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# AFFINE MANIFOLDS AND SOLVABLE GROUPS

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Let $M$ be a *compact* affine manifold. Thus $M$ has a distinguished atlas whose coordinate changes are locally in $\mathrm{Aff}(E)$, the group of affine automorphisms of Euclidean $n$-space $E$. Assume $M$ is connected and without boundary.

The universal covering $\widetilde{M}$ of $M$ has an affine immersion $D\colon \widetilde{M} \longrightarrow E$ which is unique up to composition with elements of $\mathrm{Aff}(E)$. Corresponding to $D$ there is a homomorphism $\alpha\colon \pi \longrightarrow \mathrm{Aff}(E)$, where $\pi$ is the group of deck transformations of $\widetilde{M}$, such that $D$ is equivariant for $\alpha$. Set $\alpha(\pi) = \Gamma$. Let $L\colon \mathrm{Aff}(E) \longrightarrow GL(E)$ be the natural map.

THEOREM 1. *If $\Gamma$ is nilpotent the following are equivalent*:

(a) *$M$ is complete, i.e. $D\colon \widetilde{M} \longrightarrow E$ is bijective*;

(b) *$D$ is surjective*;

(c) *no proper affine subspace of $E$ is invariant under $\Gamma$*;

(d) *$L(\Gamma)$ is unipotent*;

(e) *$M$ has parallel volume, i.e. $L(\Gamma) \subset SL(E)$*;

(f) *$M$ is affinely isomorphic to $\Gamma\backslash G$ where $G$ is a connected Lie group with a left-invariant affine structure and $\Gamma \subset G$ is a discrete subgroup*;

(g) *each de Rham cohomology class of $M$ is represented by a differential form whose components in affine charts are polynomials.*

For abelian $\Gamma$ the equivalence of (a), (d), and (e) is due to J. Smillie. We conjecture that (a), (b), (e), and (g) are equivalent even without nilpotence (if $M$ is orientable). In general (a) $\Rightarrow$ (c) and (e) $\Rightarrow$ (c); but (c) $\not\Rightarrow$ (a) even for $\Gamma$ solvable and $M$ three-dimensional.

THEOREM 2. *The following are equivalent*:

(i) *$M$ is finitely covered by a complete affine nilmanifold $M_1$ (i.e. conditions* (a) *through* (g) *of Theorem 1 hold for $M_1$)*;

(ii) *all eigenvalues of elements of $L(\Gamma)$ have norm* 1;

(iii) *$M$ has a Riemannian metric whose coefficients in affine charts are polynomials.*

L. Auslander has conjectured that if $M$ is complete then $\pi = \Gamma = \pi_1(M)$ is virtually solvable (i.e. contains a solvable subgroup of finite index); see [M] for discussion. This conjecture is true in dimension three (see [FG]).

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THEOREM 3. *If $\pi$ is virtually solvable and $M$ is complete then* (e), (f), (g) *of Theorem* 1 *hold. If* $\alpha$: $\pi \longrightarrow \Gamma$ *factors through a virtually polycyclic group of* rank $\leqslant \dim M$ *and $M$ has parallel volume, then $M$ is complete. In particular if $M$ is finitely covered by a manifold homeomorphic to a solvmanifold then parallel volume is equivalent to completeness.*

We briefly indicate the proof of Theorem 1.

(a) $\Rightarrow$ (c). This holds for any compact complete $M$. If $F \subset E$ is a $\Gamma$-invariant affine subspace then both $E/\Gamma$ and $F/\Gamma$ are Eilenberg-Mac Lane spaces of type $K(\pi, 1)$. Since they are compact manifolds their dimensions are equal; thus $F = E$.

(e) $\Rightarrow$ (c). This holds for all compact $M$. The linear holonomy $\rho = L \circ \alpha$: $\pi \longrightarrow GL(E)$ determines a $\pi$-module $E_\rho$. Let $u$: $\pi \longrightarrow E$ send $g \in \pi$ into the translational part of $\alpha(g)$. Then $u$ is a crossed homomorphism whose cohomology class $c_M \in H^1(\pi; E_\rho)$ depends only on $M$. The $n$th exterior power $\Lambda^n c_M \in H^n(\pi; \mathbf{R})$ vanishes if and only if $M$ does not have parallel volume. On the other hand $c_M$ can be expressed in de Rham cohomology of $M$ with coefficients in $E$ twisted by $\rho$. Suppose $F \subset E$ is $\Gamma$-invariant. We may assume $F$ is a linear $\rho$-invariant subspace. Then $c_M$ comes from $H^1(\pi; F_\rho)$. If $\dim F < n = \dim E$ then $\Lambda^n c_M$ comes from $H^1(\pi; \Lambda^n F) = 0$.

From now on assume $\Gamma$ is nilpotent.

(c) $\Rightarrow$ (d). Let $E_U \subset E$ be the maximal unipotent submodule. Then $H^0(\pi; E/E_U) = 0$, and nilpotent implies $H^1(\pi; E/E_U) = 0$ (Hirsch [H]). This means some coset of $E_U$ is $\Gamma$-invariant.

(b) $\Rightarrow$ (d). Suppose $E_U \neq E$. Some coset of $E_U$ is $\Gamma$-invariant; we may assume $E_U$ is $\Gamma$-invariant. There is a unique $L(\Gamma)$-invariant splitting $E = E_U \oplus F$. Let $M_1 = p(D^{-1}E_U)$ where $p$: $\widetilde{M} \longrightarrow M$ is the projection. Then $M_1$ is a compact affine manifold with unipotent holonomy, hence complete. Let $Y$ be the vector field on $\widetilde{M}$ which is $D$-related to the vector field $(x, y) \mapsto (0, y)$ on $E_U \oplus F$. Then $Y$ covers a vector field on $M$, so $Y$ is completely integrable. Every component of $p^{-1}M_1$ is a repellor for $Y$. One uses these facts to prove that $M$ is complete; but this implies (c), and hence (d).

(d) $\Rightarrow$ (a). When $L(\Gamma)$ is unipotent there is a flag $E = E_n \supset \cdots \supset E_0 = \{0\}$ of $L(\Gamma)$-invariant linear subspaces with $L(\Gamma)$ acting trivially on each $E_i/E_{i-1}$. There are nested foliations $\mathcal{Z}_n, \ldots, \mathcal{Z}_0$ on $M$ covered by foliations $\widetilde{\mathcal{Z}}_i$ on $\widetilde{M}$ such that $D$ relates $\widetilde{\mathcal{Z}}_i$ to the linear foliation $\mathcal{E}_i$ of $E$ whose leaves are cosets of $E_i$. For each $i$ there is a closed 1-form $\widetilde{\omega}_i$ on $\widetilde{\mathcal{Z}}_i$ which vanishes on $\widetilde{\mathcal{Z}}_{i-1}$, related by $D$ to a constant 1-form on $E$ vanishing on $E_{i-1}$. There are completely integrable vector fields $X_i$ in $\widetilde{\mathcal{Z}}_i$ with $\langle X_i, \omega_i \rangle = 1$. Given any $p \in \widetilde{M}$, $x \in E$ one shows that the trajectory of $X_n$ through $p$ meets a point $p_1$ such that $D(p_1)$ is the leaf of $\mathcal{E}_{n-1}$ through $x$. The trajectory of $X_{n-1}$ through $p_1$ stays in a leaf of $\widetilde{\mathcal{Z}}_{n-1}$ and eventually meets a $p_2$ such that $D(p_2)$ is the leaf of $\mathcal{E}_{n-2}$ through $x$, etc. In this way one proves that $D(\widetilde{M})$ contains a path from $D(p)$ to $x$. Hence $D$ is surjective. Injectivity is proved similarly.

(e) ⇒ (d). If $E_U \neq E$ let $F \subset E$ be a complementary submodule to $E_U$. One shows that some element of $L(\Gamma)$ expands $F$, contradicting parallel volume.

(a) ⇒ (b) and (d) ⇒ (e) are obvious.

(a) ⇒ (f). $G$ is the algebraic hull of $\Gamma$ in Aff$(E)$.

(f) ⇒ (g). By Nomizu's theorem [N] the cohomology of $M$ is represented by invariant forms on $G$; these turn out to be polynomial.

(g) ⇒ (e). If $L(\Gamma)$ is not unipotent then one proves there is no polynomial volume form.

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