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Affine Translation Surfaces with Constant Gaussian Curvature

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Abstract. We study affine translation surfaces in $\mathbb{R}^3$ and get a complete classification of such surfaces with constant Gauss-Kronecker curvature.

1. Introduction

A surface in $\mathbb{E}^3$ is called a translation surface if it is obtained as a graph of a function $F(x,y) = p(x)+q(y)$, where $p(x)$ and $q(y)$ are differentiable functions. It's well known that a minimal translation surface in the Euclidean space $\mathbb{E}^3$ must be a plane or a Scherk surface, which is the graph of the function $F(x,y) = \ln(\cos x/\cos y)$, the only doubly periodic minimal translation surface.

In this note, we study nondegenerate translation surfaces in affine space $\mathbb{R}^3$. This class of surfaces has been studied previously by many geometers. F. Manhart [3] classified all the nondegenerate affine minimal translation surfaces in affine space $\mathbb{R}^3$. Further treatments are due to H. F. Sun [5], who classified the nondegenerate affine translation surface with nonzero constant mean curvature in $\mathbb{R}^3$. Later on, Sun and Chen extended this into the case of hypersurfaces [6]. On the other hand, Binder [1] classified locally symmetric affine translation surfaces in $\mathbb{R}^3$. Here we give a complete classification of nondegenerate affine translation surfaces with constant Gaussian curvature in $\mathbb{R}^3$. Precisely, we will prove the following theorems.

Theorem 1.1. *Let M be a nondegenerate affine translation surface in $\mathbb{R}^3$ with vanishing Gaussian curvature. Then M is affinely equivalent to one of the graph of the following functions:*

$$z = x^2 + q(y); \tag{1.1}$$

$$z = e^x \pm y^{\frac{1}{2}}; \tag{1.2}$$

$$z = x\ln x \pm y \ln y; \tag{1.3}$$

$$z = \ln x \pm \ln y; \tag{1.4}$$

$$z = x^{\frac{3-2\lambda}{1-\lambda}} \pm y^{\frac{3-2\lambda}{5-3\lambda}}, \tag{1.5}$$

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where $q(y)$ is an arbitrary function and λ is a constant satisfying $\lambda \neq 1, \frac{3}{2}, \frac{5}{3}, 2$.

Theorem 1.2. *Let M be a nondegenerate affine translation surface in $\mathbb{R}^3$ with nonzero constant Gaussian curvature. Then M is affinely equivalent to the graph given by:*

$$z = \frac{1}{x^2} + q(y), \tag{1.6}$$

where $q(y)$ satisfies $(q''')^2 = q''^{\frac{13}{4}}(aq''^{-\frac{3}{4}} + b)$ for constants a, b and $a \neq 0$.

2. Preliminaries

Concerning the following basic facts of affine differential geometry, we refer to [4]. Let $f : M \to \mathbb{R}^3$ be an immersion of a connected, orientable 2-dimensional differentiable manifold into the affine space $\mathbb{R}^3$ equipped with usual flat connection D, a parallel volume element ω, and ξ be an arbitrary local field of transversal vector to $f(M)$. Thus we have the decomposition

$$D_X(f_*Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X\xi = -f_*(SX) + \tau(X)\xi. \tag{2.2}$$

Thus we have an induced affine connection ∇, a symmetric tensor h of type (0,2), a tensor S of type (1,1) and 1-form τ on M and we call h, S and τ the affine second fundamental form, the affine shape operator and the affine transversal connection form, respectively. The affine mean curvature H and the affine Gaussian curvature K are defined by

$$H = \frac{1}{2}\operatorname{Tr} S, \quad K = \det S. \tag{2.3}$$

We define a volume element θ on M by

$$\theta(X_1, X_2) = \omega(f_*(X_1), f_*(X_2), \xi),$$

for any tangent vector fields X_1, X_2 of M.

We say that f is nondegenerate if h is nondegenerate. This condition does not depend on choice of ξ. It's well known that there exists unique choice of ξ such that the corresponding induced connection ∇, the nondegenerate metric h, and the induced volume θ satisfy

(1) (∇, θ) is an equiaffine structure, that is, $\nabla\theta = 0$.

(2) θ coincides with the volume element ω_h of the nondegenerate metric h, where $\omega_h = |\det(h(X_i, X_j))|^{\frac{1}{2}}$. We call such a pair (f, ξ) a Blaschke immersion, ∇ the induced connection and h the affine metric. Condition (2) implies that $\tau = 0$.

Let $z = F(x^1, x^2)$ be a differential function on a domain G in $\mathbb{R}^3$ and consider the immersion

$$f : (x^1, x^2) \in G \mapsto (x^1, x^2, F(x^1, x^2)) \in \mathbb{R}^3.$$

We start with a tentative choice of transversal field $\xi = (0,0,1)$. Since $D_{\partial_i}\xi = 0$, we have $\tau = 0$. Denoting by ∂_i the coordinate vector field ∂/∂_i we have

$$f_*(\partial_1) = (1,0,F_1), \quad f_*(\partial_2) = (0,1,F_2),$$

where $F_j = \partial F/\partial x^j$. Thus

$$D_{\partial_i} f_*(\partial_j) = (0,0,F_{ij}) = F_{ij}\xi, \quad F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}, \tag{2.4}$$

which implies

$$\nabla_{\partial_i}\partial_j = 0, \quad h(\partial_i,\partial_j) = F_{ij}.$$

Thus the immersion is nondegenerate if and only if $\det(F_{ij}) \neq 0$. Since

$$\theta(\partial_1,\partial_2) = \det(f_*(\partial_1), f_*(\partial_2), \xi) = 1,$$

by taking $\phi = |\det(F_{ij})|^{\frac{1}{4}}$, we can find the affine normal field $\bar{\xi}$ in the form

$$\bar{\xi} = -\sum_{j,k}(F^{kj}\phi_j)f_*(\partial_k) + \phi\xi,$$

where $\phi_j = \partial\phi/\partial x_j$, (F^{ij}) is the inverse of the matrix (F_{ij}). It follows that

$$D_{\partial_i}\bar{\xi} = -\sum_{j,k}\partial_i(F^{kj}\phi_j)f_*(\partial_k), \quad S(\partial_i) = \sum_{j,k}\partial_i(F^{kj}\phi_j)\partial_k. \tag{2.5}$$

3. Proof of the theorems

Throughout this section, we assume that M is a translation surface, which is obtained by the graph of function $F(x,y) = p(x) + q(y)$ for some differential functions $p(x)$ and $q(y)$. Hence, we have

$$(F_{ij}) = (h_{ij}) = \begin{pmatrix} p''(x) & 0 \\ 0 & q''(y) \end{pmatrix}, \quad (F^{ij}) = (F_{ij})^{-1} = \begin{pmatrix} p''(x)^{-1} & 0 \\ 0 & q''(y)^{-1} \end{pmatrix},$$

and

$$\phi = |\det(F_{ij})|^{\frac{1}{4}} = |p''(x)q''(y)|^{\frac{1}{4}} \neq 0.$$

It follows from (2.4) and (2.5) that the Gaussian curvature satisfies

$$\begin{aligned} K &= \partial_1(F^{11}\phi_1)\partial_2(F^{22}\phi_2) - \partial_1(F^{22}\phi_2)\partial_2(F^{11}\phi_1) \\ &= (-\frac{7}{16}p'''^2 + \frac{1}{4}p^{(4)}p'')(-\frac{7}{16}q'''^2 + \frac{1}{4}q^{(4)}q'')(p''q'')^{-\frac{5}{2}} - \frac{1}{256}p'''^2q'''^2(p''q'')^{-\frac{5}{2}} \\ &= (\frac{12}{64}p'''^2q'''^2 - \frac{7}{64}p'''^2q''q^{(4)} - \frac{7}{64}p''p^{(4)}q'''^2 + \frac{1}{16}p''q''p^{(4)}q^{(4)})(p''q'')^{-\frac{5}{2}}. \end{aligned}$$

If we put $f(x) = p''(x)$, $g(y) = q''(y)$, then we have

$$64K = [f'^2(12g'^2 - 7gg'') + ff''(4gg'' - 7g'^2)](fg)^{-\frac{5}{2}}. \tag{3.1}$$

Firstly we consider the case when Gaussian curvature K vanishes identically, then

$$f'^2(12g'^2 - 7gg'') + ff''(4gg'' - 7g'^2) = 0. \tag{3.2}$$

From (3.2), it follows that f and g can be interchanged with each other. If $f'(x) = 0$, we can easily get that $p(x) = ax^2 + bx + c$, where a, b, c are constant. By applying an affine transformation, we get the graph of function in the form (1.1). If $g'(y) = 0$, after interchanging x and y, we also obtain (1.1).

From now on, we assume that $f'g' \neq 0$. From (3.2), we get

$$\frac{ff''}{f'^2} = \frac{12g'^2 - 7gg''}{7g'^2 - 4gg''} = \lambda, \tag{3.3}$$

which is equivalent to

$$ff'' = \lambda f'^2, \tag{3.4}$$

$$12g'^2 - 7gg'' = \lambda(7g'^2 - 4gg''), \tag{3.5}$$

where λ is a constant.

We consider the equation (3.4), which splits into two cases:

$$\lambda \neq 1, \quad f = (C_1x + C_2)^{\frac{1}{1-\lambda}}, \quad C_1 \in \mathbb{R} \setminus 0, \quad C_2 \in \mathbb{R}. \tag{3.6}$$

$$\lambda = 1, \quad f = C_3e^{C_4x}, \quad C_3 \in \mathbb{R} \setminus 0, \quad C_4 \in \mathbb{R}. \tag{3.7}$$

If $\lambda = 1$, (3.5) gives $5g'^2 = 3gg''$. Using (3.6), we get that

$$g = (C_5y + C_6)^{-\frac{3}{2}}, \quad C_5 \in \mathbb{R} \setminus 0, \quad C_6 \in \mathbb{R}.$$

After a further integral computation, we obtain the graph of the function in the form (1.2). Especially, if $\lambda = \frac{5}{3}$, the same graph can be obtained.

If $\lambda = 2$, $f = (C_1x + C_2)^{-1}$. Integrating twice, we get

$$p(x) = \frac{(C_1x + C_2)\ln|C_1x + C_2|}{C_1{}^2} - \frac{x}{C_1} + C_3x + C_4,$$

where both C_3 and C_4 are constant. And $2g'^2 = gg''$, similarly, we can get

$$q(y) = \frac{(D_1y + C_2)\ln|D_1y + D_2|}{D_1{}^2} - \frac{y}{D_1} + D_3y + D_4,$$

where all D_i are constant and $D_1 \neq 0$. By applying an affine transformation, we have the graph of function in the form (1.3).

If $\lambda = \frac{3}{2}$, similarly, after some integral computation, we obtain the function in the form (1.4).

In general cases, i.e., $\lambda \neq 1, \frac{3}{2}, \frac{5}{3}, 2$, by an appropriate affine transformation, equations (3.4)-(3.6) immediately yield case (1.5). This completes the proof of Theorem 1.1.

Next we assume that K is a nonzero constant. Differentiating (3.1) with respect to x and y, we get

(3.8) $(4ff'f'' - 5f'^3)(12g'^2 - 7gg'') + (2f^2f''' - 3ff'f'')(4gg'' - 7g'^2) = 0,$

(3.9) $(4gg'g'' - 5g'^3)(12f'^2 - 7ff'') + (2g^2g''' - 3gg'g'')(4ff'' - 7f'^2) = 0.$

In order to prove Theorem 1.2, we need the following lemma:

Lemma 3.1. *If $4ff'' \neq 5f'^2$ and $4gg'g'' \neq 5g'^3$, then $K = 0$.*

Proof. Under the hypothesis, $4ff'' \neq 5f'^2$ and $4gg'g'' \neq 5g'^3$, from (3.8) and (3.9) we see that there exist two constants λ and μ such that

$$\begin{aligned} 2f^2f''' - 3ff'f'' &= \lambda(4ff'f'' - 5f'^3), & (3.10)\\ 7gg'' - 12g'^2 &= \lambda(4gg'' - 7g'^2), & (3.11)\\ 2g^2g''' - 3gg'g'' &= \mu(4gg'g'' - 5g'^3), & (3.12)\\ 7ff'' - 12f'^2 &= \mu(4ff'' - 7f'^2). & (3.13)\end{aligned}$$

Differentiating (3.11) with respect to y, we get

$$(4\lambda - 7)gg''' = (10\lambda - 17)g'g'', \tag{3.14}$$

clearly $\lambda \neq \frac{7}{4}, \lambda \neq \frac{12}{7}$. Substituting (3.11) and (3.14) into (3.12), we get

$$\frac{2(10\lambda - 17)}{4\lambda - 7} - (4\mu + 3) + \frac{5\mu(4\lambda - 7)}{7\lambda - 12} = 0, \tag{3.15}$$

i.e.

$$(13 - 8\lambda)(12 - 7\mu + \lambda(4\mu - 7)) = 0. \tag{3.16}$$

Similarly, differentiating (3.13) with respect to x, we can get

$$(13 - 8\mu)(12 - 7\mu + \lambda(4\mu - 7)) = 0, \tag{3.17}$$

where $\mu \neq \frac{7}{4}, \mu \neq \frac{12}{7}$. If $\lambda = \mu = \frac{13}{8}$, (3.11) and (3.13) give $4ff'' = 5f'^2$ and $4gg'g'' = 5g'^3$, thus we have

$$12 - 7\mu + \lambda(4\mu - 7) = 0. \tag{3.18}$$

Substituting (3.18) into (3.11) we get $\mu g'^2 = gg''$. Then substituting (3.13) into (3.1) we can find K=0. □

Proof of Theorem 1.2. Under the hypothesis, K is a nonzero constant. From Lemma 3.1, we have either $4ff'' = 5f'^2$ or $4gg'' = 5g'^2$. Without loss of generality, we assume $4ff'' = 5f'^2$. Hence (3.1) reduces to

$$256K = [f'^2(13g'^2 - 8gg'')](fg)^{-\frac{5}{2}}. \tag{3.19}$$

Thus there exists a nonzero constant C such that

$$13g'^2 - 8gg'' = Cg^{\frac{5}{2}}. \tag{3.20}$$

If we put

$$g' = \frac{dg}{dy} = h,$$

then

$$g'' = \frac{dh}{dg}h = \frac{1}{2}\frac{dh^2}{dg}.$$

It follows from (3.20) that

$$\frac{dh^2}{dg} = \frac{13h^2}{4g} - \frac{C}{4}g^{\frac{3}{2}}. \tag{3.21}$$

Solving (3.21) gives

$$h^2 = g^{\frac{13}{4}}(ag^{-\frac{3}{4}} + b), \tag{3.22}$$

where $a = \frac{1}{3}C$ and b is a constant. Hence, by applying an affine transformation, we have the graph of the function in the form (1.6) of Theorem 1.2. This completes the proof of Theorem 1.2. □

4. A special example

In this section, we give a special example of affine translation surfaces in $\mathbb{R}^3$ with nonzero constant affine Gaussian curvature.

In view of equation (3.22), it is equivalent to

$$\frac{dg}{dy} = \pm g^{\frac{13}{8}}(ag^{-\frac{3}{4}} + b)^{\frac{1}{2}}. \tag{4.1}$$

If $b = 0$, then (4.1) becomes

$$g' = \pm a^{\frac{1}{2}} g^{\frac{5}{4}} \tag{4.2}$$

for $a > 0$. Solving (4.2) gives

$$g = \left(c \mp \frac{1}{4}a^{-\frac{1}{2}}y\right)^{-4}. \tag{4.3}$$

Integrating (4.3) twice with respect to y and by applying an affine transformation, we obtain a graph

$$z = \frac{1}{x^2} + \frac{1}{y^2}, \tag{4.4}$$

which is a special example of affine translation surfaces with nonzero constant affine Gaussian curvature.

If $b \neq 0$, we can not obtain a explicit solution of (4.1). Assume that $g^{-\frac{3}{4}} = t$, (4.1) implies that

$$\int \frac{t^{-\frac{1}{6}}}{(at+b)^{\frac{1}{2}}} dt = \int \pm \frac{3}{4} dy. \tag{4.5}$$

The left of equality (4.5) is a binomial calculous,

$$\int \frac{t^{-\frac{1}{6}}}{(at+b)^{\frac{1}{2}}} dt = \int t^{-\frac{1}{6}-\frac{1}{2}} (\frac{at+b}{t})^{-\frac{1}{2}} dt. \tag{4.6}$$

As is well known, Tchebyshev proved that this kind of integration's primary functions are not elementary functions.

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