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Affine semipolar spaces

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Abstract. Deleting a hyperplane from a polar space associated with a symplectic polarity we get a specific, symplectic, affine polar space. Similar geometry, called an affine semipolar space arises as a result of generalization of the notion of an alternating form to a semiform. Some properties of these two geometries are given and their automorphism groups are characterized.

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Introduction

Dealing with symplectic affine polar spaces we observe some regularities that lead to a new notion: *semiform*. In turn semiforms give rise to an interesting class of quite general partial linear spaces called *affine semipolar spaces*.

In [5] an affine polar space (APS in short) is derived from a polar space the same way as an affine space is derived from a projective space, i.e. by deleting a hyperplane from a polar space embedded into a projective space. Such affine polar spaces are embeddable in affine spaces and this let us think of them as of suitable reducts of affine spaces.

In general we have two types of APS'es. Structures of the first type are associated with polar spaces determined by sesquilinear forms; one can loosely say: these are "stereographical projections of quadrics". They can also be thought of as determined by sesquilinear forms defined on vector spaces which represent respective affine spaces. These structures and adjacency of their subspaces were studied in [17]. Contrary to [5], in this approach Minkowskian geometry is not excluded. In particular, the result of [17] generalizes Alexandrov–Zeeman theorems originally concerning adjacency *of points* of an affine polar space (cf. [1, 19]).

The second class of APS'es, which is included in [5] but is excluded from [17], consists of structures associated with polar spaces determined by symplectic polarities. The aim of this paper is to present in some detail the geometry of the structures in this class from view of the affine space in which they are embedded. The position of the class of thus obtained structures—let us call them symplectic affine polar spaces—is in many points a particular one.

Firstly, symplectic affine polar spaces are associated with null-systems, quite well known polarities in projective spaces with all points selfconjugate. So, symplectic APS'es have famous parents. Moreover, in each even-dimensional pappian projective space such a (projectively unique) polarity exists. Thus a symplectic APS is not an exceptional space, but conversely, it is also a "canonical" one in each admissible dimension.

A second argument refers to the position of the class of symplectic APS'es in the class of all APS'es. As an affine polar space is obtained by deleting a hyperplane from a polar space, while the latter is realized as a quadric in a metric projective space, the derived APS appears as a fragment of the derived affine space. If the underlying form that determines the polar space is symmetric then the corresponding APS can be realized *on* an affine space in one case only—when the deleted hyperplane is a tangent one. And then the affine space in question is constructed *not* as a reduct of the surrounding projective space but as a derived space as it is done in the context of chain geometry (cf. [2,8]). Moreover, such an APS can be represented without the whole machinery of polar spaces: it is the structure of isotropic lines of a metric affine space.

The only case when the point set of the reduct of a polar space *is* the point set of an affine space arises when we start from a null system i.e. in the case considered in the paper. But such an APS is not associated with a metric affine space i.e with a vector space endowed with a nondegenerate bilinear symmetric form. What is a natural analytic way in which a symplectic APS can be represented, when its point set is represented via a vector space? A way to do so is proposed in our paper: to this aim we consider a "metric", a binary scalar-valued operation defined on vectors. It is not a metric, in particular, it is not symmetric, and it is not invariant under affine translations. Nevertheless, it suffices to characterize respective geometry.

Symplectic APS'es have famous parents but they have also remarkable relatives. Although the "metric": the analytical characteristic invariant of symplectic APS'es is not a form, it is closely related to forms. Loosely speaking, it is a sum of an alternating form η defined on a subspace and an affine vector atlas defined on a vector complement of the domain of η. Immediate generalization with 'an alternating map' substituted in place of 'an alternating form' comes to mind. Such a definition of a map may seem artificial. The resulting map, that we call a *semiform*, has quite nice synthetic characterization though. Their basic properties are established in Sect. 2. A symplectic "metric" appears to be merely a special instance of such a general definition and many problems concerning it (so as to mention a characterization of the automorphism group) can be solved in this general setting easier. To illustrate and to motivate such a general definition we show in Example 2.5 a semiform associated with a vector product, that yields also an interesting geometry. On the other hand, this geometry has close connections (see Example 2.5-B) with a class of hyperbolic polar spaces.

A semiform induces an incidence geometry that we call an *affine semipolar space* [cf. (11) and (12)]. It is a Γ-space with affine spaces as its singular subspaces (cf. Theorem 3.5), and with generalized null-systems comprised by lines and planes through a fixed point (cf. Proposition 3.13; comp. a class with similar properties considered in [6]). In the paper we do not go any deeper into details of neither geometry of semiforms nor geometries other than symplectic APS'es. We rather concentrate on "APS'es and around".

Finally, we pass to our third group of arguments: that geometry of symplectic APS'es is interesting on its own right. Geometry of affine polar spaces is, by definition, an incidence geometry i.e. an APS is (as it was defined both in [5] and [17]) a partial linear space: a structure with points and lines. From the results of [4] we get that geometry of symplectic affine polar spaces can be also formulated in terms of binary collinearity of points—an analogue of the Alexandrov–Zeeman Theorem. A characterization of APS'es as suitable graphs is not known, though.

The affine polar spaces associated with metric affine spaces (as it was sketched above) can be, in a natural consequence, characterized in the "metric" language of line orthogonality or equidistance relation inherited from the underlying metric affine structure. It is impossible to investigate a line orthogonality imposed on an affine structure so as it gives rise to a symplectic APS. However, in case of a symplectic APS a "metric" mentioned above determines an "equidistance" relation which can be used as a primitive notion to characterize the geometry.

1. Definitions and preliminary results

Let S be a nonempty set, whose elements will be called *points*, and let $\mathcal{L}$ be a family of at least two element subsets of S, whose elements will be called *lines*. A point-line structure $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ is said to be a *partial linear space* whenever two of its distinct points lie in at most one line. Two points $a, b \in S$ are *collinear* if there is a line $l \in \mathcal{L}$ such that $a, b \in l$; then we write $\overline{a, b} = l$. We call $\mathfrak{M}$ a Γ-*space* if it satisfies a so called *none-one-or-all* axiom stating that for all $a \in S$ and $l \in \mathcal{L}$ the point a is collinear with none, one or all of the points of the line l. A sequence of lines $l_0, l_1, \dots, l_n$ is called a *path* in $\mathfrak{M}$ if l_i meets l_{i-1} in some point for all $i = 1, 2, \dots n$. We say that two points a, b are *joinable* in $\mathfrak{M}$ if there is a path $l_0, l_1, \dots, l_n$ such that $a \in l_0$ and $b \in l_n$. If every two points in $\mathfrak{M}$ are joinable we call $\mathfrak{M}$ *connected*.

Recall that the affine space $\mathbf{A}(\mathbb{V})$ defined over a vector space $\mathbb{V}$ has the vectors of $\mathbb{V}$ as its points and the cosets of the 1-dimensional subspaces of $\mathbb{V}$ as its lines.

We write τ_ω for the (affine) translation on the vector ω, $\tau_\omega(x) = x + \omega$.

1.1. Polar spaces

Let $\mathbb{W}$ be a vector space over a (commutative) field $\mathfrak{F}$ with characteristic $\neq 2$ and let ξ be a nondegenerate bilinear reflexive form defined on $\mathbb{W}$. Assume that the form ξ has finite index m and $n = \dim(\mathbb{W})$. We will write $\mathrm{Sub}(\mathbb{W})$ for the class of all vector subspaces of $\mathbb{W}$ and $\mathrm{Sub}_k(\mathbb{W})$ for the class of all k-dimensional subspaces. In the projective space $\mathfrak{P} = \langle \mathrm{Sub}_1(\mathbb{W}), \mathrm{Sub}_2(\mathbb{W}), \subset \rangle$ the form ξ determines the polarity $\delta = \delta_\xi$. We write $\mathrm{Q}(\xi)$ for the class of isotropic subspaces of $\mathbb{W}$:

$$\mathrm{Q}(\xi) = \{U \in \mathrm{Sub}(\mathbb{W}) : \xi(U,U) = 0\}\,; \quad \mathrm{Q}_k(\xi) = \{U \in \mathrm{Q}(\xi) : \dim(U) = k\}\,.$$

Assume that $m \geq 2$. The structure

$$\mathbf{Q}_\xi(\mathbb{W}) := \langle \mathrm{Q}_1(\xi), \mathrm{Q}_2(\xi), \subset \rangle$$

is referred to as the *polar space* determined by δ in $\mathfrak{P}$. Note that k-dimensional isotropic subspaces of $\mathbb{W}$ are $(k-1)$-dimensional singular subspaces of the polar space $\mathbf{Q}_\xi(\mathbb{W})$.

1.2. Hyperbolic polar spaces and their reducts

This section may look superfluous from view of symplectic polar spaces but it is used later in Example 2.5-B which justifies our general construction of semiforms.

Assume that the form ξ on W is symmetric and set $Y := W \times W$, $\mathbb{Y} := \mathbb{W} \oplus \mathbb{W}$, $Z := W \times \{\theta\}$ (θ being the zero vector), $H := \big\{[u,v] \in Y : u \perp_\xi v\big\}$. The form ζ on Y defined as $\zeta\big([u_1,v_1],[u_2,v_2]\big) := \xi(u_1,v_2) + \xi(v_1,u_2)$ is symmetric, nondegenerate, and $\mathrm{Sub}_1(H) = \mathrm{Q}_1(\zeta)$. Since Z is a n-dimensional maximal isotropic subspace of $2n$-dimensional vector space Y equipped with the form ζ, the projective index of $\mathfrak{Q} := \mathbf{Q}_\zeta(Y)$ is $n-1$. Hence $\mathfrak{Q}$ is a hyperbolic polar space (or a hyperbolic quadric, cf. [15, Sec. 1.3.4, p. 30]).

Now let $\mathcal{Z}$ be a maximal, i.e. $(n-1)$-dimensional, singular subspace of a hyperbolic polar space $\mathfrak{Q}$ of projective index $n-1$ and let $\mathfrak{R} = \mathfrak{R}(\mathfrak{Q}, \mathcal{Z})$ be the structure obtained by deleting the subspace $\mathcal{Z}$ from $\mathfrak{Q}$. We also write $\mathfrak{R}(W,\xi) = \mathfrak{R}(\mathbf{Q}_\zeta(Y), \mathrm{Sub}_1(Z))$.

Since $\mathcal{Z}$ is contained in a hyperplane of $\mathfrak{Q}$ the next theorem follows from [16, Theorem 3.11] which says that the ambient, thick, nondegenerate, embeddable polar space of rank at least 3 can be recovered in the complement of its subspace that is contained in a hyperplane. We give an independent, not so complex proof based on the decomposition of the hyperbolic polar space $\mathfrak{Q}$.

Theorem 1.1. *The hyperbolic polar space $\mathfrak{Q}$ is definable in its reduct $\mathfrak{R}$.*

Proof. We need to recover in $\mathfrak{R}$ points and lines of $\mathcal{Z}$ which are missing to get $\mathbf{\mathfrak{Q}}$. Let $\mathcal{C}$ be the family of all maximal singular subspaces of $\mathbf{\mathfrak{Q}}$ and let $\mathcal{R}$ be the family of all maximal singular subspaces of $\mathfrak{R}$. It is seen that $\mathcal{R} = \{\mathcal{X}\backslash\mathcal{Z} : \mathcal{X} \in \mathcal{C}\}$, and thus, every element of $\mathcal{R}$ carries the geometry of a slit space (cf. [11,12]). Write

- $\mathcal{R}_0 = \{\mathcal{X}\backslash\mathcal{Z} : \dim(\mathcal{X} \cap \mathcal{Z}) = 0\}$ for the family of punctured projective spaces, every one of which determines on $\mathcal{Z}$ its improper point,
- $\mathcal{R}_1 = \{\mathcal{X}\backslash\mathcal{Z} : \dim(\mathcal{X} \cap \mathcal{Z}) = n-2\}$ for the family of affine spaces, every one of which determines on $\mathcal{Z}$ a $(n-2)$-subspace of its improper points.

As $\mathbf{\mathfrak{Q}}$ is a dual polar space of type D_n, in view of [14, Sec. 4.2.3] the family $\mathcal{C}$ can be uniquely decomposed into the disjoint union of two subsets $\mathcal{C}_+$ and $\mathcal{C}_-$ (the half-spaces) such that

- $\dim(\mathcal{X} \cap \mathcal{Y}) = n-2, n-4, \ldots$ if $\mathcal{X} \in \mathcal{C}_+$ and $\mathcal{Y} \in \mathcal{C}_-$,
- $\dim(\mathcal{X} \cap \mathcal{Y}) = n-1, n-3, \ldots$ if $\mathcal{X}, \mathcal{Y}$ belong to the same half.

Correspondingly, the family $\mathcal{R}$ can be decomposed into $\mathcal{R}_+ = \{\mathcal{X}\backslash\mathcal{Z} : \mathcal{X} \in \mathcal{C}_+\}$ and $\mathcal{R}_- = \{\mathcal{X}\backslash\mathcal{Z} : \mathcal{X} \in \mathcal{C}_-\}$.

We assume that $\mathcal{Z} \in \mathcal{C}_-$. Then $\mathcal{R}_1 \subset \mathcal{R}_+$. If n is even, then $\mathcal{R}_0 \subset \mathcal{R}_+$, otherwise $\mathcal{R}_0 \subset \mathcal{R}_-$.

Let $\mathcal{J}_0 = \mathcal{X}_0\backslash\mathcal{Z} \in \mathcal{R}_0$ and $\mathcal{J}_1 = \mathcal{X}_1\backslash\mathcal{Z} \in \mathcal{R}_1$ where $\mathcal{X}_0, \mathcal{X}_1 \in \mathcal{C}$. Taking into account that $\mathcal{J}_0$ has a unique improper point and using the decomposition into half-spaces which gives that $1 \leq \dim(\mathcal{X}_0 \cap \mathcal{X}_1)$ independently on n we have that whenever $\mathcal{J}_0 \cap \mathcal{J}_1 \neq \emptyset$ there is a line L of $\mathfrak{R}$ such that $L \subset \mathcal{J}_0 \cap \mathcal{J}_1$. It is seen that L goes through the improper point of $\mathcal{J}_0$. Now, for $\mathcal{J}_0$, $\mathcal{J}_0' \in \mathcal{R}_0$ let us write

$$\mathcal{J}_0 \simeq \mathcal{J}_0' \quad \text{iff} \quad \text{for all } \mathcal{J}_1 \in \mathcal{R}_1 \text{ we have } (\mathcal{J}_0 \cap \mathcal{J}_1 \neq \emptyset \text{ iff } \mathcal{J}_0' \cap \mathcal{J}_1 \neq \emptyset).$$

Note that $\simeq$ is an equivalence relation and $\mathcal{J}_0 \simeq \mathcal{J}_0'$ means that $\mathcal{J}_0$, $\mathcal{J}_0'$ share the improper point. Therefore, there is a one-to-one correspondence between the equivalence classes of the relation $\simeq$ and the points of $\mathcal{Z}$.

Next, for every $(n-2)$-dimensional singular subspace N of $\mathbf{\mathfrak{Q}}$ there are precisely two maximal singular subspaces containing N, one of them is from $\mathcal{C}_+$ and the other belongs to $\mathcal{C}_-$. This gives a one-to-one correspondence between the elements of $\mathcal{R}_1$ and the hyperplanes of $\mathcal{Z}$.

That way, in terms of $\mathfrak{R}$, we get an incidence structure with points and hyperplanes of $\mathcal{Z}$. Using standard methods we are able to recover lines of $\mathcal{Z}$ which makes the proof complete for $\mathcal{Z} \in \mathcal{C}_-$. In case $\mathcal{Z} \in \mathcal{C}_+$ the reasoning runs the same way. $\square$

1.3. Symplectic affine polar spaces

From now on ξ is a nondegenerate symplectic form of index m. Then $n = \dim(\mathbb{W}) = 2m$. Assume that $m \geq 2$. The polar space

$$\mathbf{\mathfrak{Q}} := \mathbf{Q}_\xi(\mathbb{W}) = \langle \mathrm{Q}_1(\xi), \mathrm{Q}_2(\xi), \subset \rangle$$

is frequently referred to as a *null system* (cf. [3], [9, Vol. 2, Ch. 9, Sec. 3]). Since ξ is symplectic, $\mathrm{Q}_1(\xi) = \mathrm{Sub}_1(\mathbb{W})$ so, the point sets of $\mathfrak{Q}$ and of $\mathfrak{P}$ coincide.

Let $\mathcal{H}_0$ be a hyperplane of $\mathfrak{Q}$ (cf. [5]); then $\mathcal{H}_0$ is determined by a hyperplane $\mathcal{H}$ of $\mathfrak{P}$; on the other hand $\mathcal{H}$ is a polar hyperplane of a point U of $\mathfrak{P}$ i.e. $\mathcal{H} = U^{\perp}$. Finally, $\mathcal{H}_0 = \mathcal{H}$ is the set of all the points that are collinear in $\mathfrak{Q}$ with the point U of $\mathfrak{Q}$. The *affine polar space* $\mathfrak{U}$ *derived from* ($\mathfrak{Q}$,U) is the restriction of $\mathfrak{Q}$ to the complement of $\mathcal{H}$; in view of the above the point set of $\mathfrak{U}$ is the point set of the affine space $\mathfrak{A}$ obtained from $\mathfrak{P}$ by deleting its hyperplane $\mathcal{H}$. The set $\mathcal{G}$ of all the lines of $\mathfrak{U}$ is a subset of the set $\mathcal{L}$ of the lines of $\mathfrak{A}$. Moreover, the parallelism of the lines in $\mathcal{G}$ defined as in [5] (two lines are parallel iff they intersect in $\mathcal{H}_0$) coincides with the parallelism of $\mathfrak{A}$ restricted to $\mathcal{G}$. Clearly, not all the lines of $\mathfrak{P}$ that are not contained in $\mathcal{H}$ are isotropic. Furthermore, none of the lines of $\mathfrak{P}$ through U which is not contained in $\mathcal{H}$ is isotropic. For this reason, in every direction of $\mathfrak{A}$, except the one determined by U, there is a pair of parallel lines in $\mathfrak{A}$ such that one of them is isotropic and the other is not. In this exceptional direction no line is isotropic.

In [17] affine polar spaces determined in metric affine spaces associated with symmetric forms were studied. A somewhat similar interpretation of $\mathfrak{U}$ can be given here as well.

Recall that there is a basis of $\mathbb{W}$ in which the form ξ is given by the formula

$$\xi(x, y) = (x_1y_2 - x_2y_1) + (x_3y_4 - x_4y_3) + \cdots = \sum_{i=1}^{m} (x_{2i-1}y_{2i} - x_{2i}y_{2i-1}).$$

We write $\langle u, v, \dots\rangle$ for the vector subspace spanned by $u, v, \dots$ and $[x, y, z, \dots]$ for the vector with coordinates $x, y, z, \dots$ (in some cases x, y, $\dots$ may be vectors too).

Let us take $U = \langle[0, 1, 0, \dots, 0]\rangle$; then $\mathcal{H}$ is characterized by the condition $\langle[x_1, \dots, x_n]\rangle \subset \mathcal{H}$ iff $x_1 = 0$. We write $\mathbb{V}$ for the subspace of $\mathbb{W}$ characterized by $x_1 = x_2 = 0$; note that the restriction η of ξ to $\mathbb{V}$ is also a nondegenerate symplectic form. We can write $\mathbb{W} = \mathfrak{F} \oplus \mathfrak{F} \oplus \mathbb{V}$ and then for scalars a_1, a_2, b_1, b_2 and vectors u_1, u_2 of $\mathbb{V}$ we have

$$\xi([a_1, b_1, u_1], [a_2, b_2, u_2]) = a_1b_2 - a_2b_1 + \eta(u_1, u_2). \tag{1}$$

Moreover, $\mathfrak{A} = \mathbf{A}(\mathbb{Y})$ where $\mathbb{Y} = \mathfrak{F} \oplus \mathbb{V}$. A vector $[a, u]$ (a is scalar and $u \in V$) as a point of the affine space $\mathfrak{A}$ can be identified with the subspace $\langle[1, a, u]\rangle$ of $\mathbb{W}$, and the (affine) direction of the line $[a, u] + \langle[b, w]\langle$ is identified with the (projective) point $\langle[0, b, w]\rangle$.

Lemma 1.2. *Let* $p_1 = [a_1, u_1]$, $p_2 = [a_2, u_2]$ *with scalars* a_1, a_2 *and* $u_1, u_2 \in V$ *be a pair of points of* $\mathfrak{A}$. *Then*

$$p_1, p_2 \text{ are collinear in } \mathfrak{U} \quad \text{iff} \quad \eta(u_1, u_2) = a_1 - a_2.$$

Proof. Embed the points p_1, p_2 into $\mathfrak{P}$; then p_i corresponds to $U_i = \langle[1, a_i, u_i]\rangle$. Since p_1, p_2 are collinear iff the projective line which joins U_1, U_2 is in $\mathfrak{Q}$ we get that p_1, p_2 are collinear iff $\xi(U_1, U_2) = 0$. By (1) we get our claim. $\square$

2. Semiforms

2.1. Definition and examples

Formula (1) together with Lemma 1.2 suggest the following construction.

Definition 2.1. Let $\mathbb{V}$, $\mathbb{V}'$ be vector spaces over a (commutative) field $\mathfrak{F}$ with $\mathrm{char}(\mathfrak{F}) \neq 2$. Let V, V' be their sets of vectors and $\theta, \mathbf{0}$ be their zero vectors, respectively.

(i) Let $\eta : V \times V \longrightarrow V'$ be an alternating bilinear map. Then we have $\eta(u_1, u_2) = -\eta(u_2, u_1)$ and $\eta(u, u) = \mathbf{0}$ for all $u, u_1, u_2 \in V$.
(ii) Let $\delta : V' \times V' \longrightarrow V'$ be a map that satisfies the following conditions

C1. $\delta(v_1 + v, v_2 + v) = \delta(v_1, v_2)$,
C2. $\delta(\alpha v_1, \alpha v_2) = \alpha\delta(v_1, v_2)$,
C3. $\delta(v_1, v) + \delta(v, v_2) = \delta(v_1, v_2)$.

for all scalars α and $v, v_1, v_2 \in V'$.

Set $Y := V' \times V$ and $\mathbb{Y} := \mathbb{V}' \oplus \mathbb{V}$. On Y we define a binary operation $\varrho : Y \times Y \longrightarrow V'$ by the formula

$$\varrho([v_1, u_1], [v_2, u_2]) := \eta(u_1, u_2) - \delta(v_1, v_2). \tag{2}$$

The resulting map ϱ is referred to as a *semiform* defined on $\mathbb{Y}$.

If η is a bilinear map, then we write η_u for the map defined by $\eta_u(v) = \eta(u, v)$. An alternating bilinear form η, like the one considered in Definition 2.1, is *nondegenerate* when for all nonzero $u \in V$ there is $v \in V$ such that $\eta(u, v) \neq \mathbf{0}$, in other words, if $\ker(\eta_u) \neq V$. All those $u \in V$ such that $\ker(\eta_u) = V$ form the radical of η.

The formulas that are coming next are technical but quite important. They follow immediately from the definition. Let $p_i = [v_i, u_i], i = 1, 2$, $q = [v, y]$, then

$$\varrho(\alpha p_1, \alpha p_2) - \alpha\varrho(p_1, p_2) = \alpha(\alpha - 1)\eta(u_1, u_2); \tag{3}$$

$$\varrho(p_1 + q, p_2 + q) - \varrho(p_1, p_2) = \eta(u_1 - u_2, y);$$

$$\text{in particular,} \tag{4}$$

$$\varrho(p_1, p_1 + p_2) - \varrho(\theta, p_2) = \eta(u_1, u_2). \tag{5}$$

$$\varrho(q, \theta) = v, \tag{6}$$

$$\varrho(\alpha p_1, q) - \alpha\varrho(p_1, q) = (1 - \alpha)v, \tag{7}$$

$$\varrho(p_1 + p_2, q) - (\varrho(p_1, q) + \varrho(p_2, q)) = -v. \tag{8}$$

What follows are various examples of more or less natural semiforms.

Example 2.2. Let $\mathbb{V}' = \mathfrak{F}$, η be a null form on $\mathbb{V}$, and $\delta(a, b) = a - b$. Applying Definition 2.1 we get $\varrho([a_1, u_1], [a_2, u_2]) = \eta(u_1, u_2) - (a_1 - a_2)$ a semiform related to the form ξ in (1). □

Example 2.3. Every alternating map $\eta : V \times V \longrightarrow V'$ is derived from a linear map $g : \bigwedge^2 \mathbb{V} \longrightarrow V'$ by the formula

$$\eta(u_1, u_2) = g(u_1 \wedge u_2).$$

(see any standard textbook, e.g. [13, Ch. XIX]). It is known that $\dim(\bigwedge^2 \mathbb{V}) = \binom{n}{2}$, where $n = \dim(\mathbb{V})$. Note that for a fixed $u \in V$ the set $S_u := \{u \wedge y : y \in V\} = \mathrm{Im}(\wedge_u)$ is a $(n-1)$-dimensional vector subspace of $\bigwedge^2 \mathbb{V}$.

If we take $g = \mathrm{id}$, then we get a specific bilinear alternating map η with the property that $\dim(\ker(\eta_u)) = 1$ for all nonzero $u \in V$ as for linearly independent u_1, u_2 the wedge product $u_1 \wedge u_2$ cannot be zero.

Clearly, the operation $\eta = \wedge$ together with a given δ determines via (2) a semiform. □

Example 2.4. Let $\eta_i : V \times V \longrightarrow V'_i$ be an alternating bilinear map for $i = 0, \dots \nu$. Consider the map $\eta : V \times V \longrightarrow V' := \times_{i=0}^{\nu} V'_i$ given by the formula

$$\eta(u_1, u_2) = \big[\eta_0(u_1, u_2), \eta_1(u_1, u_2), \dots, \eta_\nu(u_1, u_2)\big].$$

Applying (2), the map η together with some suitable δ gives rise to a semiform.

In case $\dim(V'_i) = n < \infty$, for some i, it is possible to decompose η_i into n alternating bilinear forms $\bar{\eta}_i^j : V \times V \longrightarrow F$, where F is the ground field of V, V'_i and $j = 1, \dots, n$ so that

$$\eta_i(u_1, u_2) = \big[\bar{\eta}_i^1(u_1, u_2), \bar{\eta}_i^2(u_1, u_2), \dots, \bar{\eta}_i^n(u_1, u_2)\big].$$

If $\dim(V') < \infty$ we can do the same with η. □

Example 2.5. Let $\mathbb{V}$ be a 3-dimensional vector space. Then $\bigwedge^2 \mathbb{V} \cong \mathbb{V}$ and we can write $\eta(u_1, u_2) := u_1 \wedge u_2 = u_1 \times u_2$, where $\times : V \times V \longrightarrow V$ is a vector product. A standard formula defining $\times$ is the following:

$$[\alpha_1, \alpha_2, \alpha_3] \times [\beta_1, \beta_2, \beta_3] = \left[\varepsilon_1 \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}, \varepsilon_2 \begin{vmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{vmatrix}, \varepsilon_3 \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}\right]$$

with $\varepsilon_i = \pm 1$ (cf. [10, Ch. 2, Sec. 7, pp. 67–73], [18]). Then ϱ defined on $\mathbb{V} \oplus \mathbb{V}$ by the formula

$$\varrho([v_1, u_1], [v_2, u_2]) = u_1 \times u_2 - (v_1 - v_2)$$

is a semiform. □

2.2. Affine atlas and its characterization

In this and the forthcoming Sects. 2.3 and 2.4 most of the proofs consist in direct computations and therefore they are left for the reader.

Let us give a more explicit representation of a map δ characterized in Definition 2.1(ii).

Lemma 2.6. *Let δ meet conditions* C1–C3 *of Definition* 2.1*. Then the following conditions follow as well:*

C4. $\delta(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ *(by* C2*);*
C5. $\delta(v, v) = \mathbf{0}$ *(by* C1*,* C4*);*
C6. $\delta(v_1, v_2) = -\delta(v_2, v_1)$ *(by* C3*,* C5*);*
C7. $\delta(v_1 + v_2, \mathbf{0}) = \delta(v_1, \mathbf{0}) + \delta(v_2, \mathbf{0})$ *(by* C1*–*C3*,* C6*);*

for all $v, v_1, v_2 \in V'$.

Define $\phi : V' \longrightarrow V'$ by the formula $\phi(v) = \delta(v, \mathbf{0})$. Then ϕ is a linear map and δ is characterized by the formula

$$\delta(v_1, v_2) = \phi(v_1) - \phi(v_2) \;\; (= \phi(v_1 - v_2)). \tag{9}$$

A map δ defined by formula (9) is called *an affine atlas*. It is said to be *nondegenerate* when ϕ is an injection (i.e. if $\ker(\phi)$ is trivial). Note that when $\dim(\mathbb{V}') < \infty$ and δ is nondegenerate then the representing map ϕ is a surjection as well.

Lemma 2.7. *If $\phi : V' \longrightarrow V'$ is a linear map and δ is defined by (9), then δ meets conditions* C1*–*C3 *of Definition 2.1.*

Finally, we note that affine atlases can be equivalently characterized by another, less elegant but more convenient for our further characterizations, set of postulates.

Lemma 2.8. *If δ satisfies conditions* C1*,* C2 *of Definition 2.1 and postulates* C6*,* C7 *of Lemma 2.6, then δ satisfies* C3 *as well.*

2.3. Synthetic characterization and representations of semiforms

Let $\mathbb{Y} = \langle Y, +, \theta \rangle$, $\mathbb{Z} = \langle Z, +, \mathbf{0} \rangle$ be vector spaces with the common field $\mathfrak{F}$ of scalars. Let $\varrho : Y \times Y \longrightarrow Z$ be a map. Consider the following properties:

A1. $\varrho(p, q) = -\varrho(q, p)$ for each $p, q \in Y$.
A2. If $\varrho(\theta, p) = \mathbf{0}$ then $\varrho(\alpha q, p) = \alpha \varrho(q, p)$ for each scalar α and each vector q.
A3. If $\varrho(\theta, p) = \mathbf{0}$ then $\varrho(q_1 + q_2, p) = \varrho(q_1, p) + \varrho(q_2, p)$.
A4. If $p \neq \theta$ then there is q with $\varrho(p, q) \neq \mathbf{0}$ and $\varrho(\theta, q) = \mathbf{0}$.
A5. $\varrho(-p, -q) + \varrho(p, q) = 2(\varrho(p, p + q) - \varrho(\theta, q))$.
A6. $(\forall p)[\,\varrho(p + q, q) = \varrho(p, \theta)\,]$ implies $(\forall p_1, p_2)[\,\varrho(p_1 + q, p_2 + q) = \varrho(p_1, p_2)\,]$.
A7. $2\big(\varrho(\alpha p_1, \alpha p_2) - \alpha \varrho(p_1, p_2)\big) = \alpha(\alpha - 1)\big(\varrho(-p_1, -p_2) + \varrho(p_1, p_2)\big)$.
A8. For each $q \in Y$ there is $p \in Y$ such that $\varrho(p, \theta) = \mathbf{0}$ and $\varrho(p - q, -r) = -\varrho(q - p, r)$ for all $r \in Y$.

In view of formulas (3)–(8) it is evident that Axioms A1–A8 are satisfied by each semiform as defined in (2).

Set $M := \{p \in Y : \varrho(\theta, p) = \mathbf{0}\}$. With each $p \in Y$ we associate the map

$$\varrho_p : Y \longrightarrow Z, \qquad \varrho_p(q) = \varrho(q, p).$$

Note that if ϱ is a semiform defined in Definition 2.1 then $M = V$ and $\varrho \restriction M \times M = \eta$.

Recall that in one of the most intensively investigated cases in geometry when we consider a sesquilinear form ϱ, $M = Y$, ϱ_p is a linear map, and $p \mapsto \varrho_p$ is semilinear. Our axioms lead to a similar situation.

Lemma 2.9. *If ϱ satisfies A1 then $\varrho(p,p) = \mathbf{0}$ for each $p \in Y$. Consequently,*

$$\theta \in M.$$

Lemma 2.10. *Assume that Axiom A1 is valid. Let $p \in Y$.*

(i) *If ϱ_p is additive, then $p \in M$.*
(ii) *If ϱ_p is multiplicative, then $p \in M$.*

Consequently, if Axioms A2 and A3 are valid then the map ϱ_p is linear iff $p \in M$.

In particular (cf. Lemma 2.9), ϱ_θ is a linear map, i.e. the following hold:

$$\varrho(\alpha p, \theta) = \alpha \varrho(p, \theta),$$
$$\varrho(p_1 + p_2, \theta) = \varrho(p_1, \theta) + \varrho(p_2, \theta).$$

Clearly, $M = \ker(\varrho_\theta)$ and thus M is a subspace of $\mathbb{Y}$.

If, moreover, Axiom A4 is valid then the assignment $M \ni p \longmapsto \varrho_p$ is injective.

Lemma 2.11. (i) *Set*

$$D' := \Big\{ q \in Y : (\forall p \in Y)\big[\, \varrho(q, q+p) = \varrho(\theta, p) \,\big] \Big\}.$$

Then $\theta \in D'$ and the set D' is closed under vector addition.

(ii) *Assume that Axiom A6 is valid. Then $q \in D'$ iff $\varrho(p_1 + q, p_2 + q) = \varrho(p_1, p_2)$ for all $p_1, p_2 \in Y$.*

(iii) *Set*

$$D'' := \Big\{ q \in Y : (\forall p \in Y)\big[\, \varrho(-p, -q) = -\varrho(p, q) \,\big] \Big\}.$$

If Axiom A7 is valid, then the set D'' is closed under scalar multiplication.

If Axiom A5 is adopted, then $D' = D''$.

Consequently, if Axioms A6, A7, and A5 are valid then $D := D' = D''$ is a vector subspace of $\mathbb{Y}$.

Moreover, if Axioms A1–A7 are valid, then $M \cap D = \{\theta\}$.

Lemma 2.12. *With the Axioms A1–A7, Axiom A8 can be expressed as the following statement:*

$$\mathbb{Y} = D \oplus M.$$

Assume that the Axioms A1–A8 are valid and set $\eta := \varrho \restriction M \times M$, $\delta := \varrho \restriction D \times D$. Then η is an alternating nondegenerate vector-valued form. The map δ is a nondegenerate affine atlas; it is determined by a linear injection $\phi : D \longrightarrow D$ by the formula (9).

Lemma 2.13. *Let $q_i = p_i + r_i$ with $p_i \in M$, $r_i \in D$ for $i = 1, 2$. Then, we have [cf. (2)] the following*

$$\varrho(q_1, q_2) = \eta(p_1, p_2) - \delta(r_1, r_2).$$

Summing up the above, with not too tedious computation, we close this part by the following representation theorem

Theorem 2.14. *Let $\varrho : Y \times Y \longrightarrow Z$ be a map. The following conditions are equivalent.*

(i) *ϱ is a semiform defined in accordance with Definition 2.1, where η, δ are nondegenerate.*
(ii) *ϱ satisfies Axioms A1–A8.*

Remark 2.15. A semiform ϱ is nondegenerate and scalar-valued (i.e. $\dim(Z) = 1$) iff it is associated with a symplectic polar space.

Example 2.16. Let $\dim(\mathbb{V}) = 2$. Then the determinant is a symplectic form. Therefore the following map is a semiform (x_i, y_i are scalars).

$$\varrho([x_1, x_2, x_3], [y_1, y_2, y_3]) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - (x_1 - y_1)$$

The associated APS is determined by the so called line complex in the 3-dimensional projective space over $\mathfrak{F}$ (cf. [7, Ch. 6], [9, Vol. 2, Ch. 9, Sec. 3]).

$\square$

2.4. A simplification of semiforms

Forthcoming constructions are provided for a fixed nondegenerate semiform ϱ defined in Definition 2.1. Moreover, we assume that

$$\dim(\mathbb{V}') =: \nu < \infty.$$

Set $\mathfrak{A} = \mathbf{A}(\mathbb{Y})$. Let p_1, p_2 be vectors of $\mathbb{Y}$, so $p_i = [v_i, u_i]$, $v_i \in V'$, $u_i \in V$. By definition,

$$\varrho(p_1, p_2) = \eta(u_1, u_2) - \phi(v_1 - v_2)$$

for suitable maps η, ϕ. Recall that they need to be nondegenerate. As a consequence, $\phi \in GL(\mathbb{V}')$.

There is, generally, a great variety of semiforms. But some of them may lead to isomorphic geometries. Write $\varrho_{\eta,\phi}$ for ϱ defined by Definition 2.1 with δ defined by (9). We have evident

Proposition 2.17. (i) *There is a linear bijection $\Phi \in GL(\mathbb{Y})$ such that for any $q_1, q_2 \in Y$ it holds:*

$$\varrho_{\eta,\phi}(q_1, q_2) = \varrho_{\eta,\mathrm{id}}(\Phi(q_1), \Phi(q_2))$$

(ii) *Let* $B \in GL(\mathbb{V})$, γ *be a nonzero scalar. Then, clearly, the map* $\gamma\eta B$ *defined by* $\gamma\eta B(u_1, u_2) = \gamma\cdot\eta(B(u_1), B(u_2))$ *is an alternating form. There is a linear bijection* $\Phi \in GL(\mathbb{Y})$ *such that the following holds for any* $q_1, q_2 \in Y$

$$\varrho_{\gamma\eta B,\mathrm{id}}(q_1, q_2) = \gamma^{-1} \cdot \varrho_{\eta,\mathrm{id}}(\Phi(q_1), \Phi, (q_2)).$$

Remark. In terms of Example 2.3 we have $\eta B = g \circ (B \wedge B)$.

In view of Proposition 2.17, till the end of our paper we assume that ϱ is defined by a formula of the form

$$\varrho([v_1, u_1], [v_2, u_2]) = \eta(u_1, u_2) - (v_1 - v_2). \tag{10}$$

3. Affine semipolar spaces

3.1. Geometrical structure

Under the settings of Sect. 2.4 that $\mathfrak{A} = \mathbf{A}(\mathbb{Y})$ and p_1, p_2 are points of $\mathfrak{A}$ with $p_i = [v_i, u_i]$, $v_i \in V'$, $u_i \in V$, imitating Lemma 1.2, we put generally

$$p_1 \sim p_1 \quad \text{iff} \quad \varrho(p_1, p_2) = \mathbf{0}. \tag{11}$$

From definition it is immediate that $p_1 \sim p_2$ iff $\eta(u_1, u_2) = v_1 - v_2$.

Lemma 3.1. *Let* p_1, p_2 *be two distinct points of* $\mathfrak{A}$ *and* $L = \overline{p_1, p_2}$. *If* $p_1 \sim p_2$, *then* $q_1 \sim q_2$ *for all* $q_1, q_2 \in L$.

Proof. Let $p_1 = [v, u]$, $p_2 = p_1 + q$, $q = [v_0, u_0]$. From assumption we have $\eta(u, u + u_0) = v - (v + v_0) = -v_0$. Then, it can be directly computed that $\eta(u + \alpha u_0, u + \beta u_0) = (\alpha - \beta)\eta(u, u_0) = (\alpha - \beta)v_0 = (v + \alpha v_0) - (v + \beta v_0)$. This yields $p_1 + \alpha q \sim p_1 + \beta q$ for any scalars α, β and the proof is closed. □

In view of Lemma 3.1, the relation $\sim$ determines the class $\mathcal{G}$ of lines of $\mathfrak{A}$ by the condition

$$L \in \mathcal{G} \quad \text{iff} \quad p_1 \sim p_2 \text{ for any } p_1, p_2 \in L. \tag{12}$$

For computation purposes it is convenient to have this criteria:

$$[v_0, u_0] + \langle [v, u] \rangle \in \mathcal{G} \quad \text{iff} \quad \eta(u_0, u) = -v. \tag{13}$$

It is a straightforward consequence of (12) and (11).

The class $\mathcal{G}$ induces the partial linear space

$$\mathfrak{U} := \langle Y, \mathcal{G} \rangle$$

that we call *affine semipolar space* determined by ϱ. Let us put down some of its properties.

Remark 3.2. A triangle in an affine semipolar space $\mathfrak{U}$ is of the form:

$$[v_0, u_0], \qquad [v_0 + \eta(u, u_0), u_0 + u], \qquad [v_0 + \eta(y, u_0), u_0 + y], \tag{14}$$

where $\eta(u, y) = \mathbf{0}$.

Lemma 3.3. (i) *The class $\mathcal{G}$ is not closed under parallelism, i.e. for every $L_1 \in \mathcal{G}$ there is an affine line $L_2 \notin \mathcal{G}$ such that $L_1 \parallel L_2$.*

(ii) *Let $L_1, L_2 \in \mathcal{G}$, $L_1 \neq L_2$, and $p \in L_1 \cap L_2$. If L is an affine line through p from the affine plane $\langle L_1, L_2\rangle$, then $L \in \mathcal{G}$.*

Proof. (i): Let $L_1 = [v_0, u_0] + \langle [v, u]\rangle$, where $\eta(u, u_0) = v$. Suppose that $L_2 \in \mathcal{G}$ for all L_2 with $L_2 \parallel L_1$. This yields that $\eta(u, u_0) = v$ for all $u_0 \in V$. So take any $u_1 \in V$ and note that $\eta(u, u_1) = \eta(u, u_0 - u_2) = v - v = \mathbf{0}$ for some $u_0 \in V$ and $u_2 = u_0 - u_1$. This gives $v = \mathbf{0}$. Thus $u^{\perp} = V$, and hence $u = \theta$ as η is nondegenerate. Finally, $[v, u]$ is the zero of $\mathbb{Y}$, a contradiction.

(ii): Without loss of generality we can assume that $p = [v_0, u_0]$ and $L_i = p + \langle a_i\rangle$ where $a_i \in Y$, $i = 1, 2$. Then $L = p + \langle \alpha_1 a_1 + \alpha_2 a_2\rangle$ for some $\alpha_i \in F$. Applying (13) to L_1, L_2 and then to L we are through. □

Note that Lemma 3.3(ii) means that the family $\mathcal{G}$ is closed on pencils. This has some straightforward implication.

Corollary 3.4. *The set of points that are collinear with a given point in an affine semipolar space (determined by a semiform defined on $\mathbb{Y}$) is closed on planes and thus it is a subspace in the ambient affine space $\mathfrak{A}$.*

Theorem 3.5. *The affine semipolar space is a Γ-space and its every singular subspace carries affine geometry.*

Proof. Let $\mathfrak{U}$ be our affine semipolar space. The first part follows directly from (ii) in Lemma 3.3. The other part is a simple observation that a singular subspace of $\mathfrak{U}$, in other words, a strong subspace wrt. $\sim$ in $\mathfrak{A}$ or a subspace where every two points are $\sim$-adjacent, is an affine subspace of $\mathfrak{A}$. □

When we deal with a Γ-space a question on the form of its triangles may appear important. The following is immediate from (13) and (11).

Proposition 3.6. *If* $\dim(\ker(\eta_u)) = 1$ *for each nonzero $u \in V$, then the corresponding affine semipolar space contains no proper triangle. In that case its maximal singular subspaces are the lines.*

Example 3.7. In view of Theorem 3.5 one could expect that affine semipolar spaces are models of the system considered in [6]. In the case of wedge product considered in Example 2.3 however, and consequently in the case of vector product considered in Example 2.5, we have $\dim(\ker(\eta_u)) = 1$ for all $u \neq \theta$. By Remark 3.2 it is seen that *there are no triangles in the* affine semipolar space *determined by η with this property*. Therefore, affine semipolar spaces from Example 2.3 and Example 2.5 are not models of the system in [6]. □

Example-continuation 2.4-A Assume additionally that $V'_0 = \cdots = V'_\nu$ and η_0 is a linear combination of the other η_i, that is

$$\eta_0(u_1, u_2) = \lambda_1\eta_1(u_1, u_2) + \cdots + \lambda_\nu\eta_\nu(u_1, u_2)$$

for some scalars λ_i, $i = 1, \ldots, \nu$. By (11), in the affine semipolar space $\mathfrak{U}$ induced by η, if points $p = [v_0, v_1, \ldots, v_\nu, u]$ and $p' = [v'_0, v'_1, \ldots, v'_\nu, u']$ are collinear, then $v'_0 = v_0 + \lambda_1(v'_1 - v_1) + \cdots + \lambda_\nu(v'_\nu - v_\nu)$. Iterating, we can show that whenever $p'' = [v''_0, v''_1, \ldots, v''_\nu, u'']$ is joinable with p, then $v''_0 = v_0 + \lambda_1(v''_1 - v_1) + \cdots + \lambda_\nu(v''_\nu - v_\nu)$. This means that $\mathfrak{U}$ is not connected. $\square$

In the sequel we shall frequently consider the condition (with prescribed values u, v)

$$(\exists y)\ [\ \eta(u, y) = v\ ];$$

applying the representation given in Example 2.3 this can be read as $(\exists y)\ [\ g(u \wedge y) = v\ ]$, which is equivalent to $(\exists \omega \in S_u)\ [\ g(\omega) = v\ ]$. This observation allows us to construct quite "strange" ('locally surjective') alternating maps.

As an immediate consequence of Lemma 3.1 and the definition we have

Lemma 3.8. *Let $q = [v_0, u_0]$ be a vector of $\mathbb{Y}$. The following conditions are equivalent.*

(i) *There is no line $L \in \mathcal{G}$ with the direction q.*
(ii) *The equation*

$$\eta(u_0, u) = v_0 \tag{15}$$

is not solvable in u.

In particular, if $u_0 = \theta$ and $v_0 \neq \mathbf{0}$ then (15) is not solvable and thus there is no line $L \in \mathcal{G}$ with the direction q.

Set

$$D := \{q \in Y : \text{ no line in } \mathcal{G} \text{ has direction } q\} \tag{16}$$

Note that when $\eta_u : V \longrightarrow V'$ is onto for each nonzero u, then $D = V' \times \{\theta\}$. Directly from (11) all the points collinear with $p = [\mathbf{0}, \theta]$ form the set $\{[v, u] : v = \mathbf{0}\}$ and all the points collinear with $q = [\mathbf{0}, u_0]$ form the set $\{[v, u] : \eta(u, u_0) = v\}$. It is clear that $p \sim q$. So, if (15) has solutions for $[v, u] \in Y$, that is $[v, u] \notin D$, then $[v, u]$ is collinear with p or with q for adequate u_0. This proves the following:

Proposition 3.9. *If $D = V' \times \{\theta\}$, then $\mathfrak{U}$ is connected and the maximal distance is 2 like in a polar space.*

Example-continuation 2.5-A Let $\times$ be a vector product in a vector 3-space $\mathbb{V}$ associated with a nondegenerate bilinear symmetric form ξ and $\perp = \perp_\xi$ be the orthogonality determined by ξ. Then for $u_0, v_0 \neq \theta$ Eq. (15) is solvable iff $u_0 \perp v_0$. In that case we have:

$$D = V \times \{\theta\} \cup \big\{[v,u] \in V \times V : u \not\perp v\big\}.$$

$\square$

Lemma 3.10. *For a fixed $u_0 \in V$, $v_0 \in V'$ and a scalar α the set*

$$\mathcal{Z} = \big\{[v,u] : \eta(u_0,u) = v_0 + \alpha v\big\} \tag{17}$$

is a subspace of $\mathfrak{A}$. The class of sets of form (17) is invariant under translations of $\mathfrak{A}$.

Proof. Take $[v_1,u_1],[v_2,u_2] \in \mathcal{Z}$ and an arbitrary scalar λ. Then we compute $\eta(u_0,\lambda u_1+(1-\lambda)u_2) = \lambda\eta(u_0,u_1)+(1-\lambda)\eta(u_0,u_2) = \lambda(v_0+\alpha v_1)+(1-\lambda)(v_0+\alpha v_2) = v_0+\big(\lambda v_1+(1-\lambda)v_2\big)$ which proves that $[\lambda v_1+(1-\lambda)v_2, \lambda u_1+(1-\lambda)u_2] \in \mathcal{Z}$ and thus $\overline{[v_1,u_1],[v_2,u_2]} \subset \mathcal{Z}$. This proves that $\mathcal{Z}$ is a subspace of $\mathfrak{A}$.

Write $\mathcal{Z}_{u_0,v_0,\alpha}$ for the set defined by (17). Let $q = [x,y] \in Y$ be arbitrary. Then, for the translation τ_q we have $\tau_q([v,u]) = [v+x, u+y] \in \mathcal{Z}_{u_0,v_0,\alpha}$ iff $\eta(u_0,u+y) = v_0 + \alpha(v+x)$ iff $\eta(u_0,u) = (v_0 - \eta(u_0,y) + \alpha x) + \alpha v$ iff $[v,u] \in \mathcal{Z}_{u_0,v_0-\eta(u_0,y)+\alpha x,\alpha}$. Thus

$$\tau_q^{-1}(\mathcal{Z}_{u_0,v_0,\alpha}) = \mathcal{Z}_{u_0,v_0-\eta(u_0,y)+\alpha x,\alpha}.$$

This closes our proof. $\square$

Lemma 3.11. *If the set $\mathcal{Z}$ defined by (17) is nonempty, then it is an affine subspace of $\mathfrak{A}$ with dimension $\nu + \dim(\ker(\eta_{u_0}))$ or $\dim(\mathbb{V})$.*

Proof. If $\mathcal{Z}$ is nonempty, then by Lemma 3.10 we can assume that $[\mathbf{0},\theta] \in \mathcal{Z}$. Then $\mathcal{Z}$ is characterized by an equation $\eta(u_0,u) = \alpha_0 v$ with prescribed values of u_0, α_0 and it is the kernel of the linear map $\Psi : Y \longrightarrow V'$, $\Psi[v,u] \longmapsto \eta(u_0,u) - \alpha_0 v$.

If $\alpha_0 = 0$ then, clearly, $\mathcal{Z} = V' \times \ker(\eta_{u_0})$ and thus $\dim(\mathcal{Z}) = \dim(\ker(\eta_{u_0}) + \dim(\mathbb{V}') = \nu + \dim(\ker(\eta_{u_0})))$.

Assume that $\alpha_0 \neq 0$. Then $\mathcal{Z}$ can be considered as the kernel of the map $[v,u] \longmapsto \eta(\frac{1}{\alpha}u_0,u) - v$. Let $(d_1,\dots,d_k)$ be a linear basis of $\mathrm{Im}(\eta_{u_0})$ and $(e_1,\dots,e_m)$ be a basis of $\ker(\eta_{u_0})$. Choose one $z_i \in V$ with $\eta_{u_0}(z_i) = d_i$ for each $i = 1,\dots,k$; Then the set $\{z_1,\dots,z_k\}$ is linearly independent. Moreover, the subspaces $\langle z_1,\dots,z_k\rangle$ and $\ker(\eta_{u_0})$ have only the zero vector in common. A basis of $\mathcal{Z}$ consists of the vectors

$$\big([d_1,z_1],[d_1,z_1+e_1],\dots,[d_1,z_1+e_m],[d_2,z_2],\dots,[d_k,z_k]\big).$$

Consequently, $\dim(\mathcal{Z}) = \dim(\ker(\eta_{u_0})) + \dim(\mathrm{Im}(\eta_{u_0})) = \dim(\mathrm{Dom}(\eta_{u_0})) = \dim(\mathbb{V})$. $\square$

For $\alpha = -1$ in (17), directly by (11), the set $\mathcal{Z}$ is the set of all points collinear with $[v_0,u_0]$ in $\mathfrak{U}$. Hence, applying Lemma 3.11 we get that the subspace in Corollary 3.4 has dimension $\dim(\mathbb{V})$.

The following condition is satisfied in all our examples.

(*) If $u, v \in V$ and $u \nparallel v$, then there is $y \in V$ such that $\eta(u, y) = \mathbf{0}$ and $\eta(v, y) \neq \mathbf{0}$.

Lemma 3.12. *Let $L \in \mathcal{G}$ pass through $p = [\mathbf{0}, \theta]$ and p' be a point on L. If the condition (*) is satisfied, then $L = \bigcap \big\{\{x : x \sim q\} : q \sim p, p'\big\}$.*

Proof. $\subseteq$: Follows from Theorem 3.5.

$\supseteq$: By (13) we can write $p' = [\eta(u, \theta), u] = [\mathbf{0}, u]$ for a nonzero vector u. Let $q \sim p, p'$. From (14) $q = [\mathbf{0}, y]$ for some y such that $\eta(y, u) = \mathbf{0}$. Now, suppose that $x = [z, w] \sim [\mathbf{0}, y] = q$ for each y with $\eta(u, y) = \mathbf{0}$. By (13) we get that:

$$\eta(u, y) = \mathbf{0} \text{ implies } \eta(w, y) = z \text{ for all } y$$

Now let $\theta \neq y \in \ker(\eta_u)$. Then $\eta(w, y) = z = \eta(w, 2y) = 2z$ and thus $z = \mathbf{0}$. Hence $w \in \bigcap\big\{\ker(\eta_y) : y \in \ker(\eta_u)\big\}$. Applying the global assumptions we infer $w \parallel u$ and thus $x \in L$. $\square$

Now, we are going to make a few comments that together with Remark 3.2 will let us characterize the geometry of the lines and the planes through a point in an affine semipolar space.

Each alternating map $\eta : V \times V \longrightarrow V'$ determines the incidence substructure $\mathbf{Q}_\eta(\mathbb{V})$ of the projective space $\mathbf{P}(\mathbb{V})$ with the point set unchanged and with the class $\mathcal{L}^*$ of projective lines of the form $\langle u_1, u_2\rangle$, where $u_1, u_2 \in V$ are linearly independent and $\eta(u_1, u_2) = \mathbf{0}$ as its lines. With a fixed basis of $\mathbb{V}'$ one can write η as the (Cartesian) product of ν bilinear alternating forms $\eta_i : V \times V \longrightarrow F$:

$$\eta(u_1, u_2) = \big[\eta_1(u_1, u_2), \ldots, \eta_\nu(u_1, u_2)\big],$$

where $\nu = \dim(\mathbb{V}')$. Clearly, the η_i need not to be nondegenerate. So, each η_i determines a (possibly degenerate) null system $\mathbf{Q}_{\eta_i}(\mathbb{V})$ with the lines $\mathrm{Q}_2(\eta_i)$. The class $\mathcal{L}^*$ is simply $\bigcap_{i=1}^{\nu} \mathrm{Q}_2(\eta_i)$.

Proposition 3.13. *The geometry of the lines and planes of an affine semipolar space (determined by a semiform ϱ associated via (10) with an alternating map η) which pass through the point $[\mathbf{0}, \theta]$ is isomorphic to $\mathbf{Q}_\eta(\mathbb{V})$.*

Proof. Let $p = [\mathbf{0}, \theta]$. In view of (13) the class of lines through p is the set $\{\langle[\mathbf{0}, u]\rangle : u \neq \theta\}$ so, it can be identified with the point set of $\mathbf{P}(\mathbb{V})$ under the map $\langle[\mathbf{0}, u]\rangle \mapsto \langle u\rangle$. From Remark 3.2 two lines $\langle[\mathbf{0}, u]\rangle$, $\langle[\mathbf{0}, v]\rangle$ span a plane in the corresponding affine semipolar space iff $\eta(u, v) = \mathbf{0}$, which closes our reasoning. $\square$

3.2. Automorphisms

Recall that the *horizon* of an affine space, in other words, the set of all the points at infinity, can be endowed with an incidence structure and as such carries projective geometry.

To establish the automorphism group of the relation $\sim$ and $\mathfrak{U}$ we need some additional assumption that the set of directions $V' \times \{\theta\}$ can be characterized in terms of the projective geometry of the horizon of $\mathbf{A}(\mathbb{Y})$ with the set D [defined in (16)] distinguished. It is hard to give a formal, precise formula stating that. Let us put it this way, in the language of automorphisms.

(**) Every automorphism of the projective geometry of the horizon of $\mathbf{A}(\mathbb{Y})$ that leaves invariant the set D defined in (16) also leaves invariant $V' \times \{\theta\}$.

Clearly, in view of Lemma 3.8 this condition holds when ϱ is scalar-valued. Let us point out however, that it is not the only case. Moreover, we always have $V' \times \{\theta\} \subset D$ and in most of our examples even $D = V' \times \{\theta\}$, but in general we need (**). In Example 2.5 there is a non-scalar-valued ϱ and as seen in Example 2.5-A the set of forbidden directions D is significantly larger than $V' \times \{\theta\}$.

Example-continuation 2.5-B In addition to the notation of Example 2.5 and Example 2.5-A here we also use the notation of Sect. 1.2.

Let $f \in \Gamma L(Y)$ preserve the set of directions D. Then f preserves $V \times \{\theta\}$.

Indeed, the geometric structure of the complement of D carries the geometry of the reduct $\mathfrak{R}(\mathbb{V}, \xi)$ of the hyperbolic polar space induced by ξ as in Sect. 1.2. Our claim follows from Theorem 1.1. Consequently, the condition (**) is valid here. □

Remark 3.14. The relation $\sim$ can be characterized in terms of $\mathfrak{U}$:

$$a \sim b \quad \text{iff} \quad a = b \ \text{ or } \ a, b \text{ are on a line from } \mathcal{G}.$$

Recall by (12) that an affine line is in $\mathcal{G}$ iff any two of its distinct points are $\sim$ related. Consequently, if F is an affine transformation of $\mathfrak{A}$, then F preserves $\mathcal{G}$ iff F preserves $\sim$.

Proposition 3.15. *If F is given by the formula*

$$F([v, u]) = [\psi_1(v) + \psi_2(u) + v_0, \varphi(u) + u_0] \tag{18}$$

where $v_0 \in V'$, $u_0 \in V$, $\psi_2 : V \longrightarrow V'$ is linear, $\psi_1 : V' \longrightarrow V'$, $\varphi : V \longrightarrow V$ are linear bijections, and the following holds:

(a) $\psi_2(u) = \eta\big(\varphi(u), u_0\big)$ *for every vector u of $\mathbb{V}$, and*
(b) $\eta(\varphi(u_1), \varphi(u_2)) = \psi_1 \eta(u_1, u_2)$, *for all vectors u_1, u_2 of $\mathbb{V}$,*

then F preserves the relation $\sim$.

In that case the semiform ϱ is transformed under the rule

$$\varrho\big(F(p_1), F(p_2)\big) = \psi_1\big(\varrho(p_1, p_2)\big) \tag{19}$$

for $p_1, p_2 \in Y$. In consequence, F is an affine (linear) automorphism of $\mathfrak{U}$.

Conversely, under additional assumption that (**) is valid, each affine automorphism of $\mathfrak{U}$ is of the form (18).

Note. *If η is onto V' then given map φ, condition (b) uniquely determines ψ_1. Similarly, for a given map φ and vector u_0, condition (a) uniquely determines ψ_2.*

Proof. Assume that F is defined by the formula (18) and (a), (b) hold. It is seen by (18) that F is an affine transformation. Let $p_i = [v_i, u_i]$, $v_i \in V'$, $u_i \in V$, for $i = 1, 2$. We compute as follows: $\varrho(F(p_1), F(p_2)) = \eta(\varphi(u_1), \varphi(u_2)) + \eta(\varphi(u_1 - u_2), u_0) - \psi_1((v_1 - v_2) - \psi_2(u_1 - u_2) = \psi_1\eta(u_1, u_2) + \eta(\varphi(u_1 - u_2), u_0) - \psi_1(v_1 - v_2) - \eta(\varphi(u_1 - u_2), u_0) = \psi_1\eta(u_1, u_2) - \psi_1(v_1 - v_2) = \psi_1(\varrho(p_1, p_2))$, which proves (19). This yields that F preserves $\sim$. By Remark 3.14 we get that $F \in \mathrm{Aut}(\mathfrak{U})$.

Now, assume that F is an affine automorphism of $\mathfrak{U}$ and (**) is valid. Then, F leaves invariant the set D defined in (16) and it is a composition $\tau_{[v_0,u_0]} \circ F_0$, where $F_0 \in GL(\mathbb{Y})$ and $[v_0, u_0] \in Y$. The map F_0 can be presented in the form $F_0([v, u]) = [\psi_1(v)+\psi_2(u), \varphi_1(u)+\varphi_2(v)]$ for some linear maps $\psi_1 : V' \longrightarrow V'$, $\psi_2 : V \longrightarrow V'$, $\varphi_1 : V \longrightarrow V$, $\varphi_2 : V' \longrightarrow V$, where ψ_1 and φ_1 are bijections as F_0 is.

Notice, by (**), that the linear part F_0 of F fixes the subspace V' and thus $\varphi_2 \equiv \theta$. We write $\varphi = \varphi_1$. In view of Remark 3.14 the mapping F preserves the relation $\sim$ so, by definition we obtain the following equivalence:

$$\begin{aligned} &v_1 - v_2 = \eta(u_1, u_2) \quad \text{iff} \\ &\qquad \psi_1(v_1 - v_2) + \psi_2(u_1 - u_2) = \eta\big(\varphi(u_1), \varphi(u_2)\big) + \eta\big(\varphi(u_1 - u_2), u_0\big) \end{aligned} \tag{20}$$

for all vectors $v_1, v_2 \in V'$, $u_1, u_2 \in V$. Substituting in (20) $u_2 = \theta$ and $v_1 = v_2$ we arrive to the condition (a). In particular, from (a) we obtain $\psi_2(u_1 - u_2) = \eta(\varphi(u_1 - u_2), u_0)$ for all u_1, u_2 in $\mathbb{V}$. Thus, assuming (a), from (20) we get (b). Finally, F has form (18) as required. □

Example-continuation 2.4-B If η_i, for some $i \in \{0, \dots \nu\}$, is a linear combination of the other η_i, then, as $\mathfrak{U}$ is not connected, automorphisms of $\mathfrak{U}$ need not to be given by linear or semilinear maps. □

Proposition 3.16. *The group* $\mathrm{Aut}(\mathfrak{U})$ *is transitive.*

Proof. It suffices to compute the orbit $\mathcal{O}$ of the point $[\mathbf{0}, \theta]$ under the group of affine automorphisms of $\sim$. From Proposition 3.15, the orbit $\mathcal{O}$ contains all the vectors $[\psi_2(\theta)+v_0, \varphi(\mathbf{0})+u_0] = [v_0, u_0]$ with suitable maps ψ_2, φ. Considering $\varphi = \mathrm{id}$, $\psi_2(u) = \eta(\varphi(u), u_0)$, $\psi_1 = \mathrm{id}$ we get a class of affine automorphisms of $\sim$: those defined by the formula

$$F([v, u]) = [v + \eta(u, u_0) + v_0, u + u_0]$$

with arbitrary fixed u_0, v_0. So, each point of $\mathfrak{U}$ is in $\mathcal{O}$. □

According to Proposition 3.16 affine semipolar space is homogeneous which together with Proposition 3.13 gives the following:

Proposition 3.17. *The geometry of the lines and the planes through an arbitrary point of an affine semipolar space is a generalized null system.*

Combining Proposition 3.16 and Lemma 3.12 we get a theorem, which is important in the context of foundations of geometry of affine semipolar spaces.

Theorem 3.18. *Let $\mathfrak{U}$ be the affine semipolar space determined by a semiform that meets (*). For each pair p, q of points of $\mathfrak{U}$ such that $p \sim q$ the set*

$$\bigcap\big\{\{x : x \sim y\} : y \sim p, q\big\}$$

is the line of $\mathfrak{U}$ through p, q.

Consequently, the class of lines of $\mathfrak{U}$ is definable in terms of the binary collinearity $\sim$ of $\mathfrak{U}$.

Proof. In view of Proposition 3.16 without loss of generality we can assume that $p = [\mathbf{0}, \theta]$ and then Lemma 3.12 yields the claim directly. □

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