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AFFINE MANIFOLDS, SYZ GEOMETRY AND THE "Y" VERTEX

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Abstract

We prove the existence of a solution to the Monge–Ampère equation $\det \mathrm{Hess}(\phi) = 1$ on a cone over a thrice-punctured two-sphere. The total space of the tangent bundle is thereby a Calabi–Yau manifold with flat special Lagrangian fibers. (Each fiber can be quotiented to three-torus if the affine monodromy can be shown to lie in $\mathbf{SL}(3, \mathbb{Z}) \ltimes \mathbb{R}^3$.) Our method is through Baues and Cortés's result that a metric cone over an elliptic affine sphere has a parabolic affine sphere structure (i.e., has a Monge–Ampère solution). The elliptic affine sphere structure is determined by a semilinear PDE on $\mathbb{CP}^1$ minus three points, and we prove existence of a solution using the direct method in the calculus of variations.

1. Introduction

The basic question we would like to understand is, What does the geometry of a Calabi–Yau manifold look like near (or "at") the large complex structure limit point? In order to answer this question, one first fixes the ambiguity of rescaling the metric by an overall constant. Gromov proved that Ricci-flat manifolds with fixed diameter have a limit under the Gromov–Hausdorff metric (on the space of metric spaces).

Now by the conjecture of [**30**], one expects that near the limit, the Calabi–Yau has a fibration by special Lagrangian submanifolds which are getting smaller and smaller (than the base). The reason can be found by looking at the mirror large radius limit. Fibers are mirror to the zero brane, and the base is mirror to the $2n$ brane, which becomes large at large radius. Metrically, the Calabi–Yau geometry should be roughly a fibration over the moduli space of special Lagrangian tori (T). The dual fibration is by dual tori, Hom $(\pi_1(T), \mathbb{R})/$Hom $(\pi_1(T), \mathbb{Z})$. The flat fiber geometry of the dual torus fibration has a flat fiber dual, which is not the same as the original geometry, but should be the same after corrections by disk instantons. These should get small in the limit of small tori, though. Namely, we expect that the Gromov–Hausdorff

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limit of a fixed-diameter Calabi–Yau manifold approaching a maximal-degeneration point carries the same geometry as the moduli space of special Lagrangian tori. That is, it is a manifold (and an affine manifold at that) of half the dimension.

Further, the Calabi–Yau near the limit should be "asymptotically close" to the standard flat torus fibration over special Lagrangian tori moduli space, whose fibers are the flat tori (we get this from dual torus considerations applied to the mirror manifold). This space is a quotient of the tangent space of the moduli space (which is Hessian), and the Calabi–Yau condition means that the limiting affine manifold metric should be Monge–Ampère ($\det \operatorname{Hess} \Phi = 1$). Global considerations require that the lattice defining the torus (generated by the vectors associated to the Hessian coordinates) is well-defined, meaning that the Monge–Ampère manifold has affine transition functions in the semi-direct product of $\mathbf{SL}(n, \mathbb{Z})$ with $\mathbb{R}^n$ translations.

This well-known conjecture (see e.g., [**14**] [**19**] [**10**]) was proved by Gross–Wilson for the special case of K3 surfaces [**14**]. Their proof uses the Ooguri–Vafa [**26**] metric in the neighborhood of a torus degeneration to build an approximate Ricci-flat metric on the entirety of an elliptic K3 with 24 singular fibers (with an elliptic-fibration/stringy-cosmic-string metric outside the patches containing the degenerations).

Another aspect of this conjecture is that the limiting manifold should have singularities in codimension two, with monodromy transformations defined for each loop about the singular set. Gross has shown the existence of a limiting singular set of codimension two for (non-Lagrangian) torus fibrations on toric three-fold Calabi–Yau's, and further has shown that the limiting singular set has the structure of a trivalent graph [**12**]. Also, Ruan has constructed Lagrangian torus fibrations with codimension-two singular locus on quintic Calabi–Yau hypersurfaces [**27**]. Taking our cue from these works, then in three dimensions, a point on the limiting manifold may be a smooth point, a point near an interval singularity, or the trivalent vertex of a "Y"-shaped singularity locus (these vertices have a subclassification based on the monodromies near the vertex). Examples of explicit Monge–Ampère metrics for points of the first two types are known, the interval singularity reducing to an interval times the two-fold point singularity. The absence of a local metric model of a trivalent vertex singularity limits our ability to prove this conjecture in three dimensions. Even if we had such a model, it might not suffice to prove the conjectures about the limiting metric, just as the non-Ooguri–Vafa elliptic fibration metric does not suffice in the two-dimensional case. Still, we regard the existence of a semi-flat

Calabi–Yau metric near the "Y" vertex as an important first step in addressing these conjectures in three dimensions.[1]

We should also remark that on other fronts, there has been much progress recently in describing the proposed limit space. There are combinatorial constructions of integral affine manifolds with singularities in the works of Haase–Zharkov [**15**] and Gross–Siebert [**13**], who discuss mirror symmetry from combinatorial and algebro-geometric points of view. Haase–Zharkov [**16**] also construct affine Kähler metrics on their examples, but these do not satisfy the Monge–Ampère equation. Recently Zharkov has put forward a detailed conjectural picture of the degeneration of Calabi–Yau metrics in toric hypersurfaces [**33**].

We therefore concern ourselves with studying Monge–Ampère manifolds in low dimensions, with the goal of finding a local model for a trivalent degeneration of special Lagrangian tori. Taking our model to be a metric cone over a thrice-punctured two-sphere, we have, by an argument of Baues and Cortés, that the sphere metric should be an elliptic affine sphere on S^2 with three singularities. The singularity type of the metric at the three points is fixed by the pole behavior of a holomorphic cubic form. Our main result is a proof of the existence of such an elliptic affine sphere, hence its cone, which solves the Monge–Ampère equation with the desired singular locus.

The plan of attack is as follows. We study the Monge–Ampère equation and affine Kähler manifolds in Section 2, exhibiting a few new solutions in three dimensions. In Section 3, we review some basic notions in affine differential geometry, and recall Baues and Cortés's result relating n-dimensional elliptic affine spheres to $(n+1)$-dimensional parabolic affine spheres. In Section 4, we study the relation between elliptic fibrations ("stringy cosmic string") and affine Kähler coordinates in two dimensions. In Section 5, we recall Simon and Wang's theory of two-dimensional affine spheres, focusing on the elliptic case. In Section 6, we first study the local structure near a singularity of the elliptic affine sphere equation, and finally show the existence of an elliptic affine sphere structure on $\mathbb{CP}^1$ minus 3 points. This is our main result. The Monge–Ampère metric near the "Y" vertex is then constructed as a cone.

Remark. The key to finding an elliptic affine sphere metric on $\mathbb{CP}^1$ minus 3 points is the PDE

$$\psi_{z\bar{z}} + |U|^2 e^{-2\psi} + \tfrac{1}{2} e^{\psi} = 0,$$

[1] This vertex is not the same as the topological vertex of [**1**], which appears at a corner of the toric polyhedron describing the Calabi–Yau. The relation between the toric description of the Calabi–Yau and the singularities of the special Lagrangian torus fibration has been discussed in [**13**].

where $U\,dz^3$ is a holomorphic cubic differential and $e^\psi|dz|^2$ is the natural affine metric on an elliptic affine sphere. It is interesting to note that a similar equation,

$$\psi_{z\bar{z}} = e^{-2\psi} - e^{\psi},$$

has also come up in the construction of special Lagrangian cones in $\mathbb{C}^3$—see McIntosh [**24**]. In McIntosh's construction, the equation comes from the geometry of $\mathbf{SU}(3)$, while in the present case, the notion of an elliptic affine sphere is invariant under $\mathbf{SL}(3,\mathbb{R})$. Such equations involving $e^{-2\psi}$ and e^ψ go back to Ţiţeica [**31**], and are naturally associated to the geometry of real forms of $\mathbf{SL}(3,\mathbb{C})$.

Correction: The first author would like to take this opportunity to correct an erroneous attribution in [**23**]. The notion that parabolic affine spheres may be represented by holomorphic data does not go back to Blaschke, but seems to be originally due to Calabi [**4**]. There is also a useful Weierstrass type representation due to Ferrer–Martínez–Milán [**9**]. The first author would like to thank Professor Martínez for pointing this out to him.

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2. Affine Kähler Metrics and the Monge–Ampère Equation

We recall that a metric is of *Hessian* type if in coordinates $\{x^i\}$ it has the form $ds^2 = \Phi_{ij}dx^i \otimes dx^j$, where $\Phi_{ij} = \partial^2\Phi/\partial x^i \partial x^j$. Hitchin proved [**17**] that natural metric ("McLean" or "Weil–Petersson") on moduli space of special Lagrangian submanifolds naturally has this structure, and the semi-flat metric on the complexification (by flat bundles) defined by the Kähler potential Φ is Ricci flat if

$$\det\,(\Phi_{ij}) = 1. \tag{1}$$

A manifold whose coordinate gluing maps are all affine maps is called an *affine manifold.* A Hessian metric on an affine manifold is called *affine Kähler*. Note that our definition of a Hessian metric is more general than that of an affine Kähler metric; this distinction is not often made in the literature.

In local Hessian coordinates, we can compute the Christoffel symbols ${\Gamma^i}_{jk} = \frac{1}{2}\Phi^{il}\Phi_{jkl}$ (where $\nabla_j\partial_k = {\Gamma^i}_{jk}\partial_i$), and defining the curvature tensor ${R_{ij}}^k{}_l = \partial_i{\Gamma^k}_{jl} + {\Gamma^k}_{im}{\Gamma^m}_{jl} - (i \leftrightarrow j)$ by $[\nabla_i,\nabla_j]\partial_k = {R_{ij}}^k{}_l\partial_l$, we find

$$R_{ijkl} = -\frac{1}{4}\Phi^{ab}[\Phi_{ika}\Phi_{jlb} - \Phi_{jka}\Phi_{ilb}]. \tag{2}$$

2.1. Hessian Coordinate Transformations. One asks, what coordinate transformations preserve the Hessian form of the metric? In particular, are there non-affine coordinate changes which preserve the Hessian character of a given metric? If we try to write $ds^2 = \Phi_{ij}dx^i dx^j = \Psi_{ab}dy^a dy^b = \Psi_{ab}y^a{}_i y^b{}_j dx^i dx^j$, then the consistency equations $\Phi_{ijk} = \Phi_{kji}$ yield conditions on the coordinate transformation $y(x)$. Specifically, we have $\partial_k(\Psi_{ab}y^a{}_i y^b{}_j) = \partial_i(\Psi_{ab}y^a{}_k y^b{}_j)$, which is equivalent to

$$\Psi_{ab}(y^a{}_i y^b_{jk} - y^a{}_k y^b{}_{ij}) = 0.$$

In two dimensions, for example, there can be many solutions to these equations. In Euclidean space $\Psi_{ab} = \delta_{ab}$ with coordinates y^a, if we put $y^1 = f(x^1 + x^2) + g(x^1 - x^2)$ and $y^2 = f(x^1 + x^2) - g(x^1 - x^2)$, then the equations are solved and we can find $\Phi(x)$. For example, if $f(s) = g(s) = s^2/2$, we find $\Phi(x) = [(x^1)^4 + 6x^1x^2 + (x^2)^4]/12$.

Note that this transformation is not affine. Thus, Hessian metrics may exist on non-affine manifolds, and our notion of Hessian metric is strictly broader than our notion of affine Kähler metric. Affine Kähler manifolds can be characterized as locally having an abelian Lie algebra of gradient vector fields acting simply transitively [**28**]. Though Hessian manifolds are not the same as affine manifolds, a Hessian manifold appearing as a moduli space of special Lagrangian tori must have an affine structure. We therefore focus on affine Kähler manifolds in this paper.

2.2. An Example of a Monge–Ampère Metric. As we will see in Section 4, there are many Monge–Ampère metrics in two dimensions, but a paucity of examples in three or more dimensions. Here, we provide one detailed example and remark how a few others may be found.

Example 1. In dimension d, consider the ansatz $\Phi = \Phi(r)$, where $r = \sqrt{\sum_i (x^i)^2}$. As shown by Calabi, the equation (1) is solved if

$$\Phi(r) = \int (1 + r^d)^{1/d}. \tag{3}$$

The rescalings $r \to cr$ and $\Phi \to c\Phi$ also have constant $\det(\mathrm{Hess}\,\Phi)$. For example, in two dimensions ($d = 2$), $\Phi(r) = \sinh^{-1}(r) + r\sqrt{1 + r^2}$ is a solution.

Remark. It is also possible to find solutions in a few other cases in dimension 3 by imposing symmetry. In particular, we let Φ take the special forms for coordinates x, y, z of $\mathbb{R}^3$.

- $\Phi = A(\rho)B(z)$ for $\rho = \sqrt{x^2 + y^2}$.
- $\Phi = \Phi(xyz)$.

- $\Phi = \Phi(xy + yz + xz)$.

Solutions follow from straightforward ODE techniques.

3. Affine Spheres

Convex functions Φ satisfying the Monge–Ampère equation $\det \Phi_{ij} = 1$ have a particularly useful interpretation in terms of affine differential geometry. The graph of such a Φ in $\mathbb{R}^{n+1}$ is a *parabolic affine sphere*. In this subsection, we introduce the basic notions of affine differential geometry and recall a recent result of Baues and Cortés which allows us to find 3-dimensional solutions to the Monge–Ampère equation by constructing a 2-dimensional elliptic affine sphere.

Affine differential geometry is the study of those properties of hypersurfaces $H \subset \mathbb{R}^{n+1}$ which are invariant under volume-preserving affine transformations. For basic background on affine differential geometry, see Calabi [**5**], Cheng–Yau [**7**] and Nomizu–Sasaki [**25**]. We assume that H is a smooth locally strictly convex hypersurface. The affine normal ξ to H is a transverse vector field on H which is invariant under the action of volume-preserving affine transformations on $\mathbb{R}^{n+1}$ in the sense that if $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is such a transformation, then at all $p \in H$,

$$\Psi_*(\xi_H(p)) = \xi_{\Psi(H)}(\Psi(p)).$$

We assume ξ points inward (i.e., at p, $\xi(p)$ is on the same side of any tangent plane of H as H is itself). Given such a transverse vector field ξ, we have the following equations. Let X, Y be tangent vector fields on H and let D denote the standard flat connection on $\mathbb{R}^{n+1}$.

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{4}$$

$$D_X \xi = -S(X). \tag{5}$$

Here, ∇ is a torsion-free connection on H, h is a Riemannian metric on H (since H is convex and ξ points inward), and S is an endomorphism of the tangent space. ∇ is called the *affine connection*, h is the *affine metric*, and S is the *affine shape operator*. The trace of S divided by the dimension n is called the *affine mean curvature*. Note that there is no part of equation (5) in the span of ξ; if $D_X\xi \in T_pH$, then ξ is said to be *equiaffine*.

The affine normal ξ can be uniquely characterized as follows:

- ξ points inward on H.
- ξ is equiaffine.
- For any basis $X_1, \dots, X_n$ of the tangent space of H,

$$\det(X_1, \dots, X_n, \xi)^2 = \det h(X_i, X_j), \tag{6}$$

  where the determinant on the left is that on $\mathbb{R}^{n+1}$, and the determinant on the right is that of an $n \times n$ matrix.

Another important invariant is the *Pick form*, which may be defined as the tensor which is the difference $C = \hat{\nabla} - \nabla$ for $\hat{\nabla}$ the Levi–Civita connection of the affine metric. The Pick form satisfies the following *apolarity* condition:

$$\sum_{i=1}^{n} C^i_{ij} = 0, \qquad j = 1, \dots, n.$$

When the upper index is lowered by the affine metric, the Pick form is totally symmetric on all three indices.

A *parabolic affine sphere* is hypersurface for which ξ is a constant vector. If $\xi = (0, \dots, 0, 1)$, then a parabolic affine sphere can locally be written as a graph $(x, \Phi(x))$ for a convex function Φ which satisfies the Monge–Ampère equation $\det \Phi_{ij} = 1$. This condition may be checked by using the conditions above for the affine normal.

A hypersurface H is an *elliptic affine sphere* if all the affine normals point toward a given point in $\mathbb{R}^{n+1}$, called the *center* of H. In this case, the affine shape operator $S = \lambda I$ for $\lambda > 0$ and I the identity operator on the tangent space. By translation, we may assume the center is the origin, and by scaling, we may assume that $\lambda = 1$. In this case, the affine normal ξ is minus the position vector.

Example 2. The unit sphere in $\mathbb{R}^{n+1}$ is an elliptic affine sphere centered at the origin. In this case, we may compute for ξ equal to minus the position vector that the affine metric h is the restriction of the Euclidean inner product. It is straightforward to check (6) is satisfied, and that ξ is the affine normal.

Baues and Cortés establish a relationship between n-dimensional elliptic affine spheres and $(n+1)$-dimensional parabolic affine spheres [**3**]. We use this result to reduce the problem of finding 3-dimensional parabolic affine spheres to the problem of finding 2-dimensional elliptic affine spheres.

Theorem 1 (Baues–Cortés)**.** *Let H be an elliptic affine sphere in $\mathbb{R}^{n+1}$ centered at the origin with affine mean curvature* 1*. Shrink H if necessary so that each ray through the origin hits H only once. Let*

$$\mathcal{C} = \bigcup_{r>0} rH$$

be the cone over H. Then, $\Phi = \frac{1}{2}r^2$ is convex and solves $\det \Phi_{ij} = 1$ *on $\mathcal{C}$. Using the diffeomorphism $\mathcal{C} \cong H \times \mathbb{R}^+$, the affine Kähler metric satisfies*

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^j}\, dx^i\, dx^j = r^2 h + dr^2$$

for h the affine metric on H.

For the reader's convenience, we provide a proof of Baues and Cortés's result, along the lines of [**21**].

Remark. A similar proof shows that

$$\Phi = \int (Kr^{n+1} + A)^{\frac{1}{n+1}}$$

solves $\det \Phi_{ij} = \text{const.}$ for constants K and A. In the case where H is the standard Euclidean sphere in $\mathbb{R}^{n+1}$, we recover Calabi's Example 1 above.

Proof. Assume H is an elliptic affine sphere centered at 0 with affine mean curvature 1, and form $\mathcal{C}$ and Φ from H as above. Denote the affine Kähler metric

$$g_{ij} = \frac{\partial^2 \Phi}{\partial x^i \partial x^j}.$$

Consider the position vector field

$$X = x^i \frac{\partial}{\partial x^i} = r \frac{\partial}{\partial r}.$$

Let $\Phi = r^2/2$. Note that

$$X\Phi = x^i \frac{\partial \Phi}{\partial x^i} = r \frac{\partial}{\partial r}\left(\frac{1}{2} r^2\right) = r^2 = 2\Phi.$$

Then, take $\partial/\partial x^j$ to find

$$x^i \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = \frac{\partial \Phi}{\partial x^j}. \tag{7}$$

Consider a vector Y tangent to the hypersurface $H = \left\{\Phi = \frac{1}{2}\right\}$, so that $Y\Phi = 0$. Then (7) shows that

$$g(X,Y) = x^i \frac{\partial^2 \Phi}{\partial x^i \partial x^j} y^j = \frac{\partial \Phi}{\partial x^j} y^j = Y\Phi = 0, \tag{8}$$

and also

$$g(X,X) = x^i \frac{\partial^2 \Phi}{\partial x^i \partial x^j} x^j = X\Phi = r^2.$$

Now, we will show that the g restricts to a multiple of the affine metric on H.

Let D be the canonical flat connection on $\mathbb{R}^{n+1}$. Then our affine Kähler metric g is given by

$$g(A,B) = (D_A d\Phi, B) \tag{9}$$

where A, B are vectors and $(\cdot\,,\cdot)$ is the pairing between one forms and vectors. X is transverse to H. So at $x \in H$, $\mathbb{R}^{n+1} = T_x(\mathbb{R}^{n+1})$ splits into $T_x(H) \oplus \langle X \rangle$. Then, since $-X$ is the affine normal,

$$D_Y Z = \nabla_Y Z + h(Y,Z)(-X) \tag{10}$$

where Y, Z are tangent vectors to H, ∇ is a torsion-free connection on $T(H)$, and h is the affine metric.

Now, consider

$$\begin{aligned} 0 &= Y(d\Phi, Z) \\ &= (d\Phi, D_Y Z) + (D_Y d\Phi, Z) \\ &= -r^2\, h(Y,Z) + g(Y,Z) \end{aligned}$$

by (8), (9) and (10). Therefore, $g(Y,Z) = r^2\, h(Y,Z)$ for Y, Z tangent to H.

So far, all calculations have been at a point in $H \subset \mathcal{C}$. A simple scaling argument shows:

Proposition 1. *Under the metric g, each level set of the potential Φ is perpendicular to the radial direction X. Under the diffeomorphism*

$$\mathcal{C} \cong H \times \mathbb{R}^+,$$

we have

$$g(X,X) = r^2, \qquad g(Y,Z) = r^2 h(Y,Z),$$

where Y and Z are in the tangent space to H and h is the affine metric of H.

Now, since H is an elliptic affine sphere, we know that if $Y_1, \dots, Y_n$ is a basis of the tangent space of H at a point, then

$$\det(Y_1, \dots, Y_n, -X)^2 = \det h(Y_i, Y_j).$$

Denote $-X$ by Y_{n+1}, and let a, b be indices from 1 to $n+1$, while i, j are indices from 1 to n. Compute using Proposition 1 for the standard frame on $\mathbb{R}^{n+1}$:

$$\begin{aligned} \det g_{ab} &= \frac{\det g(Y_a, Y_b)}{\det(Y_1, \dots, Y_{n+1})^2} \\ &= \frac{g(-X,-X) \cdot \det g(Y_i, Y_j)}{\det h(Y_i, Y_j)} = r^2 \cdot r^{2n} = r^{2n+2}. \end{aligned}$$

Since on H, $r = 1$, we have that for each point on H,

$$\det g_{ab} = \det \frac{\partial^2 \Phi}{\partial x^a \partial x^b} = 1.$$

For points not on H, the Monge–Ampère equation follows since Φ scales quadratically in r. q.e.d.

Example 3. If H is the unit sphere in $\mathbb{R}^{n+1}$, then the corresponding potential function $\Phi = \frac{1}{2}\|x\|^2$ clearly satisfies $\det \Phi_{ij} = 1$ and the metric $\Phi_{ij}\, dx^i\, dx^j$ is the standard flat metric on $\mathbb{R}^{n+1}$.

4. Two-Dimensional Monge–Ampère Metrics and the Stringy-Cosmic-String

Using a hyper-Kähler rotation, we can treat any elliptic surface as a special Lagrangian fibration and try to find its associated affine coordinates and—if the fibers are flat—the corresponding solution to the Monge–Ampère equation.

In the case of the stringy-cosmic-string, we begin with a semi-flat fibration with torus fiber coordinates $t \sim t+1$ and $x \sim x+1$. As a holomorphic fibration, the stringy-cosmic-string is defined by a holomorphic modulus $\tau(z)$. One can derive the Kähler potential through the Gibbons–Hawking ansatz (with $\partial/\partial t$ as Killing vector) using connection one-form $A = -\tau_1 dx$ and potential $V = \tau_2$ (so $*dA = dV$), then solving for the holomorphic coordinate. One finds $\xi = t + \tau(z)x = t + \tau_1 x + i\tau_2 x$. The hyper-Kähler structure is specified by the forms

$$\begin{aligned} \omega_1 &= dt \wedge dx + (\frac{i}{2})\tau_2 dz \wedge d\overline{z} \\ \omega_2 + i\omega_3 &= dz \wedge d\xi. \end{aligned}$$

The stringy-cosmic-string solution starts directly from the Kähler potential $K(z,\xi) = \xi_2/\tau_2 + k(z,\overline{z})$, where $\partial_z \partial_{\overline{z}} k = \tau_2$.

We seek the affine coordinates for the base of the semi-flat special Lagranian torus fibration. In coordinates (x, t, z_1, z_2), the metric has the block diagonal form

$$Q \oplus R \equiv \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \oplus \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}. \tag{11}$$

For a semi-flat fibration over an affine Kähler manifold in affine coordinates, the base-dependent metric on the fiber looks the same as the metric on the base. Therefore, we would need to find coordinates $u_1(z,\overline{z}), u_2(z,\overline{z})$ so that the metric in u-space looks like Q in (11). This is accomplished if the change of basis matrix $M_{ij} = \partial z_i / \partial u_j$ obeys $M^T M = Q/\tau_2$. A calculation reveals the general solution to be $M = O\widetilde{M}$, where $\widetilde{M} = \frac{1}{\tau_2}\begin{pmatrix} \tau_2 & 0 \\ \tau_1 & 1 \end{pmatrix}$, and O is an orthogonal matrix. The same result can be obtained using Hitchin's method, which we now review.

Hitchin [**17**] obtains affine coordinates on the moduli space of special Lagrangian submanifolds from period integrals. In order to apply this technique here, we first make a hyper-Kähler rotation, so that the fibration is special Lagrangian, by putting $\omega = \omega_2$, $\operatorname{Im}\Omega = \omega_3$. ($\omega$ and Ω are the symplectic and holomorphic forms of the Calabi–Yau metric, respectively.) Explicitly, for each base coordinate z_i, we construct closed one forms on the Lagrangian L, defined by $\iota_{\partial/\partial z_i}\omega = \theta_i$ and compute the periods $\lambda_{ij} = \int_{A_i} \theta_j$, where $\{A_i\}$ is a basis for $H_1(L,\mathbb{Z})$. In our case, we readily find $\theta_1 = dt + \tau_1 dx$, $\theta_2 = -\tau_2 dx$, and, using the basis $A_1 = \{t \to t+1\}$, $A_2 = -\{x \to x+1\}$, we get

$$\lambda_{ij} = \begin{pmatrix} 1 & 0 \\ -\tau_1 & \tau_2 \end{pmatrix}.$$

The forms $\lambda_{ij} dz_j$ are closed on the base and we set them equal to du_i. This defines the coordinates du_i up to constants, and we find $u_1 = z_1$, $u_2 = -\mathrm{Re}\phi$, where $\partial_z \phi = \tau$. To connect with the solution above, one easily inverts the matrix $\lambda_{ij} = \partial u_i / \partial z_j$ to find the matrix $\partial z_i / \partial u_j = \widetilde{M}$.

Legendre dual coordinates v_i are defined as follows. Define $(d-1)$-forms ψ_i (here $d = 2$, so the ψ's are also one-forms) by putting $\iota_{\partial/\partial z_i} \mathrm{Im}\,\Omega = \psi_i$ and compute the periods $\mu_{ij} = \int_{B_i} \psi_j$, where $B_j \in H_{d-1}(L, \mathbb{Z})$ are Poincaré dual to the A_i. In our example, $\psi_1 = \tau_2 dx$, $\psi_2 = dt + \tau_1 dx$, $B_1 = \{x \to x+1\}$, $B_2 = \{t \to t+1\}$, and

$$\mu_{ij} = \begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1 \end{pmatrix}.$$

(Note $\lambda^T \mu$ is symmetric, as required.) Setting $dv_i = \mu_{ij} dz_j$, we find $v_1 = \mathrm{Im}\phi$ and $v_2 = z_2$.

Hitchin showed that the coordinates u_i and v_i are related by the Legendre transformation defined by the function Φ whose Hessian gives the metric. Namely, $v_i = \partial \Phi / \partial u_i$. We can think of Φ as a function of the $z_i(u_j)$ and differentiate with respect to u_j using the chain rule. (We find $\partial z_i / \partial u_j$ by inverting the matrix of derivatives $\partial u_i / \partial z_j$.) One finds

$$\Phi_{z_1} = \mathrm{Im}\,\phi - \tau_1 z_2, \qquad \Phi_{z_2} = \tau_2 z_2.$$

(For the transformed potential Ψ, we have $\Psi_{z_1} = \tau_2 z_1$ and $\Psi_{z_2} = \tau_1 z_1 - \mathrm{Re}\phi$.) The solution can be given in terms of another holomorphic antiderivative,[2] χ, such that $\partial_z \chi = \phi$.

$$\Phi = -z_2 \mathrm{Re}\phi + \mathrm{Im}\chi.$$

Note, then, that being able to write down the explicit affine Kähler potential depends only on our ability to integrate τ and invert the functions $u_i(z_j)$. The Legendre-transformed potential is $\Psi = z_1 \mathrm{Im}\phi - \mathrm{Im}\chi$.

Example 4. $\tau = 1/z$. If we put $z = re^{i\theta}$ and take $\tau = 1/z$, then $\phi = \log z$, so $u_1 = z_1$ and $u_2 = -\mathrm{Re}\phi = -\log r$. Thus, $\Phi(u_1, u_2) = -z_2 \log \sqrt{z_1^2 + z_2^2} + \int \log \sqrt{z_1^2 + z_2^2}\, dz_2$, where $z_1 = u_1$ and $z_2 =$

[2] Cortés has found a generalization of this potential as the defining function of special Kähler manifolds, which are locally special examples of parabolic affine spheres in even dimensions, described by holomorphic data [**8**]. In the present case of dimension 2, Monge–Ampère metrics were described using holomorphic data by Calabi [**4**] and Ferrer–Martínez–Milán [**9**]. Of course, all these descriptions of two-dimensional parabolic affine spheres using holomorphic data are equivalent.

$\sqrt{e^{-2u_2} - u_1^2}$. Since $v_1 = \mathrm{Im}\Phi = \tan^{-1}(z_2/z_1)$ and $v_2 = z_2$, we may solve the equations $\partial\Phi/\partial u_i = v_i$ to find

$$\Phi = u_1 \left[\tan^{-1}\left(\sqrt{(e^{-u_2}/u_1)^2 - 1}\right) - \sqrt{(e^{-u_2}/u_1)^2 - 1}\right].$$

One easily checks that $\det(\Phi_{ij}) = 1$.

To summarize, let z be a holomorphic coordinate on the base of a semi-flat elliptic fibration. Let $\tau = \tau(z)$ be the holomorphically varying modulus of the elliptic curve on the fiber. Then, we define ϕ, χ holomorphic so that

$$\phi_z = \tau, \qquad \chi_z = \phi.$$

Let $z = z_1 + iz_2$ represent real and imaginary parts, with similar notation for the real and imaginary parts of τ, ϕ, χ. Then affine flat coordinates u_1, u_2 may be chosen as

$$u_1 = z_1, \qquad u_2 = -\phi_1.$$

The metric on the base is given by

$$\tau_2 |dz|^2 = \frac{\partial^2 \Phi}{\partial u_i \partial u_j}\, du_i du_j.$$

For the affine Kähler potential Φ, which satisfies

$$\Phi = -z_2\phi_1 + \chi_2.$$

The Legendre dual coordinates $v_i = \partial\Phi/\partial u_i$ are given by

$$v_1 = \phi_2, \qquad v_2 = z_2.$$

The potential Ψ in the v coordinates is the Legendre transform of Φ:

$$\Psi = u_1 v_1 + u_2 v_2 - \Phi = z_1\phi_2 - \chi_2.$$

Φ and Ψ satisfy the Monge–Ampère equation

$$\det\left(\frac{\partial^2 \Phi}{\partial u_i \partial u_j}\right) = 1, \qquad \det\left(\frac{\partial^2 \Psi}{\partial v_i \partial v_j}\right) = 1.$$

The metric satisfies

$$\tau_2 |dz|^2 = \frac{\partial^2 \Phi}{\partial u_i \partial u_j}\, du_i du_j = \frac{\partial^2 \Psi}{\partial v_i \partial v_j}\, dv_i dv_j.$$

5. Simon and Wang's Developing Map

Simon and Wang [**29**] formulate the condition for a two-dimensional surface to be an affine sphere in terms of the conformal geometry given by the affine metric. Since we rely heavily on this work, we give a version of the arguments here for the reader's convenience. We are primarily interested in constructing three-dimensional parabolic affine spheres by writing them as cones over elliptic affine spheres in dimension two by using Baues and Cortés's Theorem 1. Therefore, we focus our discussion to the case of elliptic affine spheres in dimension two, and conclude with some remarks about two-dimensional parabolic affine spheres from this point of view. (In Section 6, it will be useful to compare elliptic and parabolic affine spheres in dimension two, particularly since two-dimensional parabolic affine spheres admit exact solutions.)

5.1. Elliptic Affine Spheres. Before we get into the construction, a few remarks are in order. We consider a parametrization $f : \mathcal{D} \to \mathbb{R}^3$ where $\mathcal{D} \subset \mathbb{C}$ is simply connected and f is conformal with respect to the affine metric. Simon and Wang's procedure involves writing the structure equations of the affine sphere as a first-order system of PDEs (an initial-value problem) in the frame $\{f, f_z, f_{\bar z}\}$—equations (19–20) below. By the Frobenius Theorem, this initial-value problem can be solved as long as certain integrability conditions are satisfied. One of these integrability conditions is a semilinear elliptic PDE in the conformal factor of the affine metric—equation (21) below. Solving this PDE on a Riemann surface Σ then provides an immersion from the universal cover $\tilde\Sigma$ to $\mathbb{R}^3$, the image being an (immersed) elliptic affine sphere. We call this immersion Simon and Wang's *developing map*.

In the case of elliptic affine spheres, we take $\Sigma = \mathbb{CP}^1$ minus 3 points. Integrating the initial value problem along a path in $\pi_1\Sigma$ computes the monodromy of an $\mathbb{RP}^2$-structure on Σ, which upon applying Baues and Cortés's cone construction, provides the monodromy of the affine flat structure on the cone over Σ, which is $\mathbb{R}^3$ minus a “Y” vertex topologically.

We have not yet completed the ODE computation of the monodromy in the present case of an elliptic affine sphere, but note that this approach has been used to compute monodromy for convex $\mathbb{RP}^2$ structures (using hyperbolic affine spheres) [**22**], and also for a global existence result for parabolic affine spheres on S^2 minus singular points [**23**].

Consider a 2-dimensional elliptic affine sphere in $\mathbb{R}^3$. Then, the affine metric gives a conformal structure, and we choose a local conformal coordinate $z = x + iy$ on the hypersurface. The affine metric is given by $h = e^\psi |dz|^2$ for some function ψ. Parametrize the surface by $f : \mathcal{D} \to \mathbb{R}^3$, with $\mathcal{D}$ a domain in $\mathbb{C}$. Since $\{e^{-\frac12\psi} f_x, e^{-\frac12\psi} f_y\}$ is an orthonormal basis for the tangent space, the affine normal ξ must satisfy this volume condition (6)

$$\det(e^{-\frac12\psi} f_x, e^{-\frac12\psi} f_y, \xi) = 1, \tag{12}$$

which implies

$$\det(f_z, f_{\bar z}, \xi) = \tfrac12 i e^\psi. \tag{13}$$

Now, only consider elliptic affine spheres centered at the origin and with affine mean curvature scaled to be 1. In this case, the affine normal is $-f$ (minus the position vector) and we have

$$\left\{ \begin{array}{c} D_X Y = \nabla_X Y + h(X,Y)(-f) \\ D_X(-f) = -X \end{array} \right. \tag{14}$$

Here, D is the canonical flat connection on $\mathbb{R}^3$, ∇ is a torsion-free connection on the affine sphere, and h is the affine metric.

It is convenient to work with complexified tangent vectors, and we extend ∇, h and D by complex linearity. Consider the frame for the tangent bundle to the surface $\{e_1 = f_z = f_*(\frac{\partial}{\partial z}), e_{\bar 1} = f_{\bar z} = f_*(\frac{\partial}{\partial \bar z})\}$. Then, we have

$$h(f_z, f_z) = h(f_{\bar z}, f_{\bar z}) = 0, \quad h(f_z, f_{\bar z}) = \tfrac{1}{2} e^{\psi}. \tag{15}$$

Consider θ the matrix of connection one-forms

$$\nabla e_i = \theta_i^j e_j, \quad i,j \in \{1, \bar 1\},$$

and $\hat\theta$ the matrix of connection one-forms for the Levi–Civita connection. By (15)

$$\hat\theta_{\bar 1}^1 = \hat\theta_1^{\bar 1} = 0, \quad \hat\theta_1^1 = \partial\psi, \quad \hat\theta_{\bar 1}^{\bar 1} = \bar\partial\psi. \tag{16}$$

The difference $\hat\theta - \theta$ is given by the Pick form. We have

$$\hat\theta_i^j - \theta_i^j = C_{ik}^j \rho^k,$$

where $\{\rho^1 = dz, \rho^{\bar 1} = d\bar z\}$ is the dual frame of one-forms. Now, we differentiate (13) and use the structure equations (14) to conclude

$$\theta_1^1 + \theta_{\bar 1}^{\bar 1} = d\psi.$$

This implies, together with (16), the apolarity condition

$$C_{1k}^1 + C_{\bar 1 k}^{\bar 1} = 0, \quad k \in \{1, \bar 1\}.$$

Then, when we lower the indices, the expression for the metric (15) implies that

$$C_{\bar 1 1 k} + C_{1 \bar 1 k} = 0.$$

Now, C_{ijk} is totally symmetric on three indices [**7**, **25**]. Therefore, the previous equation implies that all the components of C must vanish except C_{111} and $C_{\bar 1\bar 1\bar 1} = \overline{C_{111}}$.

This discussion completely determines θ:

$$\begin{pmatrix} \theta_1^1 & \theta_{\bar 1}^1 \\ \theta_1^{\bar 1} & \theta_{\bar 1}^{\bar 1} \end{pmatrix} = \begin{pmatrix} \partial\psi & C_{\bar 1\bar 1}^1 d\bar z \\ C_{11}^{\bar 1} dz & \bar\partial\psi \end{pmatrix} = \begin{pmatrix} \partial\psi & \bar U e^{-\psi} d\bar z \\ U e^{-\psi} dz & \bar\partial\psi \end{pmatrix}, \tag{17}$$

where we define $U = C_{11}^{\bar 1} e^{\psi}$.

Recall that D is the canonical flat connection induced from $\mathbb{R}^3$. (Thus, for example, $D_{f_z} f_z = D_{\frac{\partial}{\partial z}} f_z = f_{zz}$.) Using this statement, together with (15) and (17), the structure equations (14) become

$$\left\{ \begin{array}{rcl} f_{zz} &=& \psi_z f_z + U e^{-\psi} f_{\bar z} \\ f_{\bar z \bar z} &=& \bar U e^{-\psi} f_z + \psi_{\bar z} f_{\bar z} \\ f_{z \bar z} &=& -\frac12 e^{\psi} f \end{array} \right. \tag{18}$$

Then, together with the equations $(f)_z = f_z$, $(f)_{\bar z} = f_{\bar z}$, these form a linear first-order system of PDEs in f, f_z and $f_{\bar z}$:

$$\frac{\partial}{\partial z} \begin{pmatrix} f \\ f_z \\ f_{\bar z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \psi_z & U e^{-\psi} \\ -\frac12 e^{\psi} & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ f_z \\ f_{\bar z} \end{pmatrix}, \tag{19}$$

$$\frac{\partial}{\partial \bar z} \begin{pmatrix} f \\ f_z \\ f_{\bar z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac12 e^{\psi} & 0 & 0 \\ 0 & \bar U e^{-\psi} & \psi_{\bar z} \end{pmatrix} \begin{pmatrix} f \\ f_z \\ f_{\bar z} \end{pmatrix}. \tag{20}$$

In order to have a solution of the system (18), the only condition is that the mixed partials must commute (by the Frobenius theorem). Thus, we require

$$\begin{array}{rcl} \psi_{z\bar z} + |U|^2 e^{-2\psi} + \frac12 e^{\psi} &=& 0, \\ U_{\bar z} &=& 0. \end{array} \tag{21}$$

The system (18) is an initial-value problem, in that given (A) a base point z_0, (B) initial values $f(z_0) \in \mathbb{R}^3$, $f_z(z_0)$ and $f_{\bar z}(z_0) = \overline{f_z(z_0)}$, and (C) U holomorphic and ψ which satisfy (21), we have a unique solution f of (18) as long as the domain of definition $\mathcal{D}$ is simply connected. We then have that the immersion f satisfies the structure equations (14). In order for $-f$ to be the affine normal of $f(\mathcal{D})$, we must also have the volume condition (13), i.e., $\det(f_z, f_{\bar z}, -f) = \frac12 i e^{\psi}$. We require this at the base point z_0 of course:

$$\det(f_z(z_0), f_{\bar z}(z_0), -f(z_0)) = \tfrac12 i e^{\psi(z_0)}. \tag{22}$$

Then, use (18) to show that the derivatives with respect to z and $\bar z$ of $\det(f_z, f_{\bar z}, -f)e^{-\psi}$ must vanish. Therefore, the volume condition is satisfied everywhere, and $f(\mathcal{D})$ is an elliptic affine sphere with affine normal $-f$.

Using (18), we compute

$$\det(f_z, f_{zz}, -f) = \tfrac12 i U, \tag{23}$$

which implies that U transforms as a section of K^3, and $U_{\bar z} = 0$ means it is holomorphic.

Note that equation (21) is in local coordinates. In other words, if we choose a local conformal coordinate z, then the Pick form $\mathbf{U} = U\,dz^3$, and the metric is $h = e^{\psi}|dz|^2$. Then plug U, ψ into (21). In a patch with a new holomorphic coordinate $w(z)$, the metric will have the form $e^{\widetilde{\psi}}|dw|^2$, with cubic form $\widetilde{U}dw^3$. Then $\widetilde{\psi}(w), \widetilde{U}(w)$ will satisfy (21).

5.2. Parabolic Affine Spheres. Here, we very briefly recall analogues of some results of the last subsection for two-dimensional parabolic affine spheres. This is due to Simon–Wang [**29**], and there is a derivation similar to the one above in [**23**].

A smooth, strictly convex hypersurface H is a parabolic affine sphere if the affine normal ξ is a constant vector. In $\mathbb{R}^3$, we let $\xi = (0,0,1)$. Let $f\colon \mathcal{D} \to \mathbb{R}^3$ be an immersion of the parabolic affine sphere which is conformal with respect to the affine metric. Then, $\{\xi, f_z, f_{\bar z}\}$ is a complexified frame of $\mathbb{R}^3$ at each point in H. The affine structure equations lead to an initial-value problem similar to equations (19–20) above, and the integrability conditions are $U_{\bar z} = 0$ (for the Pick form U) and

$$\psi_{z\bar z} + |U|^2 e^{-2\psi} = 0 \tag{24}$$

for the affine metric $e^{\psi}|dz|^2$. This equation has many easy explicit solutions, which is an advantage over the corresponding equation (21) for elliptic affine spheres. In particular, we often treat the extra term $\frac{1}{2}e^{\psi}$ in (21) as a perturbation of equation (24).

It is also possible to find explicit solutions to the initial value problem for two-dimensional parabolic affine spheres using ODE techniques. This is not surprising, as we saw in Section 4, that the structure equations are completely integrable. For example, if $U = z^{2\alpha-3}\,dz^3$, $z = \rho e^{i\theta}$, and $A, B, \xi \in \mathbb{R}^3$ satisfy $\det(A, B, \xi) = -8$, then the following f is an immersed parabolic affine sphere in $\mathbb{R}^3$ with affine normal ξ:

$$\begin{aligned} f &= \tfrac{1}{2\alpha} A\rho^\alpha \left[\theta\cos\alpha\theta - \tfrac{1}{\alpha}\sin\alpha\theta + (\log\rho)\sin\alpha x\right] \\ &\quad - \tfrac{1}{2\alpha} B\rho^\alpha \cos\alpha\theta + \tfrac{1}{\alpha^2}\xi\rho^{2\alpha}\left[\tfrac{1}{2\alpha}\cos 2\alpha\theta + \tfrac{1}{\alpha} - \log\rho\right]. \end{aligned}$$

Note this solution has non-trivial monodromy around $z = 0$.

Remark. In equation (24), the transformation $\varphi = \log|U| - \frac{1}{2}\psi$ results in the condition that $e^{2\varphi}|dz|^2$ has constant curvature -4. We thank R. Bryant for pointing this out.

6. Elliptic Affine Spheres and the "Y" Vertex

In this section, we will prove the existence of elliptic affine two-sphere metrics with singularities—first locally near a singularity (we find a radially symmetric solution), and then globally on S^2 minus three points. The metric cone yields a parabolic affine sphere metric near the "Y" vertex.

6.1. Local Analysis. Recall that given a holomorphic cubic differential U on a domain in $\mathbb{C}$ with coordinate z, a solution ψ to

$$\psi_{z\bar z} + |U|^2 e^{-2\psi} + \tfrac{1}{2}e^{\psi} = 0$$

provides an affine metric $e^{\psi}|dz|^2$ for an elliptic affine sphere. The elliptic affine sphere can be reconstructed by Simon and Wang's developing map.

Since Baues and Cortés result gives a parabolic affine sphere on the cone over an elliptic sphere, a solution to this equation on the thrice-punctured sphere will lead to a parabolic affine sphere on $\mathbb{R}^3$ minus a Y-shaped set—whence a semiflat special Lagrangian torus fibration over this base. In the present subsection, we prove the existence of radially symmetric solutions to (21), while we discuss the more global setting of the thrice-punctured sphere in the next subsection.

For definiteness, we consider the case $U = z^{-2}$ and make the ansatz $\psi = \psi(|z|)$. We look near $z = 0$, so we make the change of variables $t = -\log|z|$, $t \in (T, \infty)$, $T \gg 0$. This leads to the equation

$$N(\psi) := \partial_t^2 \psi + 4e^{-2(\psi - t)} + 2e^{\psi - 2t} = 0. \tag{25}$$

We put $\psi = \psi_0 + \phi$, where $\psi_0 = t + \log(2t)$ is the solution to the parabolic equation (24). Note that the last term in (25) is $O(te^{-t})$ for this function. We want to solve $N(\psi_0 + \phi) = 0$, which we expand as

$$N(\psi_0 + \phi) = N(\psi_0) + dN(\phi)|_{\psi_0} + Q(\phi)|_{\psi_0}, \tag{26}$$

where $Q(\phi)$ contains quadratic and higher terms. Explicitly,

$$Q(\phi) = \frac{1}{t^2}\left(e^{-2\phi} - (1 - 2\phi)\right) + 4te^{-t}\left(e^{\phi} - (1 + \phi)\right).$$

Note that Q is not even a differential operator. One calculates

$$N(\psi_0) = 4te^{-t},$$
$$dN(\phi)|_{\psi_0} =: L\phi := \left[\partial_t^2 + V(\psi_0)\right]\phi,$$

where

$$V(\psi_0) = -8e^{-2(\psi_0 - t)} + 2e^{\psi_0 - 2t} = -\frac{2}{t^2} + 4te^{-t}.$$

Thus $L\phi = (\partial_t^2 - \frac{2}{t^2} + 4te^{-t})\phi = (L_0 + 4te^{-t})\phi$, where $L_0 = \partial_t^2 - \frac{2}{t^2}$. The equation (21) is now

$$L\phi = f - Q(\phi),$$

with $f = -4te^{-t}$. The idea will be to find an appropriate Green function G for L, in terms of which a solution to this equation becomes a fixed point of the mapping $\phi \to G(f - Q(\phi))$—then to find a range of ϕ where this is a contraction map, whence a solution by the fixed point theorem.

We claim that this map is a contraction for $\phi \sim O(te^{-t})$. More specifically, consider for a value of $T > 2$ to be determined later, the Banach space $\mathcal{B}$ of continuous functions on $[T, \infty)$ with norm

$$\|g\|_{\mathcal{B}} = \sup_{t \geq T} \frac{g(t)}{te^{-t}}.$$

Showing the map $\phi \to G(f - Q(\phi))$ is a contraction map involves estimating Gf and $GQ\phi$. In fact, since Q is a quadratic, non-derivative operator, it is easy to see that $Q\phi$ is order te^{-t} (even smaller). We then show that G preserves the condition $O(te^{-t})$ by showing $Gf \in \mathcal{B}$ (recall that $f \in \mathcal{B}$, too). To find G, we write $L = L_0 - \delta_L$, where $\delta_L = -4te^{-t}$, so that $G = L^{-1} = L_0^{-1} + L_0^{-1}\delta_L L_0^{-1} + \dots$. To solve the equation $Lu = f$, we first note that the change of variables $v = u + 1$ leads to the equation $Lv = -\frac{2}{t^2}$. Let $v_0 = L_0^{-1}(-\frac{2}{t^2}) = 1$. Then define $v_{k+1} = L_0^{-1}\delta_L v_k$. Then $v = \sum_{k=0}^{\infty} v_k$ and $Gf = u = \sum_{k=1}^{\infty} v_k$.

Lemma 2. $|v_k(t)| < (16te^{-t})^k$ *pointwise.*

Proof. It is true for $k = 0$. To compute v_{k+1} one solves the differential equation by the method of variation of parameters,[3] using the homogeneous solution t^2 or t^{-1}. We have

$$v_{k+1}(t) = t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} t_2^2(-4t_2e^{-t_2})v_k(t_2)dt_2dt_1.$$

One computes $v_1(t) = -4(t+2+2/t)e^{-t}$, and therefore $|v_1(t)| < 16te^{-t}$ for $t > 2$. Now, assume $|v_k(t)| < (16te^{-t})^k$ for some $k \geq 1$. First, compute for $a, b \in \mathbb{N}$, $t > a/b$ and $t > 2$:

$$\begin{aligned} \int_t^{\infty} s^a e^{-bs} ds &= -\frac{1}{b}\left[s^a + \frac{a}{b}s^{a-1} + \frac{a(a-1)}{b^2}s^{a-2} + \dots\right] e^{-bs}\Big|_t^{\infty} \\ &\leq \frac{1}{b}t^a(1 + (a/bt) + (a/bt)^2 + \dots)e^{-bt} \\ &\leq \frac{t}{b(t-1)}t^a e^{-bt} \leq 2t^a e^{-bt}. \end{aligned}$$

Therefore,

$$\begin{aligned} |v_{k+1}(t)| &\leq 4t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} t_2^3 e^{-t_2} v_k(t_2) dt_2 dt_1 \\ &\leq 4t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} t_2^3 e^{-t_2} (16)^k t_2^k e^{-kt_2} dt_2 dt_1 \\ &\leq (16)^k \cdot 4 \cdot 2t^2 \int_t^{\infty} t_1^{k+3-4} e^{-(k+1)t_1} dt_1 \\ &\leq (16)^{k+1} t^{k+1} e^{-(k+1)t} \end{aligned}$$

and the lemma is proven.

Note that we needed $k - 1 \geq 0$ to bound a/b from above. That the $k = 1$ term is the proper order follows from some fortuitous cancelation.

q.e.d.

[3] We can write $G_0h(t) = \int_T^{\infty} K_0(t,s)h(s)ds$, where $K_0(t,s) = \frac{1}{3}(\frac{s^2}{t} - \frac{t^2}{s})$ for $s > t$ and zero otherwise (this form of the kernel is relevant to the condition of good functional behavior at infinity). One can also use the equivalent $G_0h(t) = t^2 \int_t^{\infty} t_1^{-4} \int_{t_1}^{\infty} t_2^2 h(t_2)dt_2dt_1$, which appears in the text.

It now follows that $u < C(1-4te^{-t})^{-1}4te^{-t} \le C'te^{-t}$ for some constant C'. The space $S = \{g(t) : |g(t)| \le 2C'te^{-t}\}$ forms a closed subset of $\mathcal{B}$ on which we apply the contraction mapping theorem.

The proof of the previous lemma also shows the following.

Lemma 3. *If $|h(t)| \le C(te^{-t})^2$, then there are positive constants T and K independent of h so that if $t \ge T$, then*

$$|(Gh)(t)| \le CK(te^{-t})^2.$$

Proof. The computations above show that if $\ell \ge 2$ and $|w(t)| \le C''(te^{-t})^\ell$, then $|(L_0^{-1}w)(t)| \le 4C''(te^{-t})^\ell$. Then for our $h(t)$, compute

$$\begin{aligned} |(Gh)(t)| &\le |(L_0^{-1}h)(t)| + |(L_0^{-1}\delta_L L_0^{-1}h)(t)| + \cdots \\ &\le 4C(te^{-t})^2 + 4\cdot 16(te^{-t})^3 + 4\cdot 16^2(te^{-t})^4 + \cdots \\ &= \frac{4C(te^{-t})^2}{1-16te^{-t}}, \end{aligned}$$

and so for T large enough, we can choose $K = 4/(1-16Te^{-T})$. q.e.d.

Proposition 4. *There is a constant $T > 0$ so that for $t \ge T$, the equation* (21) *has a solution of the form* $\log(2t) + t + O(te^{-t})$.

Proof. We now show the mapping $\phi \to A\phi \equiv Gf - GQ\phi$ is a contraction. First, Gf lies within S and $Q\phi$ is small since Q is a quadratic non-differential operator. More specifically, for some $T > 0$, the sup norm of ϕ on $[T,\infty)$ can be made arbitrarily small. Therefore, by Lemma 3, $\|GQ\phi\|_{\mathcal{B}} \ll \|\phi\|_{\mathcal{B}}$ on $[T,\infty)$. As a result, A maps S to S. Further, note $\|A\phi_1 - A\phi_2\|_{\mathcal{B}} = \|GQ\phi_1 - GQ\phi_2\|_{\mathcal{B}} = \|G(Q\phi_1 - Q\phi_2)\|_{\mathcal{B}}$, since G is linear.

Since Q is quadratic and $\phi_1, \phi_2 \in S$,

$$|(Q\phi_1 - Q\phi_2)(t)| \le K'(te^{-t})^2\|\phi_1 - \phi_2\|_{\mathcal{B}},$$

for K' depending on C'. Then, Lemma 3 shows

$$|(GQ\phi_1 - GQ\phi_2)(t)| \le KK'(te^{-t})^2\|\phi_1 - \phi_2\|_{\mathcal{B}}.$$

Then clearly, by taking T large enough, there is a fixed $\theta < 1$ so that $\|A\phi_1 - A\phi_2\|_{\mathcal{B}} \le \theta\|\phi_1 - \phi_2\|_{\mathcal{B}}$ for $\phi_1, \phi_2 \in S$. By the fixed point theorem, there exists ϕ such that $A\phi = \phi$. ϕ is smooth by standard bootstrapping. Then $\log(2t) + t + \phi(t)$ solves (21). q.e.d.

In the next section, we will make an ansatz that the local form of our global function be consistent with the dominant $\log|\log|z|^2| - \log|z|$ behavior of this local solution. The dominant term is $-\log|z|$ which comes from the form of U and determines the residue at the singularity.

6.2. Global Existence. The coordinate-independent version of equation (21) for a general background metric is

$$\Delta u + 4\|U\|^2 e^{-2u} + 2e^u - 2\kappa = 0 \tag{27}$$

on S^2, where norms, the Laplacian, integrals, and the Gauss curvature κ are taken with respect to the background metric. U is a holomorphic cubic differential, which we take to have exactly 3 poles of order 2 and thus no zeroes, and u is taken to have a prescribed singularity structure such that $\int \Delta u = 6\pi$, which follows from our local analysis in Section 6.1.

Near each pole of U, there is a local coordinate z so that the pole is at $z = 0$, and $U = z^{-2}dz^3$ exactly. We call this z the *canonical holomorphic coordinate*. In a neighborhood of each pole, we take

$$u_0 = \log|\log|z|^2| - \log|z| \tag{28}$$

and the background metric to be $|dz|^2$. This u_0 is an explicit solution of the parabolic affine sphere equation (24). The background metric and u_0 are extended smoothly to the rest of $\mathbb{CP}^1$. Note that $\int \Delta u_0 = 6\pi$ (each pole contributes 2π). All integrals in this section will be evaluated with respect to the background metric.

To implement the required singularity structure, we write $u = u_0 + \eta$ for η in the Sobolev space H_1. Note this implies $\int \Delta\eta = 0$. We define the functional

$$\begin{aligned} J(\eta) \;=\; & \int \left(\tfrac{1}{2}|\nabla\eta|^2 + (2\kappa - \Delta u_0)\eta + \tfrac{1}{2}3\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta}\right) \\ & -2\pi \log \int \left(4\|U\|^2 e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}\right). \end{aligned} \tag{29}$$

(We note that it is necessary to separate η from u_0, as ∇u_0 is not in L^2.) J is not defined for all functions $\eta \in H_1$. One problem is that $\Delta u_0 \notin L^2$. The term $\int \Delta u_0 \eta$ can be taken care of by integrating by parts (see the proof of Proposition 6 below). A more serious problem is that $4\|U\|^2 e^{-2u_0} \notin L^p$ for any $p > 1$. This cannot be fixed by integrating by parts, as the example $\eta = -\frac{1}{2}\log|\log|z|^2| \in H_{1,\mathrm{loc}}$ shows. That said, there is a uniform lower bound on J among all $\eta \in H_1$ so that $\int 4\|U\|^2 e^{-2u_0}e^{-2\eta} < \infty$ (see the remark after Proposition 6). Thus, we can still talk of taking sequences of $\eta \in H_1$ to minimize J. (The term $\int 2e^{u_0}e^{\eta}$ is always finite for $\eta \in H_1$ since $e^{u_0} \in L^p$ for $p < 2$ and Moser–Trudinger shows that $e^\eta \in L^q$ for all $q < \infty$.)

We wish to show that $J(\eta)$ has a local minimum. If so, then the minimizer satisfies the Euler–Lagrange equation

$$\begin{aligned} &\Delta\eta - (2\kappa - \Delta u_0) + 3\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta} \\ &\qquad + \frac{-2\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}}{\frac{1}{2\pi}\int 4\|U\|^2 e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}} = 0. \end{aligned} \tag{30}$$

One can easily check by integrating this equation that for a solution η_0, the denominator in the last term must be equal to one. Thus, $u = \eta_0 + u_0$ satisfies the original equation (27). In this case, the equation η_0 satisfies

$$\Delta\eta - (2\kappa - \Delta u_0) + 4\|U\|^2 e^{-2u_0} e^{-2\eta} + 2e^{u_0} e^{\eta} = 0. \tag{31}$$

This is equivalent to equation (21), the equation for the metric of an elliptic affine sphere: for the background metric h, write $e^{u_0+\eta}h = e^{\psi}|dz|^2$. Then, η satisfies (31) if and only if ψ satisfies (21).

Definition 1. We call η *admissible* if $\eta \in H_1$ and $\int 4\|U\|^2 e^{-2u_0} e^{-2\eta} < \infty$.

In order to analyze the functional J, for an admissible η, consider $J(\eta + k)$ for k a constant. $J(\eta + k)$ has the form

$$(\text{indep. of } k) + 2\pi k + 3\pi A e^{-2k} - 2\pi \log[2\pi(Ae^{-2k} + Be^{k})],$$

where

$$A = A(\eta) \equiv \frac{1}{2\pi}\int 4\|U\|^2 e^{-2u_0} e^{-2\eta}, \quad B = B(\eta) \equiv \frac{1}{2\pi}\int 2e^{u_0} e^{\eta}. \tag{32}$$

Thus, upon setting $(\partial/\partial k)J(\eta + k) = 0$, we find a critical point only if $Ae^{-2k} + Be^{k} = 1$, and this can only happen if

$$AB^2 \le \frac{4}{27}.$$

If $AB^2 > 4/27$, then the infimum occurs as $k \to +\infty$, and if $AB^2 < 4/27$, there are two finite critical points: a local minimum for which $B(\eta + k) = Be^k < 2/3$ and a local maximum for which $B(\eta + k) > 2/3$. With that in mind, we formulate the following variational problem:

Let

$$Q = \{\eta \in H_1 : A + B \le 1\}.$$

We will minimize J for $\eta \in Q$. Note that this will avoid the potential problem at $k \to +\infty$, where $B(\eta + k) \to +\infty$. Also, the inequality in the definition of Q will be important. It will allow us to use the Kuhn–Tucker conditions to control the sign of the Lagrange multiplier in the Euler–Lagrange equations. The discussion above about adding a constant k can be summarized in

Lemma 5. *If $\eta \in Q$, then the minimizer of*

$$\{J(\eta + k) : k \text{ constant}, \eta + k \in Q\}$$

occurs for k so that $A(\eta+k)+B(\eta+k) = 1$, $B(\eta+k) \le 2/3$, and $k \le 0$. If $A(\eta) + B(\eta) < 1$, then the minimizer $k < 0$ and $B(\eta + k) < 2/3$. Moreover, if $A(\eta) + B(\eta) = 1$ and $B(\eta) \le 2/3$, then $k = 0$.

Proof. Compute $(\partial/\partial k)J(\eta + k)$ and use the first derivative test.

q.e.d.

Proposition 6. *There are positive constants γ and R so that for all $\eta \in Q$,*

$$J(\eta) \geq \gamma \int |\nabla \eta|^2 - R.$$

Remark. We can also prove the same result for all admissible $\eta \in H_1$. In this case, we must also control potential minimizers at $k = +\infty$. For admissible $\rho \in H_1$ so that $\int \rho = 0$, consider the functional

$$\tilde{J}(\rho) = \lim_{k \to \infty} J(\rho + k).$$

We bound $\tilde{J}$ from below much the same as the following argument, although there also is an extra term in $\tilde{J}$ that must be handled using the Moser–Trudinger estimate.

Proof. As above, $u_0 = \log|\log|z|^2| - \log|z|$ in the canonical coordinate z near each pole of U. Since $\Delta u_0 \notin L^2$, we should integrate by parts to handle the $-\int \Delta u_0 \, \eta$ term in J. Let $u_0' = \log|\log|z|^2|$ near each pole of U and smooth elsewhere. Then, $\Delta u_0 = \Delta u_0'$ near each pole and the difference $\Delta u_0 - \Delta u_0'$ is smooth on $\mathbb{CP}^1$. Then, if we let ζ be the smooth function $2\kappa - \Delta(u_0 - u_0')$,

$$\begin{aligned}
J(\eta) &= \int \left[\frac{1}{2}|\nabla \eta|^2 + \zeta\eta - \Delta u_0' \, \eta\right] + 3\pi A - 2\pi \log 2\pi(A + B) \\
&> \int \left[\frac{1}{2}|\nabla \eta|^2 + \zeta\eta + \nabla u_0' \cdot \nabla \eta\right] - 2\pi \log 2\pi \\
&\geq C + \int \left[\frac{1}{2}|\nabla \eta|^2 - \frac{1}{4\epsilon}\zeta^2 - \epsilon\eta^2 - \frac{1}{4\epsilon}|\nabla u_0'|^2 - \epsilon|\nabla \eta|^2\right] \\
&\geq C_\epsilon + \int \left(\frac{1}{2} - \delta\right)|\nabla \eta|^2
\end{aligned}$$

Here $\delta = (\frac{1}{\lambda_1} + 1)\epsilon$, for λ_1 the first non-zero eigenvalue of the Laplacian of the background metric, and we have used the facts that $A > 0$ and $A + B \leq 1$. q.e.d.

Here is another useful lemma.

Lemma 7. *For any $\eta \in H_1$,*

$$AB^2 \geq L = 2\pi^{-3} \left(\int \|U\|^{\frac{2}{3}}\right)^3.$$

If $AB^2 = L$, then there is a constant C such that

$$\eta = C + \tfrac{2}{3} \log \|U\| - u_0.$$

Proof. Let $f = (4\|U\|^2)^{\frac{1}{3}} e^{-\frac{2}{3}(u_0 + \eta)}$, $g = e^{\frac{2}{3}(u_0+\eta)}$. Apply Hölder's inequality $\int fg \leq \|f\|_3 \|g\|_{\frac{3}{2}}$. The last statement follows from the case of equality in Hölder's inequality. q.e.d.

Remark. The bound L in the previous lemma does not depend on the background metric; it depends only on the conformal structure on $\mathbb{CP}^1$ and the cubic form U.

An admissible $\eta \in H_1$ is a *weak solution* of (31) if η is a solution of (31) in the sense of distributions.

Proposition 8. *Assume that U is such that $L < 4/27$. Then, any minimizer η of $\{J(\eta) : \eta \in Q\}$ is a weak solution of* (31).

Proof. Recall $Q = \{\eta : A + B \leq 1\}$.

Case 1: The minimizer η satisfies $A + B < 1$. Since the constraint $A + B \leq 1$ is slack, η must satisfy the Euler–Lagrange equation (30). Then as above, we may integrate to find that the denominator $A + B$ in (30) must be equal to 1. Thus, this case cannot occur.

Case 2: The minimizer η satisfies $A + B = 1$. In this case, we have Lagrange multipliers $[\mu_0, \mu_1] \in \mathbb{RP}^1$ so that η weakly satisfies

$$\begin{aligned}\mu_0\bigg[\Delta\eta - (2\kappa - \Delta u_0) + 3\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta} \\ + \frac{-2\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}}{A+B}\bigg] \\ = \mu_1(-2\cdot 4\|U\|^2 e^{-2u_0}e^{-2\eta} + 2e^{u_0}e^{\eta}),\end{aligned}$$

and $A + B = 1$. Thus,

$$\mu_0[\Delta\eta - (2\kappa - \Delta u_0) + a + b] = \mu_1(-2a + b) \tag{33}$$

for

$$a = 4\|U\|^2 e^{-2u_0}e^{-2\eta}, \qquad b = 2e^{u_0}e^{\eta}.$$

Note then that $A = \int a/2\pi$, $B = \int b/2\pi$.

Also note the constraint the Kuhn–Tucker conditions place on the Lagrange multipliers. Recall that if we minimize a function f subject to the constraint $g \leq 1$, and if the minimum occurs on the boundary $g = 1$, then we have $\mu_0\nabla f = \mu_1\nabla g$ for $\mu_0\mu_1 \leq 0$. This is exactly our situation for $f = J$ and $g = A + B$.

Thus, we have three cases: if $\mu_1 = 0$, then equation (33) becomes equation (31) and we have proved the proposition.

In the second case, if $\mu_0 = 0$, then the Euler–Lagrange equation (33) may be solved explicitly for η to find

$$\eta = \tfrac{1}{3}\log(4\|U\|^2) - u_0.$$

Near each pole of U, there is a coordinate z so that $\|U\| = |z|^{-2}$ and $u_0 = \log|\log|z|^2| - \log|z|$. So

$$\eta = \tfrac{1}{3}\log 4 - \tfrac{1}{3}\log|z| - \log|\log|z|^2|$$

there and so $\eta \notin H_1$.

Finally, we consider where $\mu = \mu_1/\mu_0 < 0$. We will analyze the second variation at any critical point to show that there are no minimizers in this case.

Integrate (33) to find

$$-2\pi + 2\pi A + 2\pi B = \mu(-2 \cdot 2\pi A + 2\pi B).$$

Then since $A+B=1$, we have $2A = B$, since we are in the case $\mu \neq 0$. So $A = 1/3$ and $B = 2/3$. We analyze the second variation to show that for $L < 4/27$, there is no minimizer at $A = 1/3$, $B = 2/3$ (unless possibly if $\mu_1 = 0$).

Let η satisfy (33) and $A = 1/3$, $B = 2/3$. Consider a variation $\eta + \epsilon\alpha + \frac{\epsilon^2}{2}\beta$ so that η satisfies $A+B=1$ to second order when $\epsilon = 0$.[4]

We assume α is a constant. Then, the first variation

$$\frac{\partial}{\partial\epsilon}(A+B)\bigg|_{\epsilon=0} = -2\alpha A + \alpha B = 0$$

for $A = 1/3$, $B = 2/3$. So to first order $\eta + \alpha$ satisfies $A+B=1$ and α is tangent to $\{A+B=1\}$.

Now, we require

$$\begin{aligned} 0 &= 2\pi\,\frac{\partial^2}{\partial\epsilon^2}(A+B)\bigg|_{\epsilon=0} \\ &= \int a(4\alpha^2 - 2\beta) + b(\alpha^2+\beta) \\ &= \alpha^2 2\pi(4A+B) + \int \beta(-2a+b) \\ &= 2\pi\cdot 2\alpha^2 + \int \beta(-2a+b). \end{aligned} \tag{34}$$

Now for this variation $J = J(\eta + \epsilon\alpha + \frac{\epsilon^2}{2}\beta)$, compute

(35)

$$\frac{\partial^2 J}{\partial\epsilon^2}\bigg|_{\epsilon=0} = \int \nabla\eta\cdot\nabla\beta + |\nabla\alpha|^2 + (2\kappa - \Delta u_0)\beta + \tfrac{3}{2}\int a(4\alpha^2 - 2\beta) \\ - 2\pi\frac{\int a(4\alpha^2-2\beta) + b(\alpha^2+\beta)}{\int a+b} + 2\pi\frac{\left(\int a(-2\alpha) + b\alpha\right)^2}{\left(\int a+b\right)^2}$$

[4]This corresponds to an actual variation in Q by standard Implicit Function Theorem arguments—see [**20**]. Let X be the Banach space $H_1 \cap C^0$. Then let $g\colon X \to \mathbb{R}$, $g(\nu) = A(\eta+\nu) + B(\eta+\nu)$. It is straightforward to show that g is C^1 in the Banach space sense. Moreover, for $2a \neq b$ (which holds for any $\eta \in H_1$), we can check that $dg\colon X \to \mathbb{R}$ is non-zero. So, then $Y = g^{-1}(1) = \{A+B=1\}$ is a Banach submanifold of X near $\nu = 0$. So, for any element $\alpha \in \ker dg_0$, there is a curve in Y tangent to α. Along such a curve, we compute restrictions on the second-order term β.

$$\begin{aligned}
&= \int [\nabla\eta\cdot\nabla\beta + (2\kappa - \Delta u_0)\beta - 3a\beta] + 2\pi\cdot 6\alpha^2 A \\
&\quad - 2\pi\cdot\alpha^2(4A+B) - \int \beta(-2a+b) \\
&= \int [\nabla\eta\cdot\nabla\beta + (2\kappa - \Delta u_0)\beta - 3a\beta] + 2\pi\cdot 2\alpha^2.
\end{aligned}$$

Here, we have used the following facts to get from the first line to the second: $\nabla\alpha = 0$ since α is constant, $\int(a+b)/2\pi = A+B = 1$, and the last term vanishes since α is constant and $2A = B$. The third line follows from the second by the constraint (34) and the fact $A = 1/3$.

Now, we use the Euler–Lagrange equation (33). Recall $\mu_0 \neq 0$ and $\mu = \mu_1/\mu_0$. Then

$$\int \nabla\eta\cdot\nabla\beta = -\int(\Delta\eta)\beta = \int[-(2\kappa - \Delta u_0) + a + b - \mu(-2a+b)]\beta.$$

Plug this into (35) to find

$$\begin{aligned}
\left.\frac{\partial^2 J}{\partial\epsilon^2}\right|_{\epsilon=0} &= (1-\mu)\int(-2a+b)\beta + 2\pi\cdot 2\alpha^2 \\
&= 2\pi\cdot 2\mu\alpha^2.
\end{aligned}$$

Here, the last line follows from (34). Thus if we choose $\alpha \neq 0$, then the second variation along this path is negative since $\mu < 0$. Therefore, there is no minimizer for our variational problem satisfying $\mu < 0$. q.e.d.

Now, we show that there is a minimizer.

Lemma 9. *Assume $L < 4/27$. Then, there is a constant $\delta > 0$ so that $A, B \in (\delta, 1/\delta)$ for all $\eta \in Q$.*

Proof. Lemma 7 implies that $AB^2 \geq L$. Since $0 < L < 4/27$, $A > 0$, $B > 0$, and $A + B \leq 1$, this proves the lemma. q.e.d.

Lemma 10. *There are constants K_1, K_2 so that for all admissible $\eta \in H_1$, and for $c = (\int\eta)/(\int 1)$,*

$$\log A \geq K_1 - 2c, \qquad \log B \geq K_2 + c.$$

Proof. Since exp is convex, Jensen's inequality gives

$$\begin{aligned}
\log A &= -\log 2\pi + \log\int 4\|U\|^2 e^{-2u_0}e^{-2\eta} \\
&\geq -\log 2\pi + \frac{\int \log 4\|U\|^2 - 2u_0 - 2\eta}{\int 1} + \log\int 1.
\end{aligned}$$

The case for B is the same. q.e.d.

Lemma 11. *Let η_i be a sequence in Q so that $\lim_i J(\eta_i) = \inf_{\eta\in Q} J(\eta)$. Then, there is a positive constant C so that $\|\eta_i\|_{H_1} \leq C$ for all i.*

Proof. First, we note that Lemmas 9 and 10 show that the average value $c = (\int \eta)/(\int 1)$ is uniformly bounded above and below for all $\eta \in Q$.

Proposition 6 shows that $J(\eta) \geq \gamma \int |\nabla \eta|^2 - R$ for $\gamma, R > 0$ uniform constants. Thus for any minimizing sequence, $\int |\nabla \eta|^2$ must be uniformly bounded. Then, write $\eta = \rho + c$ for $\int \rho = 0$, c constant. Then,

$$\|\eta\|_{L^2} \leq \|\rho\|_{L^2} + \|c\|_{L^2} \leq \lambda_1^{-\frac{1}{2}} \|\nabla \rho\|_{L_2} + K = \lambda_1^{-\frac{1}{2}} \|\nabla \eta\|_{L_2} + K$$

for K a uniform constant and λ_1 the first non-zero eigenvalue of the Laplacian. This shows the H_1 norm of η in the minimizing sequence is uniformly bounded. q.e.d.

Now given a minimizing sequence $\{\eta_i\} \subset Q$, Lemma 5 shows that we can assume $A(\eta_i) + B(\eta_i) = 1$, $B(\eta_i) \leq 2/3$. Then, there is a subsequence, which we still refer to as η_i, which is weakly convergent to a function $\eta_\infty \in H_1$ (the weak compactness of the unit ball in a Hilbert space), strongly convergent to η_∞ in L^p for $p < \infty$ (Sobolev embedding), convergent pointwise almost everywhere to η_∞ (L^p convergence implies subsequential almost-everywhere convergence), and so that e^{η_i} is strongly convergent to e^{η_∞} in L^p for $p < \infty$ (Moser–Trudinger). Recall

$$J(\eta) = \int [\tfrac{1}{2}|\nabla \eta|^2 + (2\kappa - \Delta u_0)\eta] + 3\pi A - 2\pi \log 2\pi(A + B).$$

Then the second term in the integral converges by strong convergence in L^1 and weak convergence in H_1 (see the proof of Proposition 6 for the integration by parts trick). The term $\int \frac{1}{2}|\nabla \eta|^2$ is lower semicontinuous (the norm in a Hilbert space is lower semicontinuous under weak convergence). Lower semicontinuity is enough since we are seeking a minimizer. B converges by Moser–Trudinger: $e^{u_0} \in L^p$ for $p < 2$. Then since e^{η_i} converges in L^q for $\frac{1}{p} + \frac{1}{q} = 1$, $B = \int 2e^{u_0}e^{\eta}$ converges.

That leaves the term A. Fatou's lemma and the almost-everywhere convergence of η_i then show

$$A(\eta_\infty) \leq \liminf_{i \to \infty} A(\eta_i).$$

We want to rule out the case of strict inequality. Note $A(\eta_\infty) + B(\eta_\infty) \leq \lim A(\eta_i) + B(\eta_i) = 1$, and so $\eta_\infty \in Q$. Also, since $A(\eta_i) + B(\eta_i) = 1$ and $B(\eta_i) \to B(\eta_\infty)$, $\lim A(\eta_i) = 1 - B(\eta_\infty)$.

Consider the constant k so that $\eta_\infty + k$ minimizes

$$\{J(\eta_\infty + k) : \eta_\infty + k \in Q\}.$$

Note that Lemma 5 shows that $e^{-2k}A(\eta_\infty) + e^k B(\eta_\infty) = 1$. Now, compute

$$\lim_{i\to\infty} J(\eta_i) \geq \int \left[\tfrac{1}{2}|\nabla\eta_\infty|^2 + (2\kappa - \Delta u_0)\eta_\infty \right] + 3\pi \left[1 - B(\eta_\infty)\right],$$
$$J(\eta_\infty + k) = \int \left[\tfrac{1}{2}|\nabla\eta_\infty|^2 + (2\kappa - \Delta u_0)(\eta_\infty + k) \right] + 3\pi e^{-2k} A(\eta_\infty).$$

Now, substitute $e^{-2k}A(\eta_\infty) = 1 - e^k B(\eta_\infty)$ to show

$$\lim_{i\to\infty} J(\eta_i) - J(\eta_\infty + k) \geq -2\pi k + 3\pi B(\eta_\infty)(e^k - 1) \tag{36}$$

We prove $A(\eta_\infty) = \lim A(\eta_i)$ by contradiction. If on the contrary, $A(\eta_\infty) < \lim A(\eta_i)$, Lemma 5 and the fact $A(\eta_i) + B(\eta_i) = 1$ imply that $k < 0$. Then, it is straightforward to check that the right-hand side of (36) is strictly positive (it is zero if $k = 0$, and its derivative with respect to k is negative for $k < 0$—use the fact $B(\eta_\infty) = \lim B(\eta_i) \leq 2/3$). This shows $\lim J(\eta_i) > J(\eta_\infty + k)$ and so contradicts the fact that η_i is a minimizing sequence for J.

The same analysis shows that $\lim \int |\nabla\eta_i|^2 = \int |\nabla\eta_\infty|^2$. So $J(\eta_\infty) = \lim J(\eta_i)$, and η_∞ is a minimizer of $\{J(\eta) : \eta \in Q\}$.

Theorem 2. *If $L < 4/27$ then a weak solution to* (31) *exists. Conversely, if $L \geq 4/27$, then there is no weak solution to* (31).

Proof. The preceding paragraphs, together with Proposition 8, prove existence in the case $L < 4/27$. We address the non-existence in two cases:

Case $L > 4/27$. If η solves (31), then we can integrate (31) to find $A + B = 1$. On the other hand, $A > 0$, $B > 0$, and Lemma 7 shows that $AB^2 \geq L > 4/27$. Simple calculus shows that there is no such pair (A, B) in this case.

Case $L = 4/27$. As in Case 1, we must have $A + B = 1$ and $AB^2 \geq L = 4/27$. The only way this can happen is if $A = 1/3$, $B = 2/3$, so that $AB^2 = 4/27$. In this case, Lemma 7 forces $\eta = C + \frac{2}{3}\log\|U\| - u_0$ for some constant C. Since $u_0 = \log|\log|z|^2| - \log|z|$ and $\|U\| = |z|^{-2}$ near each pole of U, $\eta = C - \log|\log|z|^2| - \frac{1}{3}\log|z|$ near each pole of U. Thus, $\eta \notin H_1$. q.e.d.

Proposition 12. *Any weak solution η to* (31) *is smooth away from the poles of U.*

Proof. In a neighborhood bounded away from the poles of U, the quantities $\|U\|^2$ and u_0 are smooth and bounded. Since $\eta \in H_1$, Moser–Trudinger shows that e^η, $e^{-2\eta} \in L^p$ for all $p < \infty$. Therefore, (31) implies $\Delta\eta \in L^p_{\text{loc}}$. Since $\eta \in L^p$ by Sobolev embedding, the L^p elliptic theory [**11**] shows that $\eta \in W^{2,p}_{\text{loc}}$. Sobolev embedding shows $\eta \in C^{0,\alpha}_{\text{loc}}$, and so $\Delta\eta \in C^{0,\alpha}_{\mathrm{loc}}$. The Schauder theory then shows $\eta \in C^{2,\alpha}_{\mathrm{loc}}$. Further bootstrapping implies η is smooth. q.e.d.

Remark. It is not clear whether the solution constructed is unique. The maximum principle does not work to give uniqueness.

6.3. A metric for the "Y" vertex. Let $\widetilde{\Sigma}$ be the universal cover of $\Sigma = S^2 \setminus \{p_1, p_2, p_3\}$. Lifting the appropriate objects to the cover, we find a solution to (27) on $\widetilde{\Sigma}$. Since the equation (27) is the integrability condition for the developing map, we have a solution $\widetilde{f} : \widetilde{\Sigma} \to \mathbb{R}^3$, with monodromies of Σ acting as equiaffine deck transformations fixing the normal vector ξ and acting by isometry. The quotient by the deck transformations gives an elliptic affine sphere structure on Σ as well as the locally defined developing map f. Then, the map $F : (\Sigma \times \mathbb{R}_+) \to \mathbb{R}^3$ defined by $F(x, r) = rf(x) =: (y_1, y_2, y_3)$ maps the cone over Σ to $\mathbb{R}^3$ and is locally invertible (so we may express $r = r(y)$). The potential function $\Phi(y) = r^2/2$ defines a parabolic affine sphere on a neighborhood of the "Y" vertex, by Baues and Cortés's Theorem 1. This is our main result.

Remark. The monodromy group of this metric determines the affine flat structure. We have not yet determined this monodromy group, thus cannot verify that the metric is one predicted by Gross–Siebert [**13**] and Haase–Zharkov [**16**].

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