Comment. Math. Helvetici 56 (1981) 487–523 0010–2571/81/004487–37$01.50+0.20/0

Affine manifolds with nilpotent holonomy

DAVID FRIED, WILLIAM GOLDMAN, MORRIS W. HIRSCH

Introduction

An affine manifold is a differentiable manifold together with an atlas of coordinate charts whose coordinate changes extend to affine automorphisms of Euclidean space. These charts are called affine coordinates. A map between affine manifolds is called affine it its expression in affine coordinates is the restriction of an affine map between vector spaces. Thus we form the category of affine manifolds and affine maps.

Let M be a connected affine manifold of dimension $n \geq 1$, locally isomorphic to the vector space E. Its universal covering $\tilde{M}$ inherits a unique affine structure for which the covering projection $\tilde{M} \to M$ is an affine immersion. The group π of deck transformations acts on $\tilde{M}$ by affine automorphisms.

It is well known that there is an affine immersion $D : \tilde{M} \to E$, called the developing map. This follows, for example, from Chevalley's Monodromy Theorem; a proof is outlined in Section 2. Such an immersion is unique up to composition with an affine automorphism of E. Thus for every $g \in \pi$ there is a unique affine automorphism $\alpha(g)$ of E such that $D \circ g = \alpha(g) \circ D$. The resulting homomorphism $\alpha : \pi \to \mathrm{Aff}(E)$ from π into the group of affine automorphisms of E is called the affine holonomy representation. It is unique up to inner automorphisms of $\mathrm{Aff}(E)$. The composition $\lambda : \pi \to \mathrm{GL}(E)$ is called the linear holonomy representation. The affine structure on M is completely determined by the pair (D, α).

M is called complete when D is a homeomorphism. This is equivalent to geodesic completeness of the connection on M (in which parallel transport is locally defined by affine charts as ordinary parallel transport in E). It is notorious that compactness does not imply completeness.

The main results of this paper are about affine manifolds whose affine holonomy groups $\alpha(\pi)$ are nilpotent. An important class of such manifolds are the affine nilmanifolds $\pi \backslash G$. Here π is a discrete subgroup of a simply connected nilpoint Lie group G. It is assumed that G has a left-invariant affine structure; the space of right cosets of π then inherits an affine structure.

A tensor field on an affine manifold M is called polynomial if its components in affine cordinates are polynomial functions. In particular there is an exterior algebra of polynomial exterior differential forms on M, closed under exterior differentiation.

Our main results are summarized as follows.

THEOREM A. *Let M be a compact affine manifold whose affine holonomy group is nilpotent. Then the following conditions are equivalent:*

(a) *M is complete*
(b) *the developing map is surjective*
(c) *the linear holonomy is unipotent*
(d) *the linear holonomy preserves volume*
(e) *the affine holonomy is irreducible*
(f) *the affine holonomy is indecomposable*
(g) *M is a complete affine nilmanifold*
(h) *M has a polynomial volume form*
(i) *M is orientable and the de Rham cohomology of M is the cohomology of the complex of polynomial exterior forms.*

In Section 1 we collect several algebraic facts about affine representations and cohomology which will be used throughout the rest of the paper. In particular we introduce the notations of irreducibility and indecomposability of an affine representation. Section 2 is geometric and discusses affine structures and their holonomy and development. The parallelism on an affine manifold is used to describe several classes of tensors which have special descriptions in affine coordinates. Left-invariant structures on Lie groups, which provide many examples of affine manifolds, are introduced.

In Section 3 we define another large class of affine manifolds: the radiant manifolds. These are characterized by the property that the affine holonomy has a stationary point. By choosing the origin to be such a point we identify the affine holonomy with the linear holonomy. More generally, for any affine manifold M we define a cohomology class c_M which vanishes when precisely M is radiant. This radiance obstruction is one of our principal tools.

Section 4 uses the cohomology theory developed in Section 1 to obtain several results about nonradiant affine manifolds having nilpotent affine holonomy group.

In Section 5 we prove a technical result about nilpotent linear groups.

In Section 6, we prove the equivalence of (a)–(f) in Theorem A, using techniques of Section 4 and the technical lemma from Section 5. (See 6.11). Deformation of affine structures is discussed, and some solvable examples are given.

In Section 7 we prove that every compact complete affine manifold with nilpotent fundamental group is an affine nilmanifold, establishing (g) of Theorem A. Finally, in Section 8 we discuss polynomial tensors on affine manifolds with nilpotent holonomy, completing the proof of Theorem A.

We have greatly profited from conversations with John Smillie. Many of the results here were first proved in his thesis under the assumption of abelian affine holonomy. We are indebted to him for many valuable ideas.

1. Affine representations and cohomology of groups

Throughout this paper E denotes the real n-dimensional vector space $\mathbf{R}^n$, $n \geq 1$. The group of linear automorphisms of E is $\mathrm{GL}(E)$. An *affine map* $E \to E$ is the composition of a linear map and a translation. The group of affine automorphisms of E is denoted by $\mathrm{Aff}(E)$.

Let $g \in \mathrm{Aff}(E)$. There are unique elements $L \in \mathrm{GL}(E)$ and $b \in E$ such that $g(x) = Lx + b$. We call L the *linear part* of g and b *the translational part* of g. Notice that $L = dg(x)$, the derivative of g at x, for all $x \in E$; and $b = g(0)$.

An *affine representation* of a group G is a homomorphism $\alpha : G \to \mathrm{Aff}(E)$. For each $g \in G$ the affine automorphism $\alpha(g)$ decomposes into a linear part $\lambda(g)$ and a translational part $u(g)$. In this way E becomes a G-module via λ and $u : G \to E$ is a *crossed homomorphism*, or *cocycle*, for λ: for all $x \in E$, $g, h \in G$,

$$\alpha(g)(x) = \lambda(g)x + u(g) \tag{1}$$

$$u(gh) = u(g) + \lambda(g)u(h) \tag{2}$$

Conversely, given any (linear) representation $\lambda : G \to \mathrm{GL}(E)$, and $u : G \to E$ satisfying (2), formula (1) defines an affine representation of G having linear part λ.

It is readily verified that $y \in E$ is a stationary point for α – that is, $\alpha(g)(y) = y$ for all $g \in G$ – if and only if

$$u(g) = y - \lambda(g)y \tag{3}$$

for all $g \in G$. In this case u is called a *principal crossed homomorphism*, or a *coboundary*, and we write $u = \delta y$. It is easily proved that $u = \delta y$ if and only if α is conjugate to the linear action λ by the translation $T_y : E \to E$, $T_y(x) = x - y$ (that is, $\lambda(g) = T_y \circ \alpha(g) \circ T_y^{-1}$ for all g).

We assume familiarity with the language of G-modules and their cohomology groups: whenever a linear representation $\lambda : G \to \mathrm{GL}(E)$ is given, E is called a G-module, and vector spaces $H^i(G : E_\lambda)$ are defined for $i = 0, 1, \ldots$. Often the subscript λ will be omitted. The only G-modules we consider are defined by real finite-dimensional representations.

Of primary interest are H^0 and H^1. By definition $H^0(G : E_\lambda)$ is the space of stationary points of λ, while $H^1(G; E_\lambda)$ is the quotient space of crossed homomorphism for λ modulo principal crossed homomorphisms. The element of $H^1(G; E_\lambda)$ represented by u is the cohomology class of u, denoted by $[u]$. One may interpret $H^1(G; E_\lambda)$ as the set of translational conjugacy classes of affine representations of G having linear part λ.

The *radiance obstruction* of α is the cohomology class $c_\alpha = [u] \in H^1(G; E)$ where u is the translational part of α. The following simple observation will be very useful: $c_\alpha = 0$ *if and only if* α *is conjugate by a translation to its linear part*; *and also* $c_\alpha = 0$ *if and only if* α *has a stationary point.*

Also of interest are the homology groups $H_i(G; E_\lambda)$, of which we need only H_0. For a general definition the reader is referred to Atiyah–Wall [3]. The homology group $H_0(G; E_\lambda)$ is by definition E/K where K is the linear subspace spanned by

$$\{x - \lambda(g)x : x \in E \text{ and } g \in G\}.$$

Let $E^* = \mathrm{Hom}\,(E, \mathbf{R})$ denote the dual space of E. The *contragredient representation* of λ is

$$\lambda^t : G \to \mathrm{GL}\,(E^*)$$
$$\lambda^t(g) : w \mapsto w \circ \lambda(g^{-1}), \qquad w \in E^*.$$

It is readily proved that $H^0(G; E^*)$ and $H_0(G; E)$ are dual vector spaces under the paring

$$H^0(G, E^*) \times H_0(G, E) \to \mathbf{R}$$
$$(w, x + K) \mapsto w(x)$$

The following result, proved in Hirsch [20] (see also Dwyer [46]), will be used repeatedly:

1.1. LEMMA. *Let* G *be a nilpotent group and* E *a* G*-module. Then* $H^0(G; E) = 0$ *implies* $H^i(G; E) = 0$ *for all* $i \geq 0$.

From this we deduce:

1.2. THEOREM. *Let G be a nilpotent group, E a G-module. Then the following are equivalent*:

(a) $H^0(G; E)=0$, (b) $H^0(G; E^*)=0$,
(c) $H_0(G; E)=0$, (d) $H_0(G; E^*)=0$

Proof. Suppose $H^0(G; E)=0$. Let $H_0(G; E)=E/K$ as before. The exact sequence $0 \to K \to E \to E/K \to 0$ determines an exact sequence

$$0 \to H^0(G; K) \to H^0(G; E) \to H^0(G; E/K) \to H^1(G; K).$$

Since $H^0(G; E)=0$, by exactness, $H^0(G; K)=0$. By Lemma 1.1, $H^1(G; K)=0$, and now by exactness $H^0(G; E/K)=0$. But G acts trivially on E/K, so $H^0(G; E/K)=0$ implies $E/K=0$. Thus $H_0(G; E)=0$.

This proves (a)$\Rightarrow$(c). The other implications follow from this and the dualities

$$H^0(G; E^*)=H_0(G; E)^*,$$
$$H_0(G; E^*)=H^0(G; E)^*.$$ QED.

1.3. COROLLARY. *Let G be a nilpotent group and E a G-module. If $H^1(G; E)\neq 0$, then G leaves fixed a nonzero vector in E and a nonzero linear functional in E^*.*

A G-module E is *unipotent* if for every $g\in G$ the operator $g-I$ is a nilpotent linear endomorphism of E (i.e. $(g-I)^n=0$ where $n=\dim E$). We call an affine representation unipotent if its linear part defines a unipotent module.

It is well known that E is unipotent if and only if E has a vector space basis in terms of which G is represented by upper triangular matrices all of whose diagonal entries are 1's (see e.g. Humphreys [23]). Even if E is not unipotent there is a unique maximal submodule $E_U\subset E$ upon which G acts unipotently, called the *Fitting submodule.*

1.4. LEMMA. $H^0(G; E/E_U)=0$.

Proof. If $x+E_U\in H^0(G; E/E_U)$, then the submodule spanned by x and E_U is unipotent. Hence $x\in E_U$ by maximality. QED.

An affine representation $\alpha : G \to \mathrm{Aff}(E)$ is *reducible* if $\alpha(G)$ leaves invariant a proper affine subspace $F_0 \subset E$. Let $F \subset E$ be the linear subspace of which F_0 is a coset. Then $\alpha(G)$ permutes the cosets of F, so there is induced an affine representation $G \to \mathrm{Aff}(E/F)$, having a stationary point at the coset F_0. If α is not reducible it is called *irreducible*.

An affine representation with a stationary point is called *radiant*. This is a special case of reducible (and also of decomposable, defined below). Every linear representation is a radiant affine representation, and every radiant representation is conjugate to a linear one by a translation.

A reducible affine representation α may have the stronger property of being *decomposable* (compare Zassenhaus [45]). This means that there is a splitting $E = E_1 \oplus F$, $E_1 \neq E$, invariant under the linear part of α, with some coset of E_1 invariant under α. We call $E_1 \oplus F$ a *decomposition* of α. The radiant case $E_1 = 0$, $F = E$ is allowed. If α does not have a decomposition, α is *indecomposable*. Evidently irreducibility implies indecomposability.

1.5. THEOREM. *Let $\alpha : G \to \mathrm{Aff}(E)$ be an indecomposable affine representation of a nilpotent group G. Then α is unipotent.*

The proof of this theorem will be broken up into the following two results, 1.6 and 1.7:

1.6. LEMMA. *Let $E_U \subset E$ be the Fitting submodule of the linear part of an affine representation α of G. If G is nilpotent then some coset of E_U is invariant under $\alpha(G)$.*

Proof. Let $v : G \to E$ be the translational part of α. It is easy to verify that the composition $v' : G \xrightarrow{v} E \to E/E_U$ is a crossed homomorphism for the induced linear action λ' of G on E/E_U. Thus v' is the translational part of an affine representation $\alpha' : G \to \mathrm{Aff}(E/E_U)$ whose linear part is λ'. The natural projection $E \to E/E_U$ is equivariant respecting (α, α').

By Lemma 1.4, $H^0(G; E/E_U) = 0$, so by Lemma 1.1, $H^1(G; E/E_U) = 0$ because G is nilpotent. In particular the radiance obstruction $c_{\alpha'} \in H^1(G; E/E_U)$ vanishes. Therefore $\alpha'(G)$ fixes some $x + E_U \in E/E_U$; this means that $\alpha(G)$ leaves invariant the coset $x + E_U$. QED.

Lemma 1.6 shows that the conclusion of Theorem 1.5 holds under the stronger assumption that α is irreducible.

The proof of 1.5 is completed by the following *splitting theorem*, 1.7. The analogue for Lie algebras is well known.

1.7. THEOREM. *If G is a nilpotent group and E a G-module with Fitting submodule E_U, there exists a unique submodule F such that $E = E_U \oplus F$.*

Proof. Put $E/E_U = V$. Let $\beta : G \to \mathrm{GL}(V)$ be the induced representation. By 1.4, $H^0(G; V) = 0$.

Let $P : E \to V$ be the canonical projection, which is equivariant. Let $S : V \to E$ be a linear map such that $P \circ S = I_V$, the identity map of V.

A submodule $F \subset E$ is complementary to E_U if and only if $F = T(V)$ where $T : V \to E$ is an equivariant linear map with $P \circ T = I_V$. Thus we must prove there is a unique such T.

Any linear $T : V \to E$ with $P \circ T = I_V$ can be uniquely expressed as $T = R + S$ where $R : V \to E_U$, and T is equivariant if and only if

$$R + S = g \circ (R + S) \circ \beta(g)^{-1}$$

or

$$R = g \circ R \circ \beta(g)^{-1} + g \circ S \circ \beta(g)^{-1} - S. \tag{4}$$

We must prove there is a unique R satisfying (4).

Define a linear action γ of G on $\mathrm{Hom}(V, E_U)$ by

$$\gamma : G \to \mathrm{GL}(\mathrm{Hom}(V, E_U)), \qquad \gamma(g)(R) = g \circ R \circ \beta(g)^{-1}$$

and a cocycle u for γ by

$$u : G \to \mathrm{Hom}(V, E_U), \qquad u(g) = g \circ S \circ \beta(g)^{-1} - S.$$

To see that $u(g)$ maps V into E_U notice that $P \circ u(g) = 0$ because P is equivariant and $P \circ S = I_V$.

Now (4) says that R is a stationary point of that affine action ζ of G on $\mathrm{Hom}(V, E_U)$ defined by γ and u. Thus we must prove that ζ has a unique stationary point. This will follow from 1.1 and 1.2 if we prove that $H^0(G; \mathrm{Hom}(V, E_U)) = 0$.

To this end suppose $R : V \to E_U$ is fixed under all $\gamma(g)$, i.e., suppose

$$g \circ R = R \circ \beta(g), \quad \text{all} \quad g \in G. \tag{5}$$

Let $d = \dim E_U$. Since $g \mid E_U$ is unipotent, for all $g_1, \ldots, g_d \in G$ we have

$$(I - g_d) \circ \cdots \circ (I - g_1) \mid E_U = 0.$$

From this and (5), R vanishes on the linear span in V of all vectors of the form

$$[I-\beta(g_d)]\cdots[I-\beta(g_1)]x, \qquad x \in V. \tag{6}$$

Hence it suffices to prove every vector in V is a linear combination of vectors of the form (6). This will be evident if we show that V is spanned by vectors of the form $[I-\beta(g)]x$, i.e., that $H_0(G, V)=0$. But this follows from $H^0(G, V)=0$ and 1.2. QED.

An alternative proof of 1.7 can be based on the analogous result about Lie algebras, by passing to the Lie algebra of the identity component of the algebraic closure of G in $\mathrm{GL}(E)$.

2. Development, holonomy and parallelism

Let M denote an n-dimensional manifold, $n \geq 1$. We shall always assume M is connected and without boundary.

An *affine atlas* Φ on M is a covering of M by coordinate charts such that each coordinate change between overlapping charts in Φ extends to an affine automorphism of $E=\mathbf{R}^n$. A maximal affine atlas is an *affine structure*, on M, and M together with an affine structure is an *affine manifold*. Each chart in the affine structure defines *affine coordinates*.

Let M be an affine manifold. Let $\mathcal{S}$ denote the sheaf of germs of affine coordinate systems on M. The germ at $x \in U$ of the affine chart $f: V \to E$ is denoted by $[f]_x$.

The group $\mathrm{Aff}(E)$ acts stalkwise on $\mathcal{S}$ by composition of germs: $g \in \mathrm{Aff}(E)$ sends $[f]_x$ to $[g \circ f]_x$. It is easy to see that $\mathrm{Aff}(E)$ acts simply transitively on stalks. In fact $\mathcal{S}$ is a principal $\mathrm{Aff}(E)$-bundle over M when $\mathrm{Aff}(E)$ is given the discrete topology.

The following result, whose proof is left to the reader, gives another description of $\mathcal{S}$.

2.1. THEOREM. *$\mathcal{S}$ is a disconnected covering space of M. If $[f_1]_x, [f_2]_x \in \mathcal{S}$ are affine coordinate germs at $x \in M$, there is a deck transformation taking f_1 to f_2, induced by composition with a unique affine automorphism of E. Thus the group of deck transformations acts transitively on each fibre of $\mathcal{S}$, and the components of $\mathcal{S}$ are isomorphic as covering spaces of M. These covering spaces are regular.*

We fix one component of $\mathcal{S}$ and call it the *holonomy covering space* $\hat{M}$ of M. There is a canonical immersion $\hat{D}:\hat{M}\to E$ which assigns $f(x)$ to the germ $[f]_x$. This immersion is affine in the sense that in affine coordinates, it appears as an affine map.

Let Γ denote the group of deck transformations of $\hat{M}\to M$. An element $g\in\Gamma$ sends $[f]_x$ to $[\hat{\alpha}(g)\circ f]_x$, where $\hat{\alpha}(g)$ is the affine automorphism of E determined by g. The resulting map $\hat{\alpha}:\Gamma\to \mathrm{Aff}(E)$ is a faithful affine representation of Γ on E. It is easy to see that $\hat{D}$ is equivariant respecting α, i.e., for each $g\in\Gamma$ the following diagram commutes:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{D}} & E \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \hat{\alpha}(g)} \\ \hat{M} & \xrightarrow{\hat{D}} & E \end{array}$$

Now let $p:\tilde{M}\to M$ be a universal covering, and fix a lift of p to a covering $\hat{p}:\tilde{M}\to\hat{M}$. Denote by $D:\tilde{M}\to E$ the composition $\hat{D}\circ\hat{p}$; we call D the *developing map* of the affine manifold M. We give $\tilde{M}$ the affine structure inherited from M; then $D:\tilde{M}\to E$ is an affine immersion.

Let π denote the group of deck transformations of $\tilde{M}\to M$. Choice of a base point in $\tilde{M}$ identifies π with the fundamental group of M. Since $\tilde{M}\to\hat{M}$ is a regular covering, there is a natural surjective homomorphism $\pi\to\Gamma$ with respect to which $\tilde{M}\to\hat{M}$ is equivariant.

Denote by $\alpha:\pi\to\mathrm{Aff}(E)$ the composite homomorphism $\pi\to\Gamma\xrightarrow{\hat{\alpha}}\mathrm{Aff}(E)$. We call α the *affine holonomy* of M. The image $\alpha(\pi)\subset\mathrm{Aff}(E)$ is isomorphic to Γ; it is called the *affine holonomy group* of M. It is clear that $D:\tilde{M}\to E$ is equivariant respecting α.

The definitions of D and α depend on various choices of covering spaces. But once the universal covering $\tilde{M}\to M$ is fixed, changing $\hat{M}$ or the lift $\tilde{M}\to\hat{M}$ changes D only by composition with an element of $\mathrm{Aff}(E)$; and it changes α by conjugation with that element.

Let $\lambda:\pi\to\mathrm{GL}(E)$ be the homomorphism assigning to each $g\in\pi$ the linear part of $\alpha(g)$. We call λ the *linear holonomy*; it is well defined up to conjugation by an element of $\mathrm{GL}(E)$.

We shall assume when discussing an affine manifold M that a developing map $D:\tilde{M}\to E$ has been selected. The affine and linear holonomy representations are then determined.

This somewhat abstract description of holonomy can be brought down to earth by working through the definitions. In terms of loops on M based at $x_0\in M$, the homomorphism

$$\alpha : \pi_1(M, x_0) \to \operatorname{Aff}(E)$$

can be described as follows. Let $s:[0,1]\to M$ be a loop at x_0, representing $[s]\in \pi_1(M, x_0)$. Subdivide $[0,1]$ by $0=t_0<\cdots<t_m=1$, so that for each $i=1,\ldots,m$ there is an affine chart (f_i, U_i) with $s:[t_{i-1}, t_i]\to U_i$. (The choice of (f_0, U_0) determines a component of $\mathcal{S}$.) Let V_j be the component of $f_j(U_j\cap U_{j+1})$ that contains $f_j(s(t_j))$, $j=1,\ldots,m-1$. There is a unique $A_j\in \operatorname{Aff}(E)$ such that

$$f_{j+1}\circ f_j^{-1}=A_j \text{ on } V_j, \qquad j=1,\ldots,m.$$

Then

$$\alpha([s])=\begin{cases} A_m\circ\cdots\circ A_1 \text{ if } m\geq 2\\ \text{the identity map } I \text{ of } E \text{ if } m=1.\end{cases}$$

We emphasize that there is some freedom of choice in the developing map D. In practice we shall alter D only by composing it with a translation of E.

The affine manifold M is *complete* if D maps $\tilde{M}$ homeomorphically onto E. In this case we can take $\tilde{M}=E$. The affine holonomy is then an inclusion $\pi\hookrightarrow \operatorname{Aff}(E)$ of a discrete subgroup which acts freely and properly discontinuously on E. Thus M is identified with the orbit space E/π.

M can be compact without being complete, e.g. the 1-dimensional affine manifold $\mathbf{R}_+/\{2^n\}$. Examples have been constructed of compact affine manifolds whose developing maps fail to be covering-space projections into their images (see also Section 3).

The following result illustrates the interplay between the topology of an affine manifold, its developing map, and its affine holonomy representation:

2.2. THEOREM. *Let M be a compact complete affine manifold. Then the affine holonomy representation is irreducible.*

Proof. Let $M=E/\pi$. Suppose $F\subset E$ is an affine subspace invariant under π. Then F is the universal covering of F/π. Both F/π and E/π are Eilenberg-MacLane spaces of type $K(\pi; 1)$. The inclusion $F/\pi\to E/\pi$ induces an isomorphism of fundamental groups and is therefore a homotopy equivalence. Since F/π and E/π are compact manifolds it follows that $\dim F/\pi=\dim E/\pi$. Hence $F=E$.

QED.

On an affine manifold M there is a natural linear connection ∇ having zero curvature and torsion: in affine coordinates it appears as the standard connection on E. Covariant differentiation in M corresponds to ordinary differentiation in E.

It is proved in Auslander and Markus [9] that an affine manifold M is complete if and only if the connection ∇ is geodesically complete.

The operation of parallel transport of a tensor along a path in M depends only on the homotopy class of the path (fixing end points). This corresponds to ordinary parallel transport in E in successive affine charts.

A tensor (field) T on M is *parallel* if its covariant derivative vanishes. This means that in every connected affine chart the components of T are constant. Equivalently, the induced tensor field on $\tilde{M}$ is the pullback by the developing map of a constant tensor field T_0 on E. Thus T_0 is an element of the tensor algebra $T(E)$ which is fixed under the induced action of the linear holononomy group on $T(E)$. In every affine chart T appears as T_0.

We shall be particularly concerned with parallel vector fields, parallel 1-forms, and parallel volume elements. A parallel vector field on M corresponds to a vector in E which is stationary under the linear holonomy. The set of such vectors is $H^0(\pi; E)$. A *parallel 1-form* corresponds to a linear map $E \to \mathbf{R}$ stationary under the contragredient action, i.e. an element of $H^0(\pi; E^*)$. A *parallel volume form* is a nonzero exterior n-form on E stationary under the induced action of the linear holonomy group on $\Lambda^n E^*$. Since this action is precisely the determinant of the linear holonomy, M has a parallel volume form if and only if the linear holonomy λ has determinant identically one, i.e. when λ is a representation in the special linear group $\mathrm{SL}(E)$.

The condition that an affine manifold M have a parallel volume form is an extremely useful one. We refer to it by saying that *M has parallel volume.*

A basic conjecture, going back to Markus [28], is that a compact orientable affine manifold has parallel volume if and only if it is complete. This is trivially true for flat Riemannian manifolds. Furness and Fedida [15] prove it for flat pseudo-Riemannian tori in dimensions ≤ 3. In his thesis [38] Smillie proves the conjecture in the case of abelian affine holonomy; for nilpotent affine holonomy it follows from 6.6 and 6.8 below.

More general than parallel tensors are *polynomial tensor fields.* By these we mean tensor fields on affine manifolds whose coefficients in affine coordinates are polynomials. A tensor field T is polynomial of degree $<p$ if and only if the iterated covariant derivative $\nabla_{v_p} \circ \cdots \circ \nabla_{v_1}(T)$ vanishes for all $v_p, \ldots, v_1 \in M_x$, all $x \in M$. (Here ∇ is the covariant differentiation associated to the affine structure.)

Many examples of affine manifolds come from left-invariant affine structures on Lie groups. If G is a Lie group, an affine structure on G is *left-invariant* if for each $g \in G$ the operation $L_g : G \to G$ of left multiplication by g is an automorphism of the affine structure. In other words, in affine coordinates L_g is expressed by an affine map.

The development map of such a structure blends together with left multiplication in a remarkable way. Suppose G is simply connected; let $D : G \to E$ be the developing map. For each $g \in G$ there is a unique affine automorphism $\alpha(g)$ of E such that the following diagram commutes:

$$\begin{array}{ccc} g & \xrightarrow{D} & E \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow{D} & E \end{array}$$

Clearly $\alpha : G \to \mathrm{Aff}(E)$ is an affine representation. In particular $\alpha(G)$ preserves the connected open set $D(G) \subset E$, and acts transitively on it. Conversely if $\dim G = \dim E$ and $\alpha : G \to \mathrm{Aff}(E)$ is an affine representation having an open orbit $U = D(G)x_0$ for some $x_0 \in E$, then there is a unique left-invariant affine structure on G whose developing map is given by

$$D : G \to U \subset E, \qquad D(g) = \alpha(g)x_0.$$

Notice that when $x_0 = 0 \in E$, D is just the translational part of the affine representation.

The left-invariant affine structure on G is complete precisely when α is a simply transitive affine action of G on E. In this case α is evidently irreducible.

Let $D : G \to E$ be the developing map of a left-invariant affine structure. Then $D : G \to D(G)$ is a covering space, so the structure is complete if and only if the action of G on E is transitive. It is known that in this case G must be solvable (Auslander [6], Milnor [31]). It is conjectured that every solvable Lie group has such a structure.

For examples of left-invariant affine structures see Auslander [6], [7]; Scheuneman [35], [36]; and Section 6.

Suppose now that Γ is a discrete subgroup of G, where G has a left-invariant affine structure. Then the homogeneous space Γ/G of right cosets inherits an affine structure. When G is a simply-connected nilpotent Lie group, $\Gamma\backslash G$ is called an *affine nilmanifold.*

An affine representation $\alpha : G \to \mathrm{Aff}(E)$ is conveniently presented by the linear representation

$$\alpha' : G \to \mathrm{GL}(E \times \mathbf{R}),$$
$$\alpha' : (g) : (x, t) \mapsto (\lambda(g)x + tu(g), t)$$

where λ is the linear part and u the translational part of α.

As an example take $G=\mathbf{R}^2$ (the group) and $E=\mathbf{R}^2$ (the vector space). An affine representation $\phi : G \to \operatorname{Aff}(E)$ is defined by

$$\phi(s,t) : z \to \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} s^2/2+t \\ s \end{bmatrix},$$

for $(s,t)\in G$, $z=\begin{bmatrix} x \\ y \end{bmatrix}\in E$.

The associated linear representation on $\mathbf{R}^3$ assigns to $(s,t)\in G$ the matrix

$$\begin{bmatrix} 1 & s & \frac{s^2}{2}+t \\ \cdot & 1 & s \\ \cdot & \cdot & 1 \end{bmatrix} = \exp\left\{\begin{bmatrix} \cdot & s & \cdot \\ \cdot & \cdot & s \\ \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} \cdot & \cdot & t \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\right\}$$

It is easy to verify that $\phi(G)$ acts simply transitively on E; thus ϕ defines a left-invariant affine structure on G. The developing map $D : G \to E$ defined by evaluation at 0 is the translational part of ϕ:

$$D(s,t) = \begin{bmatrix} (s^2/2)+t \\ s \end{bmatrix}.$$

The left-invariant vector fields on G are spanned by $\partial/\partial t$ and $\partial/\partial s$; in the affine coordinates on G defined by D these correspond respectively to $\partial/\partial x$ and $y(\partial/\partial x+\partial/\partial y)$. The integral curves of $\partial/\partial x$ are horizontal lines; those of $y(\partial/\partial x+\partial/\partial y)$ are the parabolas $y^2-2x=\text{constant}$.

The functions y and y^2-2x on E transform under G by addition of constants. Therefore the 1-forms dy and $y\,dy-dx$ are the expression in affine coordinates of left-invariant 1-forms on G.

The vector field $\partial/\partial x$ and the form dy (on E) are translation invariant; hence they correspond respectively to a parallel vector field and a parallel 1-form on G.

Since the linear part of α preserves the area 2-form $dx\wedge dy$ on E, this form defines a parallel 2-form on G.

For every uniform discrete subgroup $\Gamma\subset G$ we get a complete affine 2-manifold $\Gamma\backslash G$ diffeomorphic to the 2-torus T^2. By Nagano and Yagi [32] and Kuiper [26] these are the only complete structures on T^2 other than the flat Riemannian structure $\mathbf{Z}^2\backslash\mathbf{R}^2$.

The choice of Γ, however, is crucial. For example it is easy to see that $\Gamma\backslash G$ has a closed geodesic (= 1-dimensional affine submanifold) if and only if Γ contains a pure translation $\varphi(0,t_0)$, $t_0\neq 0$.

There are many incomplete affine nilmanifold structures on T^2. For example, for each $a \geq 0$, $b > 0$ there is a radiant affine representation

$$\psi_{a,b} : G \to \mathrm{GL}(\mathbf{R}^2) \subset \mathrm{Aff}(\mathbf{R}^2),$$

$$(s, t) \to \exp \begin{bmatrix} s & as+bt \\ 0 & s \end{bmatrix}$$

$$= e^s \begin{bmatrix} 1 & as+bt \\ 0 & 1 \end{bmatrix}.$$

Each of these actions is simply transitive on the upper half-plane.

For each $\mu > 0$, $\nu > 0$ radiant actions are defined by

$$\theta_{\mu,\nu} : G \to \mathrm{GL}(\mathbf{R}^2) \subset \mathrm{Aff}(\mathbf{R}^2),$$

$$\theta_{\mu,\nu}(s, t) = \begin{bmatrix} \mu^s & 0 \\ 0 & \nu^t \end{bmatrix}$$

These are all simply transitive in the first quadrant.

Some other examples are given below after the proof of Theorem 6.4, and in 6.7.

By assuming that an affine structure is compatible with a complex structure on a manifold one defines the notion of an affine complex manifold. (An affine structure is compatible with a complex structure if and only if the associated flat affine connection is holomorphic with respect to the complex structure; alternatively a complex structure is compatible with an affine structure if and only if the associated almost complex structure is a parallel tensor of type (1, 1). It is easy to see that these two notions are equivalent.) All the results in this paper apply to such manifolds.

It is interesting to note that Fillmore and Scheuneman [11], Suwa [41], and Sakane [34] have proved, independently, that every complex surface having a complete affine complex structure has a finite cyclic covering which is a complex affine nilmanifold. For references and more information on these structures in the complex case the reader is referred to Gunning [19] and Inoue–Kobayashi–Ochiai [24].

Smillie [38], Auslander [6] and Milnor [31] relate affine structures to matrix algebras. See Matsushima [29] for a connection between affine structures on homogeneous complex manifolds and algebras.

3. Radiant manifolds and parallel tensors

In this section we study affine manifolds whose affine holonomy has a stationary point; we call such manifolds radiant.

There are many examples of radiant manifolds: certain affine nilmanifolds; the Hopf manifolds; the cartesian product of any compact surface and the circle. On the other hand the affine structures of radiant manifolds have rather special properties. There are no complete radiant manifolds (except for vector spaces), and they tend to have few parallel tensors. In a sense they are at the opposite extreme from complete affine manifolds.

We begin by defining the radiance obstruction for any affine manifold. This will play a key role in the analysis of nonradiant affine manifolds in Section 4.

M always denotes a connected n-dimensional affine manifold, $n \geq 1$, with local affine coordinates in the vector space $E = \mathbf{R}^n$. We fix a universal cover $p : \tilde{M} \to M$, with group of deck transformations π. Let $D : \tilde{M} \to E$ be a developing map for M and $\alpha : \pi \to \mathrm{Aff}(E)$ the corresponding affine holonomy representation (defined in Section 2). The linear part of α is the linear holonomy $\lambda : \pi \to \mathrm{GL}(E)$, and the translational part α is the cocycle $u : \pi \to E$ for λ.

The *radiance obstruction* of M is the cohomology class

$$c_M = c_\alpha \in H^1(\pi, E_\lambda),$$

where $c_\alpha = [u]$ is the radiance obstruction of the affine representation α (defined in Section 1). *This cohomology class depends only on the affine holonomy of M* (assuming the universal covering space $\tilde{M} \to M$ has been fixed), and not on the choice of the developing map. To see this let $\iota : \mathrm{Aff}(E) \to \mathrm{Aff}(E)$ be the identity affine representation. Let

$$\alpha^* : H^1(\mathrm{Aff}(E); E) \to H^1(\pi; E)$$

be the homomorphism induced by α. Let $c_\iota \in H^1(\mathrm{Aff}(E) : E)$ be the radiance obstruction of ι; then $\alpha^* c_\iota = c_M$. Let c'_M be the radiance obstruction corresponding to a different developing map $D' : \tilde{M} \to E$. There exists $g \in \mathrm{Aff}(E)$ such that $D' = g \circ D$, and one verifies that $c'_M = g_\# c_\iota$, where $g_\#$ is the automorphism of $H^1(\mathrm{Aff}(E); E)$ induced by conjugation by g. But it is well known that $g_\#$ is the identity, hence $c'_M = c_M$.

In a forthcoming paper [14] we shall discuss the radiance obstruction in more detail.

It follows from the results of Section 1 that $c_M = 0$ precisely when $\alpha(\pi)$ has a stationary point in E. In this case we call M a *radiant manifold*. We can compose the developing map with a translation of E so that the stationary point is the origin, i.e., α takes values in $\mathrm{GL}(E)$. In this case $\alpha = \lambda$.

Since a nontrivial group of deck transformations cannot have a stationary point[1] we see that a complete affine manifold is not radiant unless it is affinely isomorphic to a vector space.

When M is radiant we shall tacitly assume that the origin in E is stationary under $\alpha(\pi)$. In this case $\alpha(\pi)$ fixes the radiant vector field $R = \Sigma x_i \partial/\partial x_i$ on E. There is a unique vector field $\tilde{X}$ on $\tilde{M}$ which is D-related to R. Clearly $\tilde{X}$ is fixed under π; therefore $\tilde{X}$ is p-related to a vector field X on M, called the *radiant vector field* of M.

It is easy to find examples of such manifolds. The simplest examples are Hopf manifolds: these are the orbit spaces $(E-\{0\})/\pi$ where $\pi \subset \mathrm{GL}(E)$ is the cyclic subgroup generated by an *expansion* $A \in \mathrm{GL}(E)$, i.e., the eigenvalues of A all have absolute value greater than 1. For another example, the Cartesian product of a compact orientable surface Σ of genus ≥ 2 and the circle S^1 can be given a radiant affine structure as follows. Use one of the well known faithful representations of $\pi_1(\Sigma)$ as a discrete subgroup of $\mathrm{SO}(1, 2)$ to obtain a properly discontinuous free action of $\pi_1(\Sigma)$ on one sheet of a two-sheeted hyperboloid $H \subset \mathbf{R}^3 - 0$. Let $\mathbf{Z} \subset \mathrm{GL}(\mathbf{R}^3)$ be generated by kI where $0 < k < 1$. Then $\pi_1(\Sigma) \times \mathbf{Z}$ acts freely as deck transformations on a component of the interior of the light cone; the orbit space is $\Sigma \times S^1$.

There are many radiant affine structures on the product of a closed surface of genus ≥ 1 and a circle, for which the developing map is not a covering-space projection of $\tilde{M}$ onto its image. This phenomenon was discovered by Thurston (see [40]) and independently by Smillie [38]. The developing image of such a 3-torus is the complement of the three coordinate axes in $\mathbf{R}^3$; for the product of a surface of genus ≥ 2 and a circle, the developing map can be onto the complement of the origin. On the other hand, we know of no example of a compact affine manifold whose developing map is surjective but which is not complete (compare Theorem 6.9).

3.1. THEOREM. *A compact radiant manifold does not have a parallel volume form.*

Proof. Suppose M is radiant with a parallel volume form. A trivial computation in affine coordinates shows that the flow of the radiant vector field increases volume. Hence M cannot be compact. QED.

---

[1]In fact no element of the group (except the identity) can have a fixed point. It follows easily that every element of the linear holonomy of a complete affine manifold has 1 as an eigenvalue. This might be called "Hirach's principle", following Sullivan [39].

3.2. THEOREM. *A compact radiant manifold M does not have a nonzero parallel* 1-*form.*

Proof. Let $\omega : E \to R$ be a linear map invariant under the linear holonomy. Then ω is also invariant under the affine holonomy. Therefore the composite map

$$\tilde{M} \xrightarrow{D} E \xrightarrow{\omega} \mathbf{R}$$

covers a map $f : M \to \mathbf{R}$. Let $x \in \tilde{M}$ cover a local minimum point of f. Then $D(x) \in E$ is a local minimum point of ω, so ω, being linear, must be identically zero. QED.

There are numerous compact radiant manifolds having parallel *line* fields, for example a Hopf manifold $(E - \{0\})/\pi$, where $\pi \subset \mathrm{GL}(E)$ is a cyclic subgroup generated by an expansion having a real eigenvalue.

With later applications in mind we prove the following result about decomposable affine manifolds. Note that in the case of radiant manifolds ($E_1 = 0$) it shows that any point fixed by the affine holonomy action lies outside the developing image (this was first stated by Nagano–Yagi [32] but their proof is incorrect). Thus the *radiant vector field on a compact radiant manifold is nonsingular.*

3.3. THEOREM. *Suppose M is a compact affine manifold with decomposable holonomy, say $E = E_1 \oplus F$ with E_1 α-invariant and F λ-invariant, $F \neq 0$. Then each component of $M_1 = p(D^{-1}E_1)$ is incomplete.*

Proof. Clearly M_1 is an affine submanifold of M. Assume a component N of M_1 is complete.

Consider the α-invariant vector field $S(x, y) = (0, y)$ on $E = E_1 \oplus F$. Then S is D-related to a π-invariant vector field $\tilde{Q}$ on $\tilde{M}$ which induces a vector field Q on M. By the compactness of M, the vector fields Q and $\tilde{Q}$ are integrable for all time and determine flows ϕ_t and $\tilde{\phi}_t$ on M and $\tilde{M}$ respectively. If the linear flow determined by S is denoted ψ_t, we have $\psi_t \circ D = D \circ \tilde{\phi}_t$.

Suppose $N \subset M$ is a component of $p^{-1}E_1$. Then $D \mid \tilde{N}$ is a homeomorphism onto E_1 by completeness.

The submanifold $E_1 \subset E$ is a repellor for the flow ψ_t. It follows that $\tilde{N}$ is a repellor for the flow $\tilde{\phi}_t$. Let $B = \{x \in M : \phi_t x \to \tilde{N}$ as $t \to -\infty\}$ denote the repelling basin of $\tilde{N}$.

Now $D : \tilde{N} \to E_1$ is a homeomorphism and $D : \tilde{M} \to E$ is a local homeomorphism. It follows that D maps a neighborhood $B_0 \subset B$ of $\tilde{N}$

homeomorphically onto a neighborhood $W \subset C$ of E_1. Therefore D maps $\tilde{\phi}_t(B_0)$ homeomorphically onto $\psi_t(W)$ for all $t \geq 0$. Since $\bigcup_{t>0} \psi_t(w) = E$ it follows that D maps $\bigcup_{t>0} \tilde{\phi}_t(B_0)$ homeomorphically onto E. This implies that $D : \tilde{M} \to E$ is a homeomorphism. Therefore M is complete, in contradiction to Theorem 2.2.

QED.

4. Affine manifolds with nilpotent holonomy

In this section we begin to exploit the assumption of nilpotent affine holonomy group. The same notation and conventions as in Section 3 are used.

4.1. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Then the following conditions are equivalent*:

(a) *M is not radiant*;

(b) *M has both a nonzero parallel vector field and a nonzero parallel* 1-*form.*

(c) $H^1(\pi; E) \neq 0$.

Proof. (a)⇒(b): Nonradiant means $c_M \neq 0$, so $H^1(\pi; E) \neq 0$. By 1.3, $H^0(\pi; E) \neq 0$ and $H^0(\pi; E^*) \neq 0$, which is (b).

(b)⇒(a): follows from 3.2.

(c)⇔(b): follows from 1.3. QED.

Notice that the implication (a)⇒(b) in 4.1 is valid even without compactness.

4.2. THEOREM. *Let M be a compact nonradiant affine manifold having nilpotent affine holonomy group. Then M fibres over the circle S^1.*

Proof. A nonzero parallel 1-form on M is a nowhere-vanishing closed 1-form. It is well known that the existence of such form implies that M fibres over S^1 (Tischler [42]). QED.

We conjecture that 4.2 holds even if M is radiant.

As a corollary of Theorem 4.1 and the nonsingularity of radiant vector fields, we see that a compact affine manifold with nilpotent affine holonomy group has Euler characteristic zero. This also follows from Hirsch–Thurston [22]; to use this result, it suffices that the linear holonomy group be solvable. Kostant and Sullivan [25] have shown that every complete compact affine manifold has zero Euler characteristic. It has long been conjectured that every affine manifold (except $\mathbf{R}^n$) which is complete *or* compact has zero Euler characteristic. (In the noncompact case the Euler characteristic is defined as the alternating sum of the Betti numbers, which are conjectured to be finite.)

We now concentrate attention more directly on the holonomy representation.

An immediate consequence of Theorem 1.5 is:

4.3. THEOREM. *Let M be an affine manifold whose affine holonomy representation is nilpotent and indecomposable. Then the linear holonomy representation is unipotent.*

Little is known about the structure of noncompact complete affine manifolds, even those with nilpotent holonomy (but see Milnor [31]). The following result shows the importance of unipotent in this case.

4.4. THEOREM. *Let M be a noncompact complete affine manifold with nilpotent affine holonomy group. Then M is a flat affine vector bundle over a complete affine manifold with unipotent linear holonomy.*

Proof. Let $E_U \subset E$ be the Fitting component of the linear holonomy. We may assume E_U invariant under $\alpha(\pi)$. By 4.1 M has a nonzero parallel vector field, so $E_U \neq 0$.

By the splitting Theorem 1.7 there is a decomposition $E = E_U \oplus F$ of α. The vector bundle is

$$q : M = E/\alpha(\pi) \rightarrow E_U/\alpha(\pi),$$

with total space M, base space $E_U/\alpha(\pi)$, and fibre F. It is easy to see that there are local trivializations

$$f_i : q^{-1}(W_i) \rightarrow W_i \times F$$

over an open cover $\{W_i\}$ of $E_U/\alpha(\pi)$, which are affine isomorphisms, and whose transition functions $g_{ij} : W_i \rightarrow \mathrm{GL}(F)$ are constants. Thus the bundle is flat.

QED.

5. Nilpotent G-modules without expansions

The purpose of this section is to prove the following technical result. It will be used in Section 6 to prove the existence of expansions in the linear holonomy of certain compact affine manifolds.

Recall that $E = \mathbf{R}^n$, $n \geq 1$.

5.1. LEMMA. *Let G be a nilpotent group and E a G-module. Suppose that G does not contain an expansion of E. Then for every integer $r \geq 0$ there exists a C^r map $\Psi : E \to \mathbf{R}$ with the following properties:*

(a) *$\Psi > 0$ almost everywhere;*

(b) *Ψ is G-invariant;*

(c) *there exists $a > 0$ such that*

$$\Psi(e^t x) = e^{ta}\Psi(x)$$

for all $t \in \mathbf{R}$, $x \in E$.

Proof. If the lemma is true for a normal subgroup $G_0 \subset G$ of finite index, it is true for G. For let $\Psi_0 : E \to \mathbf{R}$ be a C^r G_0-invariant map satisfying (a) and (c). Let the left cosets of G_0 be $g_1 G_0, \ldots, g_\nu G_0$. Then the map

$$\Psi : E \to \mathbf{R}, \qquad \Psi(x) = \sum_{i=1}^{\nu} \Psi_0(g_i x)$$

satisfies the lemma.

Let $F = \mathbf{C} \oplus E^*$ be the complexification of the dual space E^* of E. The contragredient representation of G on E^* extends to a (complex) representation $\rho : G \to \mathrm{GL}(F)$. Let $H \subset \mathrm{GL}(F)$ be the identity component of the algebraic closure of $\rho(G)$. Then H is a connected nilpotent Lie group. Set $G_0 = \rho^{-1}(H)$. Then $G_0 \subset G$ is a normal subgroup of finite index.

From the primary decomposition of the representation of the Lie algebra of H in F induced by ρ we get a ρ-invariant splitting $F = \oplus F_k$. Each F_k has a basis $\mathcal{B}_k$ representing the operators $\rho(h) \mid F_k$, $h \in H$ as complex matrices

$$\rho_k(h) = \lambda_k(h) I + N_k(h)$$

where $N_k(h)$ is an upper-triangular nilpotent matrix. In particular the set of eigenvalues of $\rho(h)$ is $\{\lambda_k(h)\}$.

Let $f_k \in F_k$ be the first basis vector in $\mathcal{B}_k$, $k = 1, \ldots, m$. Then

$$\rho_k(h) f_k = \lambda_k(h) f_k \tag{1}$$

for all $h \in H$.

Define group homomorphisms

$$\varphi_k : G_0 \to \mathbf{R}, \qquad \varphi_k(g) = \log |\lambda_k(g)|; \qquad k = 1, \ldots, m.$$

Set

$$\varphi = (\varphi_1, \ldots, \varphi_m) : G_0 \to \mathbf{R}^m.$$

Suppose G_0 contains no expansion of E. Then $\varphi(G_0)$ is disjoint from the positive orthant $P \subset \mathbf{R}^m$,

$$P = \{y \in \mathbf{R}^m : y_k > 0,\ k = 1, \ldots, m\}.$$

Since $\varphi(G_0)$ is a subgroup of $\mathbf{R}^m$ it follows that the linear span $L \subset \mathbf{R}^m$ of $\varphi(G_0)$ is also disjoint from P. This implies that the orthogonal complement of L contains a nonzero vector $v \in \bar{P}$. This vector $v = (v_1, \ldots, v_m)$ thus has the following properties:

$$\text{each} \quad v_k \geq 0 \tag{2}$$

$$\sum_{k=1}^{m} v_k = a > 0 \tag{3}$$

$$\sum_{k=1}^{m} v_k \varphi_k(g) = 0 \quad \text{for all} \quad g \in G_0. \tag{4}$$

Evidently the vector cv, $c > 0$, has the same properties. Therefore we can also choose v to satisfy:

For each k, either $v_k = 0$ or $v_k > 2r$, where r comes from Lemma 5.1. (5)

For each k we have chosen a vector $f_k \in \operatorname{Hom}_{\mathbf{C}}(\mathbf{C} \otimes E, \mathbf{C})$ satisfying (1). We embed E in $\mathbf{C} \otimes E$ in the natural way and define

$$\Psi : E \to \mathbf{R},$$

$$\Psi(x) = \prod_{k=1}^{m} |f_k(x)|^{v_k}.$$

By (5) Ψ is differentiable of class C^r. By (2) and (3) Ψ satisfies (a), (c) of 5.1.

To show that Ψ is G_0-invariant let $g \in G_0$, $x \in E$. Then

$$\Psi(g^{-1}x) = \prod_{k=1}^{m} |f_k(g^{-1}x)|^{v_k} = \prod_{k=1}^{m} |(\lambda(g) f_k) x|^{v_k}$$

$$= \left(\prod_{k=1}^{m} |\lambda_k(g)|^{v_k}\right)\left(\prod_{k=1}^{m} |f_k(x)|^{v_k}\right) = \left(\exp \sum_{k=1}^{m} v_k \varphi_k(g)\right) \Psi(x) = \Psi(x)$$

by (4). This completes the proof of 5.1. QED.

In general Ψ in 5.1 cannot be chosen to be C^∞. For example, let

$$G=\left\{\begin{bmatrix}\lambda^{\sqrt{2}} & 0\\ 0 & \lambda^{-1}\end{bmatrix}; \lambda>0\right\} \text{ on } E=\mathbf{R}^2.$$

Any such ψ must be of the form $C(xy^{\sqrt{2}})^p$ where $C, p>0$, and will not be C^∞.

6. Compact affine manifolds with nilpotent holonomy

In this section we exploit compactness to prove three main results: 6.1, 6.8, 6.9. The proofs rely on the integration of vector fields. The first two were proved by Smillie [38] for the abelian case.

The first, Theorem 6.1, is used to find expansions of E/E_U in the linear holonomy. This turns out to be a powerful geometric tool. The second, Theorem 6.8, shows that unipotent holonomy implies completeness. This is also true for noncompact affine nilmanifolds.

As is Section 3, $D:\tilde{M}\to E$ is the developing map, π is the group of deck transformations of the universal cover $\tilde{M}\to M$, etc. Let $\Lambda\subset \mathrm{GL}(E)$ denote the linear holonomy group and $\Gamma\subset \mathrm{Aff}(E)$ the affine holonomy group.

6.1. THEOREM. *Let M be a compact affine manifold. Let $E_0\subset E$ be a proper linear subspace invariant under the affine holonomy. If the image $\Lambda_1\subset \mathrm{GL}(E/E_0)$ of Λ is nilpotent then some element of Λ_1 expands E/E_0.*

Proof. Let $q:E\to E/E_0$ denote the canonical projection. Let R be the radiant vector field on E/E_0. Let $\{(\varphi_i, U_i)\}$ be an affine atlas on M. There is a C^∞ vector field X on M which in every affine chart (φ_i, U_i) is represented by a vector field on $\varphi_i(U_i)$ which is q-related to R. For there is clearly such a vector field X_i on U_i; set $X=\sum \mu_i X_i$ where $\{\mu_i\}$ is a C^∞ partition of unity subordinate to the open cover $\{U_i\}$.

Suppose Λ_1 does not contain any expansion of E/E_0. Then by 5.1 there is a Λ_1-invariant C^1 map $\Psi:E/E_0\to\mathbf{R}$, $\Psi>0$ almost everywhere and

$$d\Psi_z R(z)=a\Psi(z) \tag{1}$$

for some $a>0$ and all $z\in E/E_0$ (this is the differential equivalent of 5.1(c)).

The composite map

$$\tilde{f}:\tilde{M}\xrightarrow{D}E\xrightarrow{q}E/E_0\xrightarrow{\Psi}\mathbf{R}$$

is π-invariant. Thus $\tilde{f}$ covers a C^1 map $f: M \to \mathbf{R}$. In affine coordinates f appears as $\Psi \circ q$. It follows that

$$df_y X(y) = af(y)$$

for all $y \in M$. This implies that if $\xi : \mathbf{R} \to M$ is an integral curve of X then

$$f(\xi(t)) = e^{ta} f(\xi(c)).$$

Now there exists $x_0 \in M$ with $f(x_0) > 0$, and the integral curve ξ_0 through x_0 is defined for all t because M is compact. Then $\lim_{t \to \infty} f(\xi_0(t)) = \infty$. But f, being continuous, is bounded. This contradiction proves 6.1. QED.

We derive several consequences from 6.1.

6.2. THEOREM. *Let M be a compact radiant manifold with nilpotent linear holonomy group Λ. Then Λ contains an expansion.*

Proof. Take $E_0 = 0$ in 6.1.

6.3. COROLLARY. *M has no nonzero parallel covariant or contravariant tensors.*

Proof. Let ω be a parallel covariant (resp. contravariant) tensor on M; let $\tilde{\omega}$ be the corresponding (constant) tensor on E invariant under Λ. By 6.2, Λ contains an expansion g. Let the eigenvalues of g be $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity), $|\lambda_i| > 1$. Now $\tilde{\omega}$ is an element of the tensor algebra $\bigoplus^l E^*$ (resp. $\bigoplus^l E$). The map on this algebra induced by g has eigenvalues $\lambda_{i_1} \cdots \lambda_{i_l}$ (resp. $\lambda_{i_1}^{-1} \cdots \lambda_{i_l}^{-1}$), which makes it an expansion (resp. a contraction). Consequently the g-invariant tensor $\tilde{\omega}$ must be zero. QED.

Conversely, suppose M is a nonradiant compact affine manifold with nilpotent affine holonomy. Then by 4.1 M has nonzero parallel vector fields and 1-forms.

Another characterization of radiance is the following:

6.4. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Then the following conditions are equivalent:*

(a) *M is radiant.*

(b) *The linear holonomy group contains an expansion.*

(c) *The linear holonomy group contains g such that $I - g$ is invertible.*

Proof. (a)$\Rightarrow$(b) by 6.2 and clearly (b)$\Rightarrow$(c). Suppose (c). By Hirsch [21] it follows that $H^1(\Gamma; E)=0$. Therefore the radiance obstruction of M vanishes, implying (a). QED.

We digress to consider deformations of affine structures.

Conditions (c) and (b) of Theorem 6.4 evidently persists under sufficiently small deformations of the linear holonomy representations. It follows that if M is a *compact radiant affine manifold with nilpotent holonomy group, then all sufficiently small deformations of the affine structure are also radiant.* In other words, for this class of manifolds radiance is an open condition. (See Goldman [16] for the definition of deformation of affine structures.)

Consider also the radiant affine structure on $M^2 \times S^1$ discussed in Section 3, where M^2 is a compact surface of genus ≥ 2. Let $g \in \pi_1(M \times S^1)$ be the image of a generator of S^1. Then g is central and $I-\lambda(g)$ is invertible. This suffices to prove that any sufficiently small deformation of the affine holonomy has a stationary point (Hirsch [21]). Thus the radiance of these affine structures is also persistent under small deformations.

In this direction D. Fried has proved [13] that there are no complete affine structures on these 3-manifolds. It is conjectured that all affine structures on these manifolds are radiant.

Corollary 6.3, and the persistence of radiance discussed above, cannot be extended to the case of solvable holonomy. To construct a counterexample we start from the subgroup $G \subset \mathrm{GL}(3, \mathbf{R})$ consisting of all the matrices of the form

$$e^{kt}\begin{bmatrix} e^t & \cdot & u \\ \cdot & e^{-t} & v \\ \cdot & \cdot & 1 \end{bmatrix} = \begin{bmatrix} e^{(k+1)t} & \cdot & u \\ \cdot & e^{(k-1)t} & v \\ \cdot & \cdot & e^{kt} \end{bmatrix}$$

where $t, u, v \in \mathbf{R}$, and $k>0$ is a real constant to be determined later. Clearly G preserves the upper half-space

$$W=\{(x, y, z) \in \mathbf{R}^3 : z>0\}$$

and acts simply transitively. Thus G inherits a left-invariant radiant (and hence incomplete) affine structure from W.

It is easily seen that G is the semidirect product of the two-dimensional vector subgroup N defined by $t=0$ with the subgroup $H \cong \mathbf{R}$ defined by $u=v=0$. The representation $\mathbf{R} \to \mathrm{Aut}(\mathbf{R}^2)$ which defines the semidirect product is given by

$$\varphi : t \to \begin{bmatrix} e^t & \cdot \\ \cdot & e^{-t} \end{bmatrix}; \qquad t \in \mathbf{R}$$

It is well-known that $G = N \rtimes H$ possesses discrete cocompact subgroups Γ. To see this take any integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{Z})$ with $a+d>2$. Then A is conjugate (in $SL_2(\mathbf{R})$) to some diagonal matrix $\begin{bmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{bmatrix}$, $\lambda \neq 0$, and therefore one may form the semidirect product $\mathbf{R}^2 \rtimes \mathbf{Z}$ defined by this representation, with a generator of $\mathbf{Z}$ corresponding to A. Since A was originally defined as an integral matrix, it preserves a lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$. (It must be emphasized, however, that this lattice is *not* generated by a basis which diagonalizes A). Then we may form the semidirect product $\Gamma = \mathbf{Z}^2 \rtimes \mathbf{Z}$ which is a discrete subgroup of $G = \mathbf{R}^2 \rtimes \mathbf{R} = N \rtimes H$, with $\Gamma \backslash G$ compact. Using the left-invariant structure on G, we thus obtain affine structures on the compact 3-manifold $M^3 = \Gamma \backslash G$; indeed, as k varies continuously a whole one-parameter family of affine structures is defined in this way.

Let us now examine the case $k = 1$. Then G is the group of matrices

$$A(u, v, t) = \begin{bmatrix} e^{2t} & \cdot & u \\ \cdot & 1 & v \\ \cdot & \cdot & e^t \end{bmatrix}; \qquad t, u, v \in \mathbf{R}.$$

It is apparent that the vector field $\partial/\partial y$ parallel to the second basis vector is fixed by G, and hence defines a parallel vector field on G, and also on $\Gamma \backslash G$. Thus we obtain a compact affine manifold with both a radiant vector field and a parallel vector field. These two vector fields generate an affine action on the group $\mathrm{Aff}(\mathbf{R})$ on the affine 3-manifold.

The existence of a parallel vector field on a radiant affine manifold is easily seen to be equivalent to the condition that the holonomy have more than one stationary point. (The stationary points of an affine action form an affine subspace, parallel to the space of parallel vector fields.)

We now describe a deformation of this affine structure. For each $s \in \mathbf{R}$, define an affine representation $A_s : G \to \mathrm{Aff}(E)$ as follows. Let $A_s(u, v, t)$ be the affine map of $E = \mathbf{R}^3$ with linear part $A(u, v, t)$ and translational part $(0, st, 0)$. For each s, A_s defines an action of G which is simply transitive on the upper half space. By passing to a quotient $\Gamma \backslash G$ we obtain new affine structures on $M^3 = \Gamma \backslash G$. These new affine manifolds are no longer radiant for $s \neq 0$, as is easily verified. Thus radiance is not an open condition on holonomy of affine structures. A glance at the circle shows that neither is radiance a closed condition.

The case $k = \pm\frac{1}{2}$ of (2) is also interesting. Taking $k = \frac{1}{2}$ we represent G by matrices

$$\begin{bmatrix} e^{3t/2} & \cdot & u \\ \cdot & e^{-t/2} & v \\ \cdot & \cdot & e^{t/2} \end{bmatrix}.$$

then G preserves the parallel 2-form $\omega = dy \wedge dz$ as well as the radiant vector field R. (Compare 6.3.) In addition to the affine vector field R the polynomial vector field $z^2(\partial/\partial x)$ and rational vector field $z^{-1}(\partial/\partial y)$ are left-invariant. The affine 1-form $\iota_R\omega = z\,dy - y\,dz$ is also left-invariant. Thus a compact radiant manifold may admit a 1-form which is affine (polynomial of degree 1) but not parallel (polynomial of degree 0). Compare 3.2.

In a forthcoming paper [14] we will show that on a compact radiant manifold all polynomial closed 1-forms and volume forms are zero.

It is interesting to note that G also carries the *complete* left-invariant affine structure corresponding to the simply transitive affine action $\alpha : G \to \mathrm{Aff}(E)$ defined as follows. Represent G as $\mathbf{R} \ltimes \mathbf{R}^2$ as above. Let $(t, \xi) \in \mathbf{R} \ltimes \mathbf{R}^2$ act on $\mathbf{R} \times \mathbf{R}^2$ by

$$(t, \xi) : (q, \eta) \to \left(q + t, \begin{bmatrix} e^t & \cdot \\ \cdot & e^{-t} \end{bmatrix} \eta + \xi\right).$$

See also Example 6.7 below. Auslander [6] gives a different complete left-invariant affine structure on G.

We return to the general theory.

Let $E_U \subset E$ be the Fitting component (= maximal unipotent submodule) of the linear holonomy. Recall that when the affine holonomy group is nilpotent we may assume (by 1.6) that E_U is invariant under the affine holonomy; and there is a unique splitting $E = E_U \oplus F$ invariant under the linear holonomy.

With this notation we have the following generalization of 6.2.

6.5. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. If $F \neq 0$ then some element of the linear holonomy group expands F.*

Proof. Apply 6.1 with $E_0 = E_U$; then $E/E_0 \approx F$ as a π-module. QED.

For the special case of abelian holonomy, 6.5 and 6.2 were first proved by John Smillie.

An important consequence of 6.5 is:

6.6. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Assume M has a parallel volume form. Then the linear holonomy is unipotent.*

Proof. Let g be any element of the linear holonomy group Λ. Then $1 = \mathrm{Det}\, g = \mathrm{Det}\,(g \mid E_U) \cdot \mathrm{Det}\,(g \mid F)$. But $g \mid E_U$ is a unipotent operator, so

$\mathrm{Det}(g \mid E_U) = 1$. Therefore $\mathrm{Deg}(g \mid F) = 1$ and so $g \mid F$ cannot be an expansion. Therefore $F = 0$ by 6.5. This means $E = E_U$. QED.

6.7. EXAMPLE. Theorem 6.6 fails for solvable holonomy. For example let $T^n = \mathbf{R}^n/\mathbf{Z}^n$ be the usual Euclidean affine torus, $\mathbf{Z}^n \subset \mathbf{R}^n$ being the integer lattice. Let $f : T^n \to T^n$ be the affine automorphism represented by a matrix $A \in \mathrm{SL}(n, \mathbf{Z})$. Let $\mathbf{Z}$ act on $T^n \times \mathbf{R}$, the generator sending (x, t) to $(f(x), t+1)$. Then $M = (T^n \times \mathbf{R})/\mathbf{Z}$ has a complete affine structure covered by the product affine structure on $T^n \times \mathbf{R}$. If A is not unipotent then the linear holonomy of M contains the nonunipotent operator $A \times I$ on $\mathbf{R}^n \times \mathbf{R}$. The affine holonomy group Γ is solvable (even polycyclic) since it embeds in the exact sequence

$$1 \to \mathbf{Z}^n \to \Gamma \to \mathbf{Z} \to 1.$$

We now turn to the problem of characterizing complete affine manifolds. The following theorem, one of the few general methods of proving completeness, shows the geometrical importance of unipotence.

6.8. THEOREM. *Let M be an affine manifold. Suppose either*

(a) *M is compact and has unipotent holonomy,*

or

(b) *M is an affine nilmanifold $\Gamma \backslash G$ and the corresponding linear representation $G \to \mathrm{GL}(E)$ is unipotent.*

Then M is complete.

Proof. First assume M is compact. From unipotence it follows that there is a flag of linear subspaces $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$, $\dim F_i = i$, invariant under the linear holonomy, such that the induced action on each F_i/F_{i-1} is trivial. Therefore there are linear maps $l_i : F_i \to \mathbf{R}$ which are invariant under the linear holonomy, and which have kernel F_{i-1}. (See e.g. Humphreys [23].)

The invariant flag on E determines a family of fields $\mathscr{F}_i$ of parallel i-planes on M, $i = 0, \ldots, n$. Each field $\mathscr{F}_i$ is integrable, as is clear in affine coordinates. The invariant linear map l_i determines a parallel 1-form ω_i defined on $\mathscr{F}_i$ and vanishing on $\mathscr{F}_{i-1}$. Using partitions of unity, one readily constructs vector fields X_i on M tangent to $\mathscr{F}_i$ with $l_i(X_i) = 1$. The X_i are covered by integrable vector fields $\tilde{X}_i$ on $\tilde{M}$, since M is compact.

We now show that $\tilde{M}$ develops *onto* E. Suppose one fixes a base point m_0 in $\tilde{M}$ which develops to the origin in E. Let $v \in E$. We construct a path in $\tilde{M}$ beginning at m_0 whose development ends at v as follows. Starting at m_0, flow along X_n for time $l_n(v)$, ending at the point $m_1 \in \tilde{M}$ with $D(m_1) = v_1 \in E$ where $v - v_1 \in \operatorname{Ker} l_n = F_{n-1}$. Then flow along X_{n-1} for time $l_{n-1}(v - v_1)$ ending at $m_2 \in \tilde{M}$ with $D(m_2) = v_2 \in E$ where $v - v_2 \in \operatorname{Ker} l_{n-1} = F_{n-1}$. Continuing this way, we reach $v = v_n$ after n steps.

We verify that the developing map is injective. Assume $\gamma_0 : [a, b] \to \tilde{M}$ is a path which develops onto a closed loop $\delta : [a, b] \to E$ with $\delta(a) = \delta(b) = 0$. We may deform γ_0 with end points fixed so that the new path γ_1 develops in F_{n-1}: we use $\gamma_t : [a, b] \to \tilde{M}$, where $\gamma_t(s)$ is the image at time $-tl_n(\delta(s))$ of $\gamma_0(s)$ under the flow of $\tilde{X}_n$, $0 < t \leq 1$, $a \leq s \leq b$. By further such deformations, we eventually obtain a path from $\gamma_0(a)$ to $\gamma_0(b)$ which develops to a point. Hence $\gamma_0(a) = \gamma_0(b)$, so the developing map is injective and M is complete.

When M is an affine nilmanifold, not assumed compact, the proof is similar. We may as well assume $M = G$. Fix a flag at the identity $e \in G$ invariant under the linear part of the affine action of G. Let $X_1, \ldots, X_n$ be tangent vectors at e forming a basis for the flag, and extend them to vector fields on G by left-multiplication. Being left-invariant, each of these vector fields is integrable for all $t \in \mathbf{R}$. The rest of the proof is analogous to the compact case. QED.

The same proof shows that in the incomplete case, the projection $E_U \oplus F \to E_U$ maps the developing image onto E_U by a fibration.

The following application of 6.8 might be true even without nilpotence.

6.9. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Suppose the developing map $D : \tilde{M} \to E$ is surjective. Then M is complete.*

Proof. Let $E = E_U \oplus F$ be the Fitting splitting of the linear holonomy. If $F \neq 0$ then Theorem 3.3 holds that each component N of $p(D^{-1}E_U)$ is incomplete. Nevertheless N has unipotent holonomy and so, by Theorem 6.8, must be complete. These facts are only compatible if $F = 0$. It follows that $M = N$ is complete. QED.

At this point it may be useful to summarize the implications proved thus far between various properties of affine manifolds:

6.10. SUMMARY. For compact affine manifolds:

(a) Surjective developing map and nilpotent affine holonomy $\Rightarrow$ indecomposable affine holonomy (6.9).

(b) Complete $\Rightarrow$ irreducible affine holonomy (2.2).

(c) Unipotent linear holonomy $\Rightarrow$ complete (6.8).

(d) Nilpotent affine holonomy group and parallel volume ⇒ unipotent linear holonomy (6.6).

(e) Nilpotent linear holonomy group and radiant ⇒ expansion in linear holonomy (6.2).

For possibly noncompact affine manifolds:

(f) Indecomposable, nilpotent affine holonomy ⇒ unipotent linear holonomy (4.3).

(g) A left-invariant affine structure on a nilpotent Lie group G is complete if and only if the corresponding affine representation of G is unipotent. (6.8, 1.5).

In a subsequent paper we will show that for compact affine manifolds, parallel volume ⇒ irreducible affine holonomy.

The following case is especially neat, as all the conditions coincide:

6.11. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Then the following are equivalent*:

(a) *M is complete*

(b) *The linear holonomy is unipotent.*

(c) *M has parallel volume.*

(d) *The affine holonomy is irreducible.*

(e) *The affine holonomy is indecomposable.*

(f) *The developing map is surjective.*

Proof. (f)⇒(e) by 6.9. (e)⇒(b) by 4.3. (b)⇒(a) by 6.8. Clearly (a)⇒(f). Also (b)⇒(c), and (c)⇒(b) by 6.6. Finally, (a)⇒(d) by 2.2, and (d)⇒(e). QED.

In the next section we adjoin another equivalent condition: that M be a complete affine nilmanifold.

In a forthcoming paper [17] it will be shown that (d)⇒(a) fails for certain compact affine 3-manifolds with *solvable* fundamental group.

7. Complete affine nilmanifolds

The following theorem follows from a more general theorem announced by Auslander [8]. Unfortunately the proof is wrong, as Auslander shows in [6]. Many of the ideas in this paper, and in particular the proof of our Theorem 7.1, come from Auslander's work. (One can, however, deduce Theorem 7.1 from the correct results in [8] (see [8], page 811).)

7.1. THEOREM. *Let M be a compact complete affine manifold with nilpotent fundamental group. Then M is an affine nilmanifold.*

Proof. We take the universal covering space of M to be vector space E. The fundamental group is the group of deck transformation $\pi \subset \mathrm{Aff}(E)$; thus $M = E/\pi$.

To prove the theorem we must find a subgroup $G \subset \mathrm{Aff}(E)$, containing π, which acts simply transitively on E.

The linear action of π on E is unipotent (Corollary 3.4). Hence for every $g \in G$ there is a unique element $\log(g)$ in the Lie algebra $\mathrm{Aff}(E)$ of $\mathrm{Aff}(E)$ whose linear part is a nilpotent transformation of E and whose exponential is g.

By a celebrated theorem of Malcev [27] there is a simply connected nilpotent Lie group H containing π as a discrete uniform subgroup. Recall that the Lie algebra $\mathfrak{h}$ of H is generated by elements $L(g) \in \mathfrak{h}$, $g \in \pi$ subject only to the relations

$$C(L(g_1), L(g_2)) = L(g_1 g_2) \tag{1}$$

where $C : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ satisfies the Baker–Campbell–Hausdorff formula (Varadarajan [43]):

$$\exp C(X, Y) = \exp X \exp Y.$$

Clearly (1) holds with $L(g) = \log(g)$, so there is a homomorphism of Lie algebras $\mathfrak{h} \to \mathrm{Aff}(E)$ and an associated homomorphism of Lie groups $H \to \mathrm{Aff}(E)$. The image of H is a connected subgroup $G \subset \mathrm{Aff}(E)$; and since for each $g \in G$ the linear part of $\log g$ is a nilpotent transformation, the image of G in $\mathrm{GL}(E)$ is unipotent. It follows that E has a basis putting this image in upper triangular unipotent form. The group of all affine automorphisms of E with such linear parts is simply connected. Thus G is simply connected. By construction, $\pi \subset G$.

Define $\tilde{f} : G \to E$ by $\tilde{f}(g) = g(0)$. Then $\tilde{f}$ is equivariant respecting the inclusion $G \subset \mathrm{Aff}(E)$. There is an induced map $f : \pi \backslash G \to M$. Now $\pi \backslash G$ and M are Eilenberg–MacLane spaces of type $K(\pi, 1)$, and f is a homotopy equivalence. Since they are compact manifolds, they have the same dimension and f must be surjective.

It follows that $\tilde{f} : G \to E$ is surjective; therefore G acts transitively on E. Since $\dim G = \dim E$, $\tilde{f}$ is a covering space projection. Since E is simply connected, $\tilde{f}$ is a homeomorphism. Therefore G acts simply transitively on E. Hence $f : \pi \backslash G \to M$ is an affine isomorphism, as claimed. QED.

In a forthcoming paper [13] we shall prove that a compact complete affine manifold with solvable holonomy has a finite covering which is an affine solvmanifold.

8. Polynomial tensors on affine manifolds with nilpotent holonomy

A tensor field on an affine manifold M is called *polynomial of degree* $\leq r$ if in affine coordinates its components are polynomial functions of (total) degree $\leq r$. It is of interest to know which fields are of this type, and which real cohomology classes are represented by polynomial exterior differential forms. (One can ask similar questions about other classes of tensors. For example: which tensors on M correspond in affine charts to real analytic tensors defined on all of E?).

The following theorem is the main result of this section:

8.1. THEOREM. *Let M be a compact affine manifold with nilpotent holonomy. If M is a complete affine nilmanifold the inclusion of the complex of polynomial exterior forms into the de Rham complex induces an isomorphism on cohomology.*

In Theorem 8.4 below a converse result is proved.

The proof of 8.1 relies on the theorem of Nomizu [33] (see also Raghunathan [47]) which identifies the real cohomology of a compact nilmanifold $\Gamma\backslash G$ with the cohomology of left-invariant differential forms on G. Therefore it suffices to prove that when G is a simply connected nilpotent Lie group with left-invariant complete affine structure, then every left-invariant tensor field on G is polynomial in affine coordinates. Now such a structure on G is defined by a simply transitive affine action of G on the vector space E; and left-invariant tensor fields on G correspond $\alpha(G)$-invariant tensor fields on E. Therefore it is enough to prove:

8.2. THEOREM. *Let G be a nilpotent Lie group and $\alpha : G \to \mathrm{Aff}(E)$ a simply transitive affine action. Then every $\alpha(G)$-invariant tensor field on E is polynomial.*

Let α be as in 8.2; then by Theorem 1.5 $\alpha(G)$ is a unipotent subgroup of $\mathrm{Aff}(E) \subset \mathrm{GL}(E \times \mathbf{R})$. Thus $\alpha(G)$ is a connected unipotent group of matrices. It is well known that such a group is a unipotent algebraic group, and in a unique way. On algebraic groups there is a natural notion of an algebraic tensor field.

The algebraic structure on G can be made explicit as follows. The exponential map $\exp : \mathfrak{G} \to G$ is a diffeomorphism, as is its inverse $\log : G \to \mathfrak{G}$. The Lie algebra $\mathfrak{G}$, being a vector space, has a natural structure as an algebraic variety. The maps exp and log are isomorphisms of algebraic varieties.

It is easy to see that every left-invariant tensor field on G is algebraic, that is, it is polynomial *in the coordinates defined by exp.* What we have to prove is that it is *also* polynomial in the coordinates defined by the developing map

$$D : G \to E, \qquad D(g) = \alpha(g)(0)$$

defined by evaluation at the origin. To this end we prove:

8.3. PROPOSITION. *The developing map $D : G \to E$ is an isomorphism of algebraic varieties. In particular the composite maps*

$$\mathfrak{G} \xrightarrow{\exp} G \xrightarrow{D} E$$

and

$$\mathfrak{G} \xleftarrow{\log} G \xleftarrow{D^{-1}} E$$

are polynomial.

Proof. As remarked above, $\alpha : G \to \alpha(G)$ is an isomorphism of algebraic groups. Now D is the composition of α with the map

$$\mathrm{Aff}\,(E) \to E, \qquad g \mapsto g(0)$$

which is evidently algebraic. This proves that $D : G \to E$ is algebraic.

To prove that $D^{-1} : E \to G$ is algebraic we induct on $\dim G = \dim E$. The case $\dim G = 0$ is trivial.

Let $\lambda : G \to \mathrm{GL}\,(E)$ be the linear part of α; clearly λ is algebraic.

Since λ is unipotent there exists a nonzero linear functional $\psi : E \to \mathbf{R}$ which is λ-invariant. Thus if $g \in G$ then $\psi \circ \alpha(g)(x) - \psi(x) = \psi \circ D(g)$ for all $x \in E$. It follows easily that $\psi \circ D$ defines a homomorphism $G \to \mathbf{R}$. Let G_1 denote the kernel of this homomorphism and let $E_1 = \mathrm{Ker}\, \psi$. Clearly $\alpha(G_1)$ acts simply transitively on E_1.

Suppose, inductively, that the inverse map $f_1 : E_1 \to G_1$ to $D_1 = D|_{G_1}$ is algebraic. The homomorphism $G \to R$ has a left-inverse so we may write G as a semidirect product $G = G_1 \rtimes \{g_t\}_{t \in \mathbf{R}}$ where $t \mapsto g_t$ is a one-parameter subgroup of G which acts on E/E_1 by translation by t. Thus in a basis of E containing a basis of E_1, $\alpha(g_t)$ is represented by

$$\left[\begin{array}{c|c} Q(t) & r(t) \\ \hline 0\cdots 0 & 1 \end{array}\right]\left[\begin{array}{c} p(t) \\ \hline t \end{array}\right]$$

where $Q:\mathbf{R}\to \mathrm{GL}(E_1)$ and $r, p:\mathbf{R}\to E_1$ are polynomial maps. Define $f: E_1\oplus\mathbf{R}=E\to G$ by

$$f(x,t)=g_t\cdot[f_1\circ Q(-t)(x-p(t))].$$

Clearly f is a composition of algebraic maps and hence algebraic. Furthermore for $t\in\mathbf{R}$ and $h\in G_1$,

$$\begin{aligned} f\circ D(g_th) &= f(\alpha(g_t)D(h)) = f((Q(t)D(h)+p(t),t)) \\ &= g_t\cdot f_1(Q(-t)((Q(t)D(h)+p(t))-p(t))) = g_t\cdot f_1(D(h)) = g_th \end{aligned}$$

so that f is inverse to D. Thus D^{-1} is algebraic, and the proof of 8.3 is complete.

QED.

Now that 8.3 has been established, 8.2 and 8.1 are consequences because of Nomizu's theorem, as explained above.

Remark. In general we do not know a sharp bound on the degree of a left-invariant tensor field. A (probably crude) bound can be obtained by estimating the degrees of polynomials in the proof of 8.3. In the notation of 8.3 we have algebraic maps

$$f:E\to G,\qquad Q:\mathbf{R}\to \mathrm{GL}(E_1),\qquad P:\mathbf{R}\to E_1.$$

We give $\mathbf{R}$ and E_1 their natural vector space structure; thus $\deg p$ is well-defined. From the definition of p it is easy to compute that

$$\deg p\le n=\dim E.$$

We give $\mathrm{GL}(E_1)$ matrix coordinates coming from linear coordinates on E_1. Then $\deg Q$ is well-defined and one sees easily that $\deg Q\le n-1$. Now G is a subgroup of $\mathrm{Aff}(E)$. In a natural way $\mathrm{Aff}(E)\subset \mathrm{GL}(E\times\mathbf{R})$, and we give $\mathrm{GL}(E\times\mathbf{R})$ matrix coordinates. Thus $\deg f$ is defined; and $\deg f_1$ is defined similarly. From the formula

$$f(x, t) = g_t \cdot [f_1 \circ Q(-t)(x - p(t))],$$

taken from the proof of 8.3, we find

$$\deg f \leq (\deg f_1)(\deg Q + \deg p) \leq (\deg f_1)(2n-1).$$

By induction on n we get

$$\deg f \leq 1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n-1)!/2^{n-1}(n-1)!.$$

Now let X be a left-invariant vector field on G, considered as an $\alpha(G)$-invariant vector field $X: E \to E$. Let $x \in E$; set $f(x) = g \in G$. Then

$$X(x) = X(D \circ f(x)) = X(D(g)) = X(\alpha(g)(0)) = \lambda(g)X(0).$$

Thus $X(x) = \lambda(f(x))X(0)$. This expresses $X: E \to E$ as the composition

$$X: E \xrightarrow{f} G \xrightarrow{\lambda} \mathrm{GL}(E) \xrightarrow{h} E$$

where h is evaluation at $X(0)$. It is clear that λ and h both have degree 1. Thus $\deg X \leq \deg f$. Similarly the degree of a left-invariant 1-form is bounded by $\deg f$.

It follows that if T is a left-invariant (p, q)-tensor field on G then as a tensor field on E

$$\deg T \leq (p+q) \deg f \leq (p+q)(2n-2)!/2^{n-1}(n-1)!$$

It would be interesting to have a sharper bound. In Fried [4] there is an example where a 4-dimensional G has a left-invariant vector field of degree 5. If $\dim G < 4$, however, every left-invariant tensor of type (p, q) has degree $\leq 2(p+q)$ (see [13]).

We note that for any Lie group with left-invariant affine structure, all *right*-invariant vector fields are polynomial of degree ≤ 1 since in affine coordinates they generate 1-parameter subgroups of $\mathrm{Aff}(E)$, so they are affine vector fields on E.

The following theorems complement 6.11 and 7.1 by characterizing compact complete affine nilmanifolds in terms of differential forms.

8.4. THEOREM. *Let M be a compact affine manifold with nilpotent affine holonomy group. Then the following conditions are equivalent*:

(a) *M is a complete affine nilmanifold.*
(b) *M is orientable and every de Rham cohomology class is represented by a polynomial differential form.*
(c) *M has a polynomial volume form.*
(d) *M has a parallel volume form.*

Proof. Theorem 8.1 shows that (a)$\Rightarrow$(b). Clearly (b)$\Rightarrow$(c); and 6.11, 7.1 show that (d)$\Rightarrow$(a).

We must show that (c)$\Rightarrow$(d). If the linear holonomy is unipotent (d) is obvious. We shall prove that if the linear holonomy is not unipotent there is no polynomial volume form.

Denote the affine holonomy by $\alpha : \pi \to \mathrm{Aff}(E)$ and the linear holonomy by $\lambda : \pi \to \mathrm{GL}(E)$. Let $E = E_U \oplus F$ be the Fitting splitting of λ. We assume λ is not unipotent, i.e., $F \neq 0$. Since M has a volume form, M is orientable. By 6.5 there exists $g_1 \in \pi$ such that $\lambda(g_1) \mid F$ is a contraction. Since $\lambda(\pi) \mid E_U$ is unipotent, and $\mathrm{Det}\, \lambda(g_1) > 0$ because M is orientable, we have

$$\mathrm{Det}\, \lambda(g_1) = \delta, \qquad 0 < \delta < 1.$$

Fix $x_0 \in E$. It follows from the unipotence of $\lambda(g_1)$ on E_U and the contracting of $\lambda(g_1)$ on F that there is a polynomial p in one variable such that for all $m \geq 0$, $|\alpha(g_1)^m x_0| \leq p(m)$.

Suppose that θ is a polynomial n-form on E invariant under the affine holonomy $\alpha(\pi)$. Let μ be a nonzero parallel n-form on E and write $\theta(x) = f(x)\mu$ where $f : E \to \mathbf{R}$ is a polynomial. Then $\theta(x) = (\mathrm{Det}\, \lambda(g) \cdot \theta(\alpha(g)x)$ for all $g \in \pi$, so $f(x) = (\mathrm{Det}\, \lambda(g))f(\alpha(g)x)$ particular, for all $m \geq 0$, $f(x) = (\mathrm{Det}\, \lambda(g_1))^m f(\alpha(g_1)^m x)$, or

$$f(x) = \delta^m f(\alpha(g_1)^m x).$$

Since f is a polynomial, $|f(\alpha(g_1)^m x_0)$ is bounded in absolute value by a polynomial $q(m)$ (which depends on x). Hence

$$|f(x_0)| \leq \delta^m q(m) \quad \text{for all} \quad m \geq 0.$$

Since $0 < \delta < 1$, it follows that $f(x_0)$ is zero. As x_0 was arbitrary, this completes the proof of Theorem 8.4.

It is to be noted that Auslander [4] and Auslander–Markus [10] have constructed many examples of compact complete affine-nilmanifolds having parallel Lorentz metrics. See also Milnor [31] for additional examples.

BIBLIOGRAPHY

[1] Arrowsmith, D. and Furness, P., *Locally symmetric spaces*, J. London Math. Soc. (2) *10* (1975), 487–499.
[2] ——. *Flat affine Klein bottles*, Geometriae Dedicata *5* (1976), 109–115.
[3] Atiyah, M. and Wall, C., *Cohomology of groups*, in *Algebraic Number Theory*, Cassels and Frohlich (Eds), 94–115, Thompson Book Co. 1967.
[4] Auslander, L., *Examples of locally affine spaces*, Annals of Math. *64* (1956), 255–259.
[5] ——, *An exposition of the theory of solvmanifolds*, Bull. Amer. Math. Soc. *79* (1973), 227–285.
[6] ——, *Simply transitive groups of affine matrices*, Amer. J. Math. *99* (1977), 809–821.
[7] ——, *Some compact solvmanifolds and locally affine spaces*, J. Math. and Mech. 7 (1958), 963–975.
[8] ——, *The structure of complete locally affine manifolds*, Topology *3* (1964), 131–139.
[9] Auslander, L. and Markus, L., *Holonomy of flat affinely connected manifolds*, Annals of Math. *62* (1955), 139–151.
[10] ——, *Flat Lorentz 3-manifolds*, Memoirs Amer. Math. Soc. 1959.
[11] Fillmore, J. and Scheuneman, J., *Fundamental groups of compact complete locally affine surfaces*, Pacific J. Math. *44* (1973), 487–496.
[12] Fried, D., *Polynomials on affine manifolds*. To appear.
[13] —— and Goldmann, W., *3-dimensional affine crystallographic groups*. To appear.
[14] Goldman, W. and Hirsch, M., *Polynomial forms on affine manifolds*, Pacific J. Math. (in press).
[15] Furness, P. and Fedida, E., *Sur les structures pseudo-riemanniennes plates des variétés compactes*, C.R. Acad. Sci, Paris, *286* (1978), 1969–171.
[16] Goldman, W., *Obstructions to deforming geometric structures*, to appear.
[17] ——, *Two examples of affine manifolds*, to appear.
[18] Goldmann, W. and Hirsch, M., *A generalization of Bieberbach's theorem*, Invent. Math. (in press).
[19] Gunning, R., *Uniformization of complex manifolds: the role of connections*, Princeton U. Press, 1978.
[20] Hirsch, M., *Flat manifolds and the cohomology of groups*, in *Algebraic and geometric topology*, Lecture Notes in Mathematics 664, Springer-Verlag, 1977.
[21] ——, *Stability of stationary points and cohomology of groups*, Proc. Amer. Math. Soc. *79* (1980), 191–196.
[22] Hirsch, M. and Thurston, W., *Foliated bundles, flat manifolds, and invariant measures*, Annals of Math. *101* (1975), 369–390.
[23] Humphreys, J., *Linear algebraic groups*, Springer-Verlag, 1975.
[24] Inoue, M., Kobayashi, S. and Ochiai, T., *Holomorphic affine connections on compact complex surfaces*, J. Fac. Sci. U. Tokyo 27 (1980), 247–264.
[25] Kostant, B. and Sullivan, D., *The Euler characteristic of an affine space form is zero*, Bull. Amer. Math. Soc. *81* (1975), 937–938.
[26] Kuiper, N., *Sur les surfaces localement affines*, Colloques de géométrie différentielle, Strasbourg (1953), 79–87.
[27] Malcev, A. *On a class of homogeneous spaces*, Izvestija Akademia Nauk SSR Ser. Math. *13* (1949); Amer. Math. Soc. Translations *9* (1962), 276–307.
[28] Markus, L., *Cosmological models in differential geometry*, U. Minnesota 1962 (mimeographed).

[29] MATSUSHIMA, Y., *Affine structures on complex manifolds*, Osaka J. Math. *5* (1968), 215–222.
[30] MILNOR, J., *Curvatures of left-invariant metrics on Lie groups*, Adv. in Math. *21* (1976), 293–329.
[31] ——, *On fundamental groups of complete affinely flat manifolds*, Adv. in Math. 25 (1977), 178–187.
[32] NAGANO, T. and YAGI, K., *The affine structures on the real two-torus, I*, Asapa J. Math. *11* (1974), 181–210.
[33] NOMIZU, K., *On the cohomology ring of compact homogeneous spaces of nilpotent Lie groups*, Annals of Maths. *59* (1954), 531–538.
[34] SAKANE, Y., *On compact complex affine manifolds*, J. Math. Soc. Japan *29* (1977), 135–145.
[35] SCHEUNEMAN, J., *Examples of compact locally affine spaces*, Bull. Amer. Math. Soc. *77* (1974), 589–592.
[36] ——, *Affine structures on three-step nilpotent Lie algebras*, Proc. Amer. Math. Soc. *46* (1974), 451–454.
[37] ——, *Translations in certain groups of affine motions*, Proc. Amer. Math. Soc. *47* (1975), 223–228.
[38] SMILLIE, J., *Affinely flat manifolds*, Doctoral thesis, U. Chicago (1977).
[39] SULLIVAN, D., *A generalization of Milnor's inequality concerning affine foliations and affine manifolds*, Comment. Math. Helv. *51* (1976), 183–189.
[40] SULLIVAN, D. and THURSTON, W., *Manifolds with canonical coordinates: some examples*. Institut des Hautes Etudes, 1979, mimeographed.
[41] SUWA, T., *Compact quotient spaces of C^2 by affine transformation groups*, J. Diff. Geom. *10* (1975), 239–252.
[42] TISCHLER, D., *On fibering certain foliated manifolds over S^1*, Topology *9* (1970), 153–154.
[43] VARADARAJAN, V., *Lie groups, Lie algebras, and their representations*, Printice Hall, 1974.
[44] VITTER, A., *Affine structures on compact complex manifolds*, Invent. Math. *17* (1972), 231–244.
[45] ZASSENHAUS, H., *Beweis eines Satzes über diskrete Gruppen*, Abhandl. Math. Sem. Univ. Hamburg *12* (1938), 289–312.
[46] DWYER, W. G., *Vanishing homology over nilpotent groups*, Proc. Amer. Math. Soc. *49* (1975), 8–12.
[47] RAGHUNATHAN, M. S. *Discrete subgroups of Lie groups*, Springer-Verlag 1972.

University of California, Santa Cruz
University of California, Berkeley
University of California, Berkeley

Received March 24, 1981