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Affine transformations in a differentiable manifold with Π-structure

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Introduction

Following M. Obata [**4**], we denote by M a manifold of even dimension $2n$ with an almost complex structure F, and by $H(x)$, $x \in M$, the homogeneous holonomy group of M with respect to a natural connection, i.e., an affine connection with respect to which F is covariant constant. $A(M)$ denotes the group of all affine transformations of M onto itself and $A_0(M)$ the connected component of the identity of $A(M)$. $QL(l, R)$ denotes the real representation of the quaternion linear group $QL(l, C)$. We assume that $H(x)$ is irreducible in R. The following theorem was proved in [**4**].

THEOREM A. *If n is even, $n=2l$, and $H(x)$ is not a subgroup of $QL(l, R)$, or if n is odd, then $A_0(M)$ preserves the almost complex structure. If n is even, $n=2l$, and $H(x)$ is a subgroup of $QL(l, R)$, then M has three independent almost complex structure F, G and H such that $FG=-GF=H$, $GH=-HG=F$, $HF=-FH=G$ and they are all parallel. $A(M)$ acts on the vector space spanned by F, G and H as a group of orthogonal transformations. Furthermore these orthogonal transformations belong to $SO(3)$ in the vector space.*

On the other hand, the notion of Π-structure on a differentiable manifold of any dimension m (not necessarily even) was introduced by D.C. Spencer [**6**]. (The name 'Π-structure' was given by G. Legrand [**1**].) It is one of the generalizations of the almost complex structure. Then the question arises if $A(M)$ preserves the Π-structure. An answer to this question will be given in § 2.1 as Theorem 2.

In § 1 we shall summarize briefly the known results on the Π-structure and the Π-connection. In § 2, we shall prove the main result.

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1. Preliminaries

Let M be a connected (real) m-dimensional differentiable manifold of class C^∞. We denote by T_x the tangent vector space of M at $x \in M$ and by T_x^C its complexification.

A *Π-structure* (or *complex almost-product structure* in the terminology of D.C. Spencer) is defined on M by giving two fields T^1 and T^2, of class C^∞, of complementary proper subspaces of T_x^C. If we set $n_1 = \dim T^1$ and $n_2 = \dim T^2$, then we have $n_1 + n_2 = m, n_1 \neq 0, n_2 \neq 0$.

Let P_1(resp. P_2) be the projection of T_x^C onto T_x^1 (resp. T_x^2) at every point $x \in M$, where T_x^1 (resp. T_x^2) denotes the value of T^1 (resp. T^2) at x. Let λ be a complex constant which is not zero. If we set

$$(1.1) \qquad \boldsymbol{F}v = \lambda(P_1 - P_2)v\,, \qquad v \in T_x^C\,,$$

then we have a linear operator $\boldsymbol{F}$ on complex vector space T_x^C such that

$$(1.2) \qquad \boldsymbol{F}^2 = \lambda^2\,(\text{Identity})\,.$$

To the operator $\boldsymbol{F}$ corresponds a complex tensor (F_j^i) defined by $(\boldsymbol{F}v)^i = F_j^i v^j$. From the relation (1.2) we have

$$(1.2)' \qquad F_h^i F_j^h = \lambda^2 \delta_j^i\,, \qquad F_j^i \neq \pm \lambda \delta_j^i\,.$$

Conversely we assume that a complex tensor field (F_j^i) of class C^∞ is given on M and that it satisfies the relation (1.2)′ with $\lambda \neq 0$ at every point of M. The linear operator $\boldsymbol{F}$ on T_x^C is defined by the tensor (F_j^i) at $x \in M$, $\boldsymbol{F}$ has the proper values λ and $-\lambda$. Let T_x^1 (resp. T_x^2) be a subspace of T_x^C generated by the proper vectors corresponding to the proper value λ (resp. $-\lambda$). Then T^1 and T^2 are obviously complementary. Thus M is given a Π-structure by the operator $\boldsymbol{F}$.

To the operator $\boldsymbol{F}$ corresponds another complex tensor $F' = -F = (-F_j^i)$, and we have (1.2)′ for this tensor. Obviously F' gives the same Π-structure as F.

In particular, if the dimension m of M is even and if T^1 and T^2 are mutually complex conjugate, then M has an almost complex structure by setting $\lambda = \sqrt{-1}$.

A base of T_x^C is called a *complex base* relative to x. The set $E^C(M)$ of complex bases relative to different points of M admits a structure of a principal fibre bundle over M with structure group $GL(m, C)$. An (infinitesimal) connection in $E^C(M)$ is called a *complex linear connection.*

Let M be a differentiable manifold with a Π-structure (T^1, T^2). A base (e_i) of T_x^C such that $e_\alpha \in T_x^1$, $e_\mu \in T_x^2$ $(\alpha = 1, \cdots, n_1\,; \mu = n_1+1, \cdots, m)$ is called a *Π-adapted base* relative to x. Let (e_i) be a Π-adapted base relative to x. The set $E_{\Pi}(M)$ of Π-adapted bases relative to different points of M admits a structure of a subbundle of $E^C(M)$ whose structure group is the subgroup $G(n_1, n_2)$ of $GL(m, C)$ consisting of matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{1.3}$$

where $A \in GL(n_1, C)$ and $B \in GL(n_2, C)$. $G(n_1, n_2)$ is isomorphic to $GL(n_1, C) \times GL(n_2, C)$. A connection in $E_{\Pi}(M)$ is called a *Π-connection.* G. Legrand [**1**] proved the following theorem. (cf. [**3**])

THEOREM 1. *In order that a complex linear connection can be identified with a Π-connection, it is necessary and sufficient that the tensor (F^i_j) is covariant constant.*

Let ∇ be the operation of the covariant differentiation with respect to a complex linear connection. A transformation φ of M onto itself is called *affine* if it satisfies the equation

$$\varphi(\nabla t) = \nabla(\varphi t), \tag{1.4}$$

for any complex tensor field t on M. $A(M)$ denotes the group of all affine transformations of M onto itself. It is a Lie group with respect to the natural topology. $A_0(M)$ denotes the connected component of the identity of $A(M)$.

Let ∇ be the operation of the covariant differentiation with respect to a Π-connection and F be the tensor field corresponding to the Π-structure (T^1, T^2). Then we have $\nabla F = 0$, i.e., F is parallel. If a transformation φ is affine, then we have

$$\nabla(\varphi F) = \varphi(\nabla F) = 0,$$

hence φF is also parallel.

We shall denote by $P^C(r, s)$ the set of all parallel tensor fields of type (r, s) on M for the Π-connection. $P^C(r, s)$ is obviously a vector space over C. Since any element of $P^C(r, s)$ is uniquely determined by its value at a point $x \in M$, $P^C(r, s)$ is isomorphic with the subspace of the tensor space of type (r, s) over T^C_x consisting of all tensors invariant under the holonomy group $H(x)$ of the Π-connection. It is easily shown that $A(M)$ leaves $P^C(r, s)$ invariant and acts on $P^C(r, s)$ as a group of automorphism. Hence we obtain a homomorphism μ of $A(M)$ into $GL(p, C)$ defined by $\mu(\varphi)t = \varphi t$ for any $t \in P^C(r, s)$, where $p = \dim P^C(r, s)$.

2. Π-connection and affine transformation

2.1. We consider a connected m-dimensional differentiable manifold M with a Π-structure (T^1, T^2). Hereafter we shall assume that M satisfies the second countability axiom, so that the principal fibre bundle over M has always a connection.

M is called *Π-irreducible* if $H(x)$ is Π-irreducible in C, i.e., if $H(x)$ leaves

T_x^i $(i=1,2)$ invariant and it is irreducible on T_x^i $(i=1,2)$. Otherwise, it is called *Π-reducible*. This notion is independent of the choice of x.

PROPOSITION 2.1. *Let M be an m-dimensional manifold with a Π-structure (T^1, T^2). If M is Π-irreducible and if $\dim T^1 \neq \dim T^2$, then the group $A(M)$ of all affine transformations of M preserves the Π-structure.*

PROOF.*) Let φ be an arbitrary element of $A(M)$. For all $x \in M$, $\varphi(T_x^i) \subset T_{\varphi(x)}^C$ $(i=1,2)$. $\varphi(T_x^i)$ $(i=1,2)$ is non-trivial. We assume $\dim T^1 > \dim T^2$. Let P_i $(i=1,2)$ be the projection of T_x^C into T_x^i $(i=1,2)$. Since the vector space $P_1 \circ \varphi(T_x^2)$ is $H(\varphi(x))$-invariant, $P_1 \circ \varphi(T_x^2)$ is either the zero space or $T_{\varphi(x)}^1$. But $P_1 \circ \varphi(T_x^2)$ can not be $T_{\varphi(x)}^1$. Hence $\varphi(T_x^2) = T_{\varphi(x)}^2$, i.e., the field T^2 is invariant by φ.

On the other hand, since $P_2 \circ \varphi(T_x^1)$ is also $H(\varphi(x))$-invariant, it is either the zero space or $T_{\varphi(x)}^2$. We assume $P_2 \circ \varphi(T_x^1) = T_{\varphi(x)}^2$. Let S be the linear subspace of $\varphi(T_x^1)$ consisting of vectors v such that $P_2 v = 0$. Hence S is an $H(\varphi(x))$-invariant subspace of $T_{\varphi(x)}^1$. We have, however, $0 < \dim S = \dim T^1 - \dim T^2 < \dim T^1$. This is contrary to the assumption. Hence $P_2 \circ \varphi(T_x^1)$ is the zero space. Hence we have $\varphi(T_x^1) = T_{\varphi x}^1$, i.e., the field T^1 is invariant by φ. Therefore $A(M)$ preserves the Π-structure (T^1, T^2). q.e.d.

Next we consider the case $\dim T^1 = \dim T^2 = n$ and $m = 2n$.

PROPOSITION 2.2. *Let M be a $2n$-dimensional differentiable manifold with a Π-structure (T^1, T^2). Assume that M is Π-irreducible and that $n = \dim T^1 = \dim T^2$. If there exists no isomorphism of T^1 onto T^2 commuting with the operations of $H(x)$ at every point x of M, then $A(M)$ preserves the Π-structure.*

PROOF. Suppose there exists an element φ of $A(M)$ ($\varphi \neq$ identity of $A(M)$) which does not preserve the given Π-structure (T^1, T^2) at $x \in M$. Since $P_2 \circ \varphi(T_x^1)$ is $H(\varphi(x))$-invariant, $P_2 \circ \varphi(T_x^1)$ is either the zero space or $T_{\varphi(x)}^2$. Similarly we have $P_1 \circ \varphi(T_x^2)$ is either the zero space or $T_{\varphi(x)}^1$. Since φ does not preserve the Π-structure at x, $P_2 \circ \varphi(T_x^1)$ and $P_1 \circ \varphi(T_x^2)$ can not be the zero spaces at the same time.

So we have either $P_2 \circ \varphi(T_x^1) = T_{\varphi(x)}^2$ or $P_1 \circ \varphi(T_x^2) = T_{\varphi(x)}^1$. In both cases, we can easily construct the isomorphism of T_x^1 onto T_x^2 commuting with the operations of $H(x)$, which is contrary to hypothesis. q.e.d.

From Propositions 2.1 and 2.2, follows:

THEOREM 2. *Let M be an m-dimensional differentiable manifold with a Π-structure (T^1, T^2). Assume that M is Π-irreducible. If there exists no isomorphism of T^1 onto T^2 commuting with the operations of $H(x)$ at every point x of M, then $A(M)$ preserves the Π-structure.*

2.2. We consider the case that there exists a differentiable field of isomor-

*) This original proof of this proposition was simplified by a remark of the referee. The author wishes to express his gratitude to the referee.

phisms $\hat{S}$ of T^1 onto T^2 commuting with the operations of the holonomy group of a Π-connection. This means that if we define a differentiable vector field v with $v(x)$ in T^C_x, $\hat{S}v(x)$ belongs to T^C_x and the vector field Sv is also differentiable.

The Π-adapted base (e_i) of T^C_x is called a *$\hat{S}$-adapted base* if we have $e_{\alpha^*}=\hat{S}e_\alpha$ for $\alpha=1,\cdots,n$; $\alpha^*=\alpha+n$. The set $E_{\hat{S}}(M)$ of $\hat{S}$-adapted bases relative to different points of M admits a structure of a subbundle of $E^C(M)$ whose structure group is subgroup $\Gamma_0(n)$ of $GL(2n,C)$ consisting of matrices

$$(2.1)\qquad \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \qquad A\in GL(n,C)$$

$\Gamma_0(n)$ is isomorphic with $GL(n,C)$.

PROPOSITION 2.3. *Let M be a $2n$-dimensional differentiable manifold with a Π-structure (T^1, T^2). Assume that M is Π-irreducible and that $n=\dim T^1=\dim T^2$. If there exists a differentiable field of isomorphisms $\hat{S}$ of T^1 onto T^2 commuting with the operations of the holonomy group of a Π-connection, then M has three independent Π-structures F, G and H such that $FG=-GF=-\sqrt{-1}\,\lambda H$, $GH=-HG=-\sqrt{-1}\,\lambda F$, $HF=-FH=-\sqrt{-1}\,\lambda G$ and they are all invariant under $H(x)$ for every $x\in M$.*

PROOF. We take an $\hat{S}$-adapted base as a Π-adapted one. Since M has a Π-structure (T^1,T^2), there exists a linear operator $\boldsymbol{F}$ satisfying (1.2). Further there exists a tensor field F satisfying (1.2)$'$. The value F_x of F at $x\in M$, which is a tensor of type (1,1), is invariant under $H(x)$ for every $x\in M$. Hence matrix F_x commutes with the operations of $H(x)$. Therefore F_x is a commutator of the representation of $H(x)$.

In general, let $\mathfrak{K}$ be a commutator algebra of the representation of $H(x)$. If $K\in\mathfrak{K}$, then K has the form

$$(2.2)\qquad K=\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

Since this is a commutator, we can get by Schur's lemma

$$K_j=\alpha_j I_n\,,\qquad \alpha_j\in C;\qquad j=1,2,3,4,$$

where I_n is the unit $(n\times n)$ matrix. Hence

$$\begin{pmatrix} \lambda I_n & 0 \\ 0 & \lambda I_n \end{pmatrix},\ \begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix},\ \begin{pmatrix} 0 & \lambda I_n \\ \lambda I_n & 0 \end{pmatrix},\ \begin{pmatrix} 0 & \sqrt{-1}\,\lambda I_n \\ -\sqrt{-1}\,\lambda I_n & 0 \end{pmatrix}$$

forms a base of $\mathfrak{K}$ relative to the $\hat{S}$-adapted base, where λ is a non-zero complex number.

Let $P^C(1,1)$ be the vector space spanned by all parallel tensor fields of type (1,1) on M and $\tilde{P}^C(1,1)$ the subset of all the elements K' of $P^C(1,1)$ such that $K'^2=\lambda^2 I_{2n}$, $K'\neq\pm\lambda I_{2n}$. We assign $K'\in\tilde{P}^C(1,1)$ to the value K'_x of K'

at x. $P^C(1,1)$ is isomorphic with the subspace of the tensor space of type (1,1) over T_x^C consisting of all tensors invariant under the operations of $H(x)$, i.e., $P^C(1,1)$ is isomorphic with the commutator algebra $\mathfrak{K}$. It is obvious that $\tilde{P}^C(1,1)$ is isomorphic with the subset $\tilde{\mathfrak{K}}$ of $\mathfrak{K}$ consisting of the commutators K such that $K^2=\lambda^2 I_{2n}$ $(K\neq \pm\lambda I_{2n})$. We denote by F, G and H the parallel tensor fields deduced from

$$\begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda I_n \\ \lambda I_n & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sqrt{-1}\,\lambda I_n \\ -\sqrt{-1}\,\lambda I_n & 0 \end{pmatrix}$$

respectively. Then we have $F^2=G^2=H^2=\lambda^2 I_{2n}$. Furthermore $FG=-GH$ $=-\sqrt{-1}\,\lambda H$, $GH=-HG=-\sqrt{-1}\,\lambda F$, $HF=-FH=-\sqrt{-1}\,\lambda G$. Thus $\tilde{P}^C(1,1)$ consists of the tensor field $K=\alpha F+\beta G+\gamma H$ such that $\alpha^2+\beta^2+\gamma^2=1$ $(\alpha, \beta, \gamma \in C)$.

A tensor field F generates the two fields of spaces of proper vectors which are identified with the given Π-structure (T^1, T^2). G and H generate the fields of spaces of proper vectors which define other Π-structures respectively. These three Π-structures are all invariant under the operations of $H(x)$ for every $x\in M$ and independent from each other. q. e. d.

2.3. Let M be a $2n$-dimensional differentiable manifold with a Π-structure (T^1, T^2) with $\dim T^1=\dim T^2=n$. We assume that there exists a differentiable field of isomorphisms S of T^1 onto $\bar{T}^2$ commuting with the operations of the holonomy group of a Π-connection, where $\bar{T}^2$ is the complex conjugate of T^2 According to G. Legrand [2], the Π-adapted base (e_i) of T_x^C is called an *S-adapted base* if we have $\bar{e}_{\alpha^*}=Se_\alpha$ for $\alpha=1,\cdots,n$; $\alpha^*=\alpha+n$. The set $E_S(M)$ of S-adapted bases relative to different points of M admits a structure of the subbundle of $E_\Pi(M)$ whose structure group is the subgroup $\hat{\Gamma}(n)$ of $GL(2n, C)$ consisting of matrices

$$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}, \qquad A\in GL(n, C)\,. \tag{2.3}$$

$\hat{\Gamma}(n)$ is isomorphic with $GL(n, C)$.

Let $QL(l, C)$ be a quaternion linear group, i.e., a subgroup of $GL(2l, C)$ composed of all the matrices A' satisfying $A'J_l=J_lA'$, where $J_l=\begin{pmatrix} 0 & -I_l \\ I_l & 0 \end{pmatrix}$. M. Obata [4] proved the following Lemma.

LEMMA. *Let G be a subgroup of $GL(m, C)$ and $\bar{G}$ its complex conjugate. Assume that G is irreducible and is conjugate to $\bar{G}$ but is not conjugate to a subgroup of $GL(m, R)$. Then we have:*

1) *There exists a matrix $S_0\in GL(m, C)$ such that*

$$L_0^{-1}AL_0=\bar{A} \qquad \textit{for all } A \textit{ in } G, \tag{2.4}$$

and $\bar{L}_0L_0=L_0\bar{L}_0=-I_m$. A matrix $L\in GL(m, C)$ satisfies (2.4) *if and only if L is written in the form $L=\alpha L_0$, α being a non-zero complex number.*

2) *m is even, $m=2l$, and there exists a matrix $K\in GL(m, C)$ such that* $K^{-1}L_0\bar{K}=J_l$.

3) *G is conjugate to a subgroup of $QL(l, C)$.*

The representation of $H(x)$ is a subgroup of $\hat{\Gamma}(n)$ and is identified with a subgroup of $GL(n, C)$. That $H(x)$ is a subgroup of $QL(l, C)$ means that the representation of $H(x)$ is a subgroup of $QL(l, C)$ under this identification.

PROPOSITION 2.4. *Let M be a $2n$-dimensional differentiable manifold with a Π-structure (T^1, T^2). Assume that M is Π-irreducible and that $n=\dim T^1=\dim T^2$ and that there exists a differentiable field of isomorphisms S of T^1 onto $\bar{T}^2$ commuting with the operations of the holonomy group $H(x)$ of a Π-connection, where $\bar{T}^2$ is the complex conjugate of T^2. If n is even, $n=2l$, and $H(x)$ is not a subgroup of $QL(l, C)$, or if n is odd, then $A(M)$ preserves the Π-structure. If n is even, $n=2l$, and $H(x)$ is a subgroup of $QL(l, C)$, then M has three independent Π-structures F, J_1 and J_2 such that $FJ_1=-J_1F=-\sqrt{-1}\lambda J_2$, $J_1J_2=-J_2J_1=-\sqrt{-1}\,\lambda F$ and $J_2F=-FJ_2=-\sqrt{-1}\,\lambda J_1$ and they are all invariant under $H(x)$ for every $x\in M$.*

PROOF. We take an S-adapted base. Since an element K of a commutator algebra $\mathfrak{K}$ of the representation of $H(x)$ has the form (2.2), we have, by a method similar as in the proof of Proposition 2.3.,

$$K_1A=AK_1\,,\quad K_2\bar{A}=AK_2\,,\quad K_3A=\bar{A}K_3\,,\quad K_4\bar{A}=\bar{A}K_4\,.$$

Since M is Π-irreducible, we have

$$K_i=\alpha_iI_n\ (\alpha_i\in C\,; i=1,4)\,,$$

$$K_j=0 \text{ or } \det K_j\neq 0\ (j=2,3)\,.$$

If $\det K_j\neq 0$ $(j=2,3)$ we have $K_j^{-1}AK_j=\bar{A}$ $(j=2,3)$. By Lemma, n is even, $n=2l$, and there exists a regular matrix L such that $L_j^{-1}K_j\bar{L}_j=\alpha_jJ_l$ $(j=2,3)$, where α_j $(j=2,3)$ is a complex number. It should be noted that such L_j can shosen independently of the special choice of K in $\mathfrak{K}$.

Therefore, in case $K_j=0$ $(j=2,3)$, we have

$$K=\begin{pmatrix}\alpha_1I_n & 0\\ 0 & \alpha_4I_n\end{pmatrix},\qquad \alpha_1,\alpha_4\in C\,. \tag{2.5}$$

In case $\det K_2\neq 0$ and $\det K_3\neq 0$, if we put $e'_\beta=L_2e_\beta$, $e'_{\beta^*}=L_3e_{\beta^*}$ $(\beta=1,\cdots,n\,;$ $\beta^*=n+1,\cdots,2n)$, where $\{e_\beta, e_{\beta^*}\}$ $(\beta=1,\cdots,n\,;\ \beta^*=n+1,\cdots,2n)$ is the S-adapted base, $\{e'_\beta, e'_{\beta^*}\}$ is another S-adapted base. Relative to this base K has the form

$$K=\begin{pmatrix}\alpha_1I_n & \alpha_2J_l\\ \alpha_3J_l & \alpha_4I_n\end{pmatrix},\qquad \alpha_i\in C\ (i=1,2,3,4)\,. \tag{2.6}$$

This implies that $H(x)$ is a subgroup of $QL(l, C)$ $(n=2l)$. Similarly, in case $K_2=0$ and $\det K_3\neq 0$, K has the form

$$K=\begin{pmatrix}\alpha_1I_n & 0\\ \alpha_3J_l & \alpha_4I_n\end{pmatrix},\qquad \alpha_1,\alpha_3,\alpha_4\in C, \tag{2.7}$$

and in case det $K_2 \neq 0$ and $K_3 = 0$

$$K = \begin{pmatrix} \alpha_1 I_n & \alpha_2 J_l \\ 0 & \alpha_4 I_n \end{pmatrix}, \qquad \alpha_1, \alpha_2, \alpha_4 \in C. \tag{2.8}$$

Conversely if $H(x)$ is a subgroup of $QL(l, C)$, then matrices of the form (2.6), (2.7) and (2.8) are commutators of $H(x)$.

Assume that n is even, $n = 2l$, and $H(x)$ is not a subgroup of $QL(l, C)$, or n is odd. The matrices of the form (2.5) are commutators of the representation of $H(x)$. Hence $\mathfrak{K}$ is generated by

$$\begin{pmatrix} \lambda I_n & 0 \\ 0 & \lambda I_n \end{pmatrix}, \quad \begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix}$$

where λ is a non-zero complex number.

Let F be the tensor field corresponding to $\begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix}$. $\tilde{P}^C(1,1)$ consists of the tensor fields $\pm F$. Hence we have $\mu(\varphi)F = \pm F$ for $\varphi \in A(M)$. F and $-F$ correspond to the given Π-structure (T^1, T^2).

Assume that n is even, $n = 2l$, and $H(x)$ is a subgroup of $QL(l, C)$. $\tilde{P}^C(1,1)$ consists of the tensor fields $K = \alpha F + \beta J_1 + \gamma J_2$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$ $(\alpha, \beta, \gamma \in C)$, where J_1 and J_2 correspond to

$$\begin{pmatrix} 0 & -\sqrt{-1}\,\lambda J_l \\ -\sqrt{-1}\,\lambda J_l & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \lambda J_l \\ -\lambda J_l & 0 \end{pmatrix}$$

respectively. And we have $J_1^2 = J_2^2 = \lambda^2 I_{2n}$, $FJ_1 = -J_1 F = -\sqrt{-1}\,\lambda J_2$, $J_1 J_2 = -J_2 J_1 = -\sqrt{-1}\,\lambda F$ and $J_2 F = -FJ_2 = -\sqrt{-1}\,\lambda J_1$. F corresponds to the given Π-structure (T^1, T^2). J_1 and J_2 correspond to other Π-structures. These three Π-structures are all invariant under the operations of $H(x)$ for every $x \in M$ and independent from each other. q. e. d.

In particular, in a $2n$-dimensional differentiable manifold with the Π-structure, if T^1 and T^2 are mutually complex conjugate and if $\lambda^2 = -1$, the Π-structure (T^1, T^2) determines an almost complex structure. In this case, the matrices F, J_1 and J_2 take the forms

$$F = \begin{pmatrix} \sqrt{-1}\, I_n & 0 \\ 0 & -\sqrt{-1}\, I_n \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \sqrt{-1}\, J_l \\ -\sqrt{-1}\, J_l & 0 \end{pmatrix}$$

respectively. These matrices are nothing but the matrices which define the three independent almost complex structure in Theorem A. We can easily see that these three almost complex structures form the quaternion structure. Hence, in this case, Proposition 2.4 reduces partially to Theorem A described in Introduction.

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