AFFINE NEAR-SEMIRINGS OVER BRANDT SEMIGROUPS

JITENDER KUMAR

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
GUWAHATI - 781039, INDIA

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By

Jitender Kumar

Department of Mathematics

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To

My Parents

Certificate

This is to certify that the thesis entitled *Affine Near-Semirings over Brandt Semigroups* submitted by *Mr. Jitender Kumar* to the Indian Institute of Technology Guwahati, for the award of the Degree of Doctor of Philosophy, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Guwahati | Dr. K. V. Krishna
March 2014 | Supervisor

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Abstract

The thesis aims at studying various structural properties of affine near-semirings over Brandt semigroups. The study considers various aspects, viz. semigroup theoretic properties, ring theoretic properties and formal language theoretic connections.

At the outset, the thesis classifies the elements of an affine near-semiring over Brandt semigroup, denoted by $A^+(B_n)$, and finds the cardinality of $A^+(B_n)$, for an arbitrary natural number n. In order to ascertain the semigroup theoretic properties of $A^+(B_n)$, the thesis completely characterizes the Green's relations on both of its semigroup reducts. The thesis reveals that the additive semigroup reduct is eventually regular and the multiplicative semigroup reduct is orthodox. Further, the rank properties of these semigroup reducts are investigated in detail. By determining all right ideals, ideals and radicals, the thesis studies the ring theoretic structure of $A^+(B_n)$. The study establishes certain formal language theoretic connections to $A^+(B_n)$ by showing that both of its semigroup reducts are syntactic semigroups.

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Introduction

The present thesis is in the broad area of general algebra. The thesis aims at studying various structural properties of affine near-semirings over Brandt semigroups.

An algebraic structure $(\mathcal{N}, +, \cdot)$ with two binary operations $+$ and $\cdot$ is said to be a near-semiring if $(\mathcal{N}, +)$ and $(\mathcal{N}, \cdot)$ are semigroups and $\cdot$ is one-side, say left, distributive over $+$, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$, for all $a, b, c \in \mathcal{N}$. The set $M(S)$ of all mappings on a semigroup $(S, +)$ and certain subsets of $M(S)$, with respect to pointwise addition and composition of mappings, are among the typical examples of near-semirings.

Since the work of van Hoorn and van Rootselaar [1967], many authors have studied the algebraic structure of near-semirings in the literature. A class of near-semirings, viz. basic process algebras, plays an important role in the construction of algebra of communicating processes [Bergstra and Klop, 1990].

The properties of near-semirings have been studied in various aspects. Some of the works are concentrated on ring theoretic properties through ideals and radicals ([Krishna, 2005; van Hoorn, 1970]). Some authors have utilized the concept of near-semiring in various applications ([Desharnais and Struth, 2008; Droste et al., 2010; Krishna and Chatterjee, 2005]). Some other works focus to study the underlying semigroups ([Gilbert and Samman, 2010a,b; Weinert, 1982]). In [Gilbert and Samman, 2010b], Gilbert and Samman have studied the Green's classes of additive semigroup reducts of endomorphism near-semirings over Brandt semigroups.

For $n \in \mathbb{N}$, write $[n] = \{1, 2, \ldots, n\}$. A Brandt semigroup $(B_n, +)$ has the underlying set $B_n = \{\vartheta\} \cup ([n] \times [n])$ with the operation $+$ given by

$$(i, j) + (k, l) = \begin{cases} (i, l), & \text{if } j = k; \\ \vartheta, & \text{if } j \neq k \end{cases}$$

and, for all $\alpha \in B_n$, $\alpha + \vartheta = \vartheta + \alpha = \vartheta$. The class of Brandt semigroups is an important subclass of completely 0-simple inverse semigroups. In this thesis, we consider a class of near-semirings, viz. affine near-semirings over Brandt semigroups.

An affine mapping over a vector space is a sum of a linear transformation and a constant map. Blackett [1956] studied the near-ring of affine mappings over a vector space. An abstract notion of affine near-rings is introduced by Gonshor [1964]. Authors have considered the study of affine near-rings in different contexts (e.g. [Feigelstock, 1985; Malone, 1969]). Holcombe [1983, 1984] studied affine near-rings, in the context of linear sequential machines. These notions are extended to near-semirings by Krishna [2005]. Further, Krishna and Chatterjee [2005] have studied affine near-semirings over generalized linear sequential machines.

A map on a semigroup $(S, +)$ is said to be an affine map if it can be written as a sum of an endomorphism and a constant map on S. The subnear-semiring of $M(S)$ generated by the set of all affine maps on S is said to be an affine near-semiring over S and it is denoted by $A^+(S)$.

In order to study the structural properties of affine near-semirings over Brandt semigroups, in the present thesis, we focus on semigroup theoretic properties of both the semigroup reducts of $A^+(B_n)$. Further, to ascertain the ring theoretic properties of $A^+(B_n)$, we study its ideals and radicals. In view of studying the formal language theoretic connections to $A^+(B_n)$, we consider the syntactic semigroup problem for both of its semigroup reducts.

After presenting the necessary preliminaries in Chapter 1, the main work of the thesis has been organized into five chapters as described below:

Chapter 2: The Near-Semiring $A^+(B_n)$

Chapter 3: Semigroup Structure

Chapter 4: Rank Properties

Chapter 5: Ideals and Radicals

Chapter 6: Syntactic Semigroups

Chapter 2. In this chapter, we obtain certain fundamental properties of the elements of $A^+(B_n)$. First we ascertain that any endomorphism over B_n is either a constant map whose image is an idempotent element or an automorphism (Theorem 2.1.4). Consequently, we characterize the set of affine maps over B_n and report its size (Theorem 2.1.6). Finally, in this chapter, along with the cardinality of $A^+(B_n)$, we classify its elements with respect to their supports (Theorem 2.2.10). In the classification theorem, we show that $A^+(B_n)$ has only the singleton support maps, constant maps and a special types of n-support maps. Most of the proofs in the thesis make use of the classification theorem.

Chapter 3. Green's relations play a vital role in structure theory of semigroups. In this chapter, we completely characterize the Green's relations for both the semigroup reducts of $A^+(B_n)$ and find the sizes of respective Green's classes. First we observe that the set of constant maps over B_n and the set of singleton support maps including the zero map form subsemigroups of the additive semigroup reduct $A^+(B_n)^+$. Indeed, these subsemigroups are isomorphic to certain semigroup constructs over B_n (Proposition 3.1.1). Consequently, we characterize the Green's relations over $A^+(B_n)^+$ (Theorems 3.1.6, 3.1.7). In case of the multiplicative semigroup reduct $A^+(B_n)^\circ$, by including the zero map to the set of singleton support maps and to the set of n-support maps, they form subsemigroups isomorphic to certain completely 0-simple inverse semigroups (Propositions 3.2.2, 3.2.4). In view of this, we characterize the Green's relations on $A^+(B_n)^\circ$ (Theorems 3.2.6, 3.2.7, 3.2.9). We also investigate the regular and idempotent elements in both the semigroup reducts of

$A^+(B_n)$ (Theorems 3.1.12, 3.2.10, 3.2.11) and find certain relevant subsemigroups (Theorems 3.1.15, 3.2.13). Further, we ascertain that the additive semigroup reduct is eventually regular (Proposition 3.1.14) and the multiplicative semigroup reduct is orthodox (Theorem 3.2.12).

The work of chapters 2 and 3 has been accepted for publication in the journal Communications in Algebra.

Chapter 4. The study of rank properties of an algebraic structure provide some insight into the structure and its concrete description. The concept of rank for general algebras is analogous to the concept of dimension in linear algebra. However, for general algebras, in particular for semigroups, the minimum size of a generating set need not be equal to the maximum size of an independent set. Accordingly, for a finite semigroup S, Howie and Ribeiro [1999, 2000] have considered the following five possible ranks, viz. small rank, lower rank, intermediate rank, upper rank and large rank of S, denoted by $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$, respectively. Since the work of Marczewski [1966], many authors have studied the rank properties of various algebraic structures (e.g. [Cameron and Cara, 2002; Gomes and Howie, 1992; Minisker, 2009; Mitchell, 2002; Ruškuc, 1994]). In Chapter 4, we investigate the rank properties of both the semigroup reducts of $A^+(B_n)$.

In order to find the large rank of a finite semigroup, we introduce a technique using prime subsets (cf. Section 4.2). While this technique is useful for semigroups of transformations, it is found to be more efficient for other semigroups too. The technique to find the large rank of finite semigroup has been accepted for publication in the journal Semigroup Forum.

In Section 4.3, we obtain small rank (Corollary 4.3.2), lower rank (Theorem 4.3.9), intermediate rank (Theorem 4.3.12) and large rank (Theorem 4.3.24) of the additive semigroup reduct $A^+(B_n)^+$. While we obtain $r_4(A^+(B_n)^+)$ for $n \geq 6$ (Theorem 4.3.21), we provide lower bounds of $r_4(A^+(B_n)^+)$ for $2 \leq n \leq 5$ (Theorems

4.3.20, 4.3.13). We conjecture that these lower bounds are indeed the upper ranks of the respective cases. The work embedded in this section has been presented at the conference General Algebra and Its Applications: GAIA 2013, Melbourne, Australia, July 15-19, 2013. Further, the work is communicated for publication in a special issue, associated to GAIA 2013, of the journal Algebra Universalis.

For the multiplicative semigroup reduct $A^+(B_n)^\circ$, we obtain small rank (Corollary 4.4.1), lower rank (Theorems 4.4.5, 4.4.6) and large rank (Theorems 4.4.13, 4.4.14). Further, we give certain lower bounds of intermediate rank (Theorem 4.4.10) and upper rank (Theorem 4.4.11) for $A^+(B_n)^\circ$. The work on rank properties of $A^+(B_n)^\circ$ has been communicated to the journal Semigroup Forum.

Chapter 5. van Hoorn and van Rootselaar [1967] have initiated the investigations on ideals in near-semirings. Consequently, van Hoorn [1970] generalized the concept of Jacobson radicals of rings to zero-symmetric near-semirings and identified fourteen types of radicals. The properties of these radicals are further investigated in [Krishna, 2005; Zulfiqar, 2009]. In this chapter, we focus on investigating the ideals and radicals of $\mathcal{N}$ – the zero-symmetric affine near-semiring over an arbitrary Brandt semigroup. In this connection, after ascertaining the right ideals of $\mathcal{N}$ (Theorem 5.2.2), we obtain all the fourteen radicals of $\mathcal{N}$ (Theorems 5.3.10, 5.3.11, 5.3.12). Finally, we determine all the congruences of $\mathcal{N}$ (Theorem 5.4.3) and, consequently, report its ideals (Corollary 5.4.4).

Chapter 6. In order to ascertain the formal language theoretic connections to $A^+(B_n)$, we consider the syntactic semigroup (monoid) problem. The syntactic semigroup problem is to decide whether a given finite semigroup is syntactic or not. The syntactic semigroup problem for various semigroups have been investigated by many authors (cf. [Goralčík and Koubek, 1998; Goralčík et al., 1982; Lallement and Milito, 1975]). In this chapter, we prove that both the semigroup reducts of $A^+(B_n)$ are syntactic semigroups by utilizing different methods available in the literature.

In this connection, we prove that the semigroup reducts of $A^+(B_1)$ are syntactic monoids by observing that they are transition monoids of some minimal automata (Theorems 6.2.3, 6.3.3). For $n \geq 2$, we prove that $A^+(B_n)^+$ is isomorphic to the syntactic semigroup of a language (Theorem 6.2.7). In case of $A^+(B_n)^\circ$, for $n \geq 2$, we show that it contains a disjunctive subset (Theorem 6.3.4).

Epilogue. The present thesis considers a study of affine near-semirings over Brandt semigroups in various aspects. In all the aspects under consideration, the thesis shows a lot of scope for further studies in the present topic. As an immediate work, one may target to extend the results of the thesis to a general class of affine near-semirings over completely 0-simple inverse semigroups. There is a lot more to know about affine near-semirings over arbitrary semigroups. One may explore the possibilities of extending any of these results to abstract affine near-semirings.

1

Preliminaries

This chapter presents fundamentals of semigroups and near-semirings which are useful throughout the thesis. The background material which is specific to certain chapters is postponed to the respective chapters. The fundamentals of semigroups are presented in Section 1.1. In this connection, we recall the notions and some useful results on Green's relations, regular semigroups and inverse semigroups in various subsections. The notion on which the entire thesis is depending, viz. Brandt semigroups, is explained in Subsection 1.1.4. The concept of support of a map given in Subsection 1.1.5 plays a vital role in the thesis. Finally, near-semirings are recalled in Section 1.2 with a special focus on affine near-semirings. This chapter also concentrates on fixing various notations that are used in the thesis.

1.1 Semigroups

In this section, we present the necessary fundamental notions of semigroups. The material of this section can be found in any standard book on semigroup theory. For example, see [Clifford and Preston, 1961; Grillet, 1995; Howie, 1976, 1995; Pin, 1986].

1.1.1 Definitions and basic results

We begin with the notion of semigroups and continue to present various required concepts on semigroups.

Definition 1.1.1. An algebraic structure $(S, *)$ is said to be a *semigroup* if $*$ is an associative binary operation on S, i.e.

$$x * (y * z) = (x * y) * z, \text{ for all } x, y, z \in S.$$

Example 1.1.2.

1. The set of natural numbers $\mathbb{N}$ with usual addition.

2. The set of natural numbers $\mathbb{N}$ with usual multiplication.

3. The set of integers $\mathbb{Z}$ with usual addition.

Example 1.1.3. Let X be a nonempty set. The set $M(X)$ of all mappings on X forms a semigroup under the composition of mappings[1], i.e., for $x \in X$ and $f, g \in M(X)$,

$$x(f \circ g) = (xf)g.$$

In the thesis, the composition $f \circ g$ will simply be denoted by fg.

[1] In the thesis, we write an argument of a function on its left, e.g. xf is the value of a function f at an argument x.

Notation 1.1.4. For $f \in M(X)$, we define

$$\mathrm{Im}(f) = \{xf \mid x \in X\}.$$

Example 1.1.5. For $m, n \in \mathbb{N}$, consider a set $R_{m,n} = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, with the multiplication given by

$$(i, j)(k, l) = (i, l),$$

is a semigroup and it is known as a *rectangular band*.

We would also require the following type of special semigroups in the thesis.

Definition 1.1.6. Let $(S_\alpha, *_\alpha)_{\alpha \in \Lambda}$ be a family of semigroups each of which has a zero element, say 0_α. Consider the disjoint union

$$S = \{0\} \cup \bigcup_{\alpha \in \Lambda} (S_\alpha \setminus \{0_\alpha\}),$$

where 0 is a new element which is not present in any S_α. Define a binary operation $*$ on S by, for $x, y \in S$,

$$x * y = \begin{cases} x *_\alpha y, & \text{if there exist } \alpha \in \Lambda \text{ such that } x, y \in S_\alpha \text{ and } xy \neq 0_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, *)$ is a semigroup known as 0-*direct union* of the semigroups S_α.

In this thesis, unless it is required, algebraic structures (such as semigroups, groups, near-semirings) will simply be referred by their underlying sets without explicit mention of their operations.

Notation 1.1.7. Additive semigroup is meant a semigroup in which the binary operation is $+$. Similarly, if $\cdot$ is a binary operation on a semigroup S, then we say S is a multiplicative semigroup. In a multiplicative semigroup, the product $x \cdot y$ will simply be denoted by xy.

In the rest of the section, we consider multiplicative semigroups.

Definition 1.1.8. A semigroup S is said to be *commutative* if $xy = yx$, for all $x, y \in S$.

Definition 1.1.9. If a semigroup S contains an element 1 with the property that

$$1x = x1 = x$$

for all $x \in S$, then 1 is called an *identity element* of S. A semigroup with an identity element is said to be a *monoid*. If a semigroup contains an element 0 such that

$$x0 = 0x = 0$$

for all $x \in S$, then 0 is called a *zero element* or *absorbing element* of S.

Remark 1.1.10. A semigroup S can be extended to a monoid by adjoining an extra element 1 together with the conditions $11 = 1$ and $1x = x1 = x$, for all $x \in S$.

Notation 1.1.11. We write S^1 to denote the monoid which is obtained from a semigroup S by adjoining an identity element, if necessary. That is, if S has an identity element, then $S^1 = S$; otherwise, S^1 is the monoid which is obtained from S as per Remark 1.1.10.

Definition 1.1.12. A nonempty subset T of a semigroup S is called a *subsemigroup* of S if it is closed under the multiplication of S.

Remark 1.1.13. Let $\{T_i : i \in \Lambda\}$ be a family of subsemigroups of a semigroup S such that $\bigcap_{i\in\Lambda} T_i$ is nonempty. Then $\bigcap_{i\in\Lambda} T_i$ is a subsemigroup of S.

Definition 1.1.14. Let X be a nonempty subset of a semigroup S. The *subsemigroup of S generated by* X, denoted by $\langle X \rangle$, is the intersection of all subsemigroups of S containing X.

Proposition 1.1.15. *Let X be a nonempty subset of a semigroup S. Then*

$$\langle X\rangle = \{x_1x_2\cdots x_n \mid x_i \in X,\ i \in \{1,2,\ldots,n\} \ \textit{ and } \ n \in \mathbb{N}\},$$

the set of finite products of elements of X.

Definition 1.1.16. A *right congruence* on a semigroup S is an equivalence relation $\equiv$ which is compatible from right, i.e. for $x, y \in S$,

$$x \equiv y \Rightarrow \forall z(xz \equiv yz).$$

Similarly, *left congruence* can be defined. A relation which is both left and right congruences is called a *congruence* on S.

Definition 1.1.17. Let S and S' be semigroups. A function $\varphi : S \longrightarrow S'$ is called a *homomorphism* if

$$(xy)\varphi = (x\varphi)(y\varphi)$$

for all $x, y \in S$. Further, if φ is a bijection, then φ is said to be an *isomorphism.* Two semigroups are said be isomorphic if there is an isomorphism between them. A homomorphism from a semigroup to itself is called an *endomorphism* and if it is an isomorphism, then it is called an *automorphism.*

Remark 1.1.18. The set of all endomorphisms of a semigroup S, denoted by $\mathrm{End}(S)$, forms a monoid with respect to composition of maps. Similarly, the set of all automorphisms of a semigroup S, denoted by $\mathrm{Aut}(S)$, forms a group.

Definition 1.1.19. Given a homomorphism $\varphi : S \longrightarrow S'$ define

$$\ker\varphi = \{(x, y) \in S \times S \mid x\varphi = y\varphi\},$$

and it is called the *kernel* of a homomorphism φ.

Remark 1.1.20. $\ker\varphi$ of a semigroup homomorphism φ is a congruence relation.

Theorem 1.1.21. *If $\varphi : S \longrightarrow S'$ is an onto-homomorphism, then the semigroup $S/_{\ker\varphi}$ is isomorphic to the semigroup S' under the assignment $[x]_{\ker\varphi} \mapsto x\varphi$.*

Notation 1.1.22. Let A and B be nonempty subsets of S. The subset

$$\{ab \mid a \in A, b \in B\}$$

of S is denoted by AB. Further, the subset $\{a\}B$ is simply denoted by aB.

Definition 1.1.23. A nonempty subset I of a semigroup S is said to be a *left ideal* of S if $SI \subseteq I$ and it is called a *right ideal* if $IS \subseteq I$. The subset I is called an *ideal* if it is both a left ideal and a right ideal of S.

Proposition 1.1.24. *Let a be an element of a semigroup S. The set*

1. *aS^1 is the smallest right ideal of S containing a,*
2. *S^1a is the smallest left ideal of S containing a, and*
3. *S^1aS^1 is the smallest ideal of S containing a.*

Definition 1.1.25. Let a be an element of a semigroup S. The right ideal aS^1, left ideal S^1a, and ideal S^1aS^1 are, respectively, known as the *principle right ideal*, the *principle left ideal*, and the *principle ideal*, generated by a.

1.1.2 Green's relations

Green's relations were first introduced in [Green, 1951]. These relations play a vital role in the structure theory of semigroups. We now present Green's relations viz. $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$ along with their necessary properties from [Howie, 1995].

Definition 1.1.26 ([Green, 1951])**.** Let S be a semigroup and $a, b \in S$.

1. $a\mathcal{L}b$ if a and b generate the same principal left ideal, i.e. $a\mathcal{L}b \Longleftrightarrow S^1a = S^1b$.
2. $a\mathcal{R}b$ if a and b generate the same principal right ideal, i.e. $a\mathcal{R}b \Longleftrightarrow aS^1 = bS^1$.

3. $a\mathcal{J}b$ if a and b generate the same principal ideal, i.e. $a\mathcal{J}b \iff S^1aS^1 = S^1bS^1$.

4. $a\mathcal{H}b$ if $a\mathcal{L}b$ and $a\mathcal{R}b$.

5. $a\mathcal{D}b$ if there exists some $x \in S$ such that $a\mathcal{L}x$ and $x\mathcal{R}b$.

The Green's relations are characterized in the following theorem.

Theorem 1.1.27. *Let S be a semigroup and $a, b \in S$. Then*

1. *$a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$.*

2. *$a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$, $by = a$.*

3. *$a\mathcal{J}b$ if and only if there exist $x, y, u, v \in S^1$ such that $xay = b$, $ubv = a$.*

Proposition 1.1.28. *The Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$ are equivalence relations. Furthermore, $\mathcal{L}$ is right congruence and $\mathcal{R}$ is a left congruence.*

Remark 1.1.29. In a commutative semigroup, we have $\mathcal{R} = \mathcal{L} = \mathcal{J} = \mathcal{D} = \mathcal{H}$.

Proposition 1.1.30. *In a finite semigroup, we have $\mathcal{D} = \mathcal{J}$.*

Notation 1.1.31. We denote the $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{D}$-class and $\mathcal{H}$-class of an element $a \in S$ by L_a, R_a, D_a and H_a, respectively.

Definition 1.1.32. A semigroup S is said to be *aperiodic*, if for every $x \in S$, there exist a nonnegative integer n such that $x^n = x^{n+1}$.

Proposition 1.1.33. *Let S be a finite semigroup. The following statements are equivalent.*

1. *S is aperiodic.*

2. *There exists $m > 0$ such that, for every $x \in S$, $x^m = x^{m+1}$.*

3. *The groups in S are trivial.*

4. *S is $\mathcal{H}$-trivial, i.e. $\mathcal{H}$ is the equality relation on S.*

1.1.3 Regular and inverse semigroups

In this section, we present some important classes of semigroups, viz. regular semigroups and inverse semigroups. Accordingly, we state relations between them. Further, we also recall some well known semigroups which are useful to understand the work embedded in the thesis.

Definition 1.1.34. An element a of a semigroup S is *regular* if there exists an element $x \in S$ such that $axa = a$. A semigroup is called *regular* if all of its elements are regular. A semigroup S is said to be *eventually regular* if for every $a \in S$ there exists a positive integer n such that a^n is regular.

Remark 1.1.35. Every regular semigroup is eventually regular.

Notation 1.1.36. For a semigroup S, we write $I(S) = \{a \in S \mid a^2 = a\}$, the set of all idempotent elements of S.

Definition 1.1.37. A semigroup S is said to be a *band* if $I(S) = S$.

Example 1.1.38. Rectangular band (cf. Example 1.1.5) is a band.

Definition 1.1.39. A semigroup S is said to be *orthodox* if it is regular and $I(S)$ is a subsemigroup of S.

Definition 1.1.40. An element b of a semigroup S is said to be an *inverse* of an element $a \in S$, if $aba = a$ and $bab = b$.

Remark 1.1.41. An element a of a semigroup S is regular if and only if a has an inverse in S.

Remark 1.1.42. An element may have more than one inverse in a semigroup. For instance, the elements $(1, 2)$ and $(2, 2)$ are inverses of the element $(1, 2)$ in the rectangular band $R_{2,2}$.

Definition 1.1.43. A semigroup S is called an *inverse semigroup* if every element of S has a unique inverse.

Theorem 1.1.44. *A regular semigroup is inverse if and only if its idempotents commute.*

1.1.4 Brandt semigroups

The Brandt semigroups is an important class of inverse semigroups and it has its own importance in semigroup theory. The work presented in the thesis is essentially based on the Brandt semigroups. In this subsection, we present fundamental definitions and results on the Brandt semigroups from Howie [1976, 1995]. In this subsection, S denotes a semigroup with zero element 0.

Definition 1.1.45. A semigroup S is called 0-*simple* if $\{0\}$ and S are its only ideals and $S^2 \neq \{0\}$.

Definition 1.1.46. A nonzero idempotent in S is said to be *primitive* if it is a minimal element in $I(S) \setminus \{0\}$ with respect to the partial order relation $\leq$ on $I(S)$ defined by, for $a, b \in I(S)$,

$$a \leq b \Longleftrightarrow ab = ba = a.$$

Definition 1.1.47. A semigroup S is said to be *completely* 0-*simple* if it is 0-simple and has a primitive idempotent.

Definition 1.1.48. Given a finite group G and a natural number n, write $[n] = \{1, 2, \ldots, n\}$ and $B(G, n) = ([n] \times G \times [n]) \cup \{\vartheta\}$. Define a binary operation (say, addition) on $B(G, n)$ by

$$(i, a, j) + (k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k; \\ \vartheta & \text{otherwise,} \end{cases}$$

$$\text{and } \ \vartheta + (i, a, j) = (i, a, j) + \vartheta = \vartheta + \vartheta = \vartheta.$$

With the above defined addition, $B(G,n)$ is a semigroup known as the *Brandt semigroup*.

When G is the trivial group, the Brandt semigroup $B(\{e\},n)$ is denoted by B_n. Instead of writing the identity element $e \in G$ in the triplets of elements of B_n, we use the following description in the definition of B_n.

Definition 1.1.49. For a natural number n, the Brandt semigroup $(B_n,+)$, where $B_n = ([n] \times [n]) \cup \{\vartheta\}$ and the operation $+$ is given by

$$(i,j)+(k,l) = \begin{cases} (i,l) & \text{if } j = k; \\ \vartheta & \text{if } j \neq k \end{cases}$$

and, for all $\alpha \in B_n$, $\alpha + \vartheta = \vartheta + \alpha = \vartheta$. Note that ϑ is the (two sided) zero element in B_n.

Remark 1.1.50.

1. For each $\alpha \in B_n$, we have $\alpha + \alpha = \alpha + \alpha + \alpha$. Thus, by Proposition 1.1.33, the Brandt semigroup B_n is aperiodic.

2. $I(B_n) = \{(k,k) \mid k \in [n]\} \cup \{\vartheta\}$.

The Green's relations in the Brandt semigroups is characterized as follows.

Lemma 1.1.51 ([Howie, 1976])**.** *Let $(i,a,j),(k,b,l) \in B(G,n)$. Then*

1. *$(i,a,j)\mathcal{R}(k,b,l)$ if and only if $i = k$.*

2. *$(i,a,j)\mathcal{L}(k,b,l)$ if and only if $j = l$.*

3. *$(i,a,j)\mathcal{H}(k,b,l)$ if and only if $i = k$ and $j = l$.*

Remark 1.1.52. All the nonzero elements of $B(G,n)$ are $\mathcal{D}$-related.

Remark 1.1.53. For nonzero elements of B_n, the relation $\mathcal{R}$ (or $\mathcal{L}$) is the equality on the first coordinate (or on the second coordinate, respectively). Hence, other than the class $\{\vartheta\}$, the number of $\mathcal{R}$ or $\mathcal{L}$ classes in B_n is n. Consequently, any two nonzero elements of B_n are $\mathcal{D}$-related.

Theorem 1.1.54. *A semigroup S is both completely 0-simple and an inverse semigroup if and only if S is isomorphic to the semigroup $B(G, n)$, for some group G and some n.*

1.1.5 Support of a map

In this subsection, we present some concepts on mappings which are useful in the subsequent chapters and fix our notation. We begin with the following definition which plays a vital role in the thesis. In this subsection, $(S, +)$ is a semigroup with zero element ϑ.

Definition 1.1.55. For $f \in M(S)$, the *support of* f, denoted by supp(f), is defined by the set

$$\mathrm{supp}(f) = \{x \in S \mid xf \neq \vartheta\}.$$

Definition 1.1.56. A function $f \in M(S)$ is said to be of k-*support* if the cardinality of supp(f) is k, i.e. $|\mathrm{supp}(f)| = k$. If $k = |S|$ or $k = 1$, then f is said to be of *full support* or *singleton support*, respectively.

Notation 1.1.57.

1. For $X \subseteq M(S)$, we write X_k to denote the set of all mappings of k-support in X, i.e.

$$X_k = \{f \in X \mid f \text{ is of } k\text{-support}\}.$$

2. For $a \in S$, the constant map ξ_a on S is defined by $x\xi_a = a$ for all $x \in S$. If we are not specific about the constant image, we may simply write ξ to denote a constant map. For $X \subseteq S$, we write

$$\mathcal{C}_X = \{\xi_a \in M(S) \mid a \in X\}.$$

We would require the following lemma at various places in the thesis.

Lemma 1.1.58. *If $f, g \in M(S)$, then $\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g)$, where $x(f+g) = xf + xg$, for all $x \in S$. Moreover, for $f \in M(S)_k$, we have*

$$|\operatorname{supp}(f+g)| \leq k \;\; \textit{and} \;\; |\operatorname{supp}(g+f)| \leq k.$$

Proof. Let $x \in \operatorname{supp}(f+g)$. Then $xf + xg \neq \vartheta$ so that $xf \neq \vartheta$ and $xg \neq \vartheta$. Thus, $x \in \operatorname{supp}(f) \cap \operatorname{supp}(g)$. Further, If $f \in M(S)_k$, then, for any $g \in M(S)$, $|\operatorname{supp}(f+g)| \leq k$ and $|\operatorname{supp}(g+f)| \leq k$. For instance, if $x \notin \operatorname{supp}(f)$, then $x \notin \operatorname{supp}(f+g)$ and also $x \notin \operatorname{supp}(g+f)$. □

1.2 Near-semirings

In this section, we present the required notions on near-semirings. For more details one may refer to [Krishna, 2005; van Hoorn and van Rootselaar, 1967].

Definition 1.2.1. An algebraic structure $(\mathcal{N}, +, \cdot)$ is said to be a *near-semiring* if

1. $(\mathcal{N}, +)$ is a semigroup,

2. $(\mathcal{N}, \cdot)$ is a semigroup, and

3. $a(b+c) = ab + ac$, for all $a, b, c \in \mathcal{N}$.

Example 1.2.2. Let $(S,+)$ be a semigroup. The algebraic structure $(M(S),+,\circ)$ is a near-semiring, where $+$ is point-wise addition and $\circ$ is composition of mappings, i.e., for $x \in S$ and $f, g \in M(S)$,

$$x(f+g) = xf + xg \quad \text{and} \quad x(f \circ g) = (xf)g.$$

Definition 1.2.3. A nonempty subset of a near-semiring $\mathcal{N}$ is called a *subnear-semiring* of $\mathcal{N}$ if it is closed under both the operations of $\mathcal{N}$.

Example 1.2.4. The set $\mathcal{C}_S$ of all constant mappings on S is a subnear-semiring of $M(S)$.

Definition 1.2.5. A near-semiring $(\mathcal{N},+,\cdot)$ is said to be *zero-symmetric*, if

1. $(\mathcal{N},+)$ is a monoid with identity 0,
2. $a0 = 0a = 0$ for all $a \in \mathcal{N}$.

1.2.1 Affine near-semiring

Now, we recall the notion of affine maps and affine near-semirings from [Krishna, 2005].

Definition 1.2.6. An element $f \in M(S)$ is said to be an *affine map* if $f = g + h$, for some $g \in \mathrm{End}(S)$ and $h \in \mathcal{C}_S$. We call this sum as an *affine decomposition* of f.

Remark 1.2.7. Sum of two affine maps on a semigroup S need not be an affine map, unless S is commutative.

Notation 1.2.8. The set $\mathrm{Aff}(S)$ denotes the set of all affine maps on a semigroup S. Further, the set of finite sums of affine maps is denoted by $A^+(S)$, i.e.

$$A^+(S) = \left\{ \sum_{i=1}^{k} f_i \;\middle|\; f_i \in \mathrm{Aff}(S),\ k \geq 1 \right\}.$$

Remark 1.2.9. The set $A^+(S)$ is the subsemigroup generated by $\mathrm{Aff}(S)$ in $(M(S), +)$.

Definition 1.2.10. The subnear-semiring generated by $\mathrm{Aff}(S)$ in $M(S)$ is said to be an *affine near-semiring* over S.

Proposition 1.2.11. *For any semigroup S, $A^+(S)$ is an affine near-semiring.*

Proof. In view of Remark 1.2.9 $A^+(S)$ is closed with respect to addition. It is a routine verification to show that $A^+(S)$ is closed with respect to composition. $\square$

2

The Near-Semiring $A^+(B_n)$

This chapter introduces the main object of study of the thesis, viz. $A^+(B_n)$, the affine near-semiring over Brandt semigroup B_n. In a systematic approach, the chapter characterizes, classifies and counts the elements of $A^+(B_n)$. In this connection, first we characterize the elements of $\mathrm{Aff}(B_n)$ in Section 2.1 and then proceed to the elements of $A^+(B_n)$ in Section 2.2. The classification is done in terms of the supports of mappings in $A^+(B_n)$. Further, the chapter reports the cardinality of $A^+(B_n)$, for an arbitrary natural number n. Certain fundamental properties of the elements of $A^+(B_n)$ are described through various results in the chapter. These results are useful throughout the thesis.

2.1 Affine maps over B_n

In this section, first we characterize the elements of endomorphisms over B_n using a result of Gilbert and Samman [2010b]. Then we ascertain that the constant functions and affine maps of n-support are only the elements of $\mathrm{Aff}(B_n)$ and find its cardinality (cf. Theorem 2.1.6).

We characterize $\mathrm{End}(B_n)$ by extending the corresponding result for $\mathrm{End}_\vartheta(B_n)$ in [Gilbert and Samman, 2010b]. As per Proposition 2.1.1, they have shown that the semigroup

$$\mathrm{End}_\vartheta(B_n) = \{f \in \mathrm{End}(B_n) \mid \vartheta f = \vartheta\},$$

with respect to composition of mappings, is isomorphic to the semigroup S_n^0, the symmetric group S_n of degree n is adjoined by the zero element 0.

Proposition 2.1.1 ([Gilbert and Samman, 2010b]). $\mathrm{End}_\vartheta(B_n)$ *is isomorphic to* S_n^0.

Sketch of the Proof. Let $f \in \mathrm{End}_\vartheta(B_n) \setminus \{\xi_\vartheta\}$. Then f determines two functions $f_1, f_2 : [n] \times [n] \longrightarrow [n]$ such that

$$(i,j)f = ((i,j)f_1, (i,j)f_2).$$

It can be observed that f_1 depends only on the first coordinate and f_2 depends only on the second coordinate, i.e. $(i,j)f_1 = (i,k)f_1$ and $(i,j)f_2 = (k,j)f_2$, for all $j,k \in [n]$. Moreover, for $i \in [n]$, $(i,k)f_1 = (l,i)f_2$. Now, define a permutation σ on $[n]$ by, for $i \in [n]$,

$$i\sigma = (i,k)f_1 = (l,i)f_2.$$

Thus, there exists a permutation $\sigma \in S_n$ such that $(i,j)f = (i\sigma, j\sigma)$. Further, for any permutation $\sigma \in S_n$, the mapping $\phi_\sigma : B_n \longrightarrow B_n$ such that $(i,j) \mapsto (i\sigma, j\sigma)$, $\vartheta \mapsto \vartheta$ is an endomorphism over B_n. Now, it can be observed that the assignment

$$0 \mapsto \xi_\vartheta \text{ and } \sigma \mapsto \phi_\sigma : S_n^0 \longrightarrow \mathrm{End}_\vartheta(B_n)$$

is an isomorphism. □

We observe the following remark regarding the function ϕ_σ given in the proof of Proposition 2.1.1.

Remark 2.1.2. For $\sigma \in S_n$, the mapping $\phi_\sigma : B_n \longrightarrow B_n$ defined by

$$(i, j)\phi_\sigma = (i\sigma, j\sigma) \text{ and } \vartheta\phi_\sigma = \vartheta$$

is an automorphism over B_n.

In view of Remark 2.1.2, we have the following corollary of Proposition 2.1.1.

Corollary 2.1.3. $\mathrm{Aut}(B_n)$ *is isomorphic to* S_n.

Now, we characterize the elements of $\mathrm{End}(B_n)$ and find its size.

Theorem 2.1.4. $\mathrm{End}(B_n) = \mathrm{Aut}(B_n) \cup \mathcal{C}_{I(B_n)}$. *Hence,* $|\mathrm{End}(B_n)| = n! + n + 1$.

Proof. Clearly, $\mathrm{Aut}(B_n) \cup \mathcal{C}_{I(B_n)} \subseteq \mathrm{End}(B_n)$. Let $f \in \mathrm{End}(B_n)$. If $f = \xi_\vartheta$, then clearly $f \in \mathcal{C}_{I(B_n)}$. Otherwise, if $\vartheta f = \vartheta$, then by Proposition 2.1.1, $f \in \mathrm{Aut}(B_n)$ and $f = \phi_\sigma$ for some $\sigma \in S_n$. If $\vartheta f \neq \vartheta$, then $\vartheta f = (k, k)$ for some $k \in [n]$. Now, for any $(i, j) \in B_n$,

$$(k, k) = \vartheta f = ((i, j) + \vartheta)f = (i, j)f + (k, k)$$

so that $(i, j)f = (k, k)$. Thus, if $\vartheta f \neq \vartheta$, then $f \in \mathcal{C}_{I(B_n)}$. Hence,

$$\mathrm{End}(B_n) = \mathrm{Aut}(B_n) \cup \mathcal{C}_{I(B_n)}.$$

Since $\mathrm{Aut}(B_n) \cap \mathcal{C}_{I(B_n)} = \varnothing$, $|\mathcal{C}_{I(B_n)}| = n + 1$ (cf. Remark 1.1.50) and $|\mathrm{Aut}(B_n)| = |S_n| = n!$ (cf. Corollary 2.1.3), we have $|\mathrm{End}(B_n)| = n! + n + 1$. □

Remark 2.1.5. Every constant map over B_n is an affine map. For instance, the zero map ξ_ϑ can be written as $\xi_{(p,p)} + \xi_\vartheta$. Similarly, a nonzero constant map $\xi_{(p,q)}$ can be decomposed as $\xi_{(p,p)} + \xi_{(p,q)}$.

Now, we are ready to classify the affine maps over B_n. Indeed, we shall prove that constant maps and n-support affine maps are precisely the affine maps over B_n.

Theorem 2.1.6. $\mathrm{Aff}(B_n) = \mathrm{Aff}(B_n)_n \cup \mathcal{C}_{B_n}$. *Moreover,* $|\mathrm{Aff}(B_n)| = (n!+1)n^2+1$.

Proof. Since every constant map over B_n is an affine map, we have

$$\mathrm{Aff}(B_n)_n \cup \mathcal{C}_{B_n} \subseteq \mathrm{Aff}(B_n).$$

Let $f \in \mathrm{Aff}(B_n)$. If $f = \xi_\vartheta$, then clearly $f \in \mathcal{C}_{B_n}$. Otherwise, write $f = g + \xi_{(p,q)}$ for some $g \in End(B_n) \setminus \{\xi_\vartheta\}$. By Theorem 2.1.4, g can be either $\xi_{(k,k)}$ for some $k \in [n]$ or ϕ_σ for some $\sigma \in S_n$.

In case $g = \xi_{(k,k)}$ for some $k \in [n]$, since $f \neq \xi_\vartheta$, we have $k = p$ so that $f = \xi_{(p,q)} \in \mathcal{C}_{B_n}$. Further, as there are n possibilities each for p and q, we have n^2 affine maps (of full support) in this case.

We may now suppose $g = \phi_\sigma$ for some $\sigma \in S_n$. Clearly, we have $\vartheta f = \vartheta$, because $\vartheta\phi_\sigma = \vartheta$. Now for $(i,j) \in B_n \setminus \{\vartheta\}$

$$(i,j)f = (i\sigma, j\sigma) + (p,q) = \begin{cases} (i\sigma, q), & \text{if } j = p\sigma^{-1}; \\ \vartheta, & \text{otherwise.} \end{cases}$$

Hence, $\mathrm{supp}(f) = \{(i, p\sigma^{-1}) : i \in [n]\}$ so that $f \in \mathrm{Aff}(B_n)_n$. Consequently,

$$\mathrm{Aff}(B_n) = \mathrm{Aff}(B_n)_n \cup \mathcal{C}_{B_n}.$$

Since the above union is disjoint and $|\mathcal{C}_{B_n}| = n^2+1$, it remains to prove that $|\mathrm{Aff}(B_n)_n| = (n!)n^2$. As shown above, every affine map of n-support is precisely of the form $\phi_\sigma + \xi_{(p,q)}$, for some $\sigma \in S_n$ and $p, q \in [n]$. Thus, $|\mathrm{Aff}(B_n)_n| \leq (n!)n^2$. Now, let $f = \phi_\sigma + \xi_{(p,q)}$ and $g = \phi_\rho + \xi_{(s,t)}$, for some $\sigma, \rho \in S_n$ and $p, q, s, t \in [n]$. If $q \neq t$, then clearly $\mathrm{Im}(f) \neq \mathrm{Im}(g)$. If $\sigma \neq \rho$, then there exists $i_0 \in [n]$ such that $i_0\sigma \neq i_0\rho$. If $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$, then $f \neq g$. Otherwise, for $(i_0, k) \in \mathrm{supp}(f) \cap \mathrm{supp}(g)$,

$$(i_0, k)f = (i_0\sigma, q) \neq (i_0\rho, t) = (i_0, k)g.$$

Assume $\sigma = \rho$ but $p \neq s$, then clearly $\mathrm{supp}(f) \neq \mathrm{supp}(g)$. Thus, distinct choices of σ and (p,q) determine distinct affine maps of n-support. Hence the result. □

Following remarks are immediate from the proof of Theorem 2.1.6.

Remark 2.1.7. If $f \in \mathrm{Aff}(B_n)$ such that $\vartheta \in \mathrm{supp}(f)$, then f is a nonzero constant map.

Remark 2.1.8. Given $f \in \mathrm{Aff}(B_n)_n$, there exist $k, q \in [n]$ and $\sigma \in S_n$ such that $\mathrm{supp}(f) = \{(i,k) \mid i \in [n]\}$, $\mathrm{Im}(f) = \{(i\sigma, q) \mid i \in [n]\} \cup \{\vartheta\}$ and $(i,k)f = (i\sigma, q)$, for all $i \in [n]$.

Definition 2.1.9. For $f \in \mathrm{Aff}(B_n)_n$, a *representation of* f is defined by a triplet $(k, q; \sigma)$, where the parameters k, q and σ are as per Remark 2.1.8.

Remark 2.1.10. An affine decomposition for $(p, q; \sigma)$ is $\phi_\sigma + \xi_{(p\sigma, q)}$, where ϕ_σ is as per Remark 2.1.2. Furthermore, for $f \in \mathrm{Aut}(B_n)$, $f + \xi_{(r,s)}$ is an n-support map represented by $(r\rho^{-1}, s; \rho)$, where $f = \phi_\rho$, for some $\rho \in S_n$.

2.2 Classification of elements in $A^+(B_n)$

In this section, first we study certain fundamental properties of elements in $A^+(B_n)$. We conclude the section by obtaining the cardinality of $A^+(B_n)$ along with a classification of its elements (cf. Theorem 2.2.10).

Proposition 2.2.1. *If $f \in A^+(B_n)$ and $\vartheta \in \mathrm{supp}(f)$, then f is a nonzero constant map.*

Proof. For $f \in A^+(B_n)$, write $f = f_1 + \cdots + f_m$ where each $f_j \in \mathrm{Aff}(B_n)$. If $\vartheta f \neq \vartheta$, then each f_j must be a nonzero constant map (cf. Remark 2.1.7) and hence f is a nonzero constant map. $\square$

It is clear that any nonzero constant map in $M(B_n)$ is of full support. The following corollary of Proposition 2.2.1 ascertains that the converse holds in case of the elements in $A^+(B_n)$.

Corollary 2.2.2. *If $f \in A^+(B_n)_{n^2+1}$, then $f \in \mathcal{C}_{B_n}$. Hence, $|A^+(B_n)_{n^2+1}| = n^2$.*

Proposition 2.2.3. *If $f, g \in \mathrm{Aff}(B_n)_n$ and $(\xi_\vartheta \neq)\ h \in \mathcal{C}_{B_n}$, then we have*

1. $|\mathrm{supp}(g + f)| = 0$ *or* 1,
2. $|\mathrm{supp}(h + f)| = 1$,
3. $|\mathrm{supp}(f + h)| = 0$ *or* n.

Proof. Let $(k, q; \sigma)$ and $(k', q'; \sigma')$ be the representations of f and g, respectively and $h = \xi_{(r,s)}$, for $r, s \in [n]$.

1. If $k \neq k'$, then $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing$ and clearly, $|\mathrm{supp}(g + f)| = 0$. Otherwise,
$$\mathrm{supp}(f) = \mathrm{supp}(g) = \{(i, k) \mid 1 \leq i \leq n\}.$$
Since σ is a permutation on $[n]$, let j be the unique element in $[n]$ such that $j\sigma = q'$. Now,
$$(j, k)(g + f) = (j\sigma', q') + (j\sigma, q) = (j\sigma', q)$$
and, for $i \in [n]$ with $i \neq j$, $(i, k)(g + f) = \vartheta$. Thus, $\mathrm{supp}(g + f) = \{(j, k)\}$.

2. Since σ is a permutation on $[n]$, there is a unique $t \in [n]$ such that $t\sigma = s$. Now,
$$(t, k)(h + f) = (t, k)\xi_{(r,s)} + (t, k)f = (r, s) + (t\sigma, q) = (r, q)$$
and, for all $\alpha \in B_n \setminus \{(t, k)\}$, $\alpha(h + f) = \vartheta$. Thus, $|\mathrm{supp}(h + f)| = 1$.

3. If $q \neq r$, then clearly, $\alpha(f + h) = \vartheta$, for all $\alpha \in B_n$, so that $|\mathrm{supp}(f + h)| = 0$. Otherwise, for all $1 \leq i \leq n$,
$$(i, k)(f + h) = (i\sigma, q) + (r, s) = (i\sigma, s)$$
and, for all $\alpha \in B_n \setminus \mathrm{supp}(f)$, $\alpha(f + h) = \vartheta$. Hence, $|\mathrm{supp}(f + h)| = n$.

$\square$

Lemma 2.2.4. *For $f \in M(B_n)$ and $k, l, p, q \in [n]$, if $(k,l)f = (p,q)$ and $\alpha f = \vartheta$ for all $\alpha \in B_n \setminus \{(k,l)\}$, then $f \in A^+(B_n)_1$. Hence, $|A^+(B_n)_1| = n^4$.*

Proof. It is sufficient to prove that f is a finite sum of affine maps. Consider a permutation σ on $[n]$ such that $k\sigma = q$ and then consider $g \in \mathrm{Aff}(B_n)_n$ whose representation is $(l, q; \sigma)$. Note that $\xi_{(p,q)} + g \in A^+(B_n)$. Moreover,

$$(k,l)(\xi_{(p,q)} + g) = (p,q) + (k\sigma, q) = (p,q)$$

and, for $(i,l) \in \mathrm{supp}(g)$ with $i \neq k$, $(i,l)(\xi_{(p,q)} + g) = (p,q) + (i\sigma, q) = \vartheta$, as $i\sigma \neq q$. Further, it is clear that $\alpha(\xi_{(p,q)} + g) = \vartheta$, for all $\alpha \notin \mathrm{supp}(g)$. Hence, $\xi_{(p,q)} + g = f$. Consequently, $|A^+(B_n)_1|$ is the number of choices of $k, l, p, q \in [n]$, as desired. □

Notation 2.2.5. We use ${}^{(k,l)}\zeta_{(p,q)}$ to denote the singleton support map f whose $\mathrm{supp}(f) = \{(k,l)\}$ and $\mathrm{Im}(f) \setminus \{\vartheta\} = \{(p,q)\}$.

In view of the proof of Lemma 2.2.4, we have the following remark.

Remark 2.2.6. Every singleton support map can be written as sum of a constant map and an n-support affine map. For instance, ${}^{(k,l)}\zeta_{(p,q)} = \xi_{(p,q)} + g$, where $g = (l, q; \sigma)$ such that $k\sigma = q$.

Lemma 2.2.7. *If $f \in A^+(B_n) \setminus \mathrm{Aff}(B_n)$, then $|\mathrm{supp}(f)| = 1$.*

Proof. Suppose that $f = f_1 + \cdots + f_m \in A^+(B_n) \setminus \mathrm{Aff}(B_n)$ with $m \geq 2$ and $f_i = g_i + h_i$, for some $g_i \in \mathrm{End}(B_n)$ and $h_i \in \mathcal{C}_{B_n}$. In view of Theorem 2.1.6, for each i, either $f_i \in \mathcal{C}_{B_n}$ or $f_i \in \mathrm{Aff}(B_n)_n$. Clearly, none of the f_i's can be ξ_ϑ. For all $i \geq 2$, if f_i's are nonzero constant maps, then $f = g_1 + \xi$, where $\xi = h_1 + f_2 + \cdots + f_m \in \mathcal{C}_{B_n}$ so that $f \in \mathrm{Aff}(B_n)$; this contradicts the choice of f. On the other hand, $f_j \in \mathrm{Aff}(B_n)_n$ for some $j \geq 2$. Note that $|\mathrm{supp}(f)| \leq |\mathrm{supp}(h_{j-1} + f_j)|$. Hence, by Proposition 2.2.3(2), $|\mathrm{supp}(f)| \leq 1$; consequently, $|\mathrm{supp}(f)| = 1$. □

In view of Theorem 2.1.6, we have the following corollaries of Lemma 2.2.7.

Corollary 2.2.8. *For $n \geq 3$ and $1 < k < n$, $A^+(B_n)_k = \varnothing$.*

Corollary 2.2.9. *For $n \geq 1$, $f \in \mathrm{Aff}(B_n)_n \iff f \in A^+(B_n)_n$. Hence, $|A^+(B_n)_n| = (n!)n^2$.*

Now, combining the results from Corollary 2.2.2 through Corollary 2.2.9, we have the following main result of the section.

Theorem 2.2.10. *For $n \geq 2$, $|A^+(B_n)| = (n! + 1)n^2 + n^4 + 1$. In fact, we have the following breakup of the elements of $A^+(B_n)$.*

1. *The number of mappings of full support is n^2.*

2. *The number of mappings of n-support is $(n!)n^2$.*

3. *The number of mappings of singleton support is n^4.*

4. *The number of mappings of 0-support is 1.*

Remark 2.2.11. For $n = 1$, $End(B_n) = \mathrm{Aff}(B_n) = A^+(B_n) = \{(1, 1; id)\} \cup \mathcal{C}_{B_n}$, where id is the identity permutation on $[n]$.

3

Semigroup Structure

In order to study the structure of the affine near-semiring over a Brandt semigroup, this chapter considers its both the semigroup reducts and ascertain their structural properties. In this connection, this work completely characterizes the Green's classes of the additive semigroup reduct in Section 3.1 and the multiplicative semigroup reduct in Section 3.2. Along with these characterizations, the sizes of all Green's classes are reported. Further, in the respective sections, idempotent elements and regular elements of the semigroup reducts of $A^+(B_n)$ have also been characterized and studied some relevant semigroups in $A^+(B_n)$. It is also ascertained that the additive semigroup reduct is eventually regular and the multiplicative semigroup reduct is orthodox.

3.1 The semigroup $A^+(B_n)^+$

In what follows, the additive semigroup reduct $(A^+(B_n), +)$ of the affine near-semiring $(A^+(B_n), +, \circ)$ is denoted by $A^+(B_n)^+$. Further, in a particular context, if there is no emphasis on the semigroup, we may simply write $A^+(B_n)$.

3.1.1 Green's classes

In this subsection, we study all the Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{H}$ on the semigroup $A^+(B_n)^+$. Since $A^+(B_n)^+$ is a finite semigroup, the Green's relations $\mathcal{J}$ and $\mathcal{D}$ coincide on $A^+(B_n)^+$ (cf. Proposition 1.1.30). We begin with the following result which is useful in characterizing the Green's classes of $A^+(B_n)^+$.

Proposition 3.1.1. *In $A^+(B_n)^+$, we have the following.*

1. *The set of constant maps $\mathcal{C}_{B_n}$ is a subsemigroup which is isomorphic to B_n.*

2. *The set $A^+(B_n)_1 \cup \{\xi_\vartheta\}$ is an ideal which is isomorphic to the 0-direct union of n^2 copies of B_n.*

Hence, both the subsemigroups are regular.

Proof.

1. The assignment $\alpha \mapsto \xi_\alpha : B_n \longrightarrow \mathcal{C}_{B_n}$ is an isomorphism.

2. Observe that, by Lemma 1.1.58, $A^+(B_n)_1 \cup \{\xi_\vartheta\}$ is an ideal. Consider the semigroup Z which is 0-direct union of the collection $\{B_n^{(i,j)} \mid (i,j) \in [n] \times [n]\}$ of n^2 copies of B_n indexed by $[n] \times [n]$. For the nonzero elements of Z, we write $(p,q)^{(i,j)}$ to denote the element (p,q) which is in the (i,j)th copy of B_n. Now, the assignment ${}^{(k,l)}\zeta_{(p,q)} \mapsto (p,q)^{(k,l)}$ and $\xi_\vartheta \mapsto \vartheta$ is clearly a semigroup isomorphism from $A^+(B_n)_1 \cup \{\xi_\vartheta\}$ to Z.

Regularity of these semigroups follows from the regularity of B_n. □

Lemma 3.1.2. *Let f and g be two mappings in the semigroup $(M(B_n), +)$. If $f\mathcal{R}g$ (or $f\mathcal{L}g$), then* $\operatorname{supp}(f) = \operatorname{supp}(g)$.

Proof. If $f = g$, the result is straightforward. Otherwise, if $f\mathcal{R}g$, then there exist $h, h' \in M(B_n)$ such that $f + h = g$ and $g + h' = f$. By Lemma 1.1.58,

$$\operatorname{supp}(g) = \operatorname{supp}(f + h) \subseteq \operatorname{supp}(f)$$

and

$$\operatorname{supp}(f) = \operatorname{supp}(g + h') \subseteq \operatorname{supp}(g).$$

Hence, $\operatorname{supp}(f) = \operatorname{supp}(g)$. Similarly, if $f\mathcal{L}g$, then $\operatorname{supp}(f) = \operatorname{supp}(g)$. $\square$

Definition 3.1.3. For $f \in M(B_n)$, an *image invariant* of f, denoted by $ii(f)$, is defined as the number $q \in [n]$, if exists, such that

$$\operatorname{Im}(f) \setminus \{\vartheta\} = \{(i, q) \mid i \in X\}$$

for some $X \subseteq [n]$.

Remark 3.1.4. From Theorem 2.2.10, it can be observed that every nonzero element of $A^+(B_n)$ has an image invariant.

Definition 3.1.5. For $1 \leq i \leq 2$, let $\pi_i : [n] \times [n] \to [n]$ be the ith projection map. That is, $(p, q)\pi_1 = p$ and $(p, q)\pi_2 = q$, for all $(p, q) \in [n] \times [n]$.

Now, we characterize the Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on $A^+(B_n)^+$ classified by the supports of its elements.

Theorem 3.1.6. *For $f, g \in A^+(B_n)_1 \cup A^+(B_n)_{n^2+1}$, we have*

1. *$f\mathcal{R}g$ if and only if* $\operatorname{supp}(f) = \operatorname{supp}(g)$ *and $\alpha f\pi_1 = \alpha g\pi_1$, $\forall \alpha \in \operatorname{supp}(f)$,*

2. *$f\mathcal{L}g$ if and only if* $\operatorname{supp}(f) = \operatorname{supp}(g)$ *and $\alpha f\pi_2 = \alpha g\pi_2$, $\forall \alpha \in \operatorname{supp}(f)$,*

3. *$f\mathcal{D}g$ if and only if* $\operatorname{supp}(f) = \operatorname{supp}(g)$.

Proof.

1. In view of Lemma 3.1.2, if both f, g are in $A^+(B_n)_{n^2+1}$ or in $A^+(B_n)_1$, the characterization follows from Proposition 3.1.1 and Remark 1.1.53.

2. Similar to (1).

3. Clearly, $f\mathcal{D}g$ implies $\mathrm{supp}(f) = \mathrm{supp}(g)$. The converse follows from Proposition 3.1.1 and Remark 1.1.53.

$\square$

Theorem 3.1.7. *For $f, g \in A^+(B_n)_n$, we have*

1. *$f\mathcal{R}g$ if and only if $\mathrm{supp}(f) = \mathrm{supp}(g)$ and $\alpha f\pi_1 = \alpha g\pi_1$, $\forall \alpha \in \mathrm{supp}(f)$,*

2. *$L_f = \{f\}$,*

3. *$f\mathcal{D}g$ if and only if $f\mathcal{R}g$.*

Proof. Clearly, (3) follows from (2). We shall prove (1) and (2) in the following.

1. For $f, g \in A^+(B_n)_n$, by Proposition 2.2.3(3), $f\mathcal{R}g$ if and only if there exist $\xi, \xi' \in \mathcal{C}_{B_n}$ such that $f = g + \xi$ and $g = f + \xi'$. This implies that $\alpha f\pi_1 = \alpha g\pi_1$, for all $\alpha \in \mathrm{supp}(f)$ ($= \mathrm{supp}(g)$, by Lemma 3.1.2). For the converse, let $ii(f) = l$ and $ii(g) = m$ (cf. Remark 3.1.4). Choose the functions $h = \xi_{(l,m)}$ and $h' = \xi_{(m,l)}$. We show, simultaneously, that $\mathrm{supp}(f + h) = \mathrm{supp}(g)$ and $\alpha(f+h) = \alpha g$, for all $\alpha \in \mathrm{supp}(g)$. Note that $\mathrm{supp}(f+h) \subseteq \mathrm{supp}(f) = \mathrm{supp}(g)$ (cf. Lemma 1.1.58). Let $\alpha \in \mathrm{supp}(g)$. Then, $\alpha f \neq \vartheta$ so that $\alpha f = (k, l)$ for some $k \in [n]$, as $ii(f) = l$. Thus, since $\alpha f\pi_1 = \alpha g\pi_1$ and $ii(g) = m$, we have $\alpha g = (k, m)$. Now,

$$\alpha(f + h) = \alpha f + \alpha h = (k, l) + (l, m) = (k, m) = \alpha g.$$

Hence, $\alpha \in \mathrm{supp}(f + h)$ and $f + h = g$. Similarly, we can prove that $g + h' = f$ so that $f\mathcal{R}g$.

2. If $n = 1$, the result is straightforward. For $n \geq 2$, let $g \in L_f$ with $g \neq f$. Then there exists $h \in A^+(B_n)$ such that $h + f = g$. However, by Proposition 2.2.3, $|\text{supp}(h + f)| \leq 1$; a contradiction. Thus, $L_f = \{f\}$.

□

In view of Theorem 2.2.10, we have the following corollary of theorems 3.1.6 and 3.1.7.

Corollary 3.1.8. *For $n \geq 2$, we have the following.*

1. *The number of $\mathcal{R}$-classes in $A^+(B_n)^+$ is $(n!)n + n^3 + n + 1$.*
2. *The number of $\mathcal{L}$-classes in $A^+(B_n)^+$ is $(n!)n^2 + n^3 + n + 1$.*
3. *The number of $\mathcal{D}$-classes in $A^+(B_n)^+$ is $(n!)n + n^2 + 2$.*

Proof.

1. Other than the class $\{\xi_\vartheta\}$, by Theorem 3.1.6(1), there are n and n^3 $\mathcal{R}$-classes containing the full support maps and singleton support maps, respectively. Further, by Theorem 3.1.7(1), $(n!)n$ $\mathcal{R}$-classes are present in $A^+(B_n)_n$ (each is of size n). Hence, we have the total number.

2. Similar to above (1), by Theorem 3.1.6(2), there are $n^3 + n + 1$ $\mathcal{L}$-classes containing singleton support and constant maps. And the remaining number $(n!)n^2$ is the number of $\mathcal{L}$-classes containing n-support maps (cf. Theorem 3.1.7(2)).

3. Other than the class $\{\xi_\vartheta\}$, by Theorem 3.1.6(3), all the full support elements form a single $\mathcal{D}$-class and there are n^2 $\mathcal{D}$-classes (each is of size n^2) containing singleton support elements. Including $(n!)n$ $\mathcal{D}$-classes which are present in $A^+(B_n)_n$ (cf. Theorem 3.1.7(3) and above (1)), we have $(n!)n + n^2 + 2$ $\mathcal{D}$-classes in $A^+(B_n)^+$.

□

Remark 3.1.9. Since $\alpha+\alpha = \alpha+\alpha+\alpha$, for all $\alpha \in B_n$, we have $f+f = f+f+f$, for all f in the semigroup $(M(B_n), +)$. Consequently, any subsemigroup of $(M(B_n), +)$ is aperiodic.

Hence, by Remark 3.1.9 and Proposition 1.1.33, we have the following proposition.

Proposition 3.1.10. *The Green's relation $\mathcal{H}$ is trivial on the semigroup $A^+(B_n)^+$.*

Remark 3.1.11. Proposition 3.3(e) in [Gilbert and Samman, 2010b], given for endomorphism near-semirings, also follows immediately from Remark 3.1.9.

3.1.2 Regular elements and idempotents

In this subsection, we characterize the regular and idempotent elements in $A^+(B_n)^+$ and ascertain that $A^+(B_n)^+$ is eventually regular. We observe that the set of regular elements in $A^+(B_n)^+$ forms an inverse semigroup.

Theorem 3.1.12. *For $n \geq 2$,*

1. *$f \in A^+(B_n)^+$ is of k-support with $k \neq n$ if and only if f is regular;*

2. *$I(A^+(B_n)^+) = \{\xi_\alpha \,|\alpha \in I(B_n)\} \cup \left\{ {}^{(i,j)}\zeta_{(k,k)} \middle| \, i, j, k \in [n] \right\}$.*

Proof.

1. In view of Proposition 3.1.1, it is sufficient to show that n-support elements are not regular. For $f \in A^+(B_n)_n$, if there is a $g \in A^+(B_n)$ such that $f+g+f = f$, then, by Proposition 2.2.3, $|\mathrm{supp}(f+g+f)| \leq 1$; a contradiction. Hence, f is not regular.

2. Since $I(B_n) = \{(k,k) \mid k \in [n]\} \cup \{\vartheta\}$, by Proposition 3.1.1,

$$I(\mathcal{C}_{B_n}) = \{\xi_\alpha \mid \alpha \in I(B_n)\}$$

and

$$I(A^+(B_n)_1) = \left\{ {}^{(i,j)}\zeta_{(k,k)} \,\middle|\, i,j,k \in [n] \right\}.$$

Further, for $f \in A^+(B_n)_n$, since $|\text{supp}(f+f)| = 1$ (cf. Proposition 2.2.3), $f+f$ cannot be f. Hence, the idempotents of $A^+(B_n)^+$ are merely in $\mathcal{C}_{B_n} \cup A^+(B_n)_1$.

$\square$

Corollary 3.1.13. *The semigroup $A^+(B_n)^+$ has n^3+n+1 idempotents and n^4+n^2+1 regular elements.*

For $n \geq 2$, since n-support elements in $A^+(B_n)^+$ are not regular, the semigroup $A^+(B_n)^+$ is not a regular semigroup. However, in the following proposition, we prove that $A^+(B_n)^+$ is eventually regular, i.e. for every $f \in A^+(B_n)^+$, we observe that there is a number m such that mf $(= f + \cdots + f$ for m times) is regular [Edwards, 1983].

Proposition 3.1.14. *The semigroup $A^+(B_n)^+$ is eventually regular.*

Proof. In view of Theorem 3.1.12, it remains to show that, for each $f \in A^+(B_n)_n$, there is a number m such that mf is regular. Now, for $f \in A^+(B_n)_n$, since $f + f \in A^+(B_n)_1$ (cf. Proposition 2.2.3), we have $2f$ is regular. Hence, the semigroup $A^+(B_n)^+$ is eventually regular. $\square$

For $n \geq 2$, let K be the set of regular elements in $A^+(B_n)^+$. By Theorem 3.1.12, $K = A^+(B_n) \setminus A^+(B_n)_n$. Further, by Theorem 3.1.12(2), all the idempotents of $A^+(B_n)^+$ are in K. We prove the following theorem concerning the set K.

Theorem 3.1.15. *The semigroup $(K, +)$ is an inverse semigroup.*

Proof. Let $f, g \in K$. If one of them is the zero map, then $|\text{supp}(f + g)| = 0$. If one of them is of singleton support, then by Lemma 1.1.58, $|\text{supp}(f + g)| \leq 1$. If both f and g are of full support, then $|\text{supp}(f + g)| = n^2 + 1$ or 0. Thus, in any case, $f + g \in K$. Hence, $(K, +)$ is a regular semigroup.

Referring to Theorem 1.1.44, it is sufficient to show that the idempotents in K commute. Let $f, g \in I(K)$. If one of them is the zero map, then $f+g = g+f = \xi_\vartheta$. Otherwise, we have the following cases.

Case 1 $|\text{supp}(f)| = |\text{supp}(g)|$. Then,

$$f+g = g+f = \begin{cases} f & \text{if } f = g; \\ \xi_\vartheta & \text{otherwise.} \end{cases}$$

Case 2 $|\text{supp}(f)| \neq |\text{supp}(g)|$. Say, $|\text{supp}(f)| = n^2+1$ and $|\text{supp}(g)| = 1$. Then,

$$f+g = g+f = \begin{cases} g & \text{if } \text{Im}(f) = \text{Im}(g) \setminus \{\vartheta\}; \\ \xi_\vartheta & \text{otherwise.} \end{cases}$$

Thus, $I(K)$ is commutative. Hence, $(K,+)$ is an inverse semigroup $\square$

3.2 The semigroup $A^+(B_n)^\circ$

In the rest of the thesis, the multiplicative semigroup reduct $(A^+(B_n), \circ)$ of the affine near-semiring $(A^+(B_n), +, \circ)$ is denoted by $A^+(B_n)^\circ$. Further, in a particular context, if there is no emphasis on the semigroup, we may simply write $A^+(B_n)$.

3.2.1 Green's classes

As mentioned earlier in the case of $A^+(B_n)^+$, the Green's relations $\mathcal{J}$ and $\mathcal{D}$ coincide also on the semigroup $A^+(B_n)^\circ$. In this subsection, we study the Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ on $A^+(B_n)^\circ$.

It is easy to observe that $A^+(B_n)_1 \cup \{\xi_\vartheta\}$ and $\mathcal{C}_{B_n}$ are subsemigroups of $A^+(B_n)^\circ$. Further, in this subsection, we prove that $A^+(B_n)_n \cup \{\xi_\vartheta\}$ is also a subsemigroup of $A^+(B_n)^\circ$. We begin with the structural properties these subsemigroups of $A^+(B_n)^\circ$.

Remark 3.2.1. The semigroup $(\mathcal{C}_{B_n}, \circ)$ has right zero multiplication, i.e. $fg = g$, for all $f, g \in \mathcal{C}_{B_n}$. Consequently, $(\mathcal{C}_{B_n}, \circ)$ is regular.

Proposition 3.2.2. *The semigroup* $(A^+(B_n)_1 \cup \{\xi_\vartheta\}, \circ)$ *is isomorphic to* $(B_{n^2}, +)$. *Hence,* $(A^+(B_n)_1 \cup \{\xi_\vartheta\}, \circ)$ *is regular.*

Proof. Clearly, the assignment ${}^{(k,l)}\zeta_{(p,q)} \mapsto ((k,l),(p,q))$ and $\xi_\vartheta \mapsto \vartheta$ is a semigroup isomorphism from $(A^+(B_n)_1 \cup \{\xi_\vartheta\}, \circ)$ to $(B_{n^2}, +)$. Hence, since the semigroup $(B_{n^2}, +)$ is regular, $(A^+(B_n)_1 \cup \{\xi_\vartheta\}, \circ)$ is regular. $\square$

Lemma 3.2.3. *For* $n \geq 2$, *let* $f, g_i (1 \leq i \leq k) \in A^+(B_n) \setminus \{\xi_\vartheta\}$ *such that* $f = g_1 g_2 \cdots g_k$. *Then,* $f \in A^+(B_n)_n$ *if and only if* $g_i \in A^+(B_n)_n$ *for all* i.

Proof. Suppose $f \in A^+(B_n)_n$. Clearly, $g_i \notin \mathcal{C}_{B_n}$ for all i. If $g_j \in A^+(B_n)_1$, for some j, then clearly $|\text{supp}(f)| \leq 1$. Hence, by Theorem 2.2.10, we have $g_i \in A^+(B_n)_n$ for all i. Conversely, for $1 \leq i \leq k$, suppose $g_i = (p_i, q_i; \sigma_i)$. For $1 \leq i \leq k-1$, note that $g_i g_{i+1}$ is either ξ_ϑ or $(p_i, q_{i+1}; \sigma_i \sigma_{i+1})$, where $q_i = p_{i+1}$. Consequently, since $f \neq \xi_\vartheta$, we have $f = (p_1, q_k; \sigma_1 \sigma_2 \cdots \sigma_k) \in A^+(B_n)_n$. $\square$

In view of Lemma 3.2.3, $A^+(B_n)_n \cup \{\xi_\vartheta\}$ is a subsemigroup of $A^+(B_n)^\circ$. Further, we have the following proposition regarding $A^+(B_n)_n \cup \{\xi_\vartheta\}$.

Proposition 3.2.4. *The semigroup* $A^+(B_n)_n \cup \{\xi_\vartheta\}$ *is isomorphic to the semigroup* $(B(S_n, n), +)$.

Proof. Note that the assignment $(i, j; \sigma) \mapsto (i, \sigma, j)$ and $\xi_\vartheta \mapsto \vartheta$, for all $i, j \in [n]$ and $\sigma \in S_n$, is an isomorphism. $\square$

Lemma 3.2.5. *If* g *is a nonconstant map in* $A^+(B_n)$, *then* $\text{supp}(fg) \subseteq \text{supp}(f)$.

Proof. If fg is the zero map, then the result is straightforward. Let $fg \neq \xi_\vartheta$ and $\alpha \in \text{supp}(fg)$. Then, $\vartheta \neq \alpha(fg) = (\alpha f)g$ so that $\alpha f \in \text{supp}(g)$. Since g is not a constant map, by Proposition 2.2.1, $\alpha f \neq \vartheta$ so that $\alpha \in \text{supp}(f)$. Hence, $\text{supp}(fg) \subseteq \text{supp}(f)$. $\square$

We present a characterization of the Green's relation $\mathcal{R}$ on $A^+(B_n)^\circ$ in the following theorem.

Theorem 3.2.6. *For $f, g \in A^+(B_n)^\circ$, we have*

1. *if $f, g \in \mathcal{C}_{B_n}$, then $f\mathcal{R}g$;*

2. *if $f, g \notin \mathcal{C}_{B_n}$, then $f\mathcal{R}g \iff \mathrm{supp}(f) = \mathrm{supp}(g)$.*

Moreover, for $n \geq 2$, the number of $\mathcal{R}$-classes in $A^+(B_n)^\circ$ is $n^2 + n + 1$.

Proof.

1. By Remark 3.2.1, any two elements of $\mathcal{C}_{B_n}$ are $\mathcal{R}$-related. Thus, $\mathcal{C}_{B_n}$ has a single $\mathcal{R}$-class containing all the constant maps.

2. If $f\mathcal{R}g$ with $f \neq g$, then there exist $h, h' \in A^+(B_n)$ such that $fh = g$ and $gh' = f$. Note that h and h' are nonconstant maps; otherwise, f and g will be constant maps. Now, by Lemma 3.2.5, $\mathrm{supp}(g) = \mathrm{supp}(fh) \subseteq \mathrm{supp}(f)$ and $\mathrm{supp}(f) = \mathrm{supp}(gh') \subseteq \mathrm{supp}(g)$. Hence, $\mathrm{supp}(f) = \mathrm{supp}(g)$.

   Conversely, suppose $\mathrm{supp}(f) = \mathrm{supp}(g)$. If $f, g \in A^+(B_n)_1$, by Remark 1.1.53 and Proposition 3.2.2, we get $f\mathcal{R}g$. Consequently, $A^+(B_n)_1$ has n^2 $\mathcal{R}$-classes each of size n^2. On the other hand, $f, g \in A^+(B_n)_n$. Then, by Proposition 3.2.4 and Lemma 1.1.51, we have $f\mathcal{R}g$. Consequently, $A^+(B_n)_n$ contains n $\mathcal{R}$-classes each of size $(n!)n$.

Hence, for $n \geq 2$, the number of $\mathcal{R}$-classes in $A^+(B_n)^\circ$ is $n^2 + n + 1$. □

We present a characterization of the Green's relation $\mathcal{L}$ on $A^+(B_n)^\circ$ in the following theorem.

Theorem 3.2.7. *For $f, g \in A^+(B_n)^\circ$, $f\mathcal{L}g \iff \mathrm{Im}(f) = \mathrm{Im}(g)$. Moreover, for $n \geq 2$, the number of $\mathcal{L}$-classes in $A^+(B_n)^\circ$ is $2n^2 + n + 1$.*

Proof. If $f\mathcal{L}g$ with $f \neq g$, then there exist $h, h' \in A^+(B_n)$ such that $hf = g$ and $h'g = f$. Since $\mathrm{Im}(g) = \mathrm{Im}(hf) \subseteq \mathrm{Im}(f)$ and $\mathrm{Im}(f) = \mathrm{Im}(h'g) \subseteq \mathrm{Im}(g)$, we have $\mathrm{Im}(f) = \mathrm{Im}(g)$.

Conversely, suppose $\mathrm{Im}(f) = \mathrm{Im}(g)$ so that $|\mathrm{supp}(f)| = |\mathrm{supp}(g)|$. Clearly, the $\mathcal{L}$-classes in $\mathcal{C}_{B_n}$ are singletons. Consequently, $\mathcal{C}_{B_n}$ has $n^2 + 1$ $\mathcal{L}$-classes. If $f, g \in A^+(B_n)_1$, by Remark 1.1.53 and Proposition 3.2.2, we have $f\mathcal{L}g$ and hence, there are n^2 $\mathcal{L}$-classes in $A^+(B_n)_1$ each is of size n^2. Otherwise, $f, g \in A^+(B_n)_n$. Then, by Proposition 3.2.4 and Lemma 1.1.51, we have $f\mathcal{L}g$. Consequently, we have n $\mathcal{R}$-classes containing n-support elements each is of size $(n!)n$.

Hence, for $n \geq 2$, the number of $\mathcal{L}$-classes in $A^+(B_n)^\circ$ is $2n^2 + n + 1$. ☐

Following characterization of the Green's relation $\mathcal{H}$ on $A^+(B_n)^\circ$ is a corollary of theorems 3.2.6 and 3.2.7.

Corollary 3.2.8. *For $f, g \in A^+(B_n)^\circ$, $f\mathcal{H}g$ if and only if $\mathrm{Im}(f) = \mathrm{Im}(g)$ and $\mathrm{supp}(f) = \mathrm{supp}(g)$. Moreover, for $n \geq 2$, the number of $\mathcal{H}$-classes in $A^+(B_n)^\circ$ is $n^4 + 2n^2 + 1$.*

We characterize the Green's relation $\mathcal{D}$ in the following theorem.

Theorem 3.2.9. *For $f, g \in A^+(B_n)^\circ$, $f\mathcal{D}g \iff |\mathrm{supp}(f)| = |\mathrm{supp}(g)|$ or $f, g \in \mathcal{C}_{B_n}$. Hence, for $n \geq 2$, the number of $\mathcal{D}$-classes in $A^+(B_n)^\circ$ is 3.*

Proof. For $f, g \in A^+(B_n)^\circ$, observe that

$$\begin{aligned}
f\mathcal{D}g \;&\Rightarrow\; \text{there exists } h \in A^+(B_n)^\circ \text{ such that } f\mathcal{L}h \text{ and } h\mathcal{R}g \\
&\Rightarrow\; \mathrm{Im}(f) = \mathrm{Im}(h) \text{ and } h\mathcal{R}g \text{ (by Theorem 3.2.7)} \\
&\Rightarrow\; |\mathrm{supp}(f)| = |\mathrm{supp}(h)| \text{ and (either } \mathrm{supp}(h) = \mathrm{supp}(g) \text{ or } h, g \in \mathcal{C}_{B_n}) \\
&\qquad \text{(by Theorem 3.2.6)} \\
&\Rightarrow\; \text{either } |\mathrm{supp}(f)| = |\mathrm{supp}(g)| \text{ or } f, g \in \mathcal{C}_{B_n}
\end{aligned}$$

Conversely, if $f, g \in \mathcal{C}_{B_n}$, then by Theorem 3.2.6, $f\mathcal{R}g$ so that $f\mathcal{D}g$. If $f, g \in A^+(B_n)_1$, then, by Remark 1.1.53 and Proposition 3.2.2, $f\mathcal{D}g$. Finally, let $f, g \in A^+(B_n)_n$. Then, by Proposition 3.2.4 and Remark 1.1.52, we have $f\mathcal{D}g$.

Hence, for $n \geq 2$, $A^+(B_n)^{\circ}$ has three $\mathcal{D}$-classes, viz. $\mathcal{C}_{B_n}$, $A^+(B_n)_1$ and $A^+(B_n)_n$. $\square$

3.2.2 Regular elements and idempotents

In this subsection, we characterize the regular and idempotent elements in $A^+(B_n)^{\circ}$ and ascertain that $A^+(B_n)^{\circ}$ is regular. Moreover, it is an orthodox semigroup. We observe that the set excluding the full support elements in $A^+(B_n)^{\circ}$ forms an inverse semigroup.

Theorem 3.2.10. *The semigroup $A^+(B_n)^{\circ}$ is regular.*

Proof. The result follows from Remark 3.2.1, Proposition 3.2.2 and Proposition 3.2.4. $\square$

Now, in the following theorem, we identify the idempotent elements in $A^+(B_n)^{\circ}$ and count their number.

Theorem 3.2.11. *For $n \geq 2$,*

$$I(A^+(B_n)^{\circ}) = \{\xi_\alpha \mid \alpha \in B_n\} \cup \{(k, k; id) \mid k \in [n]\} \cup \left\{ {}^{(i,j)}\zeta_{(i,j)} \middle| \, i, j \in [n] \right\}.$$

Hence, $|I(A^+(B_n)^{\circ})| = 2n^2 + n + 1$.

Proof. By Remark 3.2.1, clearly, all the $n^2 + 1$ elements of $(\mathcal{C}_{B_n}, \circ)$ are idempotents. Since nonzero idempotents in B_{n^2} are of the form $((i, j), (i, j))$, we have, n^2 elements of the form ${}^{(i,j)}\zeta_{(i,j)}$ are idempotents in $A^+(B_n)_1$. Observe that the idempotent elements in $A^+(B_n)_n$ are of the form $(k, k; id)$, where $k \in [n]$ and id is the identity permutation on $[n]$. Thus, there are n idempotent elements in $A^+(B_n)_n$. Hence, for $n \geq 2$, the number of idempotents in $A^+(B_n)^{\circ}$ is $2n^2 + n + 1$. $\square$

Theorem 3.2.12. *The semigroup $A^+(B_n)^\circ$ is orthodox.*

Proof. In view of Theorem 3.2.10, it is sufficient to prove that $I(A^+(B_n)^\circ)$ is a subsemigroup of $A^+(B_n)^\circ$. Let $f, g \in I(A^+(B_n)^\circ)$. Note that if f or g is a constant map, then fg is also a constant map and hence, fg is an idempotent element. Otherwise, we consider the following cases to show that $fg \in I(A^+(B_n)^\circ)$.

Case 1 $f, g \in A^+(B_n)_l$, for $l \in \{1, n\}$. It can be observed that if $f = g$, then $fg = f$; otherwise, $fg = \xi_\vartheta$.

Case 2 $f = {}^{(i,j)}\zeta_{(i,j)}$ and $g = (k, k; id)$. Observe that if $j = k$, then $fg = gf = f$; otherwise, $fg = gf = \xi_\vartheta$.

Thus, the set $I(A^+(B_n)^\circ)$ is closed with respect to composition. Hence, $A^+(B_n)^\circ$ is an orthodox semigroup. □

For $n \geq 2$, let $N = A^+(B_n) \setminus A^+(B_n)_{n^2+1}$. If $f, g \in N$, then by Proposition 2.2.1, $\vartheta \notin \mathrm{supp}(f) \cap \mathrm{supp}(g)$. Hence, $\vartheta(fg) = \vartheta$ so that $|\mathrm{supp}(fg)| \neq n^2 + 1$. Thus, N is closed with respect to composition. From the proof of Theorem 3.2.10, it can be observed that $(N, \circ)$ is regular. Also, from the proof of Theorem 3.2.12, the set $I(N)$ is closed with respect to composition. Further, note that $(I(N), \circ)$ is a commutative semigroup. Hence, we have the following theorem.

Theorem 3.2.13. *The semigroup $(N, \circ)$ is an inverse semigroup.*

3.3 An example: $A^+(B_2)$

In this section, we illustrate our results using the affine near-semiring $A^+(B_2)$. First note that $A^+(B_2)$ has 29 elements (cf. Theorem 2.2.10). The number of $\mathcal{D}$-classes in $A^+(B_2)^+$ is 10 and it is 3 in $A^+(B_2)^\circ$ (cf. Corollary 3.1.8 and Theorem 3.2.9). The number of $\mathcal{L}$-classes in $A^+(B_2)^+$ is 19 and it is 11 in $A^+(B_2)^\circ$ (cf. Corollary 3.1.8 and Theorem 3.2.7). The number of $\mathcal{R}$-classes in $A^+(B_2)^+$ is 15 and it is 7 in $A^+(B_2)^\circ$

$* \ \xi_\vartheta$

$* \ \xi_{(1,1)}$	$\xi_{(1,2)}$
$\xi_{(2,1)}$	$* \ \xi_{(2,2)}$

$(1,1;id)$	$(1,2;id)$

$(1,1;\sigma)$	$(1,2;\sigma)$

$(2,1;id)$	$(2,2;id)$

$(2,1;\sigma)$	$(2,2;\sigma)$

$* \ {}^{(1,1)}\zeta_{(1,1)}$	${}^{(1,1)}\zeta_{(1,2)}$
${}^{(1,1)}\zeta_{(2,1)}$	$* \ {}^{(1,1)}\zeta_{(2,2)}$

$* \ {}^{(1,2)}\zeta_{(1,1)}$	${}^{(1,2)}\zeta_{(1,2)}$
${}^{(1,2)}\zeta_{(2,1)}$	$* \ {}^{(1,2)}\zeta_{(2,2)}$

$* \ {}^{(2,1)}\zeta_{(1,1)}$	${}^{(2,1)}\zeta_{(1,2)}$
${}^{(2,1)}\zeta_{(2,1)}$	$* \ {}^{(2,1)}\zeta_{(2,2)}$

$* \ {}^{(2,2)}\zeta_{(1,1)}$	${}^{(2,2)}\zeta_{(1,2)}$
${}^{(2,2)}\zeta_{(2,1)}$	$* \ {}^{(2,2)}\zeta_{(2,2)}$

$A^+(B_2)^+$

$* \ \xi_\vartheta$	$* \ \xi_{(1,1)}$	$* \ \xi_{(1,2)}$	$* \ \xi_{(2,1)}$	$* \ \xi_{(2,2)}$

$(1,1;\sigma)$, $* \ (1,1;id)$	$(1,2;id)$, $(1,2;\sigma)$
$(2,1;id)$, $(2,1;\sigma)$	$(2,2;\sigma)$, $* \ (2,2;id)$

$* \ {}^{(1,1)}\zeta_{(1,1)}$	${}^{(1,1)}\zeta_{(1,2)}$	${}^{(1,1)}\zeta_{(2,1)}$	${}^{(1,1)}\zeta_{(2,2)}$
${}^{(1,2)}\zeta_{(1,1)}$	$* \ {}^{(1,2)}\zeta_{(1,2)}$	${}^{(1,2)}\zeta_{(2,1)}$	${}^{(1,2)}\zeta_{(2,2)}$
${}^{(2,1)}\zeta_{(1,1)}$	${}^{(2,1)}\zeta_{(1,2)}$	$* \ {}^{(2,1)}\zeta_{(2,1)}$	${}^{(2,1)}\zeta_{(2,2)}$
${}^{(2,2)}\zeta_{(1,1)}$	${}^{(2,2)}\zeta_{(1,2)}$	${}^{(2,2)}\zeta_{(2,1)}$	$* \ {}^{(2,2)}\zeta_{(2,2)}$

$A^+(B_2)^\circ$

The permutation σ maps $1 \mapsto 2$ and $2 \mapsto 1$; and id is the identity map on $[2]$.

FIGURE 3.1: Egg-box Diagrams for $A^+(B_2)^+$ (left) and $A^+(B_2)^\circ$ (right)

(cf. Corollary 3.1.8 and Theorem 3.2.6). Since the $\mathcal{H}$-relation is trivial on $A^+(B_2)^+$, all the 29 elements are in 29 different classes. The number of $\mathcal{H}$-classes in $A^+(B_2)^\circ$ is 25 (cf. Corollary 3.2.8). All this information along with the respective Green's classes of both the semigroups $A^+(B_2)^+$ and $A^+(B_2)^\circ$ are shown in Figure 3.1 using egg-box diagrams. Here, following the notations/representaions introduced in this thesis, the elements of $A^+(B_2)$ are displayed with their supports and images. Thus, the characterizations of the respective Green's relations can also be crosschecked in this figure. Further, in the figure, the idempotents elements in these semigroups are marked with a * on their left-top corner.

4

Rank Properties

In order to study the rank properties of a finite semigroup, Howie and Ribeiro [1999, 2000] have considered the notions of small rank, lower rank, intermediate rank, upper rank and large rank of a finite semigroup. Many authors have studied the rank properties of various semigroups (e.g. Cameron and Cara [2002]; Gomes and Howie [1987, 1992]; Minisker [2009]; Mitchell [2002]; Ruškuc [1994]). In this chapter, we investigate the rank properties of both the semigroup reducts of $A^+(B_n)$. After recalling the requisite notions in Section 4.1, we introduce a novel approach to find the large rank of a finite semigroup in Section 4.2. We obtain the five ranks for the additive semigroup reduct in Section 4.3. The rank properties of the multiplicative semigroup reduct are investigated in Section 4.4.

4.1 Ranks of a semigroup

In this section, we recall various types of ranks of a finite semigroup from [Howie and Ribeiro, 1999, 2000]. These ranks depend on the concept of independent sets in general algebras by Marczewski [1966]. We will begin with the notion of independent set in semigroups.

Definition 4.1.1. A subset U of a semigroup S is said to be *independent* if every element of U is not in the subsemigroup generated by the remaining elements of U, i.e.

$$\forall a \in U,\ a \notin \langle U \setminus \{a\} \rangle.$$

Remark 4.1.2. Though the notion of independence is analogous to that of in linear algebra, the minimum size of a generating set need not be equal to the maximum size of an independent set in general algebras, in particular in a semigroup. Please see Example 4.1.3.

Example 4.1.3. The subset $\{\overline{2}, \overline{3}\}$ of the (additive) cyclic group $\mathbb{Z}_6$ is independent, but the minimum cardinality of a generating set for $\mathbb{Z}_6$ is one.

In view of Remark 4.1.2, Howie and Ribeiro [1999, 2000] have considered the following possible definitions of ranks for a finite semigroup.

Definition 4.1.4. The *ranks* of a finite semigroup S are defined as follows.

1. $r_1(S) = \max\{k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ is independent}\}$,
2. $r_2(S) = \min\{|U| : U \subseteq S, \langle U \rangle = S\}$,
3. $r_3(S) = \max\{|U| : U \subseteq S, \langle U \rangle = S, U \text{ is independent}\}$,
4. $r_4(S) = \max\{|U| : U \subseteq S, U \text{ is independent}\}$,
5. $r_5(S) = \min\{k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ generates } S\}$.

The following proposition by Howie and Ribeiro [1999, 2000] is immediate from Definition 4.1.4.

Proposition 4.1.5. *For a finite semigroup S, we have*

$$r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S).$$

In view of Proposition 4.1.5, the following nomenclature is adopted by Howie and Ribeiro [1999, 2000] for the ranks of a finite semigroup.

Definition 4.1.6. For a finite semigroup S, the ranks $r_1(S), r_2(S), r_3(S), r_4(S)$ and $r_5(S)$ are, respectively, known as *small rank*, *lower rank*, *intermediate rank*, *upper rank* and *large rank* of S. The lower rank is commonly known as the *rank* of a semigroup.

Further, we use the following terminology in this chapter.

Notation 4.1.7.

1. An independent set with maximum cardinality is called as *maximum independent set*.

2. An independent generating with maximum cardinality is called as *maximum independent generating set*.

3. A generating set with minimum cardinality will be termed as a *minimum generating set*.

4.2 A novel approach for large rank

We consider the complementary concept of subsemigroups, called prime subsets, and give an approach to find the large rank of a finite semigroup. In this section, S denotes a finite multiplicative semigroup.

Definition 4.2.1. An element a of a semigroup S is said to be *indecomposable* if there do not exist $b, c \in S \setminus \{a\}$ such that $a = bc$.

The following result by Howie and Ribeiro provides us the connection between the size of ceratin subsemigroups and the large rank of a finite semigroup.

Theorem 4.2.2 ([Howie and Ribeiro, 2000]). *Let S be a finite semigroup and let V be a proper subsemigroup of S with the largest possible size. Then $r_5(S) = |V| + 1$. Hence, $r_5(S) = |S|$ if and only if S contains an indecomposable element.*

Using Theorem 4.2.2, Howie and Ribeiro obtained the large rank of Brandt semigroup $B(G, n)$ in a graph theoretic approach. Though their approach reveals nice connection between digraphs and Brandt semigroups ([Howie and Ribeiro, 2000]), the approach has its own limitations with respect to semigroups of transformations. In a complementary approach using prime subsets of a semigroup, we propose to overcome such a limitation.

Definition 4.2.3. A nonempty subset U of a semigroup S is said to be *prime* if, for all $a, b \in S$,

$$ab \in U \text{ implies } a \in U \text{ or } b \in U.$$

Proposition 4.2.4. *Let V be a proper subset of a finite semigroup S. Then V is a smallest prime subset of S if and only if $S \setminus V$ is a largest subsemigroup of S.*

Proof. Note that $S \setminus V$ is not a subsemigroup of S if and only if there exist $a, b \in S \setminus V$ such that $ab \notin S \setminus V$ if and only if there exist $a, b \notin V$ such that $ab \in V$ if and only if V is not a prime subset. Now, for any $X, Y \subset S$, since $|X| + |S \setminus X| = |Y| + |S \setminus Y|$, we have $|X| < |Y|$ if and only if $|S \setminus Y| < |S \setminus X|$. Hence, we have the result. $\square$

In view of Theorem 4.2.2, we have the following corollary of Proposition 4.2.4.

Corollary 4.2.5. *Let V be a smallest proper prime subset of a finite semigroup S. Then $r_5(S) = |S \setminus V| + 1$.*

Thus, the problem of finding the large rank of a finite semigroup is now reduced to the problem of finding a smallest proper prime subset of the semigroup. We use this technique to find the large ranks of both the semigroup reducts of $A^+(B_n)$ in sections 4.3 and 4.4.

Other than the semigroups of transformations considered in this thesis, using this technique, the large rank for the semigroup of order-preserving singular maps is obtained in [Kumar and Krishna, 2014b]. Further, in that paper, it is observed that the approach gives much shorter proof for the large rank of Brandt semigroup $B(G, n)$ than the proof given by Howie and Ribeiro [2000].

4.3 The semigroup $A^+(B_n)^+$

In this section, after establishing certain useful properties for minimum generating set and for maximum independent generating set, we investigates the rank properties of the semigroup $A^+(B_n)^+$.

4.3.1 Small rank and lower rank

In this subsection, after quickly ascertaining the small rank r_1 of $A^+(B_n)^+$, we will obtain its lower rank r_2. In view of Remark 2.2.11, it can be easily observed that $A^+(B_1)$ is an independent set and none of its proper subsets generates $A^+(B_1)$. Hence, for $1 \leq i \leq 5$, we have

$$r_i(A^+(B_1)^+) = |A^+(B_1)| = 3.$$

In the rest of the section, we shall investigate the ranks of $A^+(B_n)^+$, for $n > 1$.

The small rank of $A^+(B_n)^+$ comes as a consequence of the following result due to Howie and Ribeiro.

Theorem 4.3.1 ([Howie and Ribeiro, 2000]). *Let S be a finite semigroup, with $|S| \geq 2$. If S is not a band, then $r_1(S) = 1$.*

Owing to the fact that ${A^+(B_n)}^+$ (for $n \geq 2$) have some non idempotent elements, it is not a band. For instance, the constant maps $\xi_{(p,q)}$ with $p \neq q$ in ${A^+(B_n)}^+$ are not idempotent. Hence, we have the following corollary of Theorem 4.3.1.

Corollary 4.3.2. *For $n \geq 2$, $r_1({A^+(B_n)}^+) = 1$.*

Now, in the remaining subsection, we construct a minimum generating set of ${A^+(B_n)}^+$ and obtain its lower rank in Theorem 4.3.9. Consider the subsets

$$\mathcal{S} = \{\xi_{(i,i+1)} \mid i \in [n-1]\} \cup \{\xi_{(n,1)}\}$$

and

$$\mathcal{T} = \{g + h \mid g \in \mathrm{Aut}(B_n),\ h \in \mathcal{S}\}$$

of $A^+(B_n)$. We develop a proof of Theorem 4.3.9 through a sequence of lemmas by showing that the set $\mathcal{S} \cup \mathcal{T}$ serves our purpose.

Lemma 4.3.3. *For $n \geq 2$, $\langle \mathcal{S} \rangle = \mathcal{C}_{B_n}$.*

Proof. Let $f \in \mathcal{C}_{B_n}$; then, either $f = \xi_\vartheta$ or $f = \xi_{(i,j)}$. If $f = \xi_\vartheta$, then, for $p \in [n-1]$, write $\xi_\vartheta = \xi_{(p,p+1)} + \xi_{(p,p+1)}$ so that $f \in \langle \mathcal{S} \rangle$. If $f = \xi_{(i,j)}$, then, for $i < j$, we have

$$\xi_{(i,j)} = \xi_{(i,i+1)} + \xi_{(i+1,i+2)} + \cdots + \xi_{(j-1,j)},$$

and, for $i \geq j$,

$$\xi_{(i,j)} = \xi_{(i,i+1)} + \xi_{(i+1,i+2)} + \cdots + \xi_{(n-1,n)} + \xi_{(n,1)} + \xi_{(1,2)} + \cdots + \xi_{(j-1,j)}$$

so that $f \in \langle \mathcal{S} \rangle$. $\square$

Lemma 4.3.4. *If $X \subseteq A^+(B_n)$ such that $\langle X \rangle = \mathcal{C}_{B_n}$, then $\langle X \cup \mathcal{T} \rangle = {A^+(B_n)}^+$.*

Proof. Since the constant maps of $A^+(B_n)$ are generated by X, in view of Theorem 2.2.10, it is sufficient to prove that the n-support and singleton support elements are generated by $X \cup \mathcal{T}$. However, since every singleton support map is a sum of a constant map and an n-support map (cf. Remark 2.2.6), we will now observe that $X \cup \mathcal{T}$ generates the n-support maps of $A^+(B_n)$.

Let $f \in A^+(B_n)_n$. By Remark 2.1.10 and Corollary 2.2.9, $f = g + \xi_c$ for some $g \in \mathrm{Aut}(B_n)$ and $c \in B_n \setminus \{\vartheta\}$. By Lemma 4.3.3, write $\xi_c = \sum_{i=1}^{k} f_i$, for some f_i's from $\mathcal{S}$ so that

$$f = g + \sum_{i=1}^{k} f_i = g + f_1 + \sum_{i=2}^{k} f_i.$$

Note that $g + f_1 \in \mathcal{T}$ and each f_i is a sum of elements of X. Hence, $f \in \langle X \cup \mathcal{T} \rangle$. $\square$

Lemma 4.3.5. *For $g_1, g_2, g_3 \in \mathrm{Aut}(B_n)$ with $g_2 \neq g_3$ and $p, q, s, t \in [n]$ such that $p \neq s$, we have the following.*

1. *If $f_1 = g_1 + \xi_{(p,q)}$ and $f_1' = g_1 + \xi_{(s,t)}$, then $f_1 \neq f_1'$.*

2. *If $f_2 = g_2 + \xi_{(p,q)}$ and $f_3 = g_3 + \xi_{(p,q)}$, then $f_2 \neq f_3$.*

Hence, $|\mathcal{T}| = n(n!)$.

Proof. As per Corollary 2.1.3, for $1 \leq i \leq 3$, let $g_i = \phi_{\sigma_i}$ so that f_1, f_1', f_2, f_3 are n-support maps represented by

$$(p\sigma_1^{-1}, q; \sigma_1), (s\sigma_1^{-1}, t; \sigma_1), (p\sigma_2^{-1}, q; \sigma_2) \text{ and } (p\sigma_3^{-1}, q; \sigma_3),$$

respectively (cf. Remark 2.1.10).

(1) Since $p \neq s$ and σ_1 is a permutation on $[n]$, f_1 and f_1' have different support so that $f_1 \neq f_1'$.

(2) If $p\sigma_2^{-1} \neq p\sigma_3^{-1}$, then we are done. Otherwise, since $\sigma_2 \neq \sigma_3$, there exists $i_0 \in [n]$ such that $i_0\sigma_2 \neq i_0\sigma_3$. Now,

$$(i_0, p\sigma_2^{-1})f_2 = (i_0\sigma_2, q) \neq (i_0\sigma_3, q) = (i_0, p\sigma_3^{-1})f_3$$

so that $f_2 \neq f_3$.

Now, since $|\mathrm{Aut}(B_n)| = n!$ (cf. Corollary 2.1.3) and there are n elements in $\mathcal{S}$, we have $|\mathcal{T}| = n(n!)$.

$\square$

Lemma 4.3.6. *For $n \geq 2$ and $1 \leq i \leq k$, let $f, f_i \in A^+(B_n)$ such that $f = \sum_{i=1}^{k} f_i$.*

1. *If $f \in A^+(B_n)_{n^2+1}$, then $f_1 \in R_f$ and $f_k \in L_f$.*

2. *If $f \in A^+(B_n)_n$, then $f_1 \in R_f$.*

Proof. (1) If $f \in A^+(B_n)_{n^2+1}$, then $f_i \in A^+(B_n)_{n^2+1}$, for all i (cf. Lemma 1.1.58). Let $f = \xi_{(p,q)}$ and $f_i = \xi_{(p_i,q_i)}$ so that

$$\xi_{(p,q)} = \sum_{i=1}^{k} \xi_{(p_i,q_i)}.$$

Then clearly, for $1 \leq i \leq k-1$, $q_i = p_{i+1}$ and $p_1 = p$, $q_k = q$. Hence, by Theorem 3.1.6, $f_1 \in R_f$ and $f_k \in L_f$.

(2) If $f \in A^+(B_n)_n$, then $|\text{supp}(f_i)| \geq n$, for all i (cf. Lemma 1.1.58). Then, for each i, $|\text{supp}(f_i)| = n$ or n^2+1 (cf. Theorem 2.2.10). Note that, there exists j $(1 \leq j \leq k)$ such that $|\text{supp}(f_j)| = n$; otherwise, f will be a constant map. If $j \geq 2$, then, by Proposition 2.2.3, $|\text{supp}(f_{j-1} + f_j)| \leq 1$ so that $|\text{supp}(f)| \leq 1$; a contradiction. Thus, we have $j = 1$ and, for all $i > 1$, $|\text{supp}(f_i)| = n^2 + 1$.

Let $(k, q; \sigma)$ and $(k', p; \tau)$ be the representations of f and f_1, respectively, and

$$\sum_{i=2}^{k} f_i = \xi_{(r,s)}$$

for some $r, s \in [n]$. Then, note that $p = r$, $k' = k$, $\tau = \sigma$ and $s = q$. Thus, $f_1 = (k, r; \sigma)$ and $\sum_{i=2}^{k} f_i = \xi_{(r,q)}$ for some $r \in [n]$. Hence, by Theorem 3.1.7, $f_1 \in R_f$. □

Lemma 4.3.7. *Every generating subset of $A^+(B_n)^+$ contains at least*

1. *n elements of full support, and*

2. *$n(n!)$ elements of n-support.*

Proof. Let V be a generating subset of $A^+(B_n)^+$. For $f \in A^+(B_n)$, write $f = \sum_{i=1}^{k} f_i$, for some $f_i \in V$.

(1) If $f \in \mathcal{S}$, then by Lemma 4.3.6(1), $f_1 \in R_f$ so that $R_f \cap V \neq \varnothing$. Further, one can observe that if $g, h \in \mathcal{S}$ with $g \neq h$, then $R_g \cap R_h = \varnothing$ (cf. Theorem 3.1.6(1)). Hence, for every element $f \in \mathcal{S}$, the corresponding f_1 is in V with the above specified property. Consequently, since $\mathcal{S}$ has n full support elements, V will have at least n full support elements.

(2) If $f \in \mathcal{T}$, then by Lemma 4.3.6(2), we have $R_f \cap V \neq \varnothing$. For $g, h \in \mathcal{T}$ with $g \neq h$, we show that $R_g \cap R_h = \varnothing$. In view of Remark 2.1.10, by considering $n \mod n = n$, write $g = (r\sigma^{-1}, r+1 \mod n; \sigma)$ and $h = (s\rho^{-1}, s+1 \mod n; \rho)$. Since $g \neq h$, either $\sigma \neq \rho$ or $r \neq s$. Hence, by Theorem 3.1.7(1), $R_g \neq R_h$ in either case. Hence, for every element $f \in \mathcal{T}$, the corresponding f_1 is in V with the above specified property so that $|V| \geq |\mathcal{T}| = (n!)n$. □

In view of lemmas 4.3.3 and 4.3.4, we have the following corollary of Lemma 4.3.7.

Corollary 4.3.8. $|\mathcal{S} \cup \mathcal{T}|$ *is the minimum such that* $\langle \mathcal{S} \cup \mathcal{T} \rangle = A^+(B_n)^+$.

Combining the results from Lemma 4.3.3 through Corollary 4.3.8, we have the following main theorem of the subsection.

Theorem 4.3.9. *For* $n \geq 2$, $r_2(A^+(B_n)^+) = n(n! + 1)$.

4.3.2 Intermediate rank

In this subsection, after ascertaining certain relevant properties of independent generating sets of $A^+(B_n)^+$, we obtain its intermediate rank. We require the following theorem regarding the intermediate rank of B_n in the sequel.

Theorem 4.3.10 ([Howie and Ribeiro, 1999])**.** *For* $n \geq 2$, $r_3(B_n) = 2n - 2$.

Lemma 4.3.11. *Let U be an independent generating subset of $A^+(B_n)^+$; then*

1. $A^+(B_n)_1 \cap U = \varnothing$,

2. $|A^+(B_n)_n \cap U| = n(n!)$,

3. $n \leq |A^+(B_n)_{n^2+1} \cap U| \leq 2n-2$.

Hence, $|U| \leq n(n!) + 2n - 2$.

Proof.

1. Let $f = {}^{(k,l)}\zeta_{(p,q)} \in A^+(B_n)_1 \cap U$. Since U is a generating set, we have $\xi_{(p,r)} \in \langle U \setminus \{f\}\rangle$ and, for $k\sigma = r$, $(l, q; \sigma) \in \langle U \setminus \{f\}\rangle$ (cf. Lemma 1.1.58). Note that $f = \xi_{(p,r)} + (l, q; \sigma)$ so that $f \in \langle U \setminus \{f\}\rangle$; a contradiction to U is an independent set.

2. By Lemma 4.3.7(2), $|A^+(B_n)_n \cap U| \geq n(n!)$. Since $A^+(B_n)_n$ contains only $n(n!)$ $\mathcal{R}$-classes (cf. Corollary 3.1.8), if U contain more than $n(n!)$ elements of n-support, then there exist distinct $f, g \in A^+(B_n)_n \cap U$ such that $f\mathcal{R}g$. By Theorem 3.1.7(1), $f = (k, p; \sigma)$ and $g = (k, p'; \sigma)$, for some $\sigma \in S_n$. Note that, $g = f + \xi_{(p,p')}$, where $\xi_{(p,p')} \in \langle U \setminus \{g\}\rangle$, so that $g \in \langle U \setminus \{g\}\rangle$; a contradiction to independence of U. Hence, there are exactly $n(n!)$ elements of n-support in U.

3. By Lemma 4.3.7(1), $|A^+(B_n)_{n^2+1} \cap U| \geq n$. Since B_n is isomorphic to $\mathcal{C}_{B_n}$ (cf. Proposition 3.1.1), by Theorem 4.3.10, we have $|A^+(B_n)_{n^2+1} \cap U| \leq 2n-2$.

□

Theorem 4.3.12. *For $n \geq 2$, $r_3(A^+(B_n)^+) = n(n!) + 2n - 2$.*

Proof. First note that the set

$$\mathcal{S}' = \{\xi_{(1,i)} \mid 2 \leq i \leq n\} \cup \{\xi_{(j,1)} \mid 2 \leq j \leq n\}$$

generates all constant maps in $A^+(B_n)^+$. For instance, $\xi_\vartheta = \xi_{(1,2)} + \xi_{(1,2)}$ and $\xi_{(1,1)} = \xi_{(1,2)} + \xi_{(2,1)}$. Now, for $p, q \in [n]$, clearly $\xi_{(1,p)}$ and $\xi_{(q,1)} \in \langle \mathcal{S}' \rangle$. Further, since $\xi_{(p,q)} = \xi_{(p,1)} + \xi_{(1,q)}$, we have $\xi_{(p,q)} \in \langle \mathcal{S}' \rangle$.

Hence, by Lemma 4.3.4, the set $V = \mathcal{S}' \cup \mathcal{T}$ is a generating set of $A^+(B_n)^+$. We show that V is also an independent set. Since $|V| = n(n!) + 2n - 2$, by Lemma 4.3.11, the theorem follows.

V is an independent set: For $f \in V$, suppose $f = \sum_{i=1}^{k} f_i$ with $f_i \in V \setminus \{f\}$. Then, by Lemma 4.3.6, $f_1 \in R_f$. If $f \in \mathcal{S}' \cup \mathcal{T}$, in the following, we observe that $f = f_1$; which is a contradiction so that $f \notin \langle V \setminus \{f\} \rangle$.

If $f = \xi_{(q,1)} \in \mathcal{S}'$ (for some $2 \leq q \leq n$), then $f_1 = \xi_{(q,l)}$, for some $l \in [n]$ (cf. Theorem 3.1.6(1)). Hence, $f_1 = f$ (cf. construction of $\mathcal{S}'$). For $2 \leq p \leq n$, if $f = \xi_{(1,p)}$, the argument is similar.

If $f = (k, p; \sigma) \in \mathcal{T}$ (for some $k, p \in [n], \sigma \in S_n$), again by Theorem 3.1.7(1), $f_1 = (k, s; \sigma)$ for some $s \in [n]$. Consequently, $f = f_1$ (cf. construction of $\mathcal{T}$). □

4.3.3 Upper rank

It is always difficult to identify the upper rank of a semigroup and we observe that $A^+(B_n)^+$ is also not an exception. In order to investigate the upper rank $r_4(A^+(B_n)^+)$, in this subsection, first we obtain a lower bound for the upper rank and eventually we prove that this lower bound is indeed the $r_4(A^+(B_n)^+)$, for $n \geq 6$. We also report an independent set of 14 elements in $A^+(B_2)^+$.

Theorem 4.3.13. *For $n \geq 2$, $I = A^+(B_n)_n \cup \{\xi_{(i,i)} : i \in [n]\}$ is an independent set in $A^+(B_n)^+$. Hence, by Theorem 2.2.10, $r_4(A^+(B_n)^+) \geq (n!)n^2 + n$.*

Proof. For $f \in I$, suppose $f = \sum_{j=1}^{k} f_j$, for $f_j \in I$. We prove that $f_1 = f$ so that $f \notin \langle I \setminus \{f\} \rangle$. Let $f = \xi_{(i,i)}$; then clearly $f_j = f$ for all j. We may now suppose $f \in A^+(B_n)_n$ and $(k, p; \sigma)$ be the representation of f. By Lemma 4.3.6(2), $f_1 \in R_f$

and $\sum_{i=2}^{k} f_i = \xi_{(s,p)}$ for some $s \in [n]$. Note that $f_1 = (k, s; \sigma)$ (cf. Theorem 3.1.7). If $s \neq p$, then $\xi_{(s,p)} \notin \langle I \rangle$; a contradiction. Hence, $s = p$ so that $f_1 = f$. $\square$

Corollary 4.3.14. *$A^+(B_n)_n$ is an independent subset of size $(n!)n^2$ in $A^+(B_n)^+$.*

Lemma 4.3.15. *Let Q be an independent subset of B_n and*

$$Q' = \left\{ {}^{(k,l)}\zeta_\alpha \,\middle|\, k, l \in [n] \text{ and } \alpha \in Q \right\};$$

then Q' is an independent subset of $A^+(B_n)^+$.

Proof. For ${}^{(k,l)}\zeta_\alpha \in Q'$, $k_j, l_j \in [n]$ and $\alpha_j \in Q$ suppose

$${}^{(k,l)}\zeta_\alpha = \sum_{j=1}^{k} {}^{(k_j,l_j)}\zeta_{\alpha_j}.$$

Clearly $k_j = k$, $l_j = l$ for all j, and $\alpha = \sum_{j=1}^{k} \alpha_j$. Since Q is independent, we have $\alpha = \alpha_i$ for some i $(1 \leq i \leq k)$. Consequently, ${}^{(k,l)}\zeta_\alpha \notin \langle Q' \setminus \{{}^{(k,l)}\zeta_\alpha\} \rangle$ so that Q' is an independent set. $\square$

Using the upper rank of B_n given in the following theorem, we would obtain the upper rank of $\mathcal{C}_{B_n}$ in Remark 4.3.17.

Theorem 4.3.16 (Howie and Ribeiro [1999]). *For $n \geq 2$, $r_4(B_n) = \lfloor n^2/4 \rfloor + n$.*

Remark 4.3.17. Since B_n is isomorphic to the semigroup $\mathcal{C}_{B_n}$, we have $r_4(\mathcal{C}_{B_n}) = \lfloor n^2/4 \rfloor + n$.

In view of Remark 4.3.17, we have the following corollary of Lemma 4.3.15.

Corollary 4.3.18. *For $n \geq 2$, the maximum size of an independent subset in $A^+(B_n)_1$ is $n^2(\lfloor n^2/4 \rfloor + n)$.*

Remark 4.3.19. For $f_i \in A^+(B_n)$, if $\sum_{i=1}^{r} f_i + \xi_{(p,p)} + \sum_{i=r+1}^{s} f_i$ is nonzero, then the sum equals $\sum_{i=1}^{s} f_i$.

For $n = 2$, we provide a better lower bound in the following theorem.

Theorem 4.3.20. $r_4(A^+(B_2)^+) \geq 14$.

Proof. We claim that the 14-element set

$$P = \left\{ {}^{(k,l)}\zeta_\alpha \,\middle|\, k, l \in [2] \text{ and } \alpha \in Q \right\} \cup \{\xi_{(1,1)}, \xi_{(2,2)}\},$$

where $Q = \{(1,1), (1,2), (2,2)\}$, is an independent subset of the semigroup $A^+(B_2)^+$.

For $f \in P$, suppose $f = \sum_{j=1}^{k} f_j$, for $f_j \in P$. If $f = \xi_{(i,i)}$, then $f_j = f$ for all j so that $f \notin \langle P \setminus \{f\} \rangle$. Otherwise, $f = {}^{(k,l)}\zeta_\alpha$ for $\alpha \in Q$. By Remark 4.3.19, the sum for f can be reduced to a sum with only the singleton support elements of P. Hence, from the proof of Lemma 4.3.15, P is independent. □

Theorem 4.3.21. *For $n \geq 6$, $r_4(A^+(B_n)^+) = (n!)n^2 + n$.*

Proof. For $n \geq 6$, if an independent subset K of $A^+(B_n)^+$ contains a single support map or a full support map of the form $\xi_{(p,q)}$, for $p \neq q$, then

$$|K| < (n!)n^2 + n.$$

Hence, the result follows by Theorem 4.3.13.

Let K be an independent subset of $A^+(B_n)^+$. By Corollary 4.3.14, Remark 4.3.17 and Corollary 4.3.18, we have $|K| \leq \kappa$, where

$$\kappa = (n!)n^2 + \lfloor n^2/4 \rfloor + n + n^2(\lfloor n^2/4 \rfloor + n).$$

Through the following cases, we observe that, out of κ (the maximum possible number) elements, at least $(n-1)!(n-1)$ elements will not be in K. Hence, since $n \geq 6$,

$$|K| \leq \kappa - (n-1)!(n-1) < (n!)n^2 + n.$$

Case 1: $\xi_{(p,q)} \in K$ with $p \neq q$. For each $\sigma \in S_n$ and $l \in [n]$, since

$$(l, q; \sigma) = (l, p; \sigma) + \xi_{(p,q)},$$

the independent set K cannot contain $(l, q; \sigma)$ and $(l, p; \sigma)$ together. Thus, out of κ elements, at least $(n!)n$ elements will not be in K.

Case 2: ${}^{(r,s)}\zeta_{(p,q)} \in K$. For each $t \in [n]$ and $\sigma, \rho \in S_n$ such that $r\sigma = p$ and $r\rho = t$, since

$${}^{(r,s)}\zeta_{(p,q)} = (s, t; \sigma) + (s, q; \rho),$$

the independent set K cannot contain $(s, t; \sigma)$ and $(s, q; \rho)$ together.

Subcase 2.1 $p = q$. Except at $t = q$, for all other choices, none of the first terms is equal to any of the second terms in the sums $(s, t; \sigma) + (s, q; \rho)$. Thus, out of κ elements, at least $(n-1)!(n-1)$ elements (either first terms or second terms in the sums) will not be in K.

Subcase 2.2 $p \neq q$. In the similar lines of *Subcase 2.1*, at least $(n-1)!(n-2)$ elements will not be in K for the choices of $t \in [n] \setminus \{p, q\}$. If $t \in \{p, q\}$, the set of second terms of the sums for $t = p$ is equal to the set of first terms of the sums for $t = q$, which is of size $(n-1)!$. Thus, for $t \in \{p, q\}$, at least $(n-1)!$ elements will not be in K. Hence, a total of at least $(n-1)!(n-1)$ elements will not be in K.

□

4.3.4 Large rank

In this subsection, we obtain the large rank of the semigroup $A^+(B_n)^+$. In view of Theorem 4.2.2, we have the following remark regarding $r_5(A^+(B_2)^+)$.

Remark 4.3.22. Since $\xi_{(1,2)}$ is an indecomposable element in $A^+(B_2)^+$, we have

$$r_5(A^+(B_2)^+) = |A^+(B_2)| = 29.$$

However, as shown in the following proposition, there is no indecomposable element in $A^+(B_n)^+$, for $n \geq 3$.

Proposition 4.3.23. *For $n \geq 3$, all the elements of $A^+(B_n)^+$ are decomposable.*

Proof. Refereing to Theorem 2.2.10, we give a decomposition of each element $f \in A^+(B_n)^+$ in the following cases.

1. f is the zero map: $\xi_\vartheta = \xi_{(p,q)} + \xi_{(r,s)}$, for $q \neq r$.
2. f is a full or singleton support element: Let $\mathrm{Im}(f) \setminus \{\vartheta\} = \{(p,q)\}$. We have $f = g + h$, where $g, h \in A^+(B_n)$ such that $\mathrm{supp}(f) = \mathrm{supp}(g) = \mathrm{supp}(h)$ and $\mathrm{Im}(g) \setminus \{\vartheta\} = \{(p,r)\}$, $\mathrm{Im}(h) \setminus \{\vartheta\} = \{(r,q)\}$, for some $r \neq p, q$.
3. f is an n-support map: Let $f = (k,p;\sigma)$. Note that $f = (k,q;\sigma) + \xi_{(q,p)}$, for $q \neq p$.

□

Using Proposition 4.2.4 and Theorem 4.2.2, now we obtain the large rank of $A^+(B_n)^+$ in the following theorem.

Theorem 4.3.24. *For $n \geq 2$, $r_5(A^+(B_n)^+) = (n!)n^2 + n^2 + n^4 - n + 3$.*

Proof. We show that the set $V = \{\xi_{(n,k)} \mid 1 \leq k \leq n-1\}$ is a smallest prime subset of $A^+(B_n)^+$. Since, $|V| = n - 1$, the result follows from Theorem 2.2.10.

V is a prime subset: For $\xi_{(i,j)}, \xi_{(l,k)} \in A^+(B_n)$, if $\xi_{(i,j)} + \xi_{(l,k)} \in V$, then $i = n$, $j = l$ and $1 \leq k \leq n-1$. If $l = n$, then clearly $\xi_{(l,k)} \in V$; otherwise, $\xi_{(i,j)} \in V$.

V is a smallest prime subset: Let U be a prime subset of $A^+(B_n)^+$ such that $|U| < |V|$. If $U \subset V$, then let $\xi_{(n,q)} \in V \setminus U$. Now, for $\xi_{(n,p)} \in U$ and for all $i \in [n]$, clearly we have

$$\xi_{(n,p)} = \xi_{(n,i)} + \xi_{(i,p)}.$$

Note that, for $i = q$, neither $\xi_{(n,i)}$ nor $\xi_{(i,p)}$ is in U; a contradiction to U is a prime set.

Otherwise, we have $U \not\subset V$. Let $f \in U \setminus V$; then, f can be (i) ξ_ϑ, (ii) $\xi_{(n,n)}$, (iii) $\xi_{(p,q)}$, for some $p \in [n-1]$, $q \in [n]$, (iv) an n-support map, or (v) a singleton support map. In all the five cases we observe that $|U| \geq n-1$, which is a contradiction to the choice of U.

(i) $f = \xi_\vartheta$: For each $i \in [n]$, since $\xi_{(i,1)} + \xi_{(2,i)} = \xi_\vartheta$, there are at least n elements in U.

(ii) $f = \xi_{(n,n)}$: For each $i \in [n-1]$, since $\xi_{(n,i)} + \xi_{(i,n)} = \xi_{(n,n)}$, there are at least $n-1$ elements in U.

(iii) $f = \xi_{(p,q)}$, for some $p \in [n-1]$, $q \in [n]$: First note that, for each $i \in [n]$, we have $\xi_{(p,i)} + \xi_{(i,q)} = \xi_{(p,q)}$. If $p = q$, the argument is similar to above (ii). Otherwise, corresponding to $n-2$ different choices of $i \neq p, q$, there are at least $n-2$ elements in U. Now, including f, we have $|U| \geq n-1$.

(iv) f is an n-support map: Let $f = (k, p; \sigma)$. For $q \in [n] \setminus \{p\}$, since

$$(k, p; \sigma) = (k, q; \sigma) + \xi_{(q,p)},$$

there are at least $n-1$ elements in U.

(v) f is a singleton support map: Let $f = {}^{(k,l)}\zeta_{(p,q)}$. For $s \in [n] \setminus \{p, q\}$, since

$${}^{(k,l)}\zeta_{(p,q)} = {}^{(k,l)}\zeta_{(p,s)} + {}^{(k,l)}\zeta_{(s,q)},$$

there are at least $n-2$ elements in U. Since $f \in U$, we have $|U| \geq n-1$.

□

4.4 The semigroup $A^+(B_n)^\circ$

In this section, we obtain the small rank, lower rank and large rank of the semigroup $A^+(B_n)^\circ$. We also obtain lower bounds for intermediate and upper ranks of $A^+(B_n)^\circ$.

4.4.1 Small rank

In view of Remark 2.2.11, it can be easily observed that $A^+(B_1)$ is an independent set and none of its proper subsets generate $A^+(B_1)^\circ$. Hence, for $1 \leq i \leq 5$, we have

$$r_i(A^+(B_1)^\circ) = |A^+(B_1)| = 3.$$

In the rest of the section, we shall investigate the ranks of $A^+(B_n)$, for $n > 1$. We obtain the small rank of $A^+(B_n)^\circ$.

Owing to the fact that $A^+(B_n)^\circ$, for $n \geq 2$, have some non idempotent elements, it is not a band. For instance, the singleton support maps ${}^{(k,l)}\zeta_{(p,q)}$ with $(k,l) \neq (p,q)$ in $A^+(B_n)^\circ$ are not idempotents. Hence, we have the following corollary of Theorem 4.3.1.

Corollary 4.4.1. *For $n \geq 2$, $r_1(A^+(B_n)^\circ) = 1$.*

4.4.2 Lower rank

Recall our result that the set of n-support elements of $A^+(B_n)$ along with ξ_ϑ forms a subsemigroup which is isomorphic to the Brandt semigroup $B(S_n, n)$ (cf. Proposition 3.2.4). Using this key result we obtain the lower rank of the semigroup $A^+(B_n)^\circ$.

Lemma 4.4.2. *For $n \geq 2$, let $f, g_i (1 \leq i \leq k) \in A^+(B_n) \setminus \{\xi_\vartheta\}$ such that $f = g_1 g_2 \cdots g_k$. Then,*

1. *$f \in A^+(B_n)_{n^2+1}$ if and only if $g_j \in A^+(B_n)_{n^2+1}$ for some j; and*
2. *if $f \in A^+(B_n)_1$ then $g_j \in A^+(B_n)_1$ for some j.*

Proof.

1. If $g_i \notin A^+(B_n)_{n^2+1}$ for all i, then, by Proposition 2.2.1, $\vartheta \notin \mathrm{supp}(g_i)$ so that $\vartheta \notin \mathrm{supp}(f)$ and hence $f \notin A^+(B_n)_{n^2+1}$. Since the composition of a constant map with any map is a constant map, we have the converse.

2. Follows from (1) and Lemma 3.2.3.

□

In view of Lemma 3.2.3, we have the following corollary of Lemma 4.4.2.

Corollary 4.4.3. *Any generating subset of $A^+(B_n)^\circ$ contains at least a singleton support element and a full support element.*

Lemma 4.4.4. *Let σ be the cycle $(1\ 2 \cdots\ n)$ and τ be the transposition $(1\ 2)$ in S_n. The following are minimum generating subsets of the semigroups $A^+(B_n)_n \cup \{\xi_\vartheta\}$, for $n \geq 2$.*

1. *If $n \geq 3$, $\mathcal{P} = \{(1,1;\sigma), (1,2;\tau), (2,3;id), \cdots (n-1,n;id), (n,1;id)\}$.*
2. *If $n = 2$, $\mathcal{P}' = \{(1,2;\sigma), (2,1;id)\}$.*

Proof. Given a minimum generating set $\{g_1, \ldots, g_r\}$ of a finite group G with the identity element e, by [Garba, 1994b, Proposition 2.4], the set

$$\{(1,g_1,1), \ldots (1,g_{r-1},1), (1,g_r,2), (2,e,3), \ldots, (n-1,e,n), (n,e,1)\}$$

of $r+n-1$ elements is a minimum generating set of Brandt semigroup $B(G,n)$. For $n \geq 2$, since $\{\sigma, \tau\}$ is a minimum generating subset of the symmetric group S_n, by Proposition 3.2.4, the result follows. □

Theorem 4.4.5. *For $n \geq 3$, $r_2(A^+(B_n)^\circ) = n+3$.*

Proof. We prove that

$$\mathcal{Q} = \mathcal{P} \cup \left\{\xi_{(1,1)},\ {}^{(1,1)}\zeta_{(1,1)}\right\}$$

is a minimum generating set of $A^+(B_n)^\circ$ so that the result follows.

By Lemma 4.4.4(1), $\mathcal{Q}$ generates all n-support maps and the zero map in $A^+(B_n)^\circ$. For $f \in A^+(B_n)$, if $f = {}^{(k,l)}\zeta_{(p,q)}$, then write

$$ {}^{(k,l)}\zeta_{(p,q)} = (l,1;\sigma)^{(1,1)}\zeta_{(1,1)}(1,q;\rho)$$

or if $f = \xi_{(p,q)}$, then write

$$\xi_{(p,q)} = \xi_{(1,1)}(1, q; \rho),$$

where $k\sigma = 1$ and $1\rho = p$, so that $f \in \langle \mathcal{Q} \rangle$. Hence, by Theorem 2.2.10, we have $\langle \mathcal{Q} \rangle = A^+(B_n)^\circ$.

Let V be a generating subset of $A^+(B_n)^\circ$. By Proposition 3.2.4 and Lemma 4.4.4, V must contain at least $n+1$ elements of n-support to generate all n-support elements in $A^+(B_n)$. Further, by Corollary 4.4.3, V contains at least a singleton support element and a full support element so that $|V| \geq n+3$. Hence, the result. $\square$

Theorem 4.4.6. $r_2(A^+(B_2)^\circ) = 4$.

Proof. In the similar lines of the proof of Theorem 4.4.5, note that the set $\mathcal{P}' \cup \left\{\xi_{(1,1)},\ {}^{(1,1)}\zeta_{(1,1)}\right\}$ is a minimum generating set of the semigroup $A^+(B_2)^\circ$. $\square$

4.4.3 Intermediate and upper rank

In this subsection, we will only provide lower bounds for intermediate and upper ranks of $A^+(B_n)^\circ$. In view of Proposition 3.2.4, we shall rely on some known lower bounds of respective ranks for $B(G, n)$. First we recall the required results and proceed to give the lower bounds in theorems 4.4.10 and 4.4.11.

Theorem 4.4.7 ([Mitchell, 2004])**.** *Let X be a maximum independent generating set in a finite group G with identity element e and $\{I, J\}$ be a partition of the set $[n]$ such that $|I| = \lceil n/2 \rceil$ and $|J| = \lfloor n/2 \rfloor$. Then, in $B(G, n)$,*

1. *$\{(2, e, 3), (3, e, 4), \ldots, (n-1, e, n), (n, e, 1)\} \cup \{(1, x, 2) \mid x \in X\}$ is an independent generating set, and*

2. *$\{(i, e, i) \mid i \in [n]\} \cup \{(i, g, j) \mid i \in I, j \in J, g \in G\}$ is an independent set.*

Theorem 4.4.8 ([Whiston, 2000])**.** *For $n \geq 2$, the set of transpositions $\mathcal{T} = \{(1\ 2), (2\ 3), \cdots, (n-1\ n)\}$ is a maximum independent generating set in S_n and hence $r_3(S_n) = n - 1$.*

Now we prove the following lemma regarding an independent generating subset of $A^+(B_n)^\circ$.

Lemma 4.4.9. *Any independent generating subset of $A^+(B_n)^\circ$ contains*

1. *exactly one singleton support element, and*
2. *exactly one full support element.*

Proof. In view of Corollary 4.4.3, let $f = {}^{(k,l)}\zeta_{(p,q)}$ and $g = \xi_{(m,r)}$ are in U.

1. Suppose there is another singleton support map, say $f' = {}^{(s,t)}\zeta_{(u,v)} \in U$. Consider the n-support maps $h = (t, l; \sigma)$ with $s\sigma = k$ and $h' = (q, v; \tau)$ with $p\tau = u$. Since U is a generating set, we have $h, h' \in \langle U \setminus \{f'\}\rangle$. Now observe that $f' = hfh'$ so that $f' \in \langle U \setminus \{f'\}\rangle$; a contradiction to U is an independent set.

2. Suppose there is another full support map, say $g' = \xi_{(u,v)} \in U$. Consider the n-support map $h' = (r, v; \tau)$ with $m\tau = u$ and note that $h' \in \langle U \setminus \{g'\}\rangle$ (cf. Lemma 3.2.3). However, since $g' = gh'$, we have $g' \in \langle U\setminus\{g'\}\rangle$; a contradiction to U is an independent set.

□

Theorem 4.4.10. *For $n \geq 2$, $r_3(A^+(B_n)^\circ) \geq 2n$.*

Proof. Recall that $\mathcal{T} = \{(1\ 2), (2\ 3), \cdots, (n-1\ n)\}$ is a maximum independent generating subset of S_n (cf. Theorem 4.4.8). We observe that the set $\mathcal{X} = \mathcal{X}' \cup \{\xi_{(1,1)}, {}^{(1,1)}\zeta_{(1,1)}\}$, where

$$\mathcal{X}' = \{(2, 3; id), (3, 4; id), \ldots, (n-1, n; id), (n, 1; id)\} \cup \{(1, 2; \sigma) \mid \sigma \in \mathcal{T}\}$$

is an independent generating set in $A^+(B_n)^\circ$ so that $r_3(A^+(B_n)^\circ) \geq |\mathcal{X}| = 2n$. By Theorem 4.4.7(1) and Proposition 3.2.4, the set $\mathcal{X}'$ generates all n-support maps and the zero map in $A^+(B_n)^{\circ}$. Now, in the similar lines of proof of Theorem 4.4.5, one can prove that $\langle \mathcal{X} \rangle = A^+(B_n)^{\circ}$. Further, in view of Lemma 4.4.9 and Theorem 4.4.7(1), $\mathcal{X}$ is an independent subset in $A^+(B_n)^{\circ}$. □

Though Theorem 4.4.10 gives us a lower bound for upper rank of $A^+(B_n)^{\circ}$, in the following theorem we provide a better lower bound for $r_4(A^+(B_n)^{\circ})$.

Theorem 4.4.11. *For $n \geq 2$, $r_4(A^+(B_n)^{\circ}) \geq n! \lfloor n^2/4 \rfloor + n + 2$.*

Proof. Using Theorem 4.4.7(2), Proposition 3.2.4 and Lemma 4.4.9, one can observe that the set

$$\{(i, i; id) \mid i \in [n]\} \cup \{(i, j; \sigma) \mid i \in I, j \in J, \sigma \in S_n\} \cup \left\{\xi_{(1,1)},\ {}^{(1,1)}\zeta_{(1,1)}\right\},$$

where I and J are as in Theorem 4.4.7, is an independent subset in $A^+(B_n)^{\circ}$. □

4.4.4 Large rank

In this subsection, first we observe that there are no indecomposable elements in $A^+(B_n)^{\circ}$, for $n \geq 2$, so that $r_5(A^+(B_n)^{\circ}) < |A^+(B_n)|$ (cf. Theorem 4.2.2). Then, using the technique introduced in Section 4.2, we proceed to obtain the large rank of $A^+(B_n)^{\circ}$.

Proposition 4.4.12. *For $n \geq 2$, all the elements of $A^+(B_n)^{\circ}$ are decomposable.*

Proof. Refereing to Theorem 2.2.10, we give a decomposition for each element $f \in A^+(B_n)^{\circ}$ in the following cases.

1. f is the zero map: $\xi_{\vartheta} = {}^{(k,l)}\zeta_{(p,q)}{}^{(m,r)}\zeta_{(s,t)}$, for$(p, q) \neq (m, r)$.

2. f is of full support: Let $f = \xi_{(p,q)}$. Then $\xi_{(p,q)} = \xi_{(m,r)}\ {}^{(m,r)}\zeta_{(p,q)}$, for $(m, r) \neq (p, q)$.

3. f is of singleton support: Let $f = {}^{(k,l)}\zeta_{(p,q)}$. Then $f = {}^{(k,l)}\zeta_{(m,r)}\ {}^{(m,r)}\zeta_{(p,q)}$, for $(m, r) \notin \{(p, q), (k, l)\}$.

4. f is an n-support map: Let $f = (k, p; \sigma)$. Note that $f = (k, q; \tau)(q, p; \tau^{-1}\sigma)$, for $q \neq p$ and $\tau \neq id$.

$\square$

Using Proposition 4.2.4 and Theorem 4.2.2, now we obtain the large rank of $A^+(B_n)^\circ$.

Theorem 4.4.13. $r_5(A^+(B_2)^\circ) = 28$.

Proof. We show that the set $V = \{(1, 2; id), (1, 2; \sigma)\}$, where σ is the cycle $(1\ 2)$ in S_2, is a smallest prime subset of $A^+(B_2)^\circ$ so that $r_5(A^+(B_2)^\circ) = 28$. Observe that

$$(1, 2; id) = (1, 1; id)(1, 2; id) = (1, 2; id)(2, 2; id) = (1, 1; \sigma)(1, 2; \sigma) = (1, 2; \sigma)(2, 2; \sigma)$$

and

$$(1, 2; \sigma) = (1, 1; id)(1, 2; \sigma) = (1, 2; id)(2, 2; \sigma) = (1, 1; \sigma)(1, 2; id) = (1, 2; \sigma)(2, 2; id)$$

are all the possible decompositions of $(1, 2; id)$ and $(1, 2; \sigma)$, respectively, with the elements of $A^+(B_2)^\circ$. Note that, every decomposition has at least one element from V so that V is a prime subset.

If U is a prime subset of $A^+(B_2)^\circ$ with $|U| < |V|$, then $|U| = 1$. Hence, the element of U will be indecomposable; a contradiction to Proposition 4.4.12. Consequently, V is a smallest prime subset of $A^+(B_2)^\circ$. $\square$

Theorem 4.4.14. *For $n \geq 3$, $r_5(A^+(B_n)^\circ) = (n!)n^2 + n^4 + 2$.*

Proof. We prove that $V = \{\xi_{(i,j)} \mid i, j \in [n]\}$ is a smallest prime subset of $A^+(B_n)^\circ$. Since $|V| = n^2$, the result follows from Theorem 2.2.10.

By Lemma 4.4.2(1), V is a prime subset of $A^+(B_n)^\circ$. Let U be a prime subset of $A^+(B_n)^\circ$ such that $|U| < |V|$. If $U \subset V$, then let $g = \xi_{(p,q)} \in V \setminus U$. Now, for

$h = \xi_{(s,t)} \in U$, we have $h' = {}^{(p,q)}\zeta_{(s,t)}$ such that $h = gh'$; which is not possible with U. Thus, $U \not\subset V$.

Let $f \in U \setminus V$. Then, f can be (i) ξ_ϑ, (ii) a singleton support map, or (iii) an n-support map. In all the cases we observe that $|U| \geq n^2$, which is a contradiction to the choice of U.

(i) $f = \xi_\vartheta$: Note that $f = \xi_{(p,q)}{}^{(m,r)}\zeta_{(p,q)}$, where $(p,q) \neq (m,r)$. For each pair (p,q) we have one such decomposition of f and hence there are at least n^2 such decompositions. From each decomposition one of the components, viz. a map with the image set $\{(p,q)\} \cup \{\vartheta\}$ is in U so that $|U| \geq n^2$.

(ii) f is singleton support map: Let $f = {}^{(k,l)}\zeta_{(p,q)}$. Note that $f = {}^{(k,l)}\zeta_{(m,r)}{}^{(m,r)}\zeta_{(p,q)}$. For $(m,r) \neq (k,l)$, we have n^2-1 such decompositions of f. Being a prime subset, U must contain at least one component from each decomposition so that $|U| \geq n^2 - 1$. Further, consider the decomposition

$$ {}^{(k,l)}\zeta_{(p,q)} = {}^{(k,l)}\zeta_{(k,l)}(l,q;\sigma), $$

where $k\sigma = p$. One more element from the above decomposition should be in U. Thus, $|U| \geq n^2$.

(iii) f is an n-support map: Let $f = (k,p;\sigma)$. Since $n \geq 3$, for fixed $q \notin \{p,k\}$, consider the decomposition

$$ (k,p;\sigma) = (k,q;\tau)(q,p;\tau^{-1}\sigma) $$

one for each $\tau \in S_n$ so that there are $n!$ such decompositions of f. Consequently, $|U| \geq n!$ so that $|U| > n^2$, for $n \geq 4$. For $n = 3$, in addition to above-mentioned six elements, we observe that there are another three elements in U. For instance, for each $\tau \in S_3$, none of the components in the decomposition

$$ (k,p;\sigma) = (k,k;\tau)(k,p;\tau^{-1}\sigma) $$

are covered in any of the above-mentioned decompositions. Even if a left component reoccurs as a right component in any of the decompositions, at least three of them will be in U. Hence, for $n \geq 3$, $|U| \geq n^2$. $\square$

4.5 Conclusion

In this chapter, we have investigated the ranks of the semigroup reducts of $A^+(B_n)$. While we obtained the small, lower, intermediate and large ranks of $A^+(B_n)^+$, for all $n \geq 1$, the upper rank $r_4(A^+(B_n)^+)$ was found for the semigroups with $n \geq 6$. For $2 \leq n \leq 5$, through an explicit construction of an independent set, we reported a lower bound for $r_4(A^+(B_n)^+)$. While 14 is the lower bound for the case $n = 2$, it is $(n!)n^2 + n$ for the other cases. We conjecture that these lower bounds are indeed the upper ranks of the respective cases.

Further, we have investigated the ranks of multiplicative semigroup reduct $A^+(B_n)^\circ$ of $A^+(B_n)$, and obtained the small, lower and large ranks of $A^+(B_n)^\circ$, for all $n \geq 1$. While intermediate and upper ranks of $A^+(B_n)^\circ$ are not yet known, we have provided some lower bounds for these ranks.

In view of Theorem 2.1.6, the results embedded in this chapter shall give us the respective ranks of the semigroup of affine transformations $\mathrm{Aff}(B_n)$ as per the following.

Remark 4.5.1. Since $A^+(B_1) = \mathrm{Aff}(B_1)$ (cf. Remark 2.2.11), we have $r_i(\mathrm{Aff}(B_1)) = 3$, for $1 \leq i \leq 5$.

Remark 4.5.2. For $n \geq 2$, observe that the n-support elements $(k, p; \sigma)$ with $p \neq k$ are not idempotent so that $r_1(\mathrm{Aff}(B_n)) = 1$ (cf. Theorem 4.3.1).

Remark 4.5.3.

1. Lower and large ranks of $\mathrm{Aff}(B_2)$ are 3 (cf. Theorem 4.4.6) and 12 (cf. Theorem 4.4.13), respectively.

2. For $n \geq 3$, $r_2(\mathrm{Aff}(B_n)) = n + 2$ (cf. Theorem 4.4.5) and $r_5(\mathrm{Aff}(B_n)) = (n!)n^2 + 2$ (cf. Theorem 4.4.14).

5

Ideals and Radicals

The radicals J_ν, for $\nu = 0, 1, 2$, of near-rings and consequently, the Jacobson radicals of rings are extended to zero-symmetric near-semirings by van Hoorn [1970]. In that process van Hoorn found fourteen radicals and studied some relations between them. The properties of these radicals are further investigated in the literature (e.g. [Krishna, 2005; Zulfiqar, 2009]). The objective of this chapter is to study the ideals and radicals of $\mathcal{N}$, the zero-symmetric affine near-semiring over Brandt semigroup. First we present the necessary background material in Section 5.1. We obtain all the right ideals of $\mathcal{N}$ in Section 5.2. We investigate all the radicals of $\mathcal{N}$ in Section 5.3. Further, we study all the congruences on $\mathcal{N}$ and consequently, we obtain all of its ideals in Section 5.4.

5.1 Preliminaries

In this section, we present the necessary background materials on theory of near-semirings. For more details one may refer to [Krishna, 2005; van Hoorn, 1970; van Hoorn and van Rootselaar, 1967]. We begin with the definition of $\mathcal{N}$-semigroup. It is known that the concept of $\mathcal{N}$-semigroup and the representation of a near-semiring $\mathcal{N}$ are equivalent (cf. [Krishna and Chatterjee, 2007]). In this chapter, unless it is specified otherwise, $\mathcal{N}$ always denotes a zero-symmetric near-semiring.

Definition 5.1.1. A semigroup $(S, +)$ with identity 0_S is said to be an $\mathcal{N}$*-semigroup* if there exists a composition

$$(s, a) \mapsto sa : S \times \mathcal{N} \longrightarrow S$$

such that, for all $a, b \in \mathcal{N}$ and $s \in S$,

1. $s(a + b) = sa + sb$,
2. $s(ab) = (sa)b$, and
3. $s0 = 0_S$.

Example 5.1.2. The semigroup $(\mathcal{N}, +)$ of a near-semiring $(\mathcal{N}, +, \cdot)$ is an $\mathcal{N}$-semigroup. We denote this $\mathcal{N}$-semigroup by $\mathcal{N}^+$.

Definition 5.1.3. A subsemigroup T of an $\mathcal{N}$-semigroup S is said to be $\mathcal{N}$*-subsemigroup* of S if and only if $0_S \in T$ and $T\mathcal{N} \subseteq T$.

5.1.1 Ideals

Unlike the case of rings and near-rings, the concept of ideal in near-semirings is not straightforward through internal structure. Thus, it is natural to adopt universal algebraic techniques and define ideals through homomorphisms. The concept of ideals is originally from the work of van Hoorn and van Rootselaar [1967] which they defined for the case of zero-symmetric near-semirings. Later it is adopted to near-semirings by Krishna [2005].

Definition 5.1.4. Let $\mathcal{N}, \mathcal{N}'$ be near-semirings. A mapping $\phi : \mathcal{N} \longrightarrow \mathcal{N}'$ is said to be a homomorphism if, for all $a, b \in \mathcal{N}$,

$$(a+b)\phi = a\phi + b\phi,$$

$$(ab)\phi = (a\phi)(b\phi).$$

Definition 5.1.5. The kernel of a homomorphism of $\mathcal{N}$ is called an *ideal* of $\mathcal{N}$.

Definition 5.1.6. An $\mathcal{N}$*-morphism* of an $\mathcal{N}$-semigroup S is a semigroup homomorphism ϕ of S into an $\mathcal{N}$-semigroup S' such that

$$(sa)\phi = (s\phi)a$$

for all $a \in \mathcal{N}$ and $s \in S$. The kernel of an $\mathcal{N}$-morphism is called an $\mathcal{N}$*-kernel* of an $\mathcal{N}$-semigroup S.

Definition 5.1.7. The $\mathcal{N}$-kernels of the $\mathcal{N}$-semigroup $\mathcal{N}^+$ are called *right ideals* of $\mathcal{N}$.

Definition 5.1.8. A *right admissible morphism* of an $\mathcal{N}$-semigroup S is a semigroup homomorphism ϕ of S, such that

$$s\phi = s'\phi \Longrightarrow (sa)\phi = (s'a)\phi$$

for all $a \in \mathcal{N}$ and $s, s' \in S$.

Remark 5.1.9. Any $\mathcal{N}$-semigroup S contains two trivial $\mathcal{N}$-kernels $\{0\}$ and S; the first one being the kernel of the identity mapping of S and second one is the kernel of the zero morphism.

Definition 5.1.10. An equivalence relation $\sim$ is said to be a *congruence relation* on near-semiring $(\mathcal{N}, +, \cdot)$ if $\sim$ is a congruence relation on semigroup reducts $(\mathcal{N}, +)$ and $(\mathcal{N}, \cdot)$, respectively. A congruence relation other than the equality relation and the universal relation $\mathcal{N} \times \mathcal{N}$ will be called as a *nontrivial congruence relation* on $\mathcal{N}$.

Definition 5.1.11. Let S be an $\mathcal{N}$-semigroup. A semigroup congruence $\sim_r$ of S is said to be a *congruence of* $\mathcal{N}$*-semigroup* S, if for all $s, t \in S$ and $a \in \mathcal{N}$,

$$s \sim_r t \Longrightarrow sa \sim_r ta.$$

Definition 5.1.12 ([Ljapin, 1963])**.** A nonempty subset T of a semigroup S is said to be *normal subsemigroup* if for every $t_1, t_2 \in T$ or empty symbols and $s, s' \in S$ or empty symbols,

$$s + t_1 + s' \in T \Longrightarrow s + t_2 + s' \in T.$$

Remark 5.1.13.

1. If S is a group, then its normal subsemigroups are normal subgroups.
2. $0_S \in S$, then any normal subsemigroup of S contains 0_S.
3. The kernel of a semigroup homomorphism is a normal subsemigroup. Hence, right ideals of near-semiring $\mathcal{N}$ are normal subsemigroups of $\mathcal{N}^+$.

Definition 5.1.14. Let T be a normal subsemigroup of a semigroup S. The relation $\mathfrak{n}_T$ on S is defined by

$$\alpha \; \mathfrak{n}_T \; \beta \Longleftrightarrow \alpha, \beta \in s + T + s'$$

for some $s, s' \in S$.

Remark 5.1.15. The relation $\mathfrak{n}_T$ is a two sidedly stable (compatible) reflexive and symmetric relation. Then the transitive closure $\mathfrak{n}''_T$ of $\mathfrak{n}_T$ is a congruence relation on

S. Moreover, the kernel of congruence $\mathfrak{n}''_T$, i.e. the congruence class containing 0_S, is equals to T.

Notation 5.1.16. The semigroup homomorphism corresponding to $\mathfrak{n}''_T$ will be denoted by λ_T.

Remark 5.1.17. For the right ideal $\{0\}$, the congruence relation $\mathfrak{n}''_{\{0\}}$ is the equality relation on $\mathcal{N}^+$.

Definition 5.1.18. A normal subsemigroup T of an $\mathcal{N}$-semigroup S is said to have the *property* Q if the following condition holds: for all $s, s' \in S$, $t \in T$ and $a \in \mathcal{N}$,

$$(s + t + s')a \ \mathfrak{n}''_T \ (s + s')a$$

Remark 5.1.19. The right ideal $\{0\}$ satisfies property Q.

Definition 5.1.20. Strong (right) ideals of $\mathcal{N}$ are the (right) ideals with property Q.

Definition 5.1.21. A homomorphism φ of $\mathcal{N}^+$ is called *modular* if there is a left identity modulo φ, i.e. an element u such that $a\varphi = (ua)\varphi$ for any $a \in \mathcal{N}$.

Definition 5.1.22. A right ideal $D \neq \mathcal{N}$ is called *modular* if D is the kernel of a right admissible modular ϕ, and it is called λ-*modular* if λ_D is right admissible and modular.

Definition 5.1.23. Let S_1, S_2 be subsets of an $\mathcal{N}$-semigroup S. The *Noetherian quotient* $(S_1 : S_2)$ is defined as the set $\{a \in \mathcal{N} \mid S_2 a \subseteq S_1\}$.

Notation 5.1.24. The set $(\{s\} : T)$ simply be denoted by $(s : T)$. Similarly, for $(T : s)$.

Definition 5.1.25. The set $(0_S : T)$ is said to be the *annihilator* of T, and it is denoted by $A(T)$.

Remark 5.1.26. For any nonvoid $T \subseteq S, A(T) = \bigcap_{t \in T} A(t)$.

Theorem 5.1.27 ([Krishna, 2005])**.** *The annihilator $A(S)$ of an $\mathcal{N}$-semigroup S is an ideal of $\mathcal{N}$.*

5.1.2 Types of $\mathcal{N}$-semigroups and radicals

In this subsection, first we recall various types of $\mathcal{N}$-semigroups. Accordingly, we shall present the notion of radicals for near-semirings. van Hoorn [1970] introduced these radical for zero-symmetric near-semirings as a generalization of the notion of Jacobson radicals of rings.

Definition 5.1.28. An $\mathcal{N}$-semigroup S is called *monogenic* if there exists $s \in S$ such that $s\mathcal{N} = S$ (then s is called a generator for S) and it is called *strongly monogenic* if $S\mathcal{N} \neq \{0_S\}$ and for any element $s \in S$, $s\mathcal{N} = S$ or $s\mathcal{N} = \{0_S\}$ holds.

Definition 5.1.29. An $\mathcal{N}$-semigroup $S \neq \{0_S\}$ with S and $\{0_S\}$ as the only $\mathcal{N}$-subsemigroup is called *minimal* . Moreover, if $S\mathcal{N} \neq \{0_S\}$ then S is called *essentially minimal*.

Definition 5.1.30. An $\mathcal{N}$-semigroup $S \neq \{0_S\}$ with S and $\{0_S\}$ as the only $\mathcal{N}$-kernels is called *irreducible.*

Theorem 5.1.31. *A right ideal D of $\mathcal{N}$ is modular if and only if there is a monogenic $\mathcal{N}$-semigroup $S \neq \{0_S\}$ with generator s such that $D = A(s)$.*

Theorem 5.1.32. *If S is a monogenic $\mathcal{N}$-semigroup $\neq \{0_S\}$ with generator s and $A(s)$ is a maximal right ideal in $\mathcal{N}$, then S is irreducible.*

Definition 5.1.33. An $\mathcal{N}$-semigroup $S \neq \{0_S\}$ is said to be of

– type $(0, 0)$, if S is monogenic and irreducible

– type $(0, 1)$, if S is monogenic and if for any generator s, $A(s)$ is a maximal right ideal

- type $(0,2)$, if S is monogenic and if for any generator s, $A(s)$ is a maximal strong right ideal
- type $(0,3)$, if S is monogenic and if for any generator s, $A(s)$ is a strong maximal right ideal
- type $(1,0)$, if S is strongly monogenic and irreducible
- type $(1,1)$, if S is strongly monogenic and if for any generator s, $A(s)$ is a maximal right ideal
- type $(1,2)$, if S is strongly monogenic and if for any generator s, $A(s)$ is a maximal strong right ideal
- type $(1,3)$, if S is strongly monogenic and if for any generator s, $A(s)$ is a strong maximal right ideal
- type $(2,0)$, if S is essentially minimal
- type $(2,1)$, if S is monogenic and if for any generator s, $A(s)$ is a maximal $\mathcal{N}$-subsemigroup of S.

Definition 5.1.34.

1. $R_0(\mathcal{N})$ is the intersection of all maximal modular right ideals of $\mathcal{N}$.
2. $R_1(\mathcal{N})$ is the intersection of all modular maximal right ideals of $\mathcal{N}$.
3. $R_2(\mathcal{N})$ is the intersection of all maximal λ-modular right ideals of $\mathcal{N}$.
4. $R_3(\mathcal{N})$ is the intersection of all λ-modular maximal right ideals of $\mathcal{N}$.

Definition 5.1.35. For $\nu = 0, 1$ with $\mu =$ 0,1,2,3 and $\nu = 2$ with $\mu =$ 0,1

$$J_{(\nu,\mu)}(\mathcal{N}) = \bigcap_{S \text{ is of type}(\nu,\mu)} A(S).$$

In all cases, the intersection of any empty collection of subsets of $\mathcal{N}$ is $\mathcal{N}$.

$$\begin{tikzcd}
J_{(2,0)} \arrow[r] & J_{(2,1)} & & \\
J_{(1,0)} \arrow[r] \arrow[u] & J_{(1,1)} \arrow[u] \arrow[rr, bend right] & J_{(1,2)} \arrow[r] & J_{(1,3)} \\
J_{(0,0)} \arrow[r] \arrow[u] \arrow[dr] & J_{(0,1)} \arrow[u] \arrow[rr, bend right] & J_{(0,2)} \arrow[r] \arrow[u] & J_{(0,3)} \arrow[u] \\
R_0 \arrow[r] & R_1 \arrow[uu, bend right] \arrow[rr, bend right] & R_2 \arrow[r] & R_3
\end{tikzcd}$$

FIGURE 5.1: Relation between various radicals of a near-semiring

Remark 5.1.36 ([Betsch, 1963])**.** If $\mathcal{N}$ is a near-ring, then $J_{(0,\mu)}(\mathcal{N})$, $\mu = 0, 1, 2, 3$ are the radical $J_0(\mathcal{N})$; $J_{(1,\mu)}(\mathcal{N})$, μ = 0,1,2,3 are the radical $J_1(\mathcal{N})$; $J_{(2,\mu)}(\mathcal{N})$, μ = 0,1, are the radical $J_2(\mathcal{N})$; and $R_\nu(\mathcal{N})$, $\nu = 0, 1, 2, 3$ are the radical $D(\mathcal{N})$ of Betsch.

Remark 5.1.37 ([Jacobson, 1964])**.** If $\mathcal{N}$ is a ring, then all the fourteen radicals are the radical of Jacobson.

Definition 5.1.38. A zero-symmetric near-semiring $\mathcal{N}$ is called (ν, μ)-*primitive* if $\mathcal{N}$ has an $\mathcal{N}$-semigroup S of type (ν, μ) with $A(S) = \{0\}$.

Before concluding the section, in the following we state the result without proof, which gives the relation between the radicals according to van Hoorn [1970].

Theorem 5.1.39. *For the radicals of a zero-symmetric near-semiring $\mathcal{N}$ we have the relations illustrated in the Figure 5.1 ($A \to B$ means $A \subset B$).*

Now, we are ready to investigate radicals and ideals of the object of the thesis – affine near-semirings over Brandt semigroups. Since the radicals are only known in the context of zero-symmetric near-semirings, we extend the semigroup reduct

$(A^+(B_n), +)$ to monoid by adjoining 0 and make the resultant near-semiring zero-symmetric. Hereafter, in this chapter, $\mathcal{N}$ denotes this near-semiring, that is, $\mathcal{N} = A^+(B_n) \cup \{0\}$ such that

1. $(\mathcal{N}, +)$ is a monoid with identity element 0,
2. $(\mathcal{N}, \circ)$ is a semigroup,
3. $0f = f0 = 0$, for all $f \in \mathcal{N}$, and
4. $f(g + h) = fg + fh$, for all $f, g, h \in \mathcal{N}$.

5.2 Right ideals

In this section, we obtain all the right ideals of the affine near-semiring $\mathcal{N}$ by ascertaining the concerning congruences of $\mathcal{N}$-semigroups. We begin with the following lemma.

Lemma 5.2.1. *Let $\sim$ be a nontrivial congruence over the semigroup $(\mathcal{N}, +)$ and $f \in A^+(B_n)_{n^2+1}$. If $f \sim \xi_\vartheta$, then $\sim = (A^+(B_n) \times A^+(B_n)) \cup \{(0, 0)\}$.*

Proof. First note that $(A^+(B_n) \times A^+(B_n)) \cup \{(0, 0)\}$ is a congruence relation of the semigroup $(\mathcal{N}, +)$. Let $f = \xi_{(p_0,q_0)}$ and $\xi_{(p,q)}$ be an arbitrary full support map. Since

$$\xi_{(p,q)} = \xi_{(p,p_0)} + \xi_{(p_0,q_0)} + \xi_{(q_0,q)} \sim \xi_{(p,p_0)} + \xi_\vartheta + \xi_{(q_0,q)} = \xi_\vartheta,$$

we have $\xi_{(p,q)} \sim \xi_\vartheta$ for all $p, q \in [n]$. Further, given an arbitrary n-support map $(k, l; \sigma)$, since $\xi_{(p,l)} \sim \xi_\vartheta$, we have

$$(k, l; \sigma) = (k, p; \sigma) + \xi_{(p,l)} \sim (k, p; \sigma) + \xi_\vartheta = \xi_\vartheta.$$

Thus, all n-support maps are related to the absorbing map [1] under $\sim$. Similarly, given an arbitrary ${}^{(k,l)}\zeta_{(p,q)} \in A^+(B_n)_1$, since $\xi_{(p,q)} \sim \xi_\vartheta$, for $\sigma \in S_n$ such that $k\sigma = q$,

[1] To avoid any confusion between the newly adjoined zero element 0 and the existing zero map ξ_ϑ, in this chapter, we call ξ_ϑ as an absorbing map of $\mathcal{N}$.

we have

$$^{(k,l)}\zeta_{(p,q)} = \xi_{(p,q)} + (l, q; \sigma) \sim \xi_\vartheta + (l, q; \sigma) = \xi_\vartheta.$$

Hence, all elements of $A^+(B_n)$ are related to each other under $\sim$. $\square$

Now, using Lemma 5.2.1, we determine the right ideals of $\mathcal{N}$ in the following theorem.

Theorem 5.2.2. *$\mathcal{N}$ and $\{0\}$ are only the right ideals of $\mathcal{N}$.*

Proof. Let $I \neq \{0\}$ be a right ideal of $\mathcal{N}$ so that $I = \ker\varphi$, where $\varphi : \mathcal{N}^+ \longrightarrow S$ is an $\mathcal{N}$-morphism. Note that $I = [0]_{\sim_r}$, where $\sim_r$ is the congruence over the $\mathcal{N}$-semigroup $\mathcal{N}^+$ defined by $a \sim_r b$ if and only if $a\varphi = b\varphi$, i.e. the relation $\sim_r$ on $\mathcal{N}$ is compatible with respect to $+$ and if $a \sim_r b$ then $ac \sim_r bc$ for all $c \in \mathcal{N}$.

Let f be a nonzero element of $\mathcal{N}$ such that $f \sim_r 0$. First note that

$$\xi_\vartheta = f\xi_\vartheta \sim_r 0\xi_\vartheta = 0.$$

Further, for any full support map $\xi_{(p,q)}$, we have

$$\xi_{(p,q)} = f\xi_{(p,q)} \sim_r 0\xi_{(p,q)} = 0$$

so that, by transitivity, $\xi_{(p,q)} \sim_r \xi_\vartheta$. Hence, by Lemma 5.2.1, $\sim_r = \mathcal{N} \times \mathcal{N}$ so that $I = \mathcal{N}$. $\square$

Remark 5.2.3. The ideal $\{0\}$ is the maximal right ideal of $\mathcal{N}$.

5.3 Radicals

This section aims to ascertain that $\mathcal{N}$ is (ν, μ)-primitive and to find all the radicals of $\mathcal{N}$. We shall achieve the target through the additive semigroup of constant maps in $\mathcal{N}$, whose properties are systematically developed and presented in the key result Theorem 5.3.8.

Consider the subsemigroup $\mathcal{C} = \mathcal{C}_{B_n} \cup \{0\}$ of $(\mathcal{N}, +)$. We shall observe the following properties of $\mathcal{C}$.

Remark 5.3.1. The semigroup $\mathcal{C}$ is an $\mathcal{N}$-semigroup with respect to the multiplication in $\mathcal{N}$.

Lemma 5.3.2. *The $\mathcal{N}$-semigroup $\mathcal{C}$ is monogenic and its every nonzero element is a generator.*

Proof. Let $g \in \mathcal{C}_{B_n}$. We observe that g generates the $\mathcal{N}$-semigroup $\mathcal{C}$. Note that $g\mathcal{N} \subseteq \mathcal{C}$ because the product of a constant map with any map is a constant map. Conversely, for $f \in \mathcal{C}$, since $gf = f$, we have $g\mathcal{N} = \mathcal{C}$. Thus, $\mathcal{C}$ is monogenic and $g\mathcal{N} = \mathcal{C}$ for all $g \in \mathcal{C} \setminus \{0\}$. $\square$

Further, since $0\mathcal{N} = \{0\}$ and $\mathcal{C}\mathcal{N} = \mathcal{C} \neq \{0\}$, we have the following remark.

Remark 5.3.3. The $\mathcal{N}$-semigroup $\mathcal{C}$ is strongly monogenic.

Lemma 5.3.4. *The $\mathcal{N}$-semigroup $\mathcal{C}$ is essentially minimal.*

Proof. The semigroups $\mathcal{C}$ and $\{0\}$ are the only $\mathcal{N}$-subsemigroups of $\mathcal{C}$. For instance, let T be an $\mathcal{N}$-subsemigroup of $\mathcal{C}$ such that $\{0\} \neq T \subsetneq \mathcal{C}$. Then there exist $f(\neq 0) \in T$ and $g \in \mathcal{C} \setminus T$. Since $fg = g \notin T$, we have $T\mathcal{N} \nsubseteq T$; a contradiction to T is an $\mathcal{N}$-subsemigroup. Consequently, $\mathcal{C}$ is minimal. Moreover, $\mathcal{C}\mathcal{N} = \mathcal{C} \neq \{0\}$ so that $\mathcal{C}$ is essentially minimal. $\square$

Since $\mathcal{C}$ and $\{0\}$ are the only $\mathcal{N}$-subsemigroups of $\mathcal{C}$, we have the following corollary of Lemma 5.3.4.

Corollary 5.3.5. *The subsemigroup $\{0\}$ is the maximal $\mathcal{N}$-subsemigroup of $\mathcal{C}$.*

For $g \in \mathcal{C} \setminus \{0\}$, since $\{a \in \mathcal{N} \mid ga = 0\} = \{0\}$, we have the following remark.

Remark 5.3.6. $A(g) = \{0\}$ for all $g \in \mathcal{C} \setminus \{0\}$. Hence, $A(\mathcal{C}) = \{0\}$ (cf. Remark 5.1.26).

Lemma 5.3.7. *The $\mathcal{N}$-semigroup $\mathcal{C}$ is irreducible.*

Proof. By Lemma 5.3.2, the $\mathcal{N}$-semigroup $\mathcal{C}$ is monogenic with any nonzero element g as generator. Now, by Remark 5.3.6, $A(g) = \{0\}$; thus, $A(g)$ is maximal right ideal in $\mathcal{N}$ (cf. Remark 5.2.3). Hence, by Theorem 5.1.32, $\mathcal{C}$ is irreducible. □

We are now ready to present a key result of this section.

Theorem 5.3.8. *For $\nu = 0, 1$ with $\mu = 0, 1, 2, 3$ and $\nu = 2$ with $\mu = 0, 1$, the $\mathcal{N}$-semigroup $\mathcal{C}$ is of type (ν, μ).*

Proof. In view of Theorem 5.1.39, though it is sufficient to discuss for some of the types, for the sake of completeness we prove all the cases in the following.

Type $(0, \mu)$: Note that, by Lemma 5.3.2, the $\mathcal{N}$-semigroup $\mathcal{C}$ is monogenic.

1. By Lemma 5.3.7, we have $\mathcal{C}$ is irreducible. Hence, $\mathcal{C}$ is of type $(0, 0)$.
2. By Lemma 5.3.2, Remark 5.3.6 and Remark 5.2.3, for any generator g, $A(g)$ is a maximal right ideal. Hence, $\mathcal{C}$ is of type $(0, 1)$.
3. The ideal $\{0\}$ is strong right ideal (cf. Remark 5.1.19) so that, for any generator g, $A(g)$ is a strong maximal right ideal (see 2 above). Further, note that $A(g)$ is a maximal strong right ideal (cf. Remark 5.2.3). Hence, $\mathcal{C}$ is of type $(0, 2)$ and $(0, 3)$.

Type $(1, \mu)$: Note that, by Remark 5.3.3, the $\mathcal{N}$-semigroup $\mathcal{C}$ is strongly monogenic. By above 1, 2 and 3, the $\mathcal{N}$-semigroup $\mathcal{C}$ is of type $(1, 0)$, $(1, 1)$, $(1, 2)$ and $(1, 3)$.

Type $(2, \mu)$: By Lemma 5.3.4, $\mathcal{N}$-semigroup $\mathcal{C}$ is of type $(2, 0)$. Further, $\mathcal{C}$ is monogenic and, for any generator g of $\mathcal{C}$, $A(g)$ is a maximal $\mathcal{N}$-subsemigroup of $\mathcal{C}$ (cf. Corollary 5.3.5 and Remark 5.3.6). Hence, $\mathcal{C}$ is of type $(2, 1)$.

□

In view of Remark 5.3.6 and Theorem 5.3.8, we have the following:

Corollary 5.3.9. *The near-semiring $\mathcal{N}$ is (ν, μ)-primitive for all ν and μ.*

Theorem 5.3.10. *For $\nu = 0, 1$ with $\mu = 0, 1, 2, 3$ and $\nu = 2$ with $\mu = 0, 1$. We have*

$$J_{(\nu,\mu)}(\mathcal{N}) = \{0\}.$$

Theorem 5.3.11. *For $\nu = 0, 1$, we have $R_\nu(\mathcal{N}) = \{0\}$.*

Proof. We prove that result by showing that the right ideal $\{0\}$ is a modular maximal right ideal as well as a maximal modular right ideal. By Lemma 5.3.2 and 5.3.6, the $\mathcal{N}$-semigroup $\mathcal{C}$ is monogenic and has a generator g such that $A(g) = \{0\}$. Hence, the right ideal $\{0\}$ is modular (cf. Theorem 5.1.31). Further, since $\{0\}$ is a maximal right ideal (cf. Remark 5.2.3), we have $\{0\}$ is a modular maximal right ideal and also a maximal modular right ideal. □

Theorem 5.3.12. *For $\nu = 2, 3$, we have $R_\nu(\mathcal{N}) = \mathcal{N}$.*

Proof. In view Theorem 5.2.2, we prove that $\lambda_{\{0\}}$ is not modular. By Remark 5.1.17, the congruence relation $\mathfrak{n}''_{\{0\}}$ is the equality relation on $(\mathcal{N}, +)$ so that the semigroup homomorphism $\lambda_{\{0\}}$ is an identity map on $(\mathcal{N}, +)$. If the morphism $\lambda_{\{0\}}$ is modular, then there is an element $u \in \mathcal{N}$ such that $x = ux$ for all $x \in \mathcal{N}$, but there is no left identity element in $\mathcal{N}$. Consequently, $\lambda_{\{0\}}$ is not modular. Thus, there is no maximal λ-modular right ideal and λ-modular maximal right ideal of $\mathcal{N}$. Hence, for $\nu = 2, 3$, we have $R_\nu(\mathcal{N}) = \mathcal{N}$. □

5.4 Ideals

In this section, to obtain the ideals of $\mathcal{N}$, first we identify all the congruences on $\mathcal{N}$.

Lemma 5.4.1. *Let $\sim$ be a nontrivial congruence over the near-semiring $\mathcal{N}$ and $f \in \mathcal{N} \setminus \{0, \xi_\vartheta\}$. If $f \sim \xi_\vartheta$, then $\sim = (A^+(B_n) \times A^+(B_n)) \cup \{(0, 0)\}$.*

Proof. First note that $(A^+(B_n) \times A^+(B_n)) \cup \{(0, 0)\}$ is a congruence relation of the near-semiring $\mathcal{N}$. If $f \in A^+(B_n)_{n^2+1}$, since $\sim$ is a congruence of the semigroup

$(\mathcal{N},+)$, by Lemma 5.2.1, we have the result. Otherwise, we reduce the problem to Lemma 5.2.1 in the following cases.

Case 1 f is of singleton support. Let $f = {}^{(k,l)}\zeta_{(p,q)}$. Since ${}^{(k,l)}\zeta_{(p,q)} \sim \xi_\vartheta$ we have

$$\xi_{(k,l)}\ {}^{(k,l)}\zeta_{(p,q)} \sim \xi_{(k,l)}\xi_\vartheta$$

so that $\xi_{(p,q)} \sim \xi_\vartheta$.

Case 2 f is of n-support. Let $f = (p,q;\sigma)$. Since $(p,q;\sigma) \sim \xi_\vartheta$ we have

$$\xi_{(k,p)}(p,q;\sigma) \sim \xi_{(k,p)}\xi_\vartheta$$

so that $\xi_{(k\sigma,q)} \sim \xi_\vartheta$.

□

Lemma 5.4.2. *If two nonzero elements are in one class under a nontrivial congruence over* $\mathcal{N}$*, then the congruence is* $(A^+(B_n) \times A^+(B_n)) \cup \{(0,0)\}$.

Proof. Let $f, g \in \mathcal{N} \setminus \{0\}$ such that $f \sim g$ under a congruence $\sim$ over $\mathcal{N}$. If f or g is the absorbing map ξ_ϑ, then by Lemma 5.4.1, we have the result. Otherwise, we consider the following six cases classified by the supports of f and g. In each case, we show that there is an element $h \in A^+(B_n) \setminus \{\xi_\vartheta\}$ such that $h \sim \xi_\vartheta$ so that the result follows from Lemma 5.4.1.

Case 1 $f, g \in A^+(B_n)_1$. Let $f = {}^{(i,j)}\zeta_{(k,l)}$ and $g = {}^{(s,t)}\zeta_{(u,v)}$. If $(i,j) \neq (s,t)$, we have

$$\xi_\vartheta = {}^{(i,j)}\zeta_{(k,l)} + {}^{(s,t)}\zeta_{(v,v)} \sim {}^{(s,t)}\zeta_{(u,v)} + {}^{(s,t)}\zeta_{(v,v)} = {}^{(s,t)}\zeta_{(u,v)}.$$

Otherwise, $(i,j) = (s,t)$ so that $(k,l) \neq (u,v)$. Now, if $k \neq u$, then we have

$${}^{(i,j)}\zeta_{(k,l)} = {}^{(i,j)}\zeta_{(k,k)} + {}^{(i,j)}\zeta_{(k,l)} \sim {}^{(i,j)}\zeta_{(k,k)} + {}^{(i,j)}\zeta_{(u,v)} = \xi_\vartheta.$$

Similarly, if $l \neq v$, we have

$$\xi_\vartheta = {}^{(i,j)}\zeta_{(k,l)} + {}^{(i,j)}\zeta_{(v,v)} \sim {}^{(i,j)}\zeta_{(u,v)} + {}^{(i,j)}\zeta_{(v,v)} = {}^{(i,j)}\zeta_{(u,v)}.$$

Case 2 $f, g \in A^+(B_n)_{n^2+1}$. Let $f = \xi_{(k,l)}$ and $g = \xi_{(u,v)}$. By considering full support maps whose images are the same as in various subcases of *Case 1*, we can show that there is an element in $A^+(B_n) \setminus \{\xi_\vartheta\}$ that is related to ξ_ϑ under $\sim$.

Case 3 $f, g \in A^+(B_n)_n$. Let $f = (i, j; \sigma)$ and $g = (k, l; \rho)$. If $l \neq j$, then

$$(i, j; \sigma) = (i, j; \sigma) + \xi_{(j,j)} \sim (k, l; \rho) + \xi_{(j,j)} = \xi_\vartheta.$$

Otherwise, we have $(i, j; \sigma) \sim (k, j; \rho)$. Now, if $i \neq k$, then

$$\xi_\vartheta = (k, k; id)(i, j; \sigma) \sim (k, k; id)(k, j; \rho) = (k, j; \rho).$$

In case $i = k$, we have $\sigma \neq \rho$. Thus, there exists $t \in [n]$ such that $t\sigma \neq t\rho$. Now, $(i, j; \sigma) \sim (i, j; \rho)$ implies $\xi_{(k,i)}(i, j; \sigma) \sim \xi_{(k,i)}(i, j; \rho)$, i.e. $\xi_{(k\sigma,j)} \sim \xi_{(k\rho,j)}$. Consequently,

$$\xi_{(k\sigma,j)} = \xi_{(k\sigma,k\sigma)} + \xi_{(k\sigma,j)} \sim \xi_{(k\sigma,k\sigma)} + \xi_{(k\rho,j)} = \xi_\vartheta.$$

Case 4 $f \in A^+(B_n)_1, g \in A^+(B_n)_{n^2+1}$. Let $f = {}^{(k,l)}\zeta_{(p,q)}$ and $g = \xi_{(i,j)}$. Now, for $(s, t) \neq (k, l)$, we have

$$\xi_\vartheta = \xi_{(s,t)} f \sim \xi_{(s,t)} g = \xi_{(i,j)}.$$

Case 5 $f \in A^+(B_n)_{n^2+1}, g \in A^+(B_n)_n$. Let $f = \xi_{(p,q)}$ and $g = (i, j; \sigma)$. Now, for $l \neq i$, we have

$${}^{(k,l)}\zeta_{(p,q)} = {}^{(k,l)}\zeta_{(p,p)} + f \sim {}^{(k,l)}\zeta_{(p,p)} + g = \xi_\vartheta.$$

Case 6 $f \in A^+(B_n)_1, g \in A^+(B_n)_n$. Let $f = {}^{(k,l)}\zeta_{(p,q)}$ and $g = (i, j; \sigma)$. Now, for $l \neq i$, we have

$$\xi_\vartheta = \xi_{(i,i)} f \sim \xi_{(i,i)} g = \xi_{(i\sigma,j)}.$$

$\square$

In view of Lemma 5.4.1 and Lemma 5.4.2, we obtained all the congruences on $\mathcal{N}$ in the following theorem.

Theorem 5.4.3. *The near-semiring $\mathcal{N}$ has precisely the following congruences.*

1. *Equality relation.*
2. $\mathcal{N} \times \mathcal{N}$.
3. $(A^+(B_n) \times A^+(B_n)) \cup \{(0, 0)\}$.

Now, we are ready to report the ideals of $\mathcal{N}$ in the following corollary.

Corollary 5.4.4. *$\mathcal{N}$ and $\{0\}$ are the only ideals of the near-semiring $\mathcal{N}$.*

6

Syntactic Semigroups

This chapter explores formal language theoretic connections to $A^+(B_n)$. In this direction, the chapter considers the syntactic semigroup problem of $A^+(B_n)$. The syntactic semigroup problem is to decide whether a given finite semigroup is syntactic or not. The syntactic semigroup problem for various semigroups have been investigated by many authors (cf. Goralčík and Koubek [1998]; Goralčík et al. [1982]; Lallement and Milito [1975]). This chapter ascertains that both the semigroup reducts of $A^+(B_n)$ are syntactic semigroups. In this connection, Section 6.1 recalls the necessary preliminaries of the chapter. Then the syntactic semigroup problem for the additive semigroup reduct and the multiplicative semigroup reduct are studied in Section 6.2 and Section 6.3, respectively.

6.1 Preliminaries of formal languages

In this section, we present a necessary background material for subsequent sections. For more details one may refer to [Lawson, 2004].

Definition 6.1.1. Let Σ be a nonempty finite set called an *alphabet* and its elements are called *letters/symbols*. A *word* over Σ is a finite sequence of letters written by juxtaposing them. The set of all words over Σ forms a monoid with respect to concatenation of words, called the *free monoid* over Σ and it is denoted by Σ^*. The identity of Σ^* is the empty word (the empty sequence of letters), which is denoted by ε. The (free) semigroup of all nonempty words over Σ is denoted by Σ^+. A *language* L over Σ is a subset of the free monoid Σ^*.

In what follows, Σ always denotes an alphabet.

Definition 6.1.2. An *automaton* is a quintuple $\mathcal{A} = (Q, \Sigma, q_0, T, \delta)$, where Q is a nonempty finite set called the set of *states*, Σ is an *input alphabet*, $q_0 \in Q$ called the *initial state* and T is a subsets of Q, called the set of *final states*, and $\delta : Q \times \Sigma \to Q$ is a function, called the *transition function*.

Clearly, by denoting the states as vertices/nodes and the transitions as labeled arcs, an automaton can be represented by a directed graph (digraph) in which initial and final states shall be distinguished appropriately.

Definition 6.1.3. Let $\mathcal{A} = (Q, \Sigma, q_0, T, \delta)$ be an automaton. For every word $x \in \Sigma^*$, the function $f_x : Q \to Q$ is defined inductively as follows. For $q \in Q$,

(i) if $x = \varepsilon$, then $qf_x = q$;

(ii) if $x = a$ for $a \in \Sigma$, then $qf_a = (q, a)\delta$;

(iii) if $x = ay$ for $y \in \Sigma^*$ and $a \in \Sigma$, then $qf_x = qf_af_y$.

Remark 6.1.4. For an automaton $\mathcal{A}$, the set $\mathscr{T}(\mathcal{A}) = \{f_x \mid x \in \Sigma^*\}$ forms a monoid under the composition of functions.

Definition 6.1.5. The monoid $\mathscr{T}(\mathcal{A})$ is called the *transition monoid of the automaton $\mathcal{A}$.*

Definition 6.1.6. Let $\mathcal{A}$ be an automaton. The *language accepted/ recognized* by $\mathcal{A}$, denoted by $L(\mathcal{A})$, defined by

$$L(\mathcal{A}) = \{x \in \Sigma^* \mid q_0 f_x \in T\}.$$

Definition 6.1.7. An automaton $\mathcal{A}$ is said to be *minimal* if the number of states of $\mathcal{A}$ is less than or equal to the number of states of any other automaton accepting $L(\mathcal{A})$.

Definition 6.1.8. Let S be a semigroup and let $L \subseteq S$ be an arbitrary subset. The equivalence relation $\approx_L$ on S defined by

$$x \approx_L y \text{ iff } uxv \in L \Longleftrightarrow uyv \in L \text{ for all } u, v \in S$$

is a congruence known as the *syntactic congruence* of L.

Definition 6.1.9. For $L \subseteq \Sigma^+$, the quotient semigroup $\Sigma^+/_{\approx_L}$ is known as the *syntactic semigroup* of the language L. Further, the quotient monoid $\Sigma^*/_{\approx_L}$ is called the *syntactic monoid* of the language L.

Theorem 6.1.10. *Let L be a recognizable language over Σ. The syntactic monoid of L is isomorphic to the transition monoid of the minimal automaton accepting L.*

Definition 6.1.11. A subset D of a semigroup S is called disjunctive in S if the congruence $\approx_D$ is the equality relation on S.

Now we have a characterization for a finite semigroup to be a syntactic semigroup in the following theorem.

Theorem 6.1.12. *A finite semigroup is the syntactic semigroup of a recognizable language if and only if it contains a disjunctive subset.*

Definition 6.1.13. A star-free expression over an alphabet Σ is defined inductively as follows.

(i) $\varnothing$ and ε are star-free expressions representing the languages $\varnothing$ and $\{\varepsilon\}$, respectively.

(ii) For $a \in \Sigma$, a is a star-free expression representing the language $\{a\}$.

(iii) If r and s are star-free expressions representing the languages R and S, respectively, then $(r+s)$, (rs) and $r^{\complement}$ are star-free expressions representing the languages $R \cup S, RS$ and $R^{\complement}$, respectively. Here, $R^{\complement}$ denotes the complement of R in Σ^*.

A language is said to be *star-free* if it can be represented by a star-free expression.

Remark 6.1.14.

1. Σ^* is star-free and $\varnothing^{\complement}$ is its star-free expression.

2. If $B \subsetneq \Sigma$, then B^* is star-free. For instance, let $\Sigma \setminus B = \{a_1, \ldots, a_k\}$. Then

$$\left(\varnothing^{\complement}(a_1 + \cdots + a_k)\varnothing^{\complement}\right)^{\complement}$$

is a star-free expression of B^*.

Example 6.1.15. For $\Sigma = \{a, b\}$, the language $(ab)^*$ is star-free and

$$(b\varnothing^{\complement} + \varnothing^{\complement}a + \varnothing^{\complement}aa\varnothing^{\complement} + \varnothing^{\complement}bb\varnothing^{\complement})^{\complement}$$

is a star-free expression of $(ab)^*$.

Theorem 6.1.16 ([Schützenberger, 1965])**.** *A recognizable language is star-free if and only if its syntactic monoid is aperiodic.*

6.2 The semigroup $A^+(B_n)^+$

In this section, we prove that the semigroup $A^+(B_n)^+$ is syntactic. We prove the result in the following two subsections, for the case $n = 1$ and $n \geq 2$, respectively.

6.2.1 The case $n = 1$

Note that $A^+(B_1)^+$ is a monoid with the identity element $\xi_{(1,1)}$. In this subsection, we show that $A^+(B_1)^+$ is the syntactic monoid of a recognizable star-free language.

Let $\Sigma = \{a, b, c\}$ and consider the language $L_b = \{x \in \{a, b\}^* \mid x \text{ has at least one } b\}$ over Σ.

Remark 6.2.1. The language L_b is recognizable. For instance, the automaton $\mathcal{A}_b$ given in Figure 6.1 accepts the language L_b. Furthermore, $\mathcal{A}_b$ is minimal.

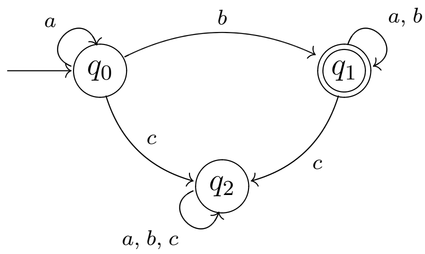

FIGURE 6.1: The automation $\mathcal{A}_b$ accepting the language L_b

Since $\mathcal{A}_b$ is minimal, by Theorem 6.1.10, we have the following remark.

Remark 6.2.2. The transition monoid $\mathscr{T}(\mathcal{A}_b)$, given in Table 6.1, of the automaton $\mathcal{A}_b$ is syntactic.

Theorem 6.2.3. *The monoid $A^+(B_1)^+$ is a syntactic monoid.*

Proof. The mapping $\phi : \mathscr{T}(\mathcal{A}) \longrightarrow A^+(B_1)^+$ which assigns $f_a \mapsto \xi_{(1,1)}, f_b \mapsto (1, 1; id)$ and $f_c \mapsto \xi_\vartheta$ is an isomorphism. Hence, the monoid $A^+(B_1)^+$ is syntactic monoid. $\square$

	f_a	f_b	f_c
f_a	f_a	f_b	f_c
f_b	f_b	f_b	f_c
f_c	f_c	f_c	f_c

Table 6.1: The Cayley table for $\mathscr{T}(\mathcal{A}_b)$

Note that $A^+(B_1)^+$ is aperiodic. Hence, by Theorem 6.1.16, the language L_b is star-free. Indeed, a star-free expression for the language L_b is given by

$$(\varnothing^{\complement} c \varnothing^{\complement})^{\complement} b (\varnothing^{\complement} c \varnothing^{\complement})^{\complement}.$$

6.2.2 The case $n \geq 2$

In this subsection, we show that the semigroup $A^+(B_n)^+$ is isomorphic to the syntactic semigroup of some language (cf. Theorem 6.2.7).

Let $\Sigma = \mathrm{Aff}(B_n)$. For $x = f_1 f_2 \cdots f_k \in \Sigma^+$, write $\hat{x} = f_1 + f_2 + \cdots + f_k$. Note that $\hat{x} \in A^+(B_n)$.

Remark 6.2.4. For $x, y \in \Sigma^+$, we have $\widehat{xy} = \hat{x} + \hat{y}$. Hence, the function

$$\varphi : \Sigma^+ \longrightarrow A^+(B_n)^+$$

defined by $x\varphi = \hat{x}$ is an onto homomorphism of semigroups.

Consider the language

$$L = \{x \in \Sigma^+ \mid \hat{x} \in P\},$$

where $P = \{\xi_{(1,2)}\} \cup \left\{ {}^{(k,l)}\zeta_{(1,1)} \mid k, l \in [n] \right\}$. Now we characterize the syntactic congruence of the language L in the following theorem.

Theorem 6.2.5. *For $x, y \in \Sigma^+$, $x \approx_L y$ if and only if $\hat{x} = \hat{y}$.*

Proof. (:$\Leftarrow$) Suppose $\hat{x} = \hat{y}$. For $u, v \in \Sigma^+$,

$$\begin{aligned} uxv \in L \Longleftrightarrow \widehat{uxv} \in P &\Longleftrightarrow \hat{u} + \hat{x} + \hat{v} \in P \\ &\Longleftrightarrow \hat{u} + \hat{y} + \hat{v} \in P \\ &\Longleftrightarrow \widehat{uyv} \in P \Longleftrightarrow uyv \in L. \end{aligned}$$

Hence, $x \approx_L y$.

($\Rightarrow$:) Suppose $\hat{x} \neq \hat{y}$. We claim that there exist u and v in Σ^+ such that $\hat{u} + \hat{x} + \hat{v} \in P$, whereas $\hat{u} + \hat{y} + \hat{v} \notin P$. Consequently, $uxv \in L$ but $uyv \notin L$ so that $x \not\approx_L y$. In view of Theorem 2.2.10, we prove our claim in the following cases.

Case 1 $\hat{x}$ is of full support, say $\xi_{(p,q)}$. Choose $u, v \in \Sigma^+$ such that $\hat{u} = \xi_{(1,p)}$ and $\hat{v} = \xi_{(q,2)}$ (cf. Remark 6.2.4). Note that

$$\hat{u} + \hat{x} + \hat{v} = \xi_{(1,p)} + \xi_{(p,q)} + \xi_{(q,2)} = \xi_{(1,2)} \in P.$$

However, as shown below, $\hat{u} + \hat{y} + \hat{v} \notin P$, for any possibility of $\hat{y} \in A^+(B_n)$.

Subcase 1.1 If $\hat{y} = \xi_\vartheta$, then clearly $\hat{u} + \hat{y} + \hat{v} = \xi_\vartheta \notin P$.

Subcase 1.2 Let $\hat{y}$ be of full support, say $\xi_{(r,s)}$. In either of the cases, $p \neq r$ or $q \neq s$, we can clearly observe that

$$\hat{u} + \hat{y} + \hat{v} = \xi_{(1,p)} + \xi_{(r,s)} + \xi_{(q,2)} = \xi_\vartheta \notin P.$$

Subcase 1.3 Let $\hat{y}$ be of n-support, say $(k, l; \sigma)$. Let $j \in [n]$ such that $j\sigma = p$. Note that

$$\hat{u} + \hat{y} + \hat{v} = \xi_{(1,p)} + (k, l; \sigma) + \xi_{(q,2)} = \begin{cases} \xi_\vartheta & \text{if } l \neq q; \\ {}^{(j,k)}\zeta_{(1,2)} & \text{if } l = q. \end{cases}$$

Subcase 1.4 Let $\hat{y}$ be of singleton support, say ${}^{(k,l)}\zeta_{(r,s)}$. Note that

$$\hat{u} + \hat{y} + \hat{v} = \xi_{(1,p)} + {}^{(k,l)}\zeta_{(r,s)} + \xi_{(q,2)} = \begin{cases} \xi_\vartheta & \text{if } p \neq r \text{ or } q \neq s; \\ {}^{(k,l)}\zeta_{(1,2)} & \text{otherwise.} \end{cases}$$

Case 2 $\hat{x}$ is of n-support, say $(p, q; \sigma)$.

Subcase 2.1 Let $\hat{y}$ be of n-support, say $(k, l; \tau)$. Choose $u, v \in \Sigma^+$ such that $\hat{u} = {}^{(l,p)}\zeta_{(1,l\sigma)}$ and $\hat{v} = \xi_{(q,1)}$. Note that

$$\hat{u} + \hat{x} + \hat{v} = {}^{(l,p)}\zeta_{(1,l\sigma)} + (p, q; \sigma) + \xi_{(q,1)} = {}^{(l,p)}\zeta_{(1,1)} \in P.$$

If $p \neq k$ or $q \neq l$, we have

$$\hat{u} + \hat{y} + \hat{v} = {}^{(l,p)}\zeta_{(1,l\sigma)} + (k, l; \tau) + \xi_{(q,1)} = \xi_\vartheta \notin P.$$

Otherwise, we have $\sigma \neq \tau$. there exists $j_0 \in [n]$ such that $j_0\sigma \neq j_0\tau$. Now choose $u, v \in \Sigma^+$ such that $\hat{u} = {}^{(j_0,p)}\zeta_{(1,j_0\sigma)}$ and $\hat{v} = \xi_{(q,1)}$. Note that $\hat{u} + \hat{x} + \hat{v} = {}^{(j_0,p)}\zeta_{(1,1)} \in P$, whereas

$$\hat{u} + \hat{y} + \hat{v} = {}^{(j_0,p)}\zeta_{(1,j_0\sigma)} + (p, q; \tau) + \xi_{(q,1)} = \xi_\vartheta \notin P$$

.

Subcase 2.2 Let $\hat{y}$ be of singleton support, say ${}^{(j,k)}\zeta_{(m,r)}$. Choose $u, v \in \Sigma^+$ such that $\hat{u} = {}^{(l,p)}\zeta_{(1,l\sigma)}$ (with $l\sigma \neq m$) and $\hat{v} = \xi_{(q,1)}$. Note that

$$\hat{u} + \hat{x} + \hat{v} = {}^{(l,p)}\zeta_{(1,1)} \in P,$$

whereas

$$\hat{u} + \hat{y} + \hat{v} = {}^{(l,p)}\zeta_{(1,l\sigma)} + {}^{(j,k)}\zeta_{(m,r)} + \xi_{(q,1)} = \xi_\vartheta \notin P.$$

Subcase 2.3 If $\hat{y} = \xi_\vartheta$, then, for $\hat{u} = {}^{(l,p)}\zeta_{(1,l\sigma)}$ and $\hat{v} = \xi_{(q,1)}$ we have $\hat{u}+\hat{x}+\hat{v} = {}^{(l,p)}\zeta_{(1,1)} \in P$, but $\hat{u} + \hat{y} + \hat{v} = \xi_\vartheta \notin P$.

Case 3 $\hat{x}$ is of singleton support, say ${}^{(p,q)}\zeta_{(r,s)}$. Choose $u, v \in \Sigma^+$ such that $\hat{u} = {}^{(p,q)}\zeta_{(1,r)}$ and $\hat{v} = {}^{(p,q)}\zeta_{(s,1)}$ so that

$$\hat{u} + \hat{x} + \hat{v} = {}^{(p,q)}\zeta_{(1,r)} + {}^{(p,q)}\zeta_{(r,s)} + {}^{(p,q)}\zeta_{(s,1)} = {}^{(p,q)}\zeta_{(1,1)} \in P.$$

Subcase 3.1 $\hat{y}$ is of singleton support, say ${}^{(j,k)}\zeta_{(l,m)}$. Since $\hat{x} \neq \hat{y}$, we have

$$\hat{u} + \hat{y} + \hat{v} = {}^{(p,q)}\zeta_{(1,r)} + {}^{(j,k)}\zeta_{(l,m)} + {}^{(p,q)}\zeta_{(s,1)} = \xi_\vartheta \notin P.$$

Subcase 3.2 If $\hat{y} = \xi_\vartheta$, then clearly $\hat{u} + \hat{y} + \hat{v} = \xi_\vartheta \notin P$.

$\square$

In view of Remark 6.2.4, we have the following corollary of Theorem 6.2.5.

Corollary 6.2.6. *The syntactic congruence $\approx_L$ and the kernel of the homomorphism φ are identical, i.e. $\approx_L = \ker\varphi$.*

Now we have the following main theorem of the section.

Theorem 6.2.7. *For $n \geq 2$, the semigroup $A^+(B_n)^+$ is syntactic.*

Proof. Using fundamental theorem of homomorphisms of semigroups, by Theorem 1.1.21 and Remark 6.2.4, we have $\Sigma^+/_{\ker\varphi}$ is isomorphic to $A^+(B_n)^+$. However, by Corollary 6.2.6, we have $\Sigma^+/_{\approx_L}$ and $A^+(B_n)^+$ are isomorphic. Hence, $A^+(B_n)^+$ is a syntactic semigroup of the language L. $\square$

In view of Remark 3.1.9 and Theorem 6.1.16, we have the following corollary of Theorem 6.2.7

Corollary 6.2.8. *The language L is star-free.*

6.3 The semigroup $A^+(B_n)^\circ$

In this section, we prove that the semigroup $A^+(B_n)^\circ$ is syntactic. We prove the result in the following two subsections, for the case $n = 1$ and $n \geq 2$, respectively.

6.3.1 The case $n = 1$

Note that $A^+(B_1)^\circ$ is monoid with the identity element $(1, 1; id)$. In this subsection, we show that $A^+(B_1)^\circ$ is the syntactic monoid of a recognizable star-free language.

Let $\Sigma = \{a, b, c\}$ and consider the language $L_a = \{xab^n \mid x \in \Sigma^* \text{ and } n \geq 0\}$ over Σ.

Remark 6.3.1. The language L_a is recognizable. For instance, the automaton $\mathcal{A}_a$ given in Figure 6.2 accepts the language L_a. Furthermore, $\mathcal{A}_a$ is minimal.

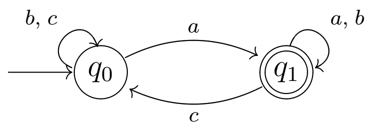

FIGURE 6.2: The automation $\mathcal{A}_a$ accepting the language L_a

Since $\mathcal{A}_a$ is minimal, by Theorem 6.1.10, we have the following remark.

Remark 6.3.2. The transition monoid $\mathscr{T}(\mathcal{A}_a)$, given in Table 6.2, of the automaton $\mathcal{A}_a$ is syntactic.

	f_a	f_b	f_c
f_a	f_a	f_a	f_c
f_b	f_a	f_b	f_c
f_c	f_a	f_c	f_c

Table 6.2: The Cayley table for $\mathscr{T}(\mathcal{A}_a)$

Theorem 6.3.3. *The monoid $A^+(B_1)^\circ$ is a syntactic monoid.*

Proof. The mapping $\phi : \mathscr{T}(\mathcal{A}) \longrightarrow A^+(B_1)^\circ$ which assigns $f_a \mapsto \xi_{(1,1)}$, $f_b \mapsto (1, 1; id)$ and $f_c \mapsto \xi_\vartheta$ is an isomorphism. Hence, the monoid $A^+(B_1)^\circ$ is syntactic monoid. $\square$

Note that $A^+(B_1)^\circ$ is aperiodic. Hence, by Theorem 6.1.16, the language L_a is star-free. Indeed, a star-free expression for the language L_a is given by

$$\varnothing^{\complement} a(\varnothing^{\complement}(a + c)\varnothing^{\complement})^{\complement}.$$

6.3.2 The case $n \geq 2$

In this subsection, we prove that the semigroup $A^+(B_n)^\circ$ is syntactic by constructing a disjunctive subset (cf. Theorem 6.1.12).

Theorem 6.3.4. *The set $D = \{(1,1;id)\} \cup \{\xi_{(p,q)} \mid p,q \in [n]\}$ is a disjunctive subset of the semigroup $A^+(B_n)^\circ$. Hence, $A^+(B_n)^\circ$ is a syntactic semigroup.*

Proof. Let $f, g \in A^+(B_n)^\circ$ such that $f \neq g$. We claim that there exist h and h' in $A^+(B_n)^\circ$ such that only one among hfh' and hgh' is in D so that $f \not\approx_D g$. Since f and g are arbitrary, it follows that $\approx_D$ is the equality relation on $A^+(B_n)^\circ$. We prove our claim in various cases on supports of f and g (cf. Theorem 2.2.10).

Case 1 f is of n-support, say $(p,q;\sigma)$. Choose $h = (1,p;id)$ and $h' = (q,1;\sigma^{-1})$. Note that

$$hfh' = (1,p;id)(p,q;\sigma)(q,1;\sigma^{-1}) = (1,1;id) \in D.$$

If $g = \xi_\vartheta$, then clearly $hgh' = \xi_\vartheta \notin D$. If g is of n-support, say $(k,l;\tau)$, then

$$hgh' = (1,p;id)(k,l;\tau)(q,1;\sigma^{-1}) = \begin{cases} \xi_\vartheta & \text{if } p \neq k \text{ or } q \neq l \\ (1,1;\tau\sigma^{-1}) & \text{otherwise} \end{cases}$$

so that $hgh' \notin D$. In case g is of singleton support, say ${}^{(r,s)}\zeta_{(u,v)}$,

$$hgh' = (1,p;id)^{(r,s)}\zeta_{(u,v)}(q,1;\sigma^{-1}) \text{ is either } \xi_\vartheta \text{ or in } A^+(B_n)_1$$

so that $hgh' \notin D$. Finally, let g be a full support element, say $g = \xi_{(k,l)}$. Now, for $h = \xi_\vartheta$ and $h' = {}^{(k,l)}\zeta_{(s,t)}$, we have

$$hgh' = \xi_\vartheta \xi_{(k,l)}\, {}^{(k,l)}\zeta_{(s,t)} = \xi_{(s,t)} \in D$$

whereas

$$hfh' = \xi_\vartheta (p,q;\sigma)^{(k,l)}\zeta_{(s,t)} = \xi_\vartheta \notin D.$$

Case 2 f is of full support, say $\xi_{(p,q)}$. If g is a constant map, then choose $h = \xi_{(p,q)}$ and $h' = {}^{(p,q)}\zeta_{(u,v)}$ so that

$$hfh' = \xi_{(p,q)}\xi_{(p,q)}\ {}^{(p,q)}\zeta_{(u,v)} = \xi_{(u,v)} \in D$$

but

$$hgh' = \xi_{(p,q)}g\ {}^{(p,q)}\zeta_{(u,v)} = \xi_{\vartheta} \notin D.$$

In case g is of singleton support, say ${}^{(r,s)}\zeta_{(u,v)}$, choose $h = \xi_{\vartheta}$ and $h' = (q, q; id)$. Note that

$$hfh' = \xi_{\vartheta}\xi_{(p,q)}(q, q; id) = \xi_{(p,q)} \in D$$

whereas

$$hgh' = \xi_{\vartheta}\ {}^{(r,s)}\zeta_{(u,v)}(q, q; id) = \xi_{\vartheta} \notin D.$$

Case 3 f is of singleton support, say ${}^{(p,q)}\zeta_{(r,s)}$. Now, we will only assume g is a singleton support map, say ${}^{(k,l)}\zeta_{(u,v)}$, or the zero map. In any case, for $h = \xi_{(p,q)}$ and $h' = {}^{(r,s)}\zeta_{(u,v)}$, we have $hgh' = \xi_{(p,q)}g{}^{(r,s)}\zeta_{(u,v)} = \xi_{\vartheta} \notin D$, whereas

$$hfh' = \xi_{(p,q)}\ {}^{(p,q)}\zeta_{(r,s)}\ {}^{(r,s)}\zeta_{(u,v)} = \xi_{(u,v)} \in D.$$

□

6.4 Conclusion

In this chapter, we have shown that both the semigroup reducts of $A^+(B_n)$ are syntactic semigroups. In addition to the basic definition of syntactic semigroup, we have deployed various techniques, viz. minimal automata and disjunctive subsets, for various cases to ascertain that these semigroups are syntactic. While $A^+(B_1)^+$ and $A^+(B_1)^\circ$ are shown as the transition monoids of some minimal automata, for $n \geq 2$, we proved that $A^+(B_n)^+$ is isomorphic to the syntactic semigroup of a language. In case of $A^+(B_n)^\circ$, for $n \geq 2$, we have shown that it contains a disjunctive subset.

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Bio-Data

Full Name	:	Jitender Kumar
<u>Education</u>		
Present	:	Research Scholar, Department of Mathematics Indian Institute of Technology Guwahati, Guwahati.
2009	:	M. Phil. in Mathematics University of Rajasthan, Jaipur.
2007	:	M. Sc. in Mathematics University of Rajasthan, Jaipur.
2005	:	B. Sc. with Mathematics, Physics & Chemistry University of Bikaner, Bikaner.
<u>Teaching Experience</u>		
July, 2009 – Till Date	:	Teaching Assistant, Department of Mathematics IIT Guwahati, Guwahati.

<u>Publications</u>

1. *Affine Near-Semirings over Brandt Semigroups*, Comm. Algebra, To appear. DOI:10.1080/00927872.2013.833211.

2. *The Large Rank of a Finite Semigroup using Prime Subsets*, Semigroup Forum, To appear. DOI:10.1007/s00233-014-9577-0.

3. *The Ranks of Additive Semigroup Reduct of Affine Near-Semiring over Brandt Semigroup*, Communicated. arXiv:1308.4087.

4. *Rank Properties of Multiplicative Semigroup Reduct of Affine Near-Semirings over B_n*, Communicated. arXiv:1311.0789.

   All the above four articles are coauthored with K. V. Krishna.

Conferences/Workshops:

1. *The Large Rank of the semigroup of order-preserving singular maps* at the International Conference on Semigroups, Algebras and Operator theory (ICSAOT-2014), Kochi, India, February 26-28, 2014.

2. Instructional Workshop on Semigroups and Applications, Kochi, India, February 22-24, 2014.

3. *On the Ranks of Additive Semigroup Reducts of Affine Near-Semirings over Brandt Semigroups* at the Conference General Algebra and Its Applications: GAIA 2013, Melbourne, Australia, July 15-19, 2013.

4. International Workshop cum Conference on Groups, Actions, Computations (GAC 2010), HRI Allahabad, India, September 1-12, 2010.