Affine Invariant Flows in the Beltrami Framework

Nir Sochen
Department of Applied Mathematics
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel
email: sochen@math.tau.ac.il
http://www.math.tau.ac.il/∼sochen

Abstract. We analyze the role of different invariant principles in image processing and analysis. A distinction between the passive and active principles is emphasized, and the geometric Beltrami framework is shown to incorporate and explain some of the known invariant flows e.g. the equi-affine invariant flow for hypersurfaces. It is also demonstrated that the new concepts put forward in this framework enable us to suggest new invariants namely the case where the codimension is greater than one.

1. Introduction

The analysis of symmetries in a problem is an important issue in computer vision. In many problems e.g. camera calibration, stereo and motion one needs to consider the Euclidean, Affine or Projective groups acting on the physical space or on the image plane. Sometimes we encounter different groups acting on the feature space, e.g. color space under change of illumination. It is desirable in all these cases to have a denoising process that can act on images without any bias towards specific feature. We say, then, that it is invariant under the transformation group of interest.

We are mostly interested in this work in the relation of the (equi-) affine transformation and diffusion-like denoising processes. The issue was first addressed in the context of linear scale-space (**?**)(**?**) where the invariance to Euclidean transformations is invoked. The reasoning was done mainly via the theory of filters rather than on the diffusion equation (**?**). The invariance to the affine group and to monotone change of the intensity values was proposed, in the seminal work (**?**), as part of the axioms that define the allowed partial differential equations (PDEs) in image processing. About the same time the same kind of problems were tackled and solved from a slightly different point of view. Researchers in shape recognition and shape evolution derived the same affine invariant equation (**?**; **?**; **?**; **?**; **?**; **?**). The main mathematical concept in these works is the differential invariant. The works (**?**; **?**; **?**) may serve as pointers to the subject. In almost all works only hypersurfaces i.e.

codimension 1 are considered. Most works focus their attention on the two dimensional case.

We present in this work a framework that provides non-linear heat-like denoising flows that act on the combined *spatial-feature* domain and are invariant under the equi-affine, i.e. $SL(n, \mathbb{R})$, flow. Our point of depart is the Beltrami framework. In this framework images are described as trinaries (Σ, M, X) where Σ and M are Riemannian manifolds and $X : \Sigma \to M$ is the embedding map. On the space of these embedding maps we define a functional whose gradient descent minimization leads to a non-linear heat-like equation. The heat-like flow generates a semi-group of transformations on the image. Constructing an invariant flow boils down to choosing a special Riemannian structure i.e. a metric, on the Σ manifold such that the flow semi-group transformation commutes with the group of spatial-feature transformations.

We derive a condition for the transformations to commute, and use our machinery to build several invariant flows. We re-derive in a unified way all known codimension 1 results (**?**; **?**; **?**; **?**; **?**; **?**; **?**; **?**). We then generalize for codimension > 1 and find the equi-affine invariant metrics and flows of a curve in $\mathbb{R}^n$ for all n. This is a generalization of the case $n = 3$ that was derived in Keriven's thesis (**?**). We find next the invariant metrics of a surface in $\mathbb{R}^n$ for $n = 4, 5, 6$, and of a three-dimensional manifold in $\mathbb{R}^5$. These flows represent the evolution of an image with vector valued pixels (e.g. color image). Note that the flows are invariant under transformation of the spatial domain and the feature domain at the same time.

The paper is organized as follows: A review of the Beltrami framework is presented in Section 2. In Section 3 we discuss general necessary conditions on the metric which are derived from considerations of passive coordinates transformations. The analysis of active coordinate change by a group of transformations and the condition on the metric that follows from the invariance requirement is treated in Section 4. Section 5 is devoted to hypersurfaces equi-affine invariant flows. We treat the codimension > 1 flows in Section 6 and conclude in Section 7.

2. The Beltrami framework

Let us briefly review the Beltrami framework for non-linear diffusion in computer vision (**?**; **?**; **?**).

We represent an image and other local features as an embedding maps of a Riemannian manifold in a higher dimensional space. The simplest example is a gray-level image which is represented as a 2D surface embedded in $\mathbb{R}^3$. We denote the map by $X : \Sigma \to \mathbb{R}^3$. Where Σ is a two-dimensional surface, and we denote the local coordinates on it by (σ^1, σ^2). The map X is given in general by $(X^1(\sigma^1, \sigma^2), X^2(\sigma^1, \sigma^2), X^3(\sigma^1, \sigma^2))$. In our example we represent it as follows $(X^1 = \sigma^1, X^2 = \sigma^2, X^3 = I(\sigma^1, \sigma^2))$. We choose on this surface a Riemannian structure, namely, a metric. The metric is a positive definite and a symmetric 2-tensor that may be defined through the local distance measurements:

$$ds^2 = g_{11}(d\sigma^1)^2 + 2g_{12}d\sigma^1 d\sigma^2 + g_{22}(d\sigma^2)^2 = g_{\mu\nu}d\sigma^\mu d\sigma^\nu. \quad (1)$$

Here and below we use the Einstein summation convention: Repeated indices are summed over. We use Greek letters to index coordinates of the manifold Σ and Latin letters to index coordinates of the embedding space. We denote the elements of the inverse of the metric by $g^{\mu\nu}$. The determinant is denoted by $g = \det(g_{\mu\nu})$. It is clear from Eq. (1) that the metric elements form a symmetric and positive definite matrix.

2.1. Polyakov Action: A measure on the space of embedding maps

Denote by $(\Sigma, (g_{\mu\nu}))$ the image manifold and its metric and by $(M, (h_{ij}))$ the space-feature manifold and its metric, then the functional S attaches a real number to a map $X : \Sigma \to M$:

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int dV \langle \nabla X^i, \nabla X^j \rangle_g h_{ij}$$

where $dV = d\sigma^1 \cdots d\sigma^D \sqrt{g}$ is the volume element of the D-dimensional manifold, and $\langle \nabla X^i, \nabla X^j \rangle_g = g^{\mu\nu}\partial_\mu X^i \partial_\nu X^j$. This functional, for $D = 2$ and $h_{ij} = \delta_{ij}$, was first proposed by Polyakov (**?**) in the context of high energy physics, and the theory known as *string theory.*

Let us formulate the Polyakov action in matrix form: (Σ, G) is the image manifold and its metric as before. Similarly, (M, H) is the spatial-feature manifold and its metric. Define

$$A^{ij} = (\nabla X^i)^t G^{-1} \nabla X^j$$

The map $X : \Sigma \to M$ has a weight

$$S[X^i, G, H] = \int dV \mathrm{Tr}(AH),$$

where m is the dimension of Σ and $g = \det(G)$.

Using standard methods in the calculus of variations the Euler-Lagrange equations with respect to the embedding (assuming Euclidean embedding space i.e. $h_{ij} = \delta_{ij}$) are (see (**?**) for explicit derivation):

$$-\frac{1}{2\sqrt{g}}\frac{\delta S}{\delta X^i} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i).$$

Or in matrix form

$$-\frac{1}{2\sqrt{g}}h^{il}\frac{\delta S}{\delta X^l} = \underbrace{\frac{1}{\sqrt{g}}\mathrm{div}\left(\sqrt{g}G^{-1}\nabla X^i\right)}_{\Delta_g X^i}. \qquad (2)$$

The analogous equations for non-Euclidean embedding space are treated in (?; ?; ?; ?). Since $(g_{\mu\nu})$ is positive definite, $g \equiv \det(g_{\mu\nu}) > 0$ for all σ^μ. This factor is the simplest one that doesn't change the minimization solution while giving a reparameterization invariant expression. The operator that is acting on X^i is the natural generalization of the Laplacian from flat spaces to manifolds and is called the Laplace-Beltrami operator and is denoted by Δ_g.

The non-linear diffusion or scale-space equation emerges as a gradient descent minimization:

$$X^i_t = \frac{\partial}{\partial t}X^i = -\frac{1}{2\sqrt{g}}\frac{\delta S}{\delta X^i} = \Delta_g X^i$$

3. Passive Transformations

We derive in this section necessary conditions for a matrix to represent a Riemannian structure i.e. a metric. We have already seen that it must be a symmetric and positive definite matrix[1]. We need to understand, next, how the metric transforms under passive coordinate change, that is, under reparameterization. This is easy to figure out since distances on the image do not depend on the coordinate system. Denote he Jacobian matrix, for the reparameterization $\sigma^\mu \to \hat{\sigma}^\mu(\sigma^1, \ldots, \sigma^n)$, and its determinant by

$$(R)^\mu_\nu = \frac{d\sigma^\mu}{d\hat{\sigma}^\nu} \quad ; \quad J = \det(R).$$

Let $d\vec{\sigma}$ stand for $(d\sigma^1, \ldots, d\sigma^D)^t$ and similarly for the transformed coordinate system $d\hat{\vec{\sigma}}$. The relation between the original and transformed infenitesimals is given componentwise by $d\sigma^\mu = \frac{d\sigma^\mu}{d\hat{\sigma}^\nu}d\hat{\sigma}^\nu$ and in matrix form $d\vec{\sigma} = Rd\hat{\vec{\sigma}}$. Let $G = (g_{\mu\nu})$ stand for the matrix whose elements are the metric coefficients $g_{\mu\nu}$. From the invariance of distances we find

$$\begin{aligned} ds^2 = g_{\mu\nu}d\sigma^\mu d\sigma^\nu &= \hat{g}_{\mu\nu}d\hat{\sigma}^\mu d\hat{\sigma}^\nu \\ = d\vec{\sigma}^t G d\vec{\sigma} &= d\hat{\vec{\sigma}}^t \hat{G} d\hat{\vec{\sigma}}. \end{aligned}$$

[1] We allow in fact semi-definite matrices.

We deduce, thus, the relation between the metric elements in the two coordinate systems

$$\hat{g}_{\mu\nu} = g_{\gamma\delta} \frac{d\sigma^\gamma}{d\hat{\sigma}^\mu} \frac{d\sigma^\delta}{d\hat{\sigma}^\nu} \tag{3}$$

In matrix form the metric should transform as

$$\hat{G} = R^t G R \tag{4}$$

and the determinant

$$\hat{g} = \det(\hat{G}) = J^2 g \tag{5}$$

The Eqs. (3,5) are referred to as the tensorial properties of the metric. They form, together with the symmetry $G = G^t$ and the (semi-) positive definiteness, a set of necessary conditions.

4. Active Transformation Acting on the Embedding Space

We assume that our embedding space is the Euclidean $\mathbb{R}^n$ space. An active transformation *change* the shape of the embedded object. It does so by the group action on the embedded space. The action of the group on the Cartesian coordinates, and the transformed coordinates, are denoted by

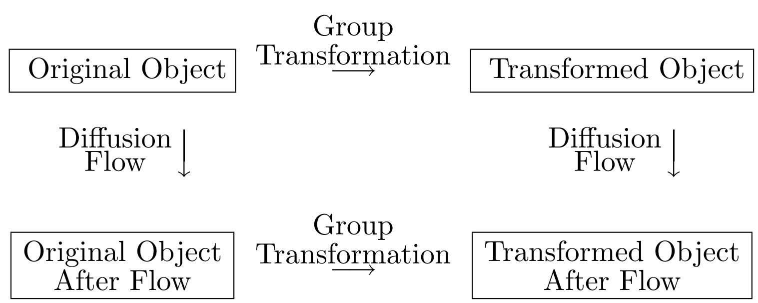
$$\tilde{X}^i = T_{\text{group}}[X^i] \qquad i = 1, \ldots, n.$$

In our case we are interested in (equi-)affine transformation of the embedding space. A generic affine transformation in n-dimensional space is

$$T_{\text{affine}}[X^i] = \tilde{X}^i = A^i_j X^j + B^i$$

where A^i_j is a non-singular constant matrix and B^i is a constant shift vector (**?**).

By group invariance we mean the commutation[2] of the following diagram:



[2] we allow commutation modulo diffeomorphism i.e. reparameterizations since we are interested in shape changing flow and the tangential part of the flow does not change the embedded shape.

To put it in a more precise form denote a transformation of the image along the diffusion (scale) parameter by $T_t[X^i]$ and the affine transformation by $T_{\mathrm{affine}}[X^i]$.

Affine invariance is a commutation relation between the transformations:

$$T_t[T_{\mathrm{affine}}[X^i]] = T_{\mathrm{affine}}[T_t[X^i]]. \tag{6}$$

It means that in the invariant flow the following is true

$$\partial_t \tilde{X}^i = \widetilde{\partial_t X^i}, \tag{7}$$

and it implies, via the Beltrami framework, that

$$\Delta_{\tilde{g}} A^i_j X^j = \Delta_{\tilde{g}} \tilde{X}^i = \widetilde{\Delta_g X^i} = A^i_j \Delta_g X^j. \tag{8}$$

It is clear now that the condition

$$(g_{\mu\nu}) = (\tilde{g}_{\mu\nu}), \quad \text{or equivalently} \quad G = \tilde{G}\ ,$$

is sufficient to insure invariance with respect to affine transformations. Note that each one of the metric's elements is affine invariant, not only the determinant.

We end this section with a comment on the invariance of diffusion-like flows with respect to more general groups. The crucial property that was used in the derivation above is the *linearity* of the affine group that acts on the embedding space. This enable us to commute the action of the group and the Laplace-Beltrami operator. For a non-linear transformation, or a local linear transformation in which the group element coefficients are locally defined by $A^i_j(\vec{X})$, the commutator is not zero and the generalized condition reads

$$\Delta_g = A^{-1} \Delta_{\tilde{g}} A\ . \tag{9}$$

One should be aware though that this is a sufficient condition and not a necessary condition. In fact we didn't take into account the freedom of reparameterization. This freedom enable us to look only on the coefficient(s) of the normal(s) direction(s) to the manifold since they are the only directions that change the shape of the manifold. Thus, for example, an invariant metric $\tilde{G} = G$ was constructed for the projective group in codimension 1. Although the condition Eq. (9) is not satisfied in this case the resulting flow is invariant. This happens since the non-invariants parts of the flow affect the reparameterization and not the shape of the manifold. One learn from this example that a better characterization of the conditions on the metric is needed. In particular the space of metrics that give rise to flows of reparameterizations only should be better studied.

5. Codimension= 1

In this, and the next, Sections we construct equi-affine invariant metrics for different embedding maps. The construction and verification of the various conditions force us to go beyond vectors and matrices and to use tensorial calculus with many indices. We introduce now some notations that will help us below (see the Appendix for explanation and examples). Define a totally antisymmetric symbol as follows

$$\mathcal{E}_{i_1 i_2 \ldots i_d} = (-1)^{s(i_1,\ldots,i_d)} \tag{10}$$

where $s(i_1,\ldots,i_d)$ is the number of basic permutations needed to bring $(i_1,\ldots i_d)$ to the form $(1,2,\ldots d)$. Although s is not well defined (there are many ways with different number of basic permutations to do that) its parity is invariant. $\mathcal{E}_{i_1 i_2 \ldots i_d} = 0$ if any of the indices appears twice.

It is convenient to use this symbol to give an explicit expression for the determinant of a matrix. Take a matrix

$$A = \begin{pmatrix} A_1^1 & A_1^2 & \ldots & A_1^d \\ A_2^1 & A_2^2 & \ldots & A_2^d \\ \vdots & \vdots & \ddots & \vdots \\ A_d^1 & A_d^2 & \ldots & A_d^d \end{pmatrix}. \tag{11}$$

The determinant is defined as follows:

$$\det A = \mathcal{E}_{i_1\ldots i_d} A_1^{i_1} A_2^{i_2} \ldots A_d^{i_d} = \mathcal{E}^{i_1\ldots i_d} A_{i_1}^1 A_{i_2}^2 \ldots A_{i_d}^d, \tag{12}$$

where summation is assumed on indices that appear twice.

It can be easily proved that

$$\mathcal{E}_{i_1 i_2 \cdots i_n} A_{p_1}^{i_1} A_{p_2}^{i_2} \cdots A_{p_d}^{i_d} = \det(A) \mathcal{E}_{p_1 p_2 \ldots p_d}. \tag{13}$$

We will use this identity time and again below. As an exercise we re-derive in the Appendix the relations (3,5) between the determinant of the metric in the original and transformed coordinate systems.

5.1. The Curve Affine Flow

We construct an affine invariant metric. Let the coordinates of $\mathbb{R}^2$ be $X^i \;\; i = 1,2$ and the curve is parameterized by σ. the line element is

$$ds^2 = g(\sigma) d\sigma^2. \tag{14}$$

Theorem
The following expression:

$$f = \mathcal{E}_{ij} X^i_\sigma X^j_{\sigma\sigma} = \det \begin{pmatrix} X^1_\sigma & X^1_{\sigma\sigma} \\ X^2_\sigma & X^2_{\sigma\sigma} \end{pmatrix} \tag{15}$$

is equi-affine invariant.
Proof:
Denote the transformed coordinates of $\mathbb{R}^2$ by tilde

$$\tilde{X}^i = A^i_j X^j, \tag{16}$$

where $\det A = 1$. Then

$$\begin{aligned} \tilde{f} &= \mathcal{E}_{ij} \tilde{X}^i_\sigma \tilde{X}^j_{\sigma\sigma} = \mathcal{E}_{ij} A^i_k X^k_\sigma A^j_l X^l_{\sigma\sigma} \\ &= (\mathcal{E}_{ij} A^i_k A^j_l) X^k_\sigma X^l_{\sigma\sigma} \\ &= \det A \mathcal{E}_{kl} X^k_\sigma X^l_{\sigma\sigma} = f \end{aligned} \tag{17}$$

where we used Eq. (16) in the second equality, rearrangement in the third equality, the identity Eq. (13) in the forth and the fact that $\det A = 1$ for equi-affine transformation in the last equality ■

Under a reparameterization of the curve $\sigma \to \hat{\sigma}(\sigma)$ it transforms as

$$\begin{aligned} \hat{f} &= \mathcal{E}_{ij} X^i_{\hat{\sigma}} X^j_{\hat{\sigma}\hat{\sigma}} = \mathcal{E}_{ij} \left(\frac{\partial \sigma}{\partial \hat{\sigma}} X^i_\sigma \right) \frac{\partial}{\partial \hat{\sigma}} \left(\left(\frac{\partial \sigma}{\partial \hat{\sigma}} \right) X^j_\sigma \right) \\ &= (\frac{\partial \sigma}{\partial \hat{\sigma}})^3 \mathcal{E}_{ij} X^i_\sigma X^j_{\sigma\sigma} + (\frac{\partial \sigma}{\partial \hat{\sigma}})(\frac{\partial^2 \sigma}{\partial \hat{\sigma}^2}) \underbrace{\mathcal{E}_{ij} X^i_\sigma X^j_\sigma}_{=0} = (\frac{\partial \sigma}{\partial \hat{\sigma}})^3 f. \end{aligned} \tag{18}$$

Define

$$g = f^{2/3} \tag{19}$$

then obviously g is invariant under equi-affine transformation and transforms as

$$\hat{g} = (\frac{\partial \sigma}{\partial \hat{\sigma}})^2 g, \tag{20}$$

under reparameterization of the curve. It follows that g is an equi-affine invariant metric for the curve. Consequently the Beltrami flow is equi-affine invariant

Theorem
The flow

$$\begin{aligned} X^i_t &= \Delta_g X^i = \frac{1}{\sqrt{g}} \partial_\sigma \sqrt{g} g^{-1} \partial_\sigma X^i \\ &= \frac{1}{\sqrt{g}} \partial_\sigma \frac{1}{\sqrt{g}} \partial_\sigma X^i = X^i_{ss} \end{aligned} \tag{21}$$

is equi-affine invariant:

$$\tilde{X}_t^i = A_j^i \Delta_g X^j \tag{22}$$

Proof:

$$\tilde{X}_t^i = \Delta_{\tilde{g}} \tilde{X}^j = \Delta_g A_j^i X^j = A_j^i \Delta_g X^j \tag{23}$$

Explicitly we get

$$\vec{X}_t = \frac{1}{g} X_{ss}^i - \frac{(\partial_s g)}{2g^2} X_s^i = a\vec{T} + b\vec{N}, \tag{24}$$

where we use the reparameterization freedom to work with the Euclidean arclength. In this parameterization $\vec{X}_s = \vec{T}$ the tangent vector and $\vec{X}_{ss} = \kappa\vec{N}$ where N is the normal and κ is the curvature. Since $\vec{X}_{ss} = \kappa\vec{N}$ for the Euclidean arc length and since $g = \kappa^{2/3}$ we get $b = \kappa^{1/3}$. The $\vec{T}$ term affect the parameterization of the curve but not its shape and can be ignored ■

5.2. The Surface Affine Flow

Following the general considerations, in Section 4, we construct an affine invariant metric for the flow of a surface in $\mathbb{R}^3$. The coordinates of the embedding space are $X^i \quad i = 1, 2, 3$. The Riemannian surface is parameterized by the local coordinates σ^1, σ^2.

Theorem

The expression

$$f_{\mu\nu} = 2\mathcal{E}_{ijk} X_{\sigma^1}^i X_{\sigma^2}^j X_{\mu\nu}^k = \mathcal{E}^{\theta\phi} \mathcal{E}_{ijk} X_\theta^i X_\phi^j X_{\mu\nu}^k \tag{25}$$

is equi-affine invariant.

Proof:

The proof is similar to the one we gave for affine curve evolution:

$$\begin{aligned} \tilde{f}_{\mu\nu} &= \mathcal{E}_{ijk} \tilde{X}_{\sigma^1}^i \tilde{X}_{\sigma^2}^j \tilde{X}_{\mu\nu}^k = \mathcal{E}_{ijk} A_l^i X_{\sigma^1}^l A_m^j X_{\sigma^2}^m A_n^k X_{\mu\nu}^n \\ &= \det A \mathcal{E}_{ijk} X_{\sigma^1}^i X_{\sigma^2}^j X_{\mu\nu}^k = f_{\mu\nu} \end{aligned} \tag{26}$$

where we used Eq. (16) in the second equality, the identity Eq. (13) in the third and the fact that $\det A = 1$ for equi-affine transformation in the last equality ■

A metric should also transform in a specific way under a change in the reparameterization $\sigma \to \hat{\sigma}(\sigma)$. Let us first rewrite $f_{\mu\nu}$ in a more convenient form. We use the antisymmetry of the $\mathcal{E}$ and the fact that i, j and k are dummy indices that are being summed over to write

$$f_{\mu\nu} = -\mathcal{E}_{ijk} X_{\sigma^2}^i X_{\sigma^1}^j X_{\mu\nu}^k = -\mathcal{E}_{jik} X_{\sigma^2}^j X_{\sigma^1}^i X_{\mu\nu}^k$$

$$= -\mathcal{E}_{jik} X^i_{\sigma^1} X^j_{\sigma^2} X^k_{\mu\nu} = \mathcal{E}_{ijk} X^i_{\sigma^1} X^j_{\sigma^2} X^k_{\mu\nu}$$

where in the first equality we use our freedom to rename the dummy indices to rename the first index of summation j and to rename the second i. We use the fact that $X^i_{\sigma^2}$ and $X^j_{\sigma^1}$ are numbers in order rewrite their multiplication in different order. The last equality follow the antisymmetry of $\mathcal{E}$, i.e. $\mathcal{E}_{jik} = -\mathcal{E}_{ijk}$.

The element $f_{\mu\nu}$ can be rewritten now (up to a multiplication by 2) as

$$f_{\mu\nu} = \mathcal{E}^{\theta\phi} \mathcal{E}_{ijk} X^i_\theta X^j_\phi X^k_{\mu\nu} = \mathcal{E}_{ijk} X^i_{\sigma^1} X^j_{\sigma^2} X^k_{\mu\nu} - \mathcal{E}_{ijk} X^i_{\sigma^2} X^j_{\sigma^1} X^k_{\mu\nu}$$

Denote by f the determinant of $f_{\mu\nu}$ then f and $f_{\mu\nu}$ transform as follows

$$\begin{aligned} \hat{f}_{\mu\nu} &= \frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu} \frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu} \mathcal{E}^{\theta\phi} \frac{\partial\sigma^a}{\partial\hat{\sigma}^\theta} \frac{\partial\sigma^b}{\partial\hat{\sigma}^\phi} \mathcal{E}_{ijk} X^i_a X^j_b X^k_{\lambda\rho} \\ &= \frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu} \frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu} J \mathcal{E}^{ab} \mathcal{E}_{ijk} X^i_a X^j_b X^k_{\lambda\rho} = J \frac{\partial\sigma^\lambda}{\partial\hat{\sigma}^\mu} \frac{\partial\sigma^\rho}{\partial\hat{\sigma}^\nu} \hat{f}_{\lambda\rho} \\ \hat{f} &= J^4 f \end{aligned} \tag{27}$$

Where we used the identity eq. (13) to write

$$\mathcal{E}^{\theta\phi} \frac{\partial\sigma^a}{\partial\hat{\sigma}^\theta} \frac{\partial\sigma^b}{\partial\hat{\sigma}^\phi} = J \mathcal{E}^{ab} \ .$$

It is clear that $g_{\mu\nu} = f_{\mu\nu}/f^{1/4}$ satisfies both the equi-affine and the metric transformation rules.

5.3. Hypersurfaces in $\mathbb{R}^{n+1}$

We construct, along the same lines, an equi-affine invariant metric for higher dimensional hypersurfaces i.e. manifolds with codimension 1. The notations are similar to the two-dimensional case. Let $X^i \quad i = 1, \ldots, n+1$ be the coordinates of $\mathbb{R}^{n+1}$, and $\sigma^1, \ldots, \sigma^n$ the local coordinates of the n-dimensional Riemannian manifold embedded in $\mathbb{R}^{n+1}$.

Theorem

The line element $g_{\mu\nu} = f_{\mu\nu}/f^{1/(n+2)}$, where

$$f_{\mu\nu} = \mathcal{E}^{\rho_1\rho_2\ldots\rho_n} \mathcal{E}_{i_1 i_2 \ldots i_n i_{n+1}} X^{i_1}_{\rho_1} X^{i_2}_{\rho_2} \cdots X^{i_n}_{\rho_n} X^{i_{n+1}}_{\mu\nu} \tag{28}$$

is equi-affine invariant.

Proof: The invariance with respect to equi-affine transformations is verified in Analogous way to eq. (26). The tensorial properties of the metric under reparameterization can be checked along the same line as the case of the surface embedded in $\mathbb{R}^3$ ■

6. Codimension > 1

The results in Section 5 are well known (**?**). They were rephrased in a language that facilitates generalizations. Several examples for such possible generalizations follow.

6.1. CURVES

Let us analyze first the equi-affine flow of a curve embedded in $\mathbb{R}^3$. The Cartesian Coordinates are $X^i \quad i = 1, 2, 3$. The curve is parameterized by σ, or by the arclength s. The Serret-Frenet structure equations are

$$\frac{\partial}{\partial s}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} \tag{29}$$

where $\vec{T}$, $\vec{N}$ and $\vec{B}$ are the tangent, normal and binormal unit vectors respectively. They form a right hand frame at point σ on the curve. κ is the curvature of the curve and τ is its torsion.

Theorem (?)

The flow

$$\vec{X}_t = (\frac{\kappa}{\tau})^{1/3}\vec{N} \tag{30}$$

is equi-affine invariant.

Proof:

Clearly the following expression

$$g(\sigma) = (\mathcal{E}_{ijk}X^i_\sigma X^j_{\sigma\sigma}X^k_{\sigma\sigma\sigma})^{1/3} \tag{31}$$

is an equi-affine invariant metric.

Since $\vec{X}_s = \vec{T}$ and $\vec{X}_{ss} = \vec{T}_s = \kappa\vec{N}$ it follows that

$$\vec{X}_{sss} = \kappa_s\vec{N} + \kappa\vec{N}_s = -\kappa^2\vec{T} + \kappa_s\vec{N} + \kappa\tau\vec{B}.$$

Using this identity we find

$$g(s) = \Big((\mathcal{E}_{ijk}T^i(\kappa N^j)(-\kappa^2T^k + \kappa_sN^k + \kappa\tau B^k)\Big)^{1/3} = \Big(\kappa^2\tau\Big)^{1/3} \tag{32}$$

where s is the Euclidean arclength. This follows from

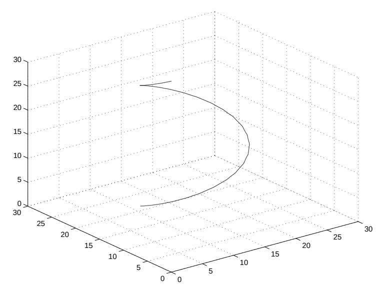
$$1 - \mathcal{E}_{ijk}T^iN^jB^k = \mathcal{E}_{ijk}T^iN^jT^k = \mathcal{E}_{ijk}T^iN^jN^k = 0\ .$$

The Beltrami flow, based on this metric, is obviously an equi-affine invariant flow. Its explicit form is

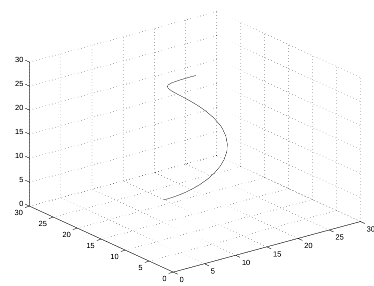
$$\vec{X}_t = \frac{1}{g}\vec{X}_{ss} - \frac{(\partial_s g)}{2g^2}\vec{X}_s = \frac{1}{(\kappa^2\tau)^{1/3}}\kappa\vec{N} - \frac{g_s}{2g^2}\vec{T}. \tag{33}$$

The assertion follows from the fact that the $\vec{T}$ term changes only the parameterization but not the shape of the curve. ■

Note that while the velocity of the curve depends on the torsion it changes its shape in the normal direction only and have zero velocity in the binormal direction. This can be seen observed easily in the example of the clockwise helix depicted in Fig. 1.



Original helix



$t = 15$

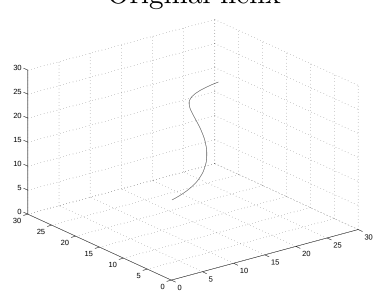
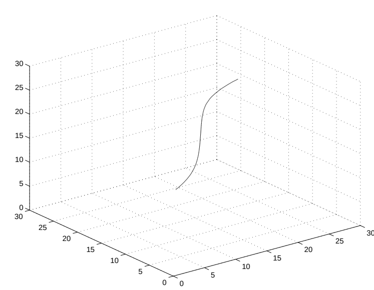


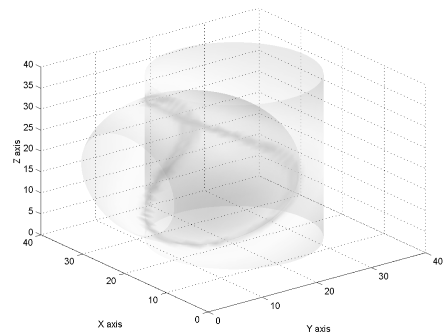
Fig. 1 $t = 30$



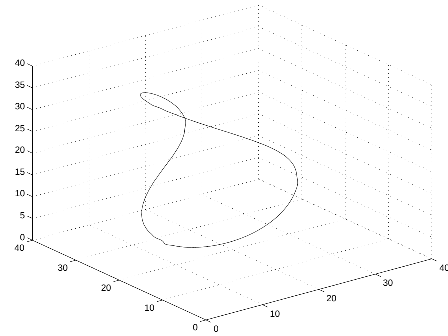
$t = 42$

We see that the helix does not shrink or expand in the z direction and the flow affect its radius only. For the clockwise helix the radius shrinks to zero and the helix converges to a straight line.

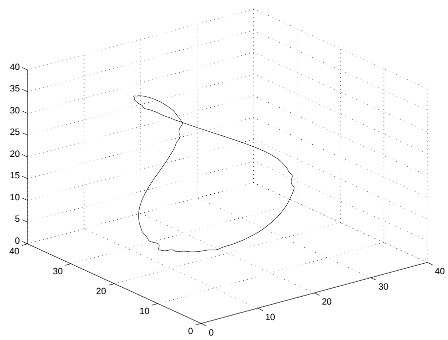
Since the torsion has a sign we get for the anticlockwise the inverse sign and the helix is expanding. This phenomenon may cause severe instability since the flow has properties of inverse diffusion. In points where the torsion change sign the two segments in its two sides flow in *inverse directions* and the flow is unstable. We show below the curve of intersection of a sphere and a cylinder. It has four point in which the torsion changes sign and we can see the instabilities that are generated in Fig. 2



Curve of intersection



$t = 5$



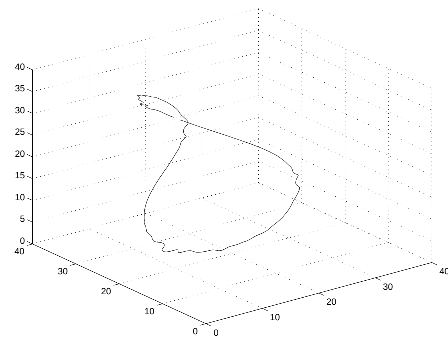


Fig. 2 $t = 15$ $t = 20$

Using the absolute value stabilizes the flow. It flows until the curve lies entirely in a plane and continues with the equi-affine curvature flow in that plane. The torsion is cutoff to a constant on that plane to regularize the division by zero[3].

The generalization to an n-dimensional curve is straightforward. The generalized Serret-Frenet relations for a curve in $\mathbb{R}^n$ are

$$\frac{\partial}{\partial s}\begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \\ \vdots \\ \vec{B}_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 & \cdots & \cdots & 0 \\ -\kappa & 0 & \tau_1 & 0 & \cdots & 0 \\ 0 & -\tau_1 & 0 & \tau_2 & \cdots & 0 \\ \vdots & 0 & -\tau_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \tau_{n-2} \\ 0 & 0 & 0 & \cdots & -\tau_{n-2} & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \\ \vdots \\ \vec{B}_{n-2} \end{pmatrix}. \tag{34}$$

The vectors $\vec{T}$, $\vec{N}$ and $\{\vec{B}_i\}_{i=1}^{n-2}$ form a positively oriented basis of $\mathbb{R}^n$ i.e. $\mathcal{E}_{i_1\cdots i_n} T^{i_1} N^{i_2} B_1^{i_3} \cdots B_{n-2}^{i_n} = 1$. The coefficients k and τ_i are the curvature and the torsions respectively.

[3] See http://www.math.tau.ac.il/∼sochen/MIA/jmiv.html

Theorem

The equi-affine invariant metric is

$$g(s) = \left(\mathcal{E}_{i_1 i_2 \dots i_n} \frac{\partial X^{i_1}}{\partial s} \frac{\partial^2 X^{i_2}}{\partial s^2} \cdots \frac{\partial^n X^{i_n}}{\partial s^n} \right)^{\frac{4}{n(n+1)}}$$
$$= \left(k^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} \right)^{4/n(n+1)} . \quad (35)$$

Proof

Notice that $X^i_s = T^i$. In the second expression X^i_{ss} only the coefficient of the normal N^i is contributing because of the antisymmetric tensor. In the third expression only the coefficient of B^i_1 is contributing since the terms of $\vec{T}$ and $\vec{N}$ are being cancelled in the alternating summation. In general, in the r-th derivative $X_{s \cdots s}$ only the coefficient of B_{r-2} is contributing. We claim that this coefficient has the general form $k \prod_{i=1}^{r-2} \tau_i$. The proof is by induction: for $r = 2$ the coefficient is k by definition i.e. $X^i_{ss} = kN^i$. For $r = 3$ we saw that it is $k\tau$. Assume now that this is the expression for the r-th derivative and check the r+1 derivative.

$$X^j_{\underbrace{s \dots s}_{r+1}} = \partial_s X^j_{\underbrace{s \dots s}_{r}} = \partial s \left[(\cdots) T^j + (\cdots) N^j + \cdots + k \prod_{i=1}^{r-2} \tau_i B^j_{r-2} \right]$$
$$= \left[(\cdots) T^j + (\cdots) N^j + \cdots + k \prod_{i=1}^{r-2} \tau_i \partial_s B^j_{r-2} \right]$$
$$= \left[(\cdots) T^j + (\cdots) N^j + \cdots + \left(k \prod_{i=1}^{r-2} \tau_i \right) \tau_{r-1} \ B^j_{r-1} \right] \quad (36)$$

where the coefficients of the lower normals and the tangent vector are changing from one side of an equality sign to the other. They are not calculated since they do not contribute to the final answer anyway. We can now proceed by direct calculation

$$g(s) = \left(\mathcal{E}_{i_1 i_2 \dots i_n} \frac{\partial X^{i_1}}{\partial s} \frac{\partial^2 X^{i_2}}{\partial s^2} \cdots \frac{\partial^n X^{i_n}}{\partial s^n} \right)^{\frac{4}{n(n+1)}}$$
$$= \left(\prod_{r=2}^{n} \left(k \prod_{i=1}^{r-2} \tau_i \right) \mathcal{E}_{i_1 i_2 \dots i_n} T^{i_1} N^{i_2} B^{i_3}_1 \cdots B^{i_n}_{n-2} \right)^{\frac{4}{n(n+1)}}$$
$$= \left(k^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} \right)^{4/n(n+1)} . \quad (37)$$

since $\mathcal{E}_{i_1 i_2 \dots i_n} T^{i_1} N^{i_2} B_1^{i_3} \cdots B_{n-2}^{i_n} = 1$ by the fact the $\vec{T}$, $\vec{N}$ and the $\vec{B}_i$s form a positively oriented basis ■

Theorem

The affine invariant flow of a curve in $\mathbb{R}^n$ is

$$\vec{X}_t = \left(\frac{k^{\frac{n^2-3n+4}{4}}}{\tau_1^{n-2} \tau_2^{n-3} \cdots \tau_{n-2}} \right)^{\frac{4}{n(n+1)}} \vec{N} \tag{38}$$

Proof

By direct computation

$$X_t^i = g^{-1} X_{ss}^i + bT^i$$

where b is some coefficient. Since the flow in the T direction is a reparameterization only we ignore it and concentrate on the first term.

$$X_t^i = g^{-1} k N^i = \left(k^{n-1} \prod_{i=1}^{n-2} \tau_i^{n-i-1} \right)^{-4/n(n+1)} k N^i$$

and the assertion follows ■

6.2. Surfaces

For codimension greater then 1 we proceed by a case by case analysis.

6.2.1. *A surface embedded in* $\mathbb{R}^4$

Consider first the case of two-dimensional surface embedded in $\mathbb{R}^4$. Define

$$f_{\mu\nu} = \mathcal{E}^{ab} \mathcal{E}^{\lambda\rho} \mathcal{E}_{i_1 i_2 i_3 i_4} X_a^{i_1} X_b^{i_2} X_{\lambda\mu}^{i_3} X_{\rho\nu}^{i_4}. \tag{39}$$

Obviously $f_{\mu\nu}$ is equi-affine invariant. Under reparametrization $f_{\mu\nu}$ and its determinant f transform as

$$\begin{aligned} \hat{f}_{\mu\nu} &= J^2 \frac{\partial \sigma^\lambda}{\partial \hat{\sigma}^\mu} \frac{\partial \sigma^\rho}{\partial \hat{\sigma}^\nu} \hat{f}_{\lambda\rho} \\ \hat{f} &= J^6 f. \end{aligned} \tag{40}$$

It is easy to show that $g_{\mu\nu} = f_{\mu\nu}/f^{1/3}$ transforms properly under a change of local coordinates and is equi-affine invariant.

6.2.2. *A surface embedded in* $\mathbb{R}^5$

The only equi-affine invariant object, that is also reparameterization invariant, up to a multiplicative function, is

$$f = \mathcal{E}_{i_1 i_2 i_3 i_4 i_5} X_1^{i_1} X_2^{i_2} X_{11}^{i_3} X_{12}^{i_4} X_{22}^{i_5}. \tag{41}$$

Clearly, it is an equi-affine invariant expression. Under a change of parameterization it transforms as

$$\hat{f} = J^4 f. \tag{42}$$

Note that it is the only possible expression and there is no second expression (like the determinant in all other cases) to cancel out the multiplicative factor. It is impossible, therefore, to write down a metric similar in form to those of the previous subsections.

6.2.3. *Surface embedded in* $\mathbb{R}^6$

Denote

$$(abc) = \mathcal{E}_{i_1 i_2 i_3 i_4 i_5 i_6} X_1^{i_1} X_2^{i_2} X_{11}^{i_3} X_{12}^{i_4} X_{22}^{i_5} X_{abc}^{i_6}$$

the following expression is equi-affine invariant:

$$f_{\mu\nu} = \mathcal{E}^{ac}\mathcal{E}^{bd}(ab\mu)(cd\nu)$$

Clearly this expression is symmetric in μ and ν.

It transforms as follows under reparameterization:

$$\hat{f}_{\mu\nu} = J^{10} \frac{\partial \sigma^\lambda}{\partial \hat{\sigma}^\mu} \frac{\partial \sigma^\rho}{\partial \hat{\sigma}^\nu} f_{\lambda\rho}$$

and the determinant transforms as $f = J^{22} f$. It follows that

$$g_{\mu\nu} = \frac{f_{\mu\nu}}{f^{5/11}}$$

transforms as a metric and is equi-affine invariant.

6.3. Volumetric data

Volumetric medical images and movies are examples of three-dimensional manifolds embedded in a higher dimensional spatial-feature space.

6.3.1. *3D manifold embedded in* $\mathbb{R}^5$

Equi-affine invariant metric for a three-dimensional manifold embedded in $\mathbb{R}^5$ is defined in terms of

$$(abcd) = \mathcal{E}_{i_1 i_2 i_3 i_4 i_5} X_1^{i_1} X_2^{i_2} X_3^{i_3} X_{ab}^{i_4} X_{cd}^{i_5}$$

the following expression is equi-affine invariant:

$$f_{\mu\nu} = \mathcal{E}^{acq}\mathcal{E}^{bpr}(abc\mu)(pqr\nu) \ .$$

Clearly this expression is symmetric in μ and ν.

It transforms as follows under reparameterization:

$$\hat{f}_{\mu\nu} = J^4 \frac{\partial \sigma^\lambda}{\partial \hat{\sigma}^\mu} \frac{\partial \sigma^\rho}{\partial \hat{\sigma}^\nu} f_{\lambda\rho}$$

and the determinant transforms as $f = J^{14} f$. It follows that

$$g_{\mu\nu} = \frac{f_{\mu\nu}}{f^{2/7}}$$

transforms as a metric and is equi-affine invariant.

6.4. The meaning of spatial-feature transformations

While spatial equi-affine transformations and monotone change of the intensity function were studied in the past, and are well understood, the coupled spatial intensity transformations were neglected. We try to provide, in this section, an intuition for the action of these transformations on images by simple examples.

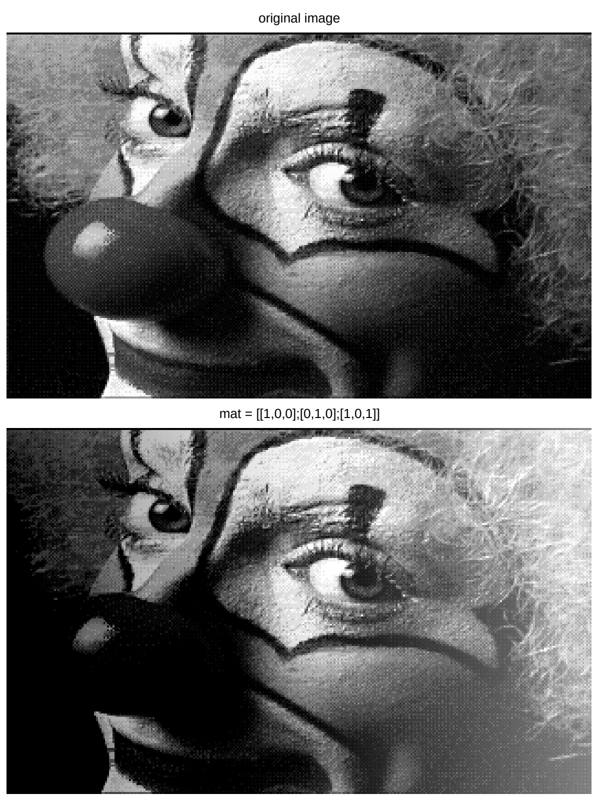


Fig. 4 Top: original image. Bottom: After “rotation” in the x-I plane.

As a simple exercise we apply various transformations on the clown image. Fig. 4 top depict the original image. We now "rotate" it in the x-I direction by the following equi-affine transformation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

The transformed image is shown in Fig. 4 bottom. This transformation leaves the spatial coordinates intact while transforming $I \to x + I$. This has the effect of changing the *direction* of the illumination.

One can, of course, include spatial transformations together with the illumination direction transformations. Few examples are shown on the clown image Fig. 5.

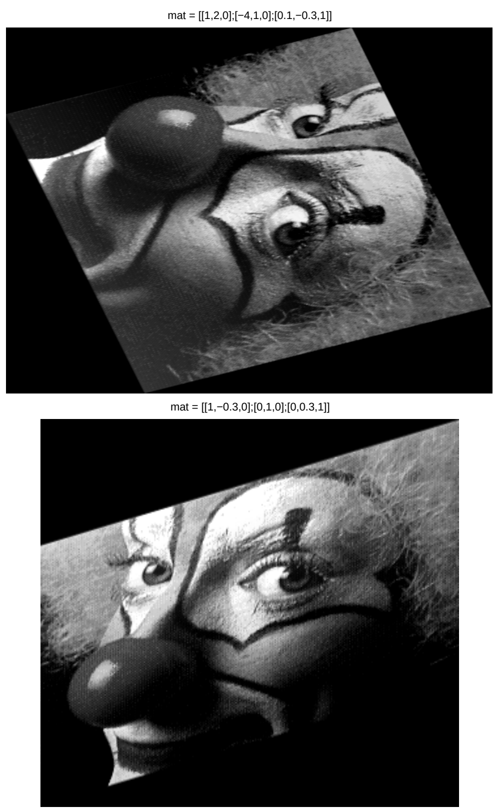


Fig. 5 Two example of spatial-feature transformations.

7. Concluding Remarks

The question of non-linear diffusion flows which are invariant under groups of transformation is studied. The Beltrami viewpoint, which separates between the image manifold and the embedding space, makes it easier to notice the difference between passive and active transformations. We analyze the conditions on the metric in order to construct an equi-affine invariant flow. We are able to generalize results from codimension 1 to higher codimension and in particular to construct an equi-affine invariant flow for a curve in any codimension and equi-affine invariant metrics for a surface in codimensions 2 and 4. These flows can be applied, in principle, to any two or four features defined on a 2D image. Given the metric, the invariant flow can be computed and projected on the normal subspace. A detailed analysis of the corresponding flows will be treated in a future publication.

We call the attention of the reader to the meaning of these transformations. Take the codimension 2 surface for example. The coordinates of the embedding space are (x, y, C_1, C_2) where C_i can be, for example, chromatic channel. The metric that we present is invariant under shifts trivially, and also under the full four-dimensional group $SL(4, \mathbb{R})$ – the group of 4×4 non-singular matrices with determinant $= 1$. This means that the metric is invariant under the combined transformations of the spatial and the color (or the features in other cases) spaces! In other words the flow is invariant under spatial and illumination transformations at the same time.

While the subject of invariant flow was extansively studied in the past, the flows of codimensions greater than 1 were not known. The present study presents a unifying framework for all previous results and opens the way to the construction of new invariants. This paper points to new possibilities and leaves many open questions. The directions for further research, that we wish to follow, are the construction of general equi-affine invariant metric for a surface in any codimension and the construction of the corresponding flows. Three dimensional manifolds are of interest as well since they represent volumetric medical images, or movies. The analytical study of the resulting flow should follow. In particular the questions of existence, uniqueness and of extremum principle (or at least stability) are of special interest. Other groups of transformations such as the full affine group and the projective group are of special interest in computer vision. The Beltrami flow allow the question of invariance in an embedding space which is not Euclidean as well. We hope to address part of these questions in the future.

Acknowledgments

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Appendix

In this appendix we explain and give simple example for the computational techniques presented in Section 4 and used extensively in the rest of the paper. We introduced in Section 4 the totally antisymmetric symbol

$$\mathcal{E}_{i_1 i_2 \dots i_d} = (-1)^{s(i_1,\dots,i_d)} \tag{43}$$

where $s(i_1, \dots, i_d)$ is the number of basic permutations needed to bring $(i_1, \dots i_d)$ to the form $(1, 2, \dots d)$. Although s is not well defined (there are many ways with different number of basic permutations to do that) its parity is invariant. $\mathcal{E}_{i_1 i_2 \dots i_d} = 0$ if any of the indices appears twice. For $d = 2$ it can be written as follows

$$(\mathcal{E}_{i_1 i_2}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

for $d = 3$ it is given by

$$\mathcal{E}_{i_1 i_2 i_3} = \begin{cases} 1 & (\mathrm{i}_1 i_2 i_3) = \text{even permutation of } (123) \\ -1 & (\mathrm{i}_1 i_2 i_3) = \text{odd permutation of } (123) \\ 0 & \text{otherwise} \end{cases}$$

We use this symbol to give an explicit expression for the determinant of a matrix. Take a matrix

$$A = \begin{pmatrix} A^1_1 & A^2_1 & \dots & A^d_1 \\ A^1_2 & A^2_2 & \dots & A^d_2 \\ \vdots & \vdots & \ddots & \vdots \\ A^1_d & A^2_d & \dots & A^d_d \end{pmatrix}. \tag{44}$$

The determinant is defined as follows:

$$\det A = \mathcal{E}_{i_1 \dots i_d} A^{i_1}_1 A^{i_2}_2 \dots A^{i_d}_d = \mathcal{E}^{i_1 \dots i_d} A^1_{i_1} A^2_{i_2} \dots A^d_{i_d}, \tag{45}$$

where summation is assumed on indices that appear twice.

The proof is by induction. Take first the determinant of a 2x2 matrix

$$\det \begin{pmatrix} A^1_1 & A^2_1 \\ A^1_2 & A^2_2 \end{pmatrix} = \mathcal{E}_{ij} A^i_1 A^j_2$$

$$
\begin{aligned}
&= \mathcal{E}_{11} A_1^1 A_2^1 + \mathcal{E}_{12} A_1^1 A_2^2 + \mathcal{E}_{21} A_1^2 A_2^1 + \mathcal{E}_{22} A_1^2 A_2^2 \\
&= A_1^1 A_2^2 - A_1^2 A_2^1. \qquad (46)
\end{aligned}
$$

It is more illuminating at this stage to work out the example of d=3 than to jump to the general case. For a general 3x3 matrix

$$
A = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \\ A_3^1 & A_3^2 & A_3^3 \end{pmatrix}. \qquad (47)
$$

The determinant is developed along a row where each element of the row is multiplied by a determinant of a 2x2 matrix for which we already proved the formula. Explicitly

$$
\det A = A_1^1 \mathcal{E}_{i_2 i_3} A_2^{i_2} A_3^{i_3} - A_1^2 \mathcal{E}_{i_1 i_3} A_2^{i_1} A_3^{i_3} + A_2^3 \mathcal{E}_{i_1 i_2} A_2^{i_1} A_3^{i_2} \qquad (48)
$$

Look now on the first term. We can rewrite it as

$$
A_1^1 \mathcal{E}_{i_2 i_3} A_2^{i_2} A_3^{i_3} = \mathcal{E}_{1 i_2 i_3} A_1^1 A_2^{i_2} A_3^{i_3}
$$

This follows from the fact that $\mathcal{E}_{ij} = \mathcal{E}_{1ij}$ for i and j that take the values $2, 3$. Similarly

$$
A_1^2 \mathcal{E}_{i_1 i_3} A_2^{i_1} A_3^{i_3} = -\mathcal{E}_{2 i_1 i_3} A_1^2 A_2^{i_1} A_3^{i_3}
$$

and

$$
A_1^3 \mathcal{E}_{i_1 i_2} A_2^{i_1} A_3^{i_2} = \mathcal{E}_{3 i_1 i_2} A_1^3 A_2^{i_1} A_3^{i_2}
$$

and the formula follows.

As an exercise we re-derive the relation between the determinant of the metric in the original and the reparameterized coordinate systems. The Jacobian matrix of the reparameterization is the matrix whose elements are $\partial\sigma^\mu / \partial\hat{\sigma}^\nu$. The Jacobian is the determinant of this matrix:

$$
J = \det(\partial\sigma^\mu / \partial\hat{\sigma}^\nu) = \mathcal{E}^{\mu_1 \mu_2 \ldots \mu_d} \frac{\partial\sigma^1}{\partial\hat{\sigma}^{\mu_1}} \frac{\partial\sigma^2}{\partial\hat{\sigma}^{\mu_2}} \cdots \frac{\partial\sigma^d}{\partial\hat{\sigma}^{\mu_d}} \qquad (49)
$$

Using the identity Eq. (13) we get

$$
\mathcal{E}^{\mu_1 \mu_2 \ldots \mu_d} \frac{\partial\sigma^{\nu_1}}{\partial\hat{\sigma}^{\mu_1}} \frac{\partial\sigma^{\nu_2}}{\partial\hat{\sigma}^{\mu_2}} \cdots \frac{\partial\sigma^{\nu_d}}{\partial\hat{\sigma}^{\mu_d}} = J \mathcal{E}^{\nu_1 \nu_2 \ldots \nu_d}. \qquad (50)
$$

The determinant g transforms as follows:

$$
\begin{aligned}
\hat{g} &= \mathcal{E}^{\mu_1 \mu_2 \ldots \mu_d} \hat{g}_{1\mu_1} \hat{g}_{2\mu_2} \cdots \hat{g}_{d\mu_d} \\
&= \mathcal{E}^{\mu_1 \mu_2 \ldots \mu_d} g_{\gamma_1 \delta_1} \frac{d\sigma^{\gamma_1}}{d\hat{\sigma}^1} \frac{d\sigma^{\delta_1}}{d\hat{\sigma}^{\mu_1}} \cdots g_{\gamma_d \delta_d} \frac{d\sigma^{\gamma_d}}{d\hat{\sigma}^d} \frac{d\sigma^{\delta_d}}{d\hat{\sigma}^{\mu_d}} \\
&= J \mathcal{E}^{\delta_1 \delta_2 \ldots \delta_d} g_{\gamma_1 \delta_1} \frac{d\sigma^{\gamma_1}}{d\hat{\sigma}^1} \cdots g_{\gamma_d \delta_d} \frac{d\sigma^{\gamma_d}}{d\hat{\sigma}^d}
\end{aligned}
$$

$$= Jg\mathcal{E}_{\gamma_1\gamma_2\ldots\gamma_d}\frac{d\sigma^{\gamma_1}}{d\hat{\sigma}^1}\frac{d\sigma^{\gamma_2}}{d\hat{\sigma}^2}\ldots\frac{d\sigma^{\gamma_d}}{d\hat{\sigma}^d} = (J^2)g. \tag{51}$$

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