Affine permutations and rational slope parking functions

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1. Topology of homogeneous affine Springer fibers (enumeration of cells).
2. Finite dimensional representations of DAHA and non-symmetric Macdonald polynomials.
3. Shuffle conjecture and its generalizations.

Shi Arrangements

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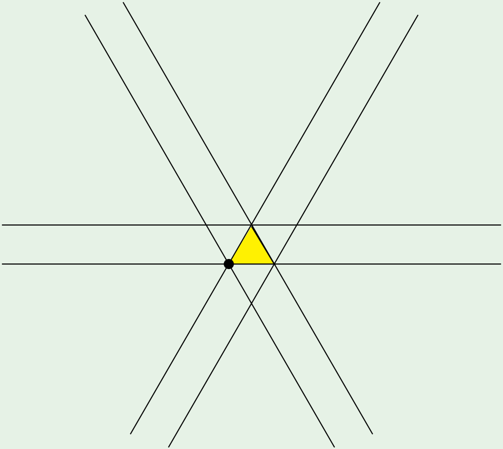
k-Shi arrangement: $Sh_n^k := \{\{x_i - x_j = l\} : 0 < j < i \leq n,\ -k < l \leq k\}$
Reg_n^k denote the set of connected components (regions) of the complement to Sh_n^k.

Example

$n = 3$, $k = 1, 2$.



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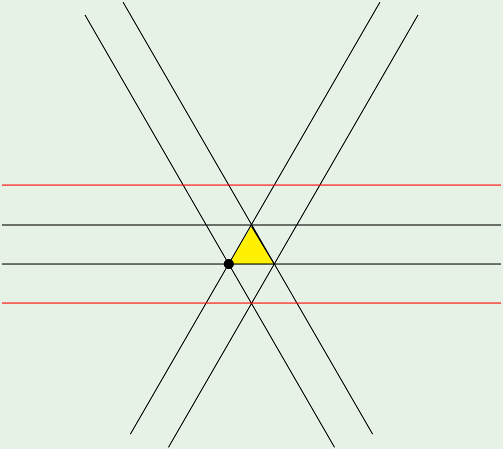

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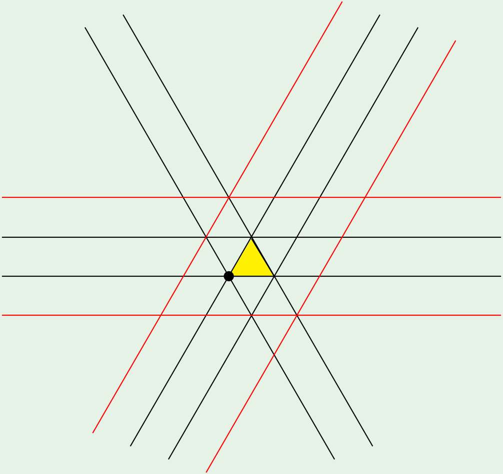

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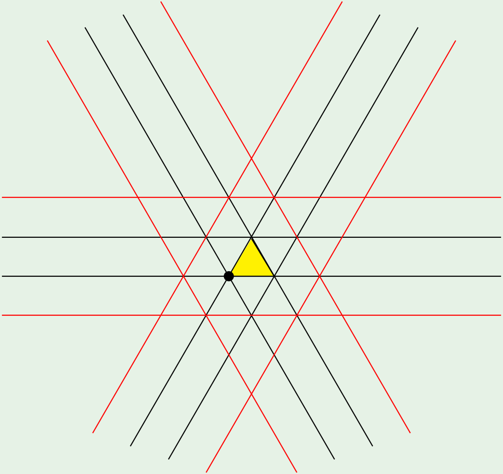

Rational Slope Parking Functions

Example

Consider the function $f : \{1, 2, 3, 4\} \to \mathbb{Z}$ given by $f(1) = 2$, $f(2) = 0$, $f(3) = 4$, and $f(4) = 0$. The corresponding Young diagram fits under diagonal in a 4×7 rectangle:

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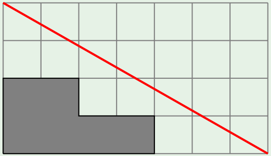

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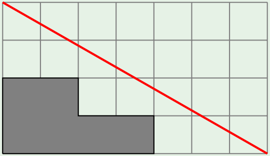

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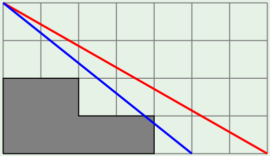

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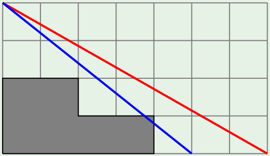

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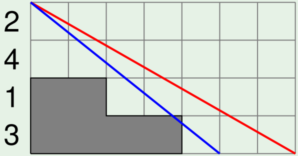

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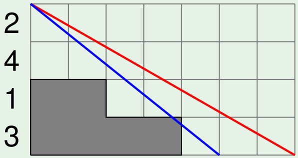

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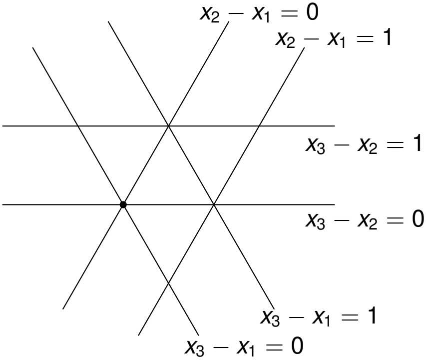
Remark

Note that for $m = n + 1$ the set $\mathrm{PF}_{m/n}$ is exactly the set of classical parking functions PF, and for $m = kn + 1$ it is the set of k-parking functions PF_k.

Pak-Stanley Labeling



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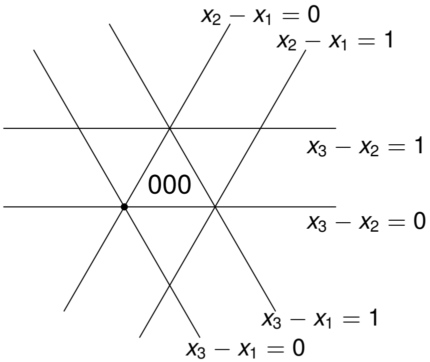

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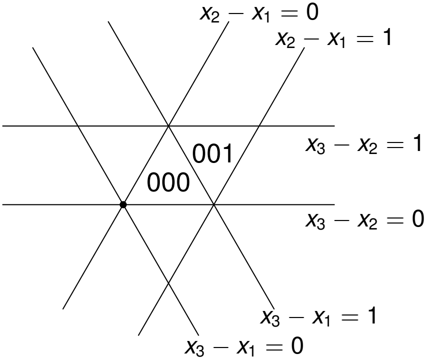

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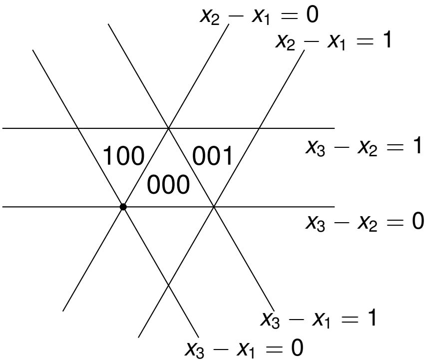

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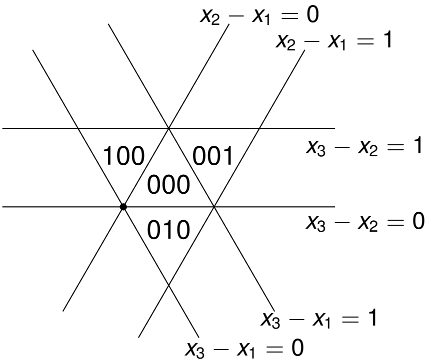

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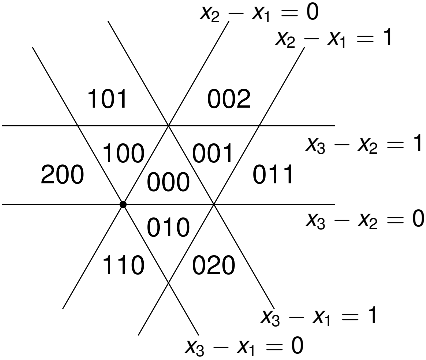

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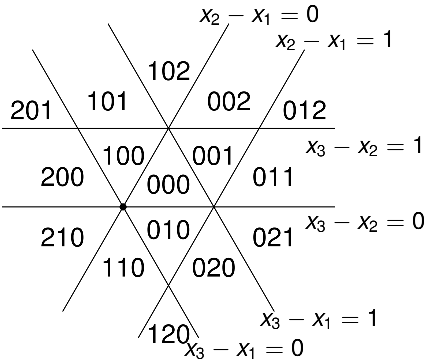

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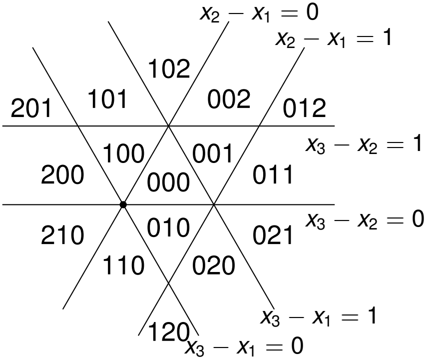

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Theorem (Pak-Stanley)

The map $\lambda : Reg_n^k \to \mathrm{PF}_k = \mathrm{PF}_{(kn+1)/n}$ is a bijection.

Affine Symmetric Group

Definition

The *affine symmetric group* $\widehat{S}_n$ is generated by $s_1, \ldots, s_{n-1}, s_0$ subject to

1. $s_i^2 = 1$,
2. $s_i s_j = s_j s_i$ for $i - j \not\equiv \pm 1 \bmod n$,
3. $s_i s_j s_i = s_j s_i s_j$ for $i - j \equiv \pm 1 \bmod n$.

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$\widehat{S}_n$ acts on V with generators s_i acting by reflections in hypersurfaces $x_{i+1} - x_i = 0$ for $i > 0$, and s_0 acting by reflection in the hypersurface $x_n - x_1 = 1$.

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The induced action on the set alcoves is free and transitive, so that the map $\omega \mapsto \omega(A_0)$ provides a bijection from $\widehat{S}_n$ to the set of alcoves.

Affine Permutations

Definition

A bijection $\omega : \mathbb{Z} \to \mathbb{Z}$ is called an affine S_n-permutation, if $\omega(x+n) = \omega(x) + n$ for all x, and $\sum_{i=1}^{n} \omega(i) = \frac{n(n+1)}{2}$.

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In this presentation the operation is composition and the generators $s_1, \ldots, s_{n-1}, s_0$ are given by

1. $s_i(x) = x + 1$ for $x \equiv i \bmod n$,
2. $s_i(x) = x - 1$ for $x \equiv i + 1 \bmod n$,
3. $s_i(x) = x$ otherwise.

Stable Affine Permutations

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Definition

An affine permutation $\omega \in \widehat{S}_n$ is called *m*-stable, if for all *x* the inequality $\omega(x+m) > \omega(x)$ holds, i.e. there are no inversions of height *m*. The set of all *m*-stable affine permutations is denoted by $\widehat{S}_n^m$.

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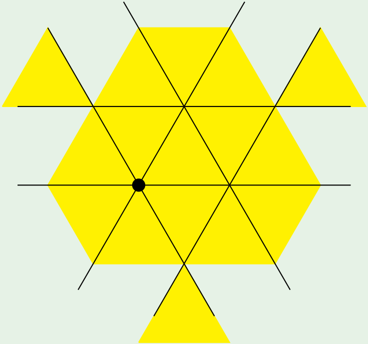
Lemma

Let $m = kn + r$, where $0 < r < n$. The set of alcoves $\{\omega^{-1}(\mathrm{A}_0) : \omega \in \widehat{S}_n^m\}$ coincides with the set of alcoves that fit inside the region $D_n^m \subset V$ defined by the inequalities:

1. $x_i - x_{i+r} \geq -k$ for $1 \leq i \leq n - r$,
2. $x_{i+r-n} - x_i \leq k + 1$ for $n - r + 1 \leq i \leq n$.

Example

$n = 3$, $k = 1$, and $m = kn + 1 = 4$.



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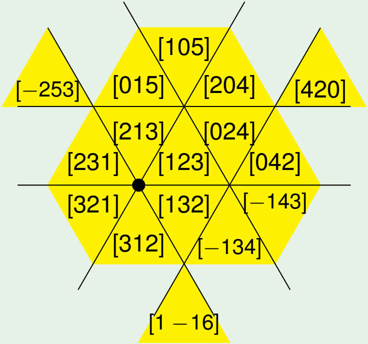

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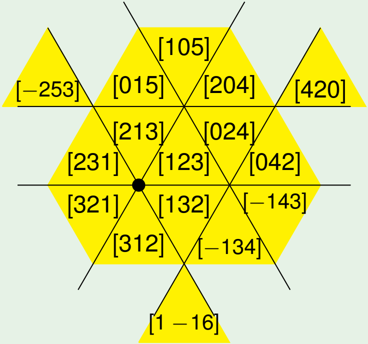

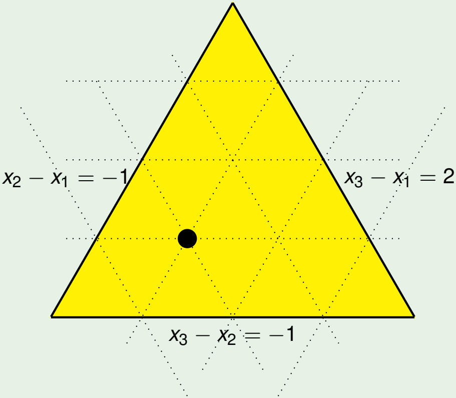

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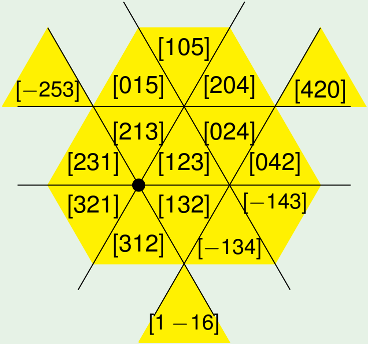

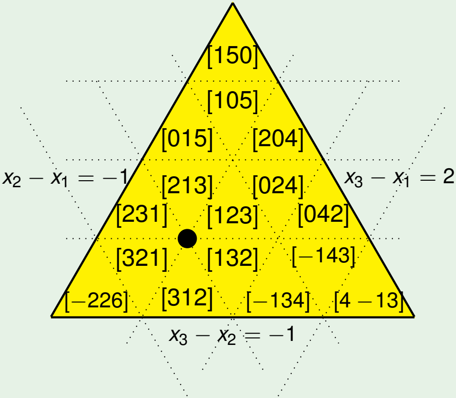

Map *PS*

Definition

Let $\omega \in \widehat{S}_n^m$. Then the map $\mathrm{PS}_\omega : \{1, \ldots, n\} \to \mathbb{Z}$ is given by:

$$\mathrm{PS}_\omega(\alpha) := \sharp\left\{\beta \mid \beta > \alpha, 0 < \omega^{-1}(\alpha) - \omega^{-1}(\beta) < m\right\}.$$

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Conjecture

The map $\mathrm{PS} : \omega \mapsto \mathrm{PS}_\omega$ *is a bijection between* $\widehat{S}_n^m$ *and* $\mathrm{PF}_{m/n}$.

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The case $m = kn - 1$ can be covered by a small modification of the above.

Bijection A : $\widehat{S}_n^m \to \mathrm{PF}_{m/n}$

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$$\begin{array}{cccccccccccccc} x & \ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\ \omega(x) & \ldots & -4 & 2 & -1 & -3 & 0 & 6 & 3 & 1 & 4 & 10 & 7 & 5 & \ldots \end{array}$$

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Note that there are no inversions of height 7, so ω is 7-stable. Consider the set $\Delta_\omega := \{i \in \mathbb{Z} : \omega(i) > 0\} = \{-2, 2, 3, 4, \dots\}$. Note that it is invariant under addition of 4 and 7. The parking function A_ω is constructed as follows:

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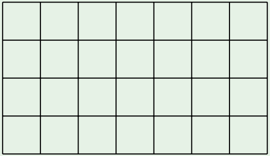

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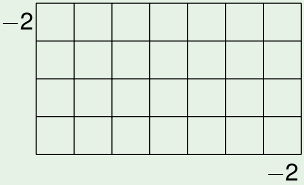
$\omega(x)$	...	-4	2	-1	-3	0	6	3	1	4	10	7	5	...

Note that there are no inversions of height 7, so ω is 7-stable. Consider the set $\Delta_\omega := \{i \in \mathbb{Z} : \omega(i) > 0\} = \{-2, 2, 3, 4, \ldots\}$. Note that it is invariant under addition of 4 and 7. The parking function A_ω is constructed as follows:



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In particular, the "weak symmetry" holds for $m = kn \pm 1$.

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Thank you!