant. Hence it preserves the ring $\mathbb{K}[x]^W$ of symmetric polynomials.*

Proof. We work in the quotient module $\mathcal{D}_{q,x}[W]/\sum_{w\in W}\mathcal{D}_{q,x}[W](w-1)$ denoting by $f(x)$ the modulo class of 1, as in Section 6. Then, for an operator $P\in\mathcal{D}_{q,x}[W]$, the W-invariance of $(P)_{\rm sym}$ is equivalent to

$$s_iPf(x)=f(x)\quad(i=1,\cdots,n-1). \tag{7.8}$$

By the definition (2.11) of T_i, it is also equivalent to the condition

$$T_iPf(x)=tf(x),\quad\text{i.e.,}\quad \overline{T}_iPf(x)=f(x), \tag{7.9}$$

for $i=1,\cdots,n-1$. We now consider the operator $\mathcal{P}_m(u)$ of (7.5). Note here that the multiplication operators $x_1,\cdots,x_n$ have commutation relations

$$\overline{T}_ix_i\overline{T}_i=t^{-1}x_{i+1},\quad \overline{T}_ix_j=x_j\overline{T}_i\quad(j\neq i,i+1) \tag{7.10}$$

with $\overline{T}_i$ $(i=1,\cdots,n-1)$, similar to (6.2) of the Dunkl operators. For a fixed i, the subsets J of $[1,n]$ with $|J|=m$ are classified into the three groups under the action of s_i:

$$\begin{aligned}&\text{(i)}\quad i,i+1\notin J,\\ &\text{(ii)}\quad i\in J,i+1\notin J\quad\text{or}\quad i\notin J,i+1\in J,\\ &\text{(iii)}\quad i,i+1\in J.\end{aligned} \tag{7.11}$$

If J satisfies (i), it is clear that $\overline{T}_ix_J(1-\mathbf{u}\mathbf{Y})_J=x_J(1-\mathbf{u}\mathbf{Y})_J\overline{T}_i$. The other two cases are reduced essentially to

$$\begin{aligned}&(\overline{T}_i-1)[x_i(1-uY_i)+x_{i+1}(1-uY_{i+1})]f(x)=0,\quad\text{and}\\ &(\overline{T}_i-1)x_ix_{i+1}(1-utY_i)(1-uY_{i+1})f(x)=0,\end{aligned} \tag{7.12}$$

respectively, which can be checked directly by using (7.10), (6.2) and the quadratic equation $(\overline{T}_i-1)(\overline{T}_i+t^{-1})=0$. (Note that x_i+x_{i+1},x_ix_{i+1} and tY_i+Y_{i+1},tY_iY_{i+1} commute with $\overline{T}_i$.) □

Proposition 7.3. *For each $m=0,1,\cdots,n$, we have*

$$\mathcal{P}_m(u)_{\rm sym}=\sum_{r=0}^{m}(-u)^rt^{\binom{r}{2}+(m-n+1)r}\sum_{\substack{I\subset[1,n]\\|I|=r}}x_Ie_{m-r}(x_{[1,n]\backslash I})a_I(x)\tau_I. \tag{7.13}$$

Proof. By Proposition 6.1, we already have the formula

$$\mathcal{P}(u)f(x)=\sum_{r=0}^{m}\sum_{I\subset J\subset[1,n]}(-\mathbf{u})_{I|J}\,x_J\overline{T}_{w_I}\tau_{[n-r+1,n]}f(x) \tag{7.14}$$

for the general symmetric function $f(x)$. Hence we see that the q-difference operator $\mathcal{P}_m(u)_{\mathrm{sym}}$ can be written in the form

$$\mathcal{P}_m(u)_{\mathrm{sym}} = \sum_{r=0}^{m} \sum_{\substack{I\subset[1,n]\\ |I|=r}} p_I(u;x)\tau_I. \tag{7.15}$$

Since $P(u)_{\mathrm{sym}}$ is W-invariant by Proposition 7.2, we conclude that, for each $I \subset [1,n]$ with $|I| = r$, $p_I(u;x) = w(p_{[1,r]}(u;x))$ if $w \in W$ and $w([1,r]) = I$. Hence we have only to determine the coefficients $p_{[1,r]}(u;x)$ for each $r = 0, 1, \cdots, m$. In formula (7.14), we look at the product

(7.16)
$$\begin{aligned}&\overline{T}_{w_I}\tau_{[n-r+1,n]}f(x)\\ &= t^{-\ell(I)}(t - c(\alpha_{i_r}) + c(\alpha_{i_r})s_{i_r})\cdots(t - c(\alpha_{n-1}) + c(\alpha_{n-1})s_{n-1})\\ &\quad\cdots(t - c(\alpha_{i_1}) + c(\alpha_{i_1})s_{i_1})\cdots(t - c(\alpha_{n-r}) + c(\alpha_{n-r})s_{n-r})\tau_{[n-r+1,n]}f(x),\end{aligned}$$

for each $I = \{i_1 < \cdots < i_r\}$. Here we used the notation

$$c(\alpha) = c(\epsilon_i - \epsilon_j) = \frac{1 - tx_i/x_j}{1 - x_i/x_j} \tag{7.17}$$

for each positive root $\alpha = \epsilon_i - \epsilon_j$ $(1 \le i < j \le n)$. From expression (7.14), it is clear that the q-shift operator $\tau_{[1,r]}$ appears only when $I = [1,r]$ and all the terms containing s_i are picked up in the expansion. In the case of $I = [1,r]$, the product of terms containing s_i is given by

$$c(\alpha_r)s_r\cdots c(\alpha_{n-1})s_{n-1}\cdots c(\alpha_1)s_1\cdots c(\alpha_{n-r})s_{n-r} = \left(\prod_{k=1}^{N} c(\beta_k)\right) w_I, \tag{7.18}$$

where $N = \ell(w_{[1,r]}) = r(n-r)$ and $\{\beta_1, \cdots, \beta_N\}$ is the sequence of positive roots associated with the reduced decomposition (4.6) of $w_{[1,r]} = s_r\cdots s_{n-1}\cdots s_1\cdots s_{n-r}$:

$$\begin{aligned}\{\beta_1, \cdots, \beta_N\} &= \{\alpha_r, s_r(\alpha_{r+1}), \ldots, s_r\cdots s_{n-2}(\alpha_{n-1}), \ldots,\\ &\qquad \ldots, s_r\cdots s_{n-1}\cdots s_1\cdots s_{n-r-1}(\alpha_{n-r})\}\\ &= \{\epsilon_i - \epsilon_j\,;\, 1 \le i \le r, r+1 \le j \le n\}.\end{aligned} \tag{7.19}$$

Namely the coefficient of $\tau_{[1,r]}$ arising from $\overline{T}_{w_{[1,r]}}\tau_{[n-r+1,n]}f(x)$ is given by

$$t^{-r(n-r)} \prod_{\substack{i\in[1,r]\\ j\in[r+1,n]}} \frac{1 - tx_i/x_j}{1 - x_i/x_j} = t^{-r(n-r)}a_{[1,r]}(x). \tag{7.20}$$

In order to get the coefficient $p_{[1,r]}(u;x)$, we have to take the summation

(7.21)
$$p_{[1,r]}(u;x) = \sum_{\substack{J:[1,r]\subset J\\ |J|=m}} (-\mathbf{u})_{[1,r]|J}\, x_J t^{-r(n-r)} a_{[1,r]}(x)$$

Note here that $(-\mathbf{u})_{[1,r]|J} = (-u)^r t^{mr-\binom{r}{2}}$ for any J with $[1,r] \subset J$, $|J| = m$. From the W-invariance of $\mathcal{P}_m(u)_{\rm sym}$, we have

$$p_I(u;x) = (-u)^r t^{mr-\binom{r}{2}-r(n-r)} x_I e_{m-r}(x_{[1,n]\setminus I}) a_I(x), \tag{7.22}$$

for each $I \subset [1,n]$ with $|I| = r$. This proves (7.13). $\square$

Let us see some consequences of Proposition 7.3. We apply the operator $\mathcal{P}_m(u)$ to the constant function 1. By the definition (7.5) of $\mathcal{P}_m(u)$, we have

$$\mathcal{P}_m(u)(1) = (ut;t)_m e_m(x). \tag{7.23}$$

On the other hand, by (7.13) we have

$$\mathcal{P}_m(u)(1) = \sum_{r=0}^{m} (-u)^r t^{\binom{r}{2}+(m-n+1)r} \sum_{\substack{I\subset[1,n]\\|I|=r}} x_I e_{m-r}(x_{[1,n]\setminus I}) a_I(x). \tag{7.24}$$

By comparing the coefficients of u^r in (7.23) and (7.24), we obtain

Corollary. *If $0 \le r \le m \le n$, we have*

$$\sum_{\substack{I\subset[1,n]\\|I|=r}} x_I e_{m-r}(x_{[1,n]\setminus I}) \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} = t^{(n-m)r} \begin{bmatrix} m \\ r \end{bmatrix}_t e_m(x), \tag{7.25}$$

$$\sum_{\substack{I\subset[1,n]\\|I|=r}} x_I e_{m-r}(x_{[1,n]\setminus I}) \prod_{\substack{i\in I\\ j\notin I}} \frac{x_i - tx_j}{x_i - x_j} = \begin{bmatrix} m \\ r \end{bmatrix}_t e_m(x). \tag{7.26}$$

Formula (7.26) is obtained from (7.25) by the transformation $t \to t^{-1}$. These formulas give a refinement of the equality

$$B_m(1) = (t;t)_m e_m(x) = J_{(1^m)}(x) \tag{7.27}$$

We also remark that formulas (2.21) and (2.22) in Section 2 can be obtained from Proposition 7.3 for $m = n$. In fact, we have

$$P_n(u) = x_1 \cdots x_n (1 - ut^n Y_1) \cdots (1 - utY_n) \quad \text{and} \tag{7.28}$$

$$P_n(u)_{\rm sym} = x_1 \cdots x_n \sum_{r=0}^{n} (-ut)^r t^{\binom{r}{2}} \sum_{\substack{I\subset[1,n]\\|I|=r}} a_I(x) \tau_I.$$

This implies (2.21).

§8: A double analogue of the multinomial coefficients.

In this section, we will give an application of our results to combinatorics of the multinomial coefficients, in terms of the so-called modified Macdonald polynomials. The modified Macdonald polynomials were introduced by A.M. Garsia and M. Haiman [GH] in their study of a graded representation model for Macdonald polynomials (see also [Ma1], p.358).

Following [GH], we define the *modified Macdonald polynomials in infinite number of variables* by

$$\widetilde{P}_\lambda(x;q,t) = P_\lambda(\frac{x}{1-t};q,t) \quad \text{and} \quad \widetilde{J}_\lambda(x;q,t) = J_\lambda(\frac{x}{1-t};q,t), \tag{8.1}$$

in the λ-ring notation. Given a symmetric function $f(x) = f(x_1, x_2, \cdots)$ in infinite variables $x = (x_1, x_2, \cdots)$, the symbol $f(\frac{x}{1-t})$ in the λ-ring notation stands for the symmetric function $f(y(x))$ obtained by the transformation of variables $y(x) = (x_i t^j)_{i\geq 1, j\geq 0}$. In infinite variables, the symmetric function $f(x)$ can be written uniquely in the form $f(x) = \varphi(p_1(x), p_2(x), \cdots)$ as a polynomial of the power sums $p_k(x) = \sum_{j=1}^\infty x_j^k$ $(k = 1, 2, \cdots)$. Then the symbol $f(\frac{x}{1-t})$ represents the symmetric function

$$f(\frac{x}{1-t}) = \varphi(\frac{p_1(x)}{1-t}, \frac{p_2(x)}{1-t^2}, \cdots). \tag{8.2}$$

obtained by the transformation $p_k(x) \to p_k(x)/(1-t^k)$ $(k = 1, 2, \cdots)$. When we consider the modified Macdonald polynomials in n variables $x = (x_1, \cdots, x_n)$, each of $\widetilde{P}_\lambda(x;q,t)$ and $\widetilde{J}_\lambda(x;q,t)$ should be understood as the one obtained from the corresponding symmetric function in infinite variables by setting $x_{n+1} = x_{n+2} = \cdots = 0$. It follows from the orthogonality property of Macdonald polynomials (see (1.7) or [Ma1], (VI.4.13)) that

$$\sum_\lambda b_\lambda \widetilde{P}_\lambda(x)\, P_\lambda(y) = \widetilde{\Pi}(x,y), \quad \text{where} \quad \widetilde{\Pi}(x,y) = \prod_{i,j} \frac{1}{(x_i y_j;q)_\infty}. \tag{8.3}$$

An advantage of modified Macdonald polynomials is that they have nice transition coefficients with classical Schur functions $s_\lambda(x)$:

$$\widetilde{J}_\mu(x;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t)\, s_\lambda(x), \tag{8.4}$$

where $K_{\lambda,\mu}(q,t)$ are the double Kostka coefficients. By Theorem 3.2, we already know that $K_{\lambda,\mu}(q,t) \in \mathbb{Z}[q,t]$ for all λ and μ.

We now introduce a family of functions $B_{\lambda,\mu}(q,t)$ via decomposition

$$\widetilde{J}_\lambda(x;q,t) = \sum_\lambda B_{\lambda,\mu}(q,t)\, m_\mu(x) \tag{8.5}$$

in terms of monomial symmetric functions. Note that $B_{\lambda,\mu}(q,t) = 0$ unless $|\lambda| = |\mu|$. Using the Kostka numbers $K_{\lambda,\mu}$ defined by

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu}\, m_\mu(x), \tag{8.6}$$

one can express the coefficient $B_{\lambda,\mu}(q,t)$ as

$$B_{\lambda,\mu}(q,t) = \sum_\nu K_{\nu,\lambda}(q,t)\, K_{\nu,\mu}. \tag{8.7}$$

Theorem 8.1. *For any partitions λ and μ of a given natural number n, we have*

(1) $B_{\lambda,\mu}(q,t) \in \mathbb{Z}[q,t]$,

(2) $B_{\lambda,\mu}(1,1) = \dfrac{n!}{\mu_1!\mu_2!\cdots}$,

(3) $B_{(n),\mu}(q,t) = q^{n(\mu')}\dfrac{(q;q)_n}{(q;q)_{\mu_1}(q;q)_{\mu_2}\cdots}$,

(4) $B_{\lambda',\mu}(q,t) = q^{n(\lambda')}t^{n(\lambda)}B_{\lambda,\mu}(t^{-1},q^{-1})$ *(Duality)*,

where $n(\lambda') = \sum_{s\in\lambda} a(s)$ and $n(\lambda) = \sum_{s\in\lambda} \ell(s)$.

Conjecture 8.2?. $B_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$ *for any partitions λ and μ.*

We remark that Conjecture 8.2? follows from the positivity conjecture of Macdonald [Ma1], (VI.8.18?) on the double Kostka polynomials. Hence it would be natural to consider the polynomials $B_{\lambda,\mu}(q,t)$ as a two-parameter deformation of the multinomial coefficients.

Proof of Theorem 8.1. The first statement follows from Theorem 3.2.(2) since $K_{\nu,\mu} \in \mathbb{N}$. As for the second statement, let us remark that $K_{\nu,\lambda}(1,1) = f^{\nu}$ is the number of standard Young tableaux of shape ν ([Ma1], (VI.8.16)). Hence

$$B_{\lambda,\mu}(1,1) = \sum_{\nu} f^{\nu}K_{\nu,\mu} = \frac{n!}{\mu_1!\mu_2!\cdots}. \tag{8.8}$$

The last equality is well-known and in fact follows for example from the Robinson-Schensted-Knuth correspondence (see e.g. [S]). Similarly, if $\lambda = (n)$, then one has $K_{\nu,(n)}(q,t) = K_{\nu,(1^n)}(q)$ (see [Ma1], p.362). Hence

$$B_{(n),\mu}(q,t) = \sum_{\nu} K_{\nu,(1^n)}(q)K_{\nu,\mu} = q^{n(\mu')}\frac{(q;q)_n}{(q;q)_{\mu_1}(q;q)_{\mu_2}\cdots}. \tag{8.9}$$

The last equality is also well-known and is proven for example in [DJKMO] or [K], §2.4. Finally, statement (4) follows from the corresponding duality theorem for the double Kostka polynomials (see [Ma1], (VI.8.15) and (VI.8.5)). □

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