Journal of Algebraic Combinatorics **15** (2002), 151–187

Affine Type A Crystal Structure on Tensor Products of Rectangles, Demazure Characters, and Nilpotent Varieties*

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Received February 8, 1999; Revised September 13, 2001

Abstract. Answering a question of Kuniba, Misra, Okado, Takagi, and Uchiyama, it is shown that certain higher level Demazure characters of affine type A, coincide with the graded characters of coordinate rings of closures of conjugacy classes of nilpotent matrices.

Keywords: crystal graph, tableau, Kostka polynomial, Littlewood-Richardson coefficient

1. Introduction

In [14, Theorem 5.2] it was shown that the graded characters of certain A_{n-1}-stable Demazure submodules of level one integrable highest weight modules of type $A^{(1)}_{n-1}$, have graded A_{n-1}-multiplicities given by Kostka-Foulkes polynomials in the variable $q = e^{-\delta}$ where δ is the null root. The proof consists of showing the equality of four families of polynomials.

(1) Graded multiplicities of A_{n-1}-irreducibles in suitable A_{n-1}-stable level one Demazure characters of type $A^{(1)}_{n-1}$.
(2) Graded multiplicities of A_{n-1}-irreducibles in tensor products of finite $A^{(1)}_{n-1}$-crystals indexed by fundamental A_{n-1}-weights.
(3) Generating functions of Young tableaux of a given shape and weight, by the charge statistic.
(4) The Kostka-Foulkes polynomials (see [22] for their definition).

The first two were shown to coincide by [14], the next two by [24], and the last two by [18].

The main result of this paper is a "higher level" generalization of [14, Theorem 5.2]. The Kostka-Foulkes polynomials are replaced by the generalized Kostka polynomials. They are indexed by the isotypic component (a partition λ with at most n rows) and a sequence $R = (R_1, R_2, \ldots, R_t)$ of A_{n-1}-weights that are multiples of fundamental weights. Here R_j

*Supported in part by the National Science Foundation.

is a rectangular partition with μ_j columns and η_j rows with $\eta_j < n$. The generalizations of the above four polynomials are:

(1) The λ-th graded multiplicity in an A_{n-1}-stable type $A_{n-1}^{(1)}$ Demazure submodule of higher level.
(2) $E_{\lambda;R}(q)$, the graded multiplicity of the λ-th irreducible in the finite crystal of type $A_{n-1}^{(1)}$ given by $B^R = B^{R_t} \otimes \cdots \otimes B^{R_1}$, where B^{R_j} (defined in [7]) is the crystal of type $A_{n-1}^{(1)}$ whose underlying A_{n-1}-module is irreducible of highest weight R_j. The grading is given by the energy function of [24].
(3) $\mathrm{LRT}_{\lambda;R}(q)$, the so-called generalized Kostka polynomials, which are defined combinatorially using Littlewood-Richardson (LR) tableaux with a generalized charge statistic [30].
(4) $K_{\lambda;R}(q)$, the graded multiplicity of the λ-th isotypic component of the graded gl_n-module given by the coordinate ring of a nilpotent adjoint orbit closure, twisted by a line bundle [34].

The polynomials $K_{\lambda;R}(q)$, which possess many properties generalizing those of the Kostka-Foulkes, have been studied extensively using algebro-geometric and combinatorial methods [11, 12, 30, 31, 33, 34].

Under suitable restrictions, we show that the first two families of polynomials coincide using the methods of [14]. The last two families coincide by [30, Theorem 10], where, analogously to [18], it is shown that the LR tableau generating function satisfies a defining recurrence of Weyman [34, 36] for $K_{\lambda;R}(q)$ that generalizes Morris' recurrence for Kostka-Foulkes polynomials [23].

Our main task is to establish the equality of the middle two families. It is well-known that they agree at $q = 1$. So it must be shown that the natural grading on B^R given by the energy function, coincides with the graded poset structure on the set of LR tableaux which parametrize the multiplicity space of B^R viewed as an A_{n-1}-crystal. In particular, using the language of tableaux and the Robinson-Schensted-Knuth correspondence, explicit descriptions are given for the following constructions.

- The affine crystal raising operator $\tilde{e}_0$ acting on B^R, which involves the generalized cyclage operators of [30] on LR tableaux and the promotion operator on tableaux.
- The combinatorial R-matrix on a tensor product of the form $B^{R_2} \otimes B^{R_1}$, which corresponds to a generalized automorphism of conjugation acting on Littlewood-Richardson tableaux [30].
- The energy function on B^R [24], which is shown to coincide with the generalized charge on LR tableaux [11, 30].

Moreover it is shown that every generalized cocyclage relation [30] on LR tableaux may be realized by $\tilde{e}_0$ on B^R.

As an application, a simple proof is given for a monotonicity property of $K_{\lambda;R}(q)$ (conjectured by A. N. Kirillov) that extends a property of the Kostka-Foulkes polynomials that was proved by Han [4].

The connection between the Demazure modules and the nilpotent adjoint orbit closures has a geometric explanation in the following special case. Let η be a partition of n and X_η the Zariski closure of the conjugacy class of the nilpotent $n \times n$ Jordan matrix with block sizes given by the transpose partition η^t of η:

$$X_\eta = \{A \in \mathrm{gl}_n(\mathbb{C}) \mid \dim\ker A^i \geq \eta_1 + \cdots + \eta_i \text{ for all } i.\}$$

We consider the graded $gl(n)$-module given by the coordinate ring $\mathbb{C}[X_\eta]$ of X_η. Lusztig gave an embedding of the variety X_η as an open dense subset of a P-stable Schubert variety Y_η in $\widehat{SL}_n/P$, where $P \cong SL_n$ is the parabolic subgroup given by "omitting the reflection r_0" [21]. The relevant level l Demazure module, viewed as an sl_n-module, is isomorphic to the dual of the space of global sections $H^0(Y_\eta, \mathcal{L}_{l\Lambda_0})$, where $\mathcal{L}_{l\Lambda_0}$ is the restriction to Y_η of the homogeneous line bundle on $\widehat{SL}_n/P$ affording the fundamental weight $l\Lambda_0$. As l goes to infinity, $H^0(Y_\eta, \mathcal{L}_{l\Lambda_0})$ tends to $\mathbb{C}[X_\eta]$ as a graded sl_n-module.

Thanks to M. Okado for pointing out the preprint math.QA/9802111, (which became the paper [27]), which has considerable overlap with this paper (which is a revision of the 1998 preprint math.QA/9804039) and [30, 31] (math.QA/9804037 and math.QA/9804038 respectively).

1.1. Quantized universal enveloping algebras

We only require the following three algebras: $U_q(sl_n) \subset U'_q(\widehat{sl}_n) \subset U_q(\widehat{sl}_n)$.

Let us recall some definitions for quantized universal enveloping algebras taken from [5] and [6]. Consider the following data: a finitely generated $\mathbb{Z}$-module P (weight lattice), a set I (index set for the Dynkin diagram), elements $\{\alpha_i \mid i \in I\}$ (basic roots) and $\{h_i \in P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \mid i \in I\}$ (basic coroots) such that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a generalized Cartan matrix, and a symmetric form $(\cdot,\cdot) : P \times P \to \mathbb{Q}$ such that $(\alpha_i, \alpha_i) \in \mathbb{Z}$ is positive, $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in \mathbb{Q} \otimes P$. Let $\mathfrak{g} = \mathfrak{g}(I, P)$ be the Kac-Moody Lie algebra defined by the above data, and $U_q(\mathfrak{g})$ the quantized universal enveloping algebra, the $\mathbb{Q}(q)$-algebra with generators $\{e_i, f_i \mid i \in I\}$ and $\{q^h \mid h \in P^*\}$ and relations as in [6, Section 2].

For $\mathfrak{g} = \widehat{sl}_n$, let $I = \{0, 1, 2, \ldots, n-1\}$, $(a_{ij})_{i,j\in I}$ the Cartan matrix of type $A^{(1)}_{n-1}$, P the free $\mathbb{Z}$-module with basis $\{\Lambda_i \mid i \in I\} \cup \{\delta\}$ (fundamental weights) and let P^* have dual basis $\{h_i \mid i \in I\} \cup \{d\}$. Define the elements $\{\alpha_i \in P \mid i \in I\}$ by

$$\alpha_i = \delta_{0i}\delta + \sum_{j\in I} a_{ij}\Lambda_j,$$

so that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is the Cartan matrix of type $A^{(1)}_{n-1}$ and $\delta = \sum_{i\in I}\alpha_i$. Define the symmetric $\mathbb{Q}$-valued form $(\cdot,\cdot)$ by $(\alpha_i, \alpha_j) = a_{ij}$ for $i, j \in I$, $(\alpha_i, \Lambda_0) = \delta_{i0}$ for $i \in I$, and $(\Lambda_0, \Lambda_0) = 0$. This defines the data for $\mathfrak{g} = \widehat{sl}_n$. Let $c = \sum_{i\in I} h_i \in P^*$.

Let $P^+ = \mathbb{Z}\delta \oplus \bigoplus_{i\in I}\mathbb{Z}_+\Lambda_i$ be the dominant weights. For $\Lambda \in P^+$ let $\mathcal{V}(\Lambda)$ be the irreducible integrable highest weight $U_q(\widehat{sl}_n)$-module of highest weight Λ, $\mathcal{B}(\Lambda)$ its crystal graph, and $u_\Lambda \in \mathcal{B}(\Lambda)$ the highest weight vector.

For $U'_q(\widehat{sl}_n)$, the same Dynkin index set I and Cartan matrix (a_{ij}) are used as for $U_q(\widehat{sl}_n)$, but P is replaced by the "classical weight lattice" $P_{cl} = P/\mathbb{Z}\delta = \bigoplus_{i\in I}\mathbb{Z}\Lambda_i$ (where by abuse of notation the image of Λ_i in P_{cl} is also denoted Λ_i). The simple coroots $\{h_i \mid i \in I\}$ form a $\mathbb{Z}$-basis of P_{cl}^*. The simple roots are $\{\bar{\alpha}_i \mid i \in I\}$ where $\bar{\alpha}_i$ denotes the image of α_i in P_{cl} for $i \in I$. In P_{cl} the simple roots are linearly dependent. The pairing and symmetric form are induced by those above. The quantum algebra for this data is denoted $U_q'(\widehat{sl_n})$. It may be viewed as a subalgebra of $U_q(\widehat{sl_n})$. Let $P_{cl}^+ = \bigoplus_{i \in I} \mathbb{Z}_+ \Lambda_i$. For $\Lambda \in P^+$, the $U_q(\widehat{sl_n})$-module $\mathcal{V}(\Lambda)$ is a $U_q'(\widehat{sl_n})$-module by restriction, with weights taken modulo δ.

For $U_q(sl_n)$, the index set for the Dynkin diagram is $J = \{1, 2, \dots, n-1\} \subset I$, the Cartan matrix $(a_{ij})_{i,j \in J}$ is the restriction of the above Cartan matrix to $J \times J$, the weight lattice is $\bar{P}_{cl} = P_{cl}/\mathbb{Z}\Lambda_0$. The simple coroots $\{h_i \mid i \in J\}$ form a $\mathbb{Z}$-basis of $\bar{P}_{cl}^* = \bigoplus_{i \in J} \mathbb{Z} h_i$. The simple roots are $\{\bar{\alpha}_i \in \bar{P}_{cl} \mid i \in J\}$. The algebra for this data is $U_q(sl_n)$, which can be viewed as a subalgebra of $U_q'(\widehat{sl_n})$. Denote by $\{\bar{\Lambda}_i \mid i \in J\}$ the fundamental weights. Often we view these as elements of P_{cl} by $\bar{\Lambda}_i = \Lambda_i - \Lambda_0$. We shall occasionally refer to an element of $\mathbb{Z}^n$ instead of its image in $\bar{P}_{cl}$ under the projection $\mathrm{wt}_{\mathrm{sl}} : \mathbb{Z}^n \to \bar{P}_{cl}$ defined by

$$\mathrm{wt}_{\mathrm{sl}}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n-1} (a_i - a_{i+1}) \bar{\Lambda}_i. \tag{1.1}$$

The kernel of $\mathrm{wt}_{\mathrm{sl}}$ is generated by the vector (1^n). Let $\bar{P}_{cl}^+ = \bigoplus_{i \in J} \mathbb{Z}_+ \bar{\Lambda}_i$ be the dominant integral weights. For $\lambda \in \bar{P}_{cl}^+$ let V^λ be the irreducible $U_q(sl_n)$-module of highest weight λ, and B^λ its crystal graph.

Denote by W and $\bar{W}$ the Weyl groups of $\widehat{sl_n}$ and sl_n respectively. W (resp. $\bar{W}$) is the subgroup of $\mathrm{Aut}(P)$ generated by the simple reflections $\{r_i \mid i \in I (\text{resp. J})\}$ where

$$r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \, \alpha_i.$$

Let $\bar{Q} = \bigoplus_{i \in J} \mathbb{Z} \bar{\alpha}_i \subset P_{cl}$. W acts faithfully on the affine subspace $X = \Lambda_0 + \bar{Q} \subset P_{cl}$. For $\mu \in \bar{Q}$ let $t_\mu : X \to X$ be translation by μ. Then $W \cong \bar{Q} \rtimes \bar{W}$ where $\mu \in \bar{Q}$ acts by t_μ. Let $\theta = \sum_{i \in J} \bar{\alpha}_i \in \bar{P}_{cl}$ be the highest root of sl_n and $r_\theta \in \bar{W}$ the reflection through the hyperplane orthogonal to θ, given in simple reflections by

$$r_\theta = r_1 \dots r_{n-2} r_{n-1} r_{n-2} \dots r_1.$$

Then $r_0 = t_\theta r_\theta$ acts on X by

$$r_0(\Lambda_0 + x) = \Lambda_0 + \theta + r_\theta x$$

and

$$r_i(\Lambda_0 + x) = \Lambda_0 + r_i x \quad \text{for } i \in J$$

for all $x \in \bar{Q}$.

For $\Lambda \in P^+$ and $w \in W$ the Demazure module of lowest weight $w\Lambda$ is defined by $\mathcal{V}_w(\Lambda) = U_q(\mathfrak{b}) v_{w\Lambda}$ where $v_{w\Lambda}$ is a generator of the (one dimensional) extremal weight space in $\mathcal{V}(\Lambda)$ of weight $w\Lambda$ and $U_q(\mathfrak{b})$ is the upper triangular subalgebra of $U_q(\widehat{sl_n})$ generated by the e_i and $h \in P^*$.

1.2. Main result

Let η be a partition of n and X_η as in the introduction. The coordinate ring $\mathbb{C}[X_\eta]$ has a graded sl_n-action induced by matrix conjugation on X_η. For $\lambda \in \bar{P}^+_{cl}$, define the graded multiplicity

$$K_{\lambda;\eta}(q) = \sum_{d\geq 0} q^d \dim \mathrm{Hom}_{sl_n}(V^\lambda, \mathbb{C}[X_\eta]_d) \tag{1.2}$$

where $\mathbb{C}[X_\eta]_d$ is the homogeneous component of degree d.

Remark 1.1 $K_{\lambda;\eta}(q)$ is not the Kostka polynomial, but a generalization; see [34]. If λ is a partition of n with at most n parts then $K_{\mathrm{wt}_{\mathrm{sl}}\lambda;\eta}(q) = \tilde{K}_{\lambda^t\eta}(q)$ which is a renormalization of the Kostka-Foulkes polynomial with indices λ^t and μ.

Theorem 1.2 *Let l be a positive integer, η a partition of n, w_0 the longest element of $\bar{W}$, and $w_\eta = t_{w_0\mathrm{wt}_{\mathrm{sl}}(\eta^t)} \in W$. Then*

$$e^{-l\Lambda_0}\,\mathrm{ch}\mathcal{V}_{w_\eta}(l\Lambda_0) = \sum_\lambda K_{\mathrm{wt}_{\mathrm{sl}}(\lambda);\eta}(q)\,\mathrm{ch}V^{\mathrm{wt}_{\mathrm{sl}}(\lambda)}$$

where λ runs over the partitions having at most n parts and $|\lambda| = ln$.

2. Littlewood-Richardson tableaux

In this section we recall the theory of Littlewood-Richardson tableaux [30].

2.1. Tableaux and RSK

For definitions regarding partitions and tableaux, see [2], which uses the English convention. A horizontal strip is a skew shape with at most one cell per column. If D and E are skew shapes, let $D \otimes E$ be the skew shape obtained by placing a translate of D to the southwest of a translate of E (this is denoted $D * E$ in [2, Section 5.1]). A (skew) tableau b is identified with its row-reading word, denoted here by word(b) (called $w(b)$ in [2, Section 2.1]). A row word is a weakly increasing one, and a column word is a strictly decreasing one. Write $m_i(u)$ to be the number of occurrences of the letter i in the word u. Let

$$\mathrm{content}(u) = (m_1(u), m_2(u), \ldots).$$

Denote by $\mathrm{T}(D)$ the set of tableaux of shape D and by $\mathrm{T}(D, \beta)$ the set of tableaux of shape D and content β.

Knuth's relation $\equiv$ [13] is the equivalence relation on words, generated by the so-called elementary transformations, where x, y, z are letters and u, v are words:

$$\begin{aligned} uxzyv &\equiv uzxyv \quad \text{for } x \leq y < z \\ uyzxv &\equiv uyxzv \quad \text{for } x < y \leq z. \end{aligned} \tag{2.1}$$

Say that a skew shape is normal (resp. antinormal) if it has a unique northwest (resp. southeast) corner cell [3].

Theorem 2.1 [13] *For any word v,*

(1) *There is a unique (up to translation) tableau $P(v)$ of normal shape such that $v \equiv P(v)$.*
(2) *There is a unique (up to translation) skew tableau $\mathrm{P}_{\searrow}(v)$ of antinormal shape such that $v \equiv \mathrm{P}_{\searrow}(v)$.*

$P(v)$ may be computed by Schensted's column insertion algorithm [2, Appendix A.2]. Both $P(v)$ and $\mathrm{P}_{\searrow}(v)$ may be computed by Schützenberger's jeu-de-taquin sliding algorithm [2, Section 1.2].

Write $[n] = \{1, 2, \dots, n\}$. Given a word u and a subalphabet A, let $u|_A$ denote the word obtained from u by erasing all letters not in A. Let v_j be a row word (almost all empty) for $j \geq 1$ and write $v = (\dots, v_2, v_1)$. Define the pair of tableaux $(\mathbb{P}(v), \mathbb{Q}(v))$ by

$$\begin{aligned} \mathbb{P}(v) &= P(\cdots v_2 v_1) \\ \mathrm{shape}\big(\mathbb{Q}(v)|_{[j]}\big) &= \mathrm{shape}(P(v_j \dots v_1)) \quad \text{for all } j \geq 0. \end{aligned} \tag{2.2}$$

Theorem 2.2 [13] *The map $v \mapsto (\mathbb{P}(v), \mathbb{Q}(v))$ is a bijection from the set of sequences of row words (almost all empty) to pairs of tableaux of the same shape, such that* $\mathrm{content}(v) = \mathrm{content}(\mathbb{P}(v))$ *and $m_i(\mathbb{Q}(v))$ is the length of v_i for all $i \geq 1$.*

This bijection is a version of the celebrated RSK correspondence.

2.2. *R-LR tableaux*

This section follows [30, Section 2.3]. Let $R = (R_1, R_2, \dots, R_t)$ be a sequence of partitions such that R_j has η_j parts. Let $A_1 = [1, \eta_1]$ be the first η_1 positive integers, $A_2 = [\eta_1 + 1, \eta_1 + \eta_2]$ the interval consisting of the next η_2 integers, and so on. Let $R_t \otimes \cdots \otimes R_1$ be the skew shape embedded in the plane in such a way that A_j is the set of row indices for R_j. Let $\gamma = (\gamma_1, \gamma_2, \dots)$ be the sequence of integers given by juxtaposing the parts of R_1 through R_t.

For a finite sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of positive integers, define the key tableau of content α by $\mathrm{Key}(\alpha) = P(\cdots 2^{\alpha_2} 1^{\alpha_1})$. This is the unique tableau of content α, whose shape α^+ is the partition obtained by sorting the parts of α. If α is a partition then $\mathrm{Key}(\alpha)$ is the tableau of shape α whose r-th row consists of α_r copies of the letter r for all r.

Let $Y_j = \mathrm{Key}(R_j)$ in the alphabet A_j.

Example 2.3 Let $R = ((2, 2), (3, 3, 3), (3, 3))$. Then $\gamma = (2, 2, 3, 3, 3, 3, 3)$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, and $A_3 = \{6, 7\}$. The tableaux Y_j are given by

$$Y_1 = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \qquad Y_2 = \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 4 & 4 & 4 \\ \hline 5 & 5 & 5 \\ \hline \end{array} \qquad Y_3 = \begin{array}{|c|c|c|} \hline 6 & 6 & 6 \\ \hline 7 & 7 & 7 \\ \hline \end{array}$$

Remark 2.4 The notation used here coincides with that in [30] except that here n is used as in A_{n-1} and is not assumed to coincide with the quantity $\sum_{j=1}^{t} \eta_j$ as it is in [30].

Say that a word u in the alphabet $A_1 \cup \cdots \cup A_t$ is R-LR (short for R-Littlewood-Richardson) if $P(u|_{A_j}) = Y_j$ for all $1 \le j \le t$. Denote by $W(R)$ the set of R-LR words. Denote by $\mathrm{LRT}(D; R) = W(R) \cap \mathrm{T}(D)$ the set of R-LR tableaux of (skew) shape D and $\mathrm{LRT}(R) = \bigcup_\lambda \mathrm{LRT}(\lambda; R)$ where λ runs over partitions.

Theorem 2.5 [37] *The map* $v \mapsto (\mathbb{P}(v), \mathbb{Q}(v))$ *gives a bijection*

$$\mathrm{T}(R_t \otimes \cdots \otimes R_1) \cong \bigcup_\lambda \mathrm{T}(\lambda) \times \mathrm{LRT}(\lambda; R). \tag{2.3}$$

Remark 2.6 If $R_j = (\gamma_j)$ is a single-rowed partition for all j, then $\mathrm{LRT}(D; R) = \mathrm{T}(D, \gamma)$, and $W(R)$ is the set of words of content γ.

2.3. Symmetry bijections

From now on, R is assumed to be a sequence of rectangular partitions $R_j = (\mu_j^{\eta_j})$ for $1 \le j \le t$. The symmetric group on the set $[t]$ acts on sequences of t rectangles by reordering. In [30, Section 2.4] bijections $\tau_p : W(R) \to W(r_p R)$ are explicitly defined. This notation is not precise as τ_p depends on the sequence of rectangles R.

Given any word w, let $Q(w) = \mathbb{Q}(w)$ where $\mathbb{Q}(w)$ is computed with respect to the factorization of w into the row words given by its individual letters.

Theorem 2.7 [30, Theorem 8] *The bijections* $\tau_p : W(R) \to W(r_p R)$ *satisfy the following properties*:

(A0) τ_p *restricts to a bijection* $\tau_p : \mathrm{LRT}(D; R) \to \mathrm{LRT}(D; r_p R)$ *for any skew shape* D.
(A1) $P(\tau_p w) = \tau_p P(w)$ *for all* $w \in W(R)$.
(A2) $Q(\tau_p w) = Q(w)$ *for all* $w \in W(R)$.
(A3) *The bijections* $\{\tau_p\}$ *define an action of the symmetric group on LR words.*
(A4) *Suppose the permutation* τ *stabilizes the interval* $I \subset [t]$ *and fixes its complement. Let* $J = \bigcup_{i \in I} A_i$. *Then for all* $w \in W(R)$, *the words* w *and* τw *agree in all positions occupied by letters not in* J.
(A5) *If* $R_p = R_{p+1}$ *then* τ_p *acts as the identity on* $W(R)$.

Remark 2.8 In the case that R_j is a single row for all j, the bijections τ_p were defined in [19] and are called automorphisms of conjugation.

2.4. Generalized charge

Define the statistic charge_R: $W(R) \to \mathbb{Z}_+$ as follows. If R has less than two rectangles then charge_R is identically zero. Let $R = (R_1, R_2)$. Define $d_{R_2,R_1} : W(R_1, R_2) \to \mathbb{Z}_+$ by

$$d_{R_2,R_1}(u) = \begin{array}{l}\text{number of cells in shape}(P(u))\text{ strictly}\\ \text{to the right of the }\max(\mu_1, \mu_2)\text{-th column.}\end{array} \tag{2.4}$$

For a general sequence of rectangles R, for $u \in W(R)$ define

$$\mathrm{charge}_R(u) = \frac{1}{t!} \sum_{\tau \in S_t} \sum_{i=1}^{t-1} (t-i) d_i(\tau u)$$

where $d_i(\tau u)$ is the statistic (2.4) on the restriction of τu to the union of the i-th and $(i+1)$-th alphabets for τR.

Remark 2.9 In the case that $R_j = (\gamma_j)$ for all j, charge_R is exactly the formula given in [17] for the charge statistic first defined in [18, 19].

Remark 2.10 Suppose $\mu_1 \geq \mu_2$, $T \in \mathrm{LRT}(\lambda; (R_1, R_2))$ and λ is a partition such that $\lambda_1 = \mu_1$. Then λ is the shape consisting of R_1 sitting atop R_2, $T = P(Y_2 Y_1)$, and $d_1(T) = 0$. Of course $d_1(\tau T) = 0$ as well.

Define the polynomial

$$\mathrm{LRT}_{\lambda; R}(q) = \sum_{T \in \mathrm{LRT}(\lambda; R)} q^{\mathrm{charge}_R(T)}. \tag{2.5}$$

The following is a special case of [30, Theorem 10].

Theorem 2.11 *Let η be a partition of n and let R_j have η_j rows and l columns. Then for every partition λ with $|\lambda| = ln$ having at most n parts,*

$$K_{\lambda;\eta}(q) = \mathrm{LRT}_{\lambda; R}(q),$$

with $K_{\lambda;\eta}(q)$ as in (1.2).

2.5. *R-cocyclage poset*

The set $\mathrm{LRT}(R)$ admits a graded poset structure called the R-cocyclage. Let $w = ux \in W(R)$ with x a letter. By [30, Section 3.2], there is a bijection $\chi_R : W(R) \to W(R)$ defined by

$$\chi_R(w) = \left(w_0^R x\right)\left(w_0^R u\right) \tag{2.6}$$

where w_0^R is the automorphism of conjugation associated with the longest element in the Young subgroup $S_{A_1} \times \cdots \times S_{A_t}$.

Given $S, T \in \mathrm{LRT}(R)$, write $S \leftarrow_R T$ if there is a $w \in W(R)$ such that $T = P(w)$ and $S = P(\chi_R(w))$. More specifically, if in addition $w = ux$ with x a letter and $\nu = \mathrm{shape}(P(u))$, then write $S \leftarrow_{R,\nu} T$. Write $S \lessdot_{R,\nu} T$ if $S \leftarrow_{R,\nu} T$ and the column index of the cell $s = \mathrm{shape}(T)/\nu$ is strictly greater than μ_j for all j. Write $S \lessdot_R T$ if $S \lessdot_{R,\nu} T$ for some ν. This is called the R-cocyclage relation.

Theorem 2.12 [30. Theorem 20]
(1) $\mathrm{LRT}(R)$ *is a graded poset with covering relation* $\lessdot_R$. *Let* $\leq_R$ *be the associated partial order.*
(2) *An element of* $\mathrm{LRT}(R)$ *is* $\leq_R$*-minimal if and only if it has exactly* $\max_j \mu_j$ *columns.*
(3) *Let* $R^\wedge = (R_2, R_3, \ldots)$, $T \in \mathrm{LRT}(R)$ *be* $\leq_R$*-minimal and* $\mu_1 = \max_j \mu_j$. *Then* $T = T^\wedge Y_1$ *and* $T^\wedge \in \mathrm{LRT}(R^\wedge)$ *in the alphabet* $A_2 \cup A_3 \cup \ldots$.

The generalized charge has the following intrinsic characterization, which implies that it is a grading function for the poset $(\mathrm{LRT}(R), \leq_R)$.

Theorem 2.13 [30, Theorem 22] *There is a unique function* $\mathrm{charge}_R : \mathrm{LRT}(R) \to \mathbb{Z}_+$ *such that*:
(C1) *If* $R = \emptyset$ *then* $\mathrm{charge}_R = 0$.
(C2) *If* $S \lessdot_R T$ *for* $S, T \in \mathrm{LRT}(R)$ *then* $\mathrm{charge}_R(S) = \mathrm{charge}_R(T) - 1$.
(C3) *Suppose* $T \in \mathrm{LRT}(\lambda; R)$ *with* $\lambda_1 = \mu_1 = \max_j \mu_j$. *Then in the notation in Theorem* 2.12(3), $\mathrm{charge}_R(T) = \mathrm{charge}_{R^\wedge}(T^\wedge)$.
(C4) *For any permutation* τ *of* $[t]$, $\mathrm{charge}_{\tau R}(\tau T) = \mathrm{charge}_R(T)$.

3. Crystal structure on tensor products of rectangles

We give an explicit description of the affine crystal structure on tensor products of rectangular crystals. This is accomplished by translating the theory of such crystals [6, 7, 10, 24] into the language of Young tableaux and the Robinson-Schensted-Knuth (RSK) correspondence.

3.1. *Crystals*

A P-weighted I-crystal is a weighted I-colored directed graph B, that is, a vertex set B equipped with a weight function $\mathrm{wt} : B \to P$ and directed edges colored by the set I, satisfying the following properties.

(1) There are no multiple edges; that is, for each $i \in I$ and $b, b' \in B$ there is at most one edge colored i from b to b'.

If such an edge exists, this is denoted $b' = \tilde{f}_i(b)$ or equivalently $b = \tilde{e}_i(b')$, by abuse of the notation of a function $B \to B$. In this case it is said that $\tilde{f}_i(b)$ is defined or equivalently that $\tilde{e}_i(b')$ is defined. Define $\varphi_i, \varepsilon_i : B \to \mathbb{Z}_+$ by

$$\varphi_i(b) = \max\{m \in \mathbb{Z}_+ \mid \tilde{f}_i^m(b) \text{ is defined}\}$$
$$\varepsilon_i(b) = \max\{m \in \mathbb{Z}_+ \mid \tilde{e}_i^m(b) \text{ is defined}\}$$

(1) If $\tilde{f}_i(b)$ is defined then $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$. Equivalently, $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$.
(2) $\langle h_i, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$.

If B_j is a P-weighted I-crystal for $1 \leq j \leq t$, the Cartesian product $B_t \times \cdots \times B_1$ can be given a crystal structure, denoted $B = B_t \otimes \cdots \otimes B_1$. The convention used here is opposite

Kashiwara's but is consistent with the traditional notation for tableaux. Let $b_j \in B_j$ and $b = b_t \otimes \cdots \otimes b_1 \in B$. The weight function on B is given by

$$\mathrm{wt}(b) = \sum_{j=1}^{t} \mathrm{wt}(b_j).$$

The operators $\tilde{f}_i$ are defined by the signature rule [10]. Given $b \in B$ and $i \in I$, construct a biword (sequence of pairs of letters) consisting of $\varphi_i(b_j)$ copies of the biletter $\binom{j}{-}$ and $\varepsilon_i(b_j)$ copies of the biletter $\binom{j}{+}$ for all j, sorted in weakly increasing order by the order $\binom{j}{\pm} < \binom{j'}{\pm}$ if $j > j'$ and $\binom{j}{-} < \binom{j}{+}$. This biword is now repeatedly reduced by removing adjacent biletters whose lower letters are $+-$ in that order. If $+$ and $-$ are viewed as left and right parentheses then this removes matching pairs of parentheses. At the end one obtains a biword whose lower word has the form $-^s+^t$. If $s > 0$ (resp. $t > 0$) let j_- (resp. j_+) be the upper letter corresponding to the rightmost $-$ (resp. leftmost $+$) in the reduced biword, and define

$$\tilde{f}_i(b) = b_m \otimes \cdots \otimes b_{1+j_-} \otimes \tilde{f}_i(b_{j_-}) \otimes b_{-1+j_-} \otimes \cdots \otimes b_1$$

and respectively

$$\tilde{e}_i(b) = b_m \otimes \cdots \otimes b_{1+j_+} \otimes \tilde{e}_i(b_{j_+}) \otimes b_{-1+j_+} \otimes \cdots \otimes b_1.$$

Then $\varphi_i(b) = s$ and $\varepsilon_i(b) = t$.

A morphism $g : B \to B'$ of P-weighted I-crystals is a map g that preserves weights and satisfies $g(\tilde{f}_i(b)) = \tilde{f}_i(g(b))$ for all $i \in I$ and $b \in B$, that is, $\tilde{f}_i(b)$ is defined if and only if $\tilde{f}_i(g(b))$ is, and in that case, the above equality holds.

Using this tensor operation, P-weighted I-crystals form a tensor category.

Let B be a P-weighted I-crystal, $b \in B$, and $p = \varphi_i(b) - \varepsilon_i(b)$. Define [5]

$$\tilde{r}_i(b) = \begin{cases} \tilde{f}_i^{\,p}(b) & \text{if } p > 0 \\ b & \text{if } p = 0 \\ \tilde{e}_i^{\,-p}(b) & \text{if } p < 0 \end{cases} \tag{3.1}$$

Suppose B is an I-crystal and $K \subset I$. Say that $b \in B$ is a K-highest weight vector if $\varepsilon_i(b) = 0$ for all $i \in K$.

We only require the following kinds of crystals.

(1) The crystal graphs of integrable highest weight $U_q(\widehat{sl}_n)$-modules are P-weighted I-crystals and are called $U_q(A_{n-1}^{(1)})$-crystals.
(2) The crystal graphs of integrable $U'_q(\widehat{sl}_n)$-modules that are either highest weight or finite-dimensional, are P_{cl}-weighted I-crystals and are called $U'_q(A_{n-1}^{(1)})$-crystals.
(3) The crystal graphs of integrable highest weight $U_q(sl_n)$-modules are $\bar{P}_{cl}$-weighted J-crystals and are called $U_q(A_{n-1})$-crystals.

For elements of such crystals, an affine highest weight vector is an I-highest weight vector and a classical highest weight vector is a J-highest weight vector.

3.2. $U_q(A_{n-1})$-*crystals and RSK*

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}_+^n$ be a partition of length at most n. Let V^λ be the irreducible $U_q(sl_n)$-module of highest weight λ and B^λ its crystal graph. The structure of the $U_q(A_{n-1})$-crystal B^λ is given explicitly in [10] using Young tableaux.

As a set $B^\lambda = \mathrm{T}(\lambda)$, the set of semistandard Young tableaux of shape λ with entries in the set $[n]$. The $U_q(A_{n-1})$-crystal graph structure is given as follows.

(1) Let $\lambda = (1)$. $B^{(1)} = [n]$. $\tilde{f}_i(i) = i + 1$ for $1 \leq i \leq n - 1$ and $\tilde{f}_i(j)$ is undefined for $j \neq i$. For $1 \leq i \leq n$, $\mathrm{wt}(i) = \mathrm{wt}_{\mathrm{sl}}(\epsilon_i)$ where ϵ_i is the i-th standard basis vector in $\mathbb{Z}^n$.
(2) The set of words $(B^{(1)})^{\otimes N}$ of length N in the alphabet $[n]$, has a $U_q(A_{n-1})$-crystal structure given by the signature rule. For $u \in (B^{(1)})^{\otimes N}$, $\mathrm{wt}(u) = \mathrm{wt}_{\mathrm{sl}}(\mathrm{content}(u))$.
(3) Let D be a skew shape having N cells. Consider the embedding $\mathrm{T}(D) \hookrightarrow (B^{(1)})^{\otimes N}$ given by $b \mapsto \mathrm{word}(b)$. The image of this embedding is stable under $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in J$. Thus $\mathrm{T}(D)$ has the structure of a $U_q(A_{n-1})$-crystal (call it B^D) by declaring that this embedding is a $U_q(A_{n-1})$-crystal morphism. If D is a partition λ this defines the crystal B^λ.

We give a streamlined version of the signature rule for the $U_q(A_{n-1})$-crystal structure on words. Let u be a word in the alphabet $[n]$ and $i \in J$. To compute $\tilde{e}_i(u)$, $\tilde{f}_i(u)$, and $\tilde{r}_i(u)$, place a left parenthesis below each letter $i + 1$ in u and a right parenthesis beneath each letter i in u. Match the parentheses in the usual way. The unmatched parentheses indicate a subword of the form $i^s(i + 1)^t$. Then $\tilde{e}_i(u)$, $\tilde{f}_i(u)$, and $\tilde{r}_i(u)$, are obtained from u by replacing the unmatched subword $i^s(i+1)^t$ by $i^{s+1}(i+1)^{t-1}$, $i^{s-1}(i+1)^{t+1}$, and $i^t(i+1)^s$ respectively. $\tilde{e}_i(u)$ is defined if and only if $t > 0$, and $\tilde{f}_i(u)$ is defined if and only if $s > 0$.

Example 3.1 Let $i = 2$. The unmatched subword is underlined.

$$
\begin{aligned}
u &= 1\underline{2}431132431224\underline{2}4\underline{23}\\
\tilde{e}_2(u) &= 1\underline{2}431132431224\underline{2}4\underline{22}\\
\tilde{f}_2(u) &= 1\underline{2}431132431224\underline{2}4\underline{33}\\
\tilde{r}_2(u) &= 1\underline{2}431132431224\underline{3}4\underline{33}
\end{aligned}
$$

Remark 3.2 The bijections $\tilde{r}_i$ are the automorphisms of conjugation (see Remark 2.8). More generally, by theorem of Kashiwara [9], the bijections $\tilde{r}_i$ generate an action of the Weyl group.

The RSK map is a morphism of $U_q(A_{n-1})$-crystals in the following sense.

Proposition 3.3 *If g is any of $\tilde{e}_i$, $\tilde{f}_i$, and $\tilde{r}_i$ for $i \in J$, then*

$$
\begin{aligned}
\mathbb{P}(g(v)) &= g(\mathbb{P}(v))\\
\mathbb{Q}(g(v)) &= \mathbb{Q}(v)
\end{aligned}
\tag{3.2}
$$

See [19] for a proof in the case $g = \tilde{r}_i$; the other cases are similar.

Let $\mathrm{LRT}_{\leq n}(R)$ (resp. $\mathrm{LRT}_{\leq n}(\lambda;\, R)$) be the subset of $\mathrm{LRT}(R)$ (resp. $\mathrm{LRT}(\lambda;\, R)$) consisting of tableaux whose shape has at most n rows. The next result is an immediate consequence of Theorem 2.5, given that elements of B^R have alphabet $[n]$.

Proposition 3.4 *There is a bijection*

$$B^R \cong \bigcup_{\lambda} B^\lambda \times \mathrm{LRT}_{\leq n}(\lambda;\, R)$$

given by $b \mapsto (\mathbb{P}(b), \mathbb{Q}(b))$.

Say that a word is Yamanouchi if each of its right factors has partition content.

Proposition 3.5 *A word is a classical highest weight vector if and only if it is Yamanouchi.*

Proof: Suppose $w = uxv$ is not Yamanouchi, with u, v words and x a letter, such that xv is the shortest right factor of w whose content is not a partition. Then w admits $\tilde{e}_{x-1}$ as this copy of x cannot be paired. Conversely, if $w = u'xv'$ admits $\tilde{e}_{x-1}$ and $\tilde{e}_{x-1}(w) = u'(x-1)v'$, then $m_x(xv') > m_{x-1}(xv')$. □

Remark 3.6

(1) $\mathrm{Key}(\lambda)$ is the unique Yamanouchi tableau of shape λ, or equivalently, the unique highest weight vector in B^λ.
(2) For $w \in \bar{W}$, $\mathrm{Key}(w\lambda)$ is the unique vector in B^λ of extremal weight $w\lambda$.
(3) B^λ is a connected $U_q(A_{n-1})$-crystal. For if $b \in B^\lambda$ is not $\mathrm{Key}(\lambda)$ then it admits a raising operator $\tilde{e}_i$ for $i \in J$. Since B^λ is finite, one must eventually reach the classical highest weight vector $\mathrm{Key}(\lambda)$.

Remark 3.7 By Proposition 3.5, the RSK map (2.2) may be defined by the property (3.2) for $g = \tilde{f}_i$ and all $i \in J$, together with its value on each classical highest weight vector. The latter can be given explicitly. Let $v = \cdots v_2 v_1$ where v_j is a row word for all j such that v is a classical highest weight vector. By Proposition 3.5 content(v) is a partition, say λ. By Proposition 3.3 and Remark 3.6 $\mathbb{P}(v) = \mathrm{Key}(\lambda)$. $\mathbb{Q}(v)$ is the tableau whose i-th row contains $m_i(v_j)$ copies of the letter j, for all i and j.

This way of defining the RSK correspondence essentially appears in the 1938 paper of G. de B. Robinson, [25, Section 5], who had devised the RSK map to prove the Littlewood-Richardson rule [20]. He defines operations on tableaux which turn out to be canonical sequences of operators $\tilde{e}_i$ and $\tilde{f}_i$. See Macdonald's rigorous version of Robinson's proof [22, I.9].

Later we shall require a characterization of $\equiv$ in terms of $U_q(A_{n-1})$-crystals.

Lemma 3.8 *Let v and v' be two words of the same length in the alphabet $[n]$. Then $\mathbb{P}(v) = \mathbb{P}(v')$ if and only if v and v' admit the same sequences of raising and lowering operators taken from $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \leq i \leq n-1$.*

Proof: The forward direction follows immediately from Proposition 3.3. For the converse, again by Proposition 3.3 it may be assumed that v and v' are tableaux of partition shape and it must be shown that $v = v'$. Let shape$(v) = \lambda$ and shape$(v') = \lambda'$. Let g be a sequence of raising operators such that $g(v)$ is a classical highest weight vector. By Remark 3.6 $g(v) = \text{Key}(\lambda)$. By hypothesis it follows that $g(v')$ is a classical highest weight vector, namely, $g(v') = \text{Key}(\lambda')$. Since g can be "undone" by a suitable sequence of lowering operators, it may be assumed that $v = \text{Key}(\lambda)$ and $v' = \text{Key}(\lambda')$. By direct computation $\varphi_i(\text{Key}(\lambda)) = \lambda_i - \lambda_{i+1}$ for $1 \le i \le n-1$. Again by hypothesis, $\lambda_i - \lambda_{i+1} = \lambda'_i - \lambda'_{i+1}$ for $1 \le i \le n-1$. But λ and λ' are partitions of the same size having at most n parts, so $\lambda = \lambda'$. □

3.3. Affine crystal structure on rectangular crystals

Theorem 3.9 [7] *For each* $0 < k < n$ *and* $l > 0$, *there is a* $U'_q(\widehat{sl}_n)$*-module* $W^{k,l}$ *with crystal graph* $B^{k,l}$, *which is isomorphic to* $B^{(l^k)}$ *as a* $U_q(A_{n-1})$*-crystal.*

In [7] the affine crystal operators $\tilde{e}_0$ and $\tilde{f}_0$ are defined on $B^{k,l}$, but even the well-definedness of the given operators is not at all clear and no proofs are given. We give an explicit algorithm to calculate $\tilde{e}_0$ on $B^{k,l}$ in terms of well-known tableau operations, and prove it is correct. We shall find properties that uniquely define $\tilde{e}_0$ on $B^{k,l}$, and then show our operators satisfies these properties. The main tool (used repeatedly in [7] for various root systems) is the existence of an automorphism ψ of $B^{k,l}$. If λ is not a rectangle then there is no way to extend the $U_q(A_{n-1})$-crystal structure on B^λ to a $U'_q(A^{(1)}_{n-1})$-crystal structure. For if there were, then as argued below, B^λ would have an automorphism ψ but such an automorphism does not exist if λ is not a rectangle.

The affine Dynkin diagram $A^{(1)}_{n-1}$ admits the rotation automorphism that sends i to $i+1$ modulo n. This induces an automorphism ψ of the $U'_q(A^{(1)}_{n-1})$-crystal $B^{k,l}$ such that

$$\psi \circ \tilde{f}_i = \tilde{f}_{i+1} \circ \psi \quad \text{for all } i \in I \tag{3.3}$$

where subscripts are taken modulo n. Equivalently $\tilde{f}_i$ may be replaced by $\tilde{e}_i$. Moreover, if $\psi : P_{cl} \to P_{cl}$ is the automorphism of P_{cl} defined by $\psi(\Lambda_i) = \Lambda_{i+1}$ for all $i \in I$, then

$$\text{wt}(\psi(b)) = \psi(\text{wt}(b)) \quad \text{for all } b \in B^{k,l}. \tag{3.4}$$

Equivalently,

$$m_{i+1}(\psi(b)) = m_i(b) \quad \text{for all } b \in B^{k,l} \quad \text{and } i \in I. \tag{3.5}$$

Lemma 3.10 *ψ is uniquely defined by* (3.3) *and* (3.5).

Proof: Recall that $B^{k,l} = B^{(l^k)}$ as $U_q(A_{n-1})$-crystals. By Remark 3.6 $B^{k,l}$ has a unique classical highest weight vector, namely, $y = \text{Key}(l^k)$. By (3.3) and the connectedness of $B^{k,l}$

it is enough to show that $\psi(y)$ is uniquely determined. By definition $\text{shape}(\psi(y)) = (l^k)$. By (3.5) $\text{content}(\psi(y)) = (0, l^k, 0^{n-1-k})$. By Remark 3.6 $\psi(y) = \text{Key}(0, l^k, 0^{n-1-k})$. $\square$

If u is a word or tableau and p is an integer, denote by $u + p$ the word or tableau whose entries are obtained from those of u by adding p.

Lemma 3.11 *ψ is uniquely defined by* (3.3) *for* $1 \le i \le n-2$ *and* (3.5).

Proof: Let $b \in B^{k,l}$, $c = b|_{[2,n-1]}$, $b' = \psi(b)$, and $c' = b'_{[3,n]}$. Since (3.3) holds for $1 \le i \le n-2$, c admits a sequence of lowering operators $e_{i_1} \dots e_{i_p}$ for $2 \le i_j \le n-2$, if and only if $c' - 1$ does. Since c and $c' - 1$ are words in the alphabet $[2, n-1]$, $P(c) = P(c'-1)$ by Lemma 3.8. Since the skew tableau c' has antinormal shape, $\mathrm{P}_{\searrow}(c) = c' - 1$ by Theorem 2.1. This specifies c'.

It remains to show that the subtableau $b'|_{[1,2]}$ is uniquely specified. Its shape is the partition whose diagram is the set difference $(l^k) - \text{shape}(c')$. Since $b'|_{[1,2]}$ is a tableau of partition shape that contains only ones and twos, it must have at most two rows. It is uniquely determined by its shape and content. But its content is specified by (3.5). $\square$

Remark 3.12 Observe that a skew shape is both normal and antinormal if and only if it is a translate of a rectangle. This property of rectangles is used prominently in the proof of Lemma 3.11.

The following operation is Schützenberger's promotion operator pr, which was defined on standard tableaux but has an obvious extension to tableaux [3, 28]. Let D be a skew shape. $\text{pr} : B^D \to B^D$ is defined as follows.

(1) Remove all the letters n in b, which removes from D a horizontal strip H.
(2) Slide (using Schützenberger's jeu-de-taquin [3, 29] [2, Section 1.2]) the remaining subtableau $b|_{[n-1]}$ to the southeast into the horizontal strip H, entering the cells of H from left to right.
(3) Add one to each entry of the resulting skew tableau.
(4) Fill the set H' of vacated cells with ones.

Let $\text{pr}(b)$ be the resulting object. The set of cells H' is a horizontal strip that was created from left to right. It follows that $\text{pr}(b)$ is a tableau. Note that a tableau b is uniquely specified by its restriction $b|_{[n-1]}$ and its shape. These two pieces of data must also uniquely specify $\text{pr}(b)$. And they do, in the following manner.

Lemma 3.13 *Let b be a tableau of partition shape. Then* $\text{pr}(b)$ *is uniquely defined by conditions that* $P(\text{pr}(b)|_{[2,n]}) = b|_{[n-1]} + 1$ *and* $\text{shape}(\text{pr}(b)) = \text{shape}(b)$.

Lemma 3.14 *Let $b \in B^{(l^k)}$. Then* $\text{pr}(b)$ *is the unique element of* $B^{(l^k)}$ *such that*

$$\text{pr}(b)|_{[2,n]} = \mathrm{P}_{\searrow}\big(b|_{[n-1]}\big) + 1. \tag{3.6}$$

Proof: In this case H consists of the last $m_n(b)$ cells in the last row of the rectangle (l^k). Note that

$$\mathrm{P}_{\searrow}\big(b|_{[n-1]}\big) + 1 \equiv b|_{[n-1]} + 1 \equiv \mathrm{pr}(b)|_{[2,n]}$$

by Theorem 2.1 and Lemma 3.13. Both tableaux in (3.6) are of antinormal shape and are Knuth equivalent. Hence they are equal by Theorem 2.1. There is no choice for the location of letters 1 in $\mathrm{pr}(b)$, so all of $\mathrm{pr}(b)$ is determined. $\square$

Proposition 3.15 $\psi = \mathrm{pr}$.

Proof: It is enough to check the hypotheses of Lemma 3.11. Equation (3.5) holds by definition. Let $1 \leq i \leq n-2$.

$$\begin{aligned}
\mathrm{pr}(\tilde{f}_i(b))|_{[2,n]} &= \mathrm{P}_{\searrow}\big(\tilde{f}_i(b)|_{[n-1]}\big) + 1 \\
&= \mathrm{P}_{\searrow}\big(\tilde{f}_i\big(b|_{[n-1]}\big)\big) + 1 \\
&= \tilde{f}_i\big(\mathrm{P}_{\searrow}\big(b|_{[n-1]}\big)\big) + 1 \\
&= \tilde{f}_i\big(\mathrm{pr}(b)|_{[2,n]} - 1\big) + 1 \\
&= \tilde{f}_{i+1}\big(\mathrm{pr}(b)|_{[2,n]}\big) \\
&= \tilde{f}_{i+1}(\mathrm{pr}(b))|_{[2,n]}.
\end{aligned}$$

These equalities hold by Lemma 3.14, the explicit constructions for $\tilde{f}_i$, and the fact that $\mathrm{P}_{\searrow}$ respects $\tilde{f}_i$ (Theorem 2.1 and (3.2)). This gives (3.3) for $1 \leq i \leq n-2$. $\square$

By (3.3), the operators $\tilde{e}_0$ and $\tilde{f}_0$ on $B^{k,l}$ are given explicitly by

$$\begin{aligned}
\tilde{e}_0 &= \mathrm{pr}^{-1} \circ \tilde{e}_1 \circ \mathrm{pr} \\
\tilde{f}_0 &= \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}.
\end{aligned} \tag{3.7}$$

From now on it is assumed that $R = (R_1, R_2, \ldots, R_t)$ is a sequence of rectangular partitions $R_j = (\mu_j^{\eta_j})$. We shall also write B^{R_j} to mean B^{η_j,μ_j}.

The signature rule makes B^R a $U'_q(A^{(1)}_{n-1})$-crystal, but this method of computing $\tilde{e}_0$ on B^R requires the computation of ε_0 and φ_0 for each tensor factor. It is more efficient to compute $\tilde{e}_0$ on B^R as follows. Let $b = b_t \otimes \cdots \otimes b_1 \in B^R$ with $b_j \in B^{R_j}$. Define $\mathrm{pr} : B^R \to B^R$ by

$$\mathrm{pr}(b) = \mathrm{pr}(b_t) \otimes \cdots \otimes \mathrm{pr}(b_1).$$

By the signature rule and (3.7) it follows that

$$\begin{aligned}
\tilde{e}_0 &= \mathrm{pr}^{-1} \circ \tilde{e}_1 \circ \mathrm{pr} \\
\tilde{f}_0 &= \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}
\end{aligned} \tag{3.8}$$

as operators on B^R.

Example 3.16 Let $n = 7$, $R = ((2, 2), (3, 3, 3), (3, 3))$, and $b \in B^R$ given by the following skew tableau of shape $R_3 \otimes R_2 \otimes R_1$:

$$\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & 1 & 1 \\ \times & \times & \times & \times & \times & \times & 2 & 2 \\ \times & \times & \times & 1 & 1 & 3 & & \\ \times & \times & \times & 2 & 3 & 4 & & \\ \times & \times & \times & 3 & 4 & 5 & & \\ 2 & 4 & 6 & & & & & \\ 3 & 5 & 7 & & & & & \end{array}$$

We wish to compute $\tilde{e}_0(b) = \mathrm{pr}^{-1}(\tilde{e}_1(\mathrm{pr}(b)))$. The element $\mathrm{pr}(b)$ is given by

$$\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & 2 & 2 \\ \times & \times & \times & \times & \times & \times & 3 & 3 \\ \times & \times & \times & 2 & 2 & 4 & & \\ \times & \times & \times & 3 & 4 & 5 & & \\ \times & \times & \times & 4 & 5 & 6 & & \\ 1 & 3 & 5 & & & & & \\ 4 & 6 & 7 & & & & & \end{array}$$

Let us apply $\tilde{e}_1$ to $\mathrm{pr}(b)$. It is simplest to use the $U_q(A_{n-1})$-crystal embedding of B^R into the set of words. We have

$$\begin{aligned} \mathrm{word}(\mathrm{pr}(b)) &= 4671354563452243322 \\ \tilde{e}_1(\mathrm{word}(\mathrm{pr}(b))) &= 4671354563451243322. \end{aligned}$$

So $\tilde{e}_1(\mathrm{pr}(b))$ is given by

$$\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & 2 & 2 \\ \times & \times & \times & \times & \times & \times & 3 & 3 \\ \times & \times & \times & 1 & 2 & 4 & & \\ \times & \times & \times & 3 & 4 & 5 & & \\ \times & \times & \times & 4 & 5 & 6 & & \\ 1 & 3 & 5 & & & & & \\ 4 & 6 & 7 & & & & & \end{array}$$

Finally $\tilde{e}_0(b) = \mathrm{pr}^{-1}(\tilde{e}_1(\mathrm{pr}(b)))$ is given by:

$$\begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & 1 & 1 \\
\times & \times & \times & \times & \times & \times & 2 & 2 \\
\times & \times & \times & 1 & 3 & 3 & & \\
\times & \times & \times & 2 & 4 & 4 & & \\
\times & \times & \times & 3 & 5 & 7 & & \\
2 & 4 & 6 & & & & & \\
3 & 5 & 7 & & & & &
\end{array}$$

Observe that $\tilde{e}_0(b_3 \otimes b_2 \otimes b_1) = b_3 \otimes \tilde{e}_0(b_2) \otimes b_1$.

3.4. Connectedness

In [1] the theory of simple crystals is used to prove the following important result. It says that the underlying $U'_q(\widehat{sl}_n)$-module of B^R, is irreducible.

Theorem 3.17 [1] *B^R is a connected $U'_q(A^{(1)}_{n-1})$-crystal.*

3.5. Combinatorial R-matrices

Again we quote a basic theorem of affine crystal theory, applied to the type $A^{(1)}_{n-1}$ case.

Theorem 3.18 [6, Proposition 4.3.1] *There is a $U'_q(A^{(1)}_{n-1})$-crystal isomorphism*

$$\sigma = \sigma_{R_2,R_1} : B^{R_2} \otimes B^{R_1} \to B^{R_1} \otimes B^{R_2} \tag{3.9}$$

called the combinatorial R-matrix.

The proof uses the existence of the algebraic R-matrix acting on the affinizations (in the sense of [6, Section 3.2]) of modules of the form $W^{k,l}$.

Theorem 3.19 *Let R be a sequence of rectangles and R' a reordering of R. Then there is a unique $U'_q(A^{(1)}_{n-1})$-crystal isomorphism*

$$\sigma_R^{R'} : B^R \to B^{R'}. \tag{3.10}$$

Proof: Existence is given by composing isomorphisms of the form (3.9) acting on adjacent tensor factors. For uniqueness, recall from Theorem 3.17 that B^R is a connected $U'_q(A^{(1)}_{n-1})$-crystal and $\sigma_R^{R'}$ is a $U'_q(A^{(1)}_{n-1})$-crystal morphism. This implies that $\sigma_R^{R'}$ is uniquely determined by its value on any single element of B^R. Now B^R (resp. $B^{R'}$) contains a unique element y (resp. y') of content $\sum_j R_j = \sum_j R'_j$ where R_j is viewed as an element of $\mathbb{Z}^n$. Explicitly, y and y' are tensor products of classical highest weight vectors in each factor. Since $\sigma_R^{R'}$ is a $U'_q(A^{(1)}_{n-1})$-crystal morphism, it must preserve weight. It follows that $\sigma_R^{R'}(y) = y'$. □

In the special case $R = (R_1, R_2, R_3)$ and $R' = (R_3, R_2, R_1)$ the uniqueness of $\sigma_R^{R'}$ implies the Yang-Baxter equation for combinatorial R-matrices:

$$\begin{aligned}&\left(1 \otimes \sigma_{R_3,R_2}\right) \circ \left(\sigma_{R_3,R_1} \otimes 1\right) \circ \left(1 \otimes \sigma_{R_2,R_1}\right)\\ &\quad = \left(\sigma_{R_2,R_1} \otimes 1\right) \circ \left(1 \otimes \sigma_{R_3,R_1}\right) \circ \left(\sigma_{R_3,R_2} \otimes 1\right)\end{aligned} \tag{3.11}$$

as maps $B^R \to B^{R'}$.

We wish to describe the bijection σ of (3.9) in terms of tableaux. By 3.2 σ preserves the $\mathbb{P}$ tableau. By Proposition 3.4, σ induces a shape-preserving bijection $\tau = \tau_{R_2,R_1}$: $\mathrm{LRT}_{\le n}((R_2, R_1)) \to \mathrm{LRT}_{\le n}((R_1, R_2))$ such that

$$\mathbb{Q}(\sigma(b)) = \tau(\mathbb{Q}(b)) \quad \text{for all } \ b \in B^{R_2} \otimes B^{R_1}. \tag{3.12}$$

Since $B^{R_2} \otimes B^{R_1}$ is multiplicity-free as a $U_q(A_{n-1})$-crystal (see for example [35]), it follows that τ is unique. This is none other than the bijection τ_1 of Section 2.3, restricted to shapes with at most n rows. See the proof of [30, Proposition 34] for a very explicit description of τ.

By the same kind of abuse of notation used in Section 2.3, σ_p denotes any instance of a combinatorial R-matrix that exchanges the p-th and $(p+1)$-th tensor factors in a tensor product of the form B^R. In this notation (3.11) becomes a braid relation for the symmetric group on $[t]$. Thus the proof of Theorem 2.7(A3) gives a purely combinatorial argument that the combinatorial R-matrices satisfy (3.11).

Remark 3.20 Suppose R_j is a single row (γ_j) for all j. Let $b = \cdots \otimes b_2 \otimes b_1$ with $b_j \in B^{R_j}$, so that b_j is a row word of length γ_j for all j. Fix $i \ge 1$. Let $R' = r_i R$ and $b' = \sigma_i(b) = \cdots \otimes b'_2 \otimes b'_1$. In this case τ_i coincides with the automorphism of conjugation $\tilde{r}_i$, this time acting in the multiplicity space (see Remark 3.2).

Remark 3.21 The map σ was determined only by the requirement that it be an isomorphism of $U_q(A_{n-1})$-crystals. It can be shown combinatorially that the map so defined, is also an isomorphism of $U'_q(A^{(1)}_{n-1})$-crystals; the proof was included in the preprint version of this paper [32].

3.6. *Energy function*

Affine crystal theory gives a natural grading on B^R, by the so-called energy function.

Theorem 3.22 [6] *There is a function $H = H_{R_2,R_1} : B^{R_2} \otimes B^{R_1} \to \mathbb{Z}$, defined uniquely up to a global additive constant, by the following property. Let $b_2 \otimes b_1 \in B^{R_2} \otimes B^{R_1}$ and*

$\sigma(b_2 \otimes b_1) = b_1' \otimes b_2'$ *where* σ *is the combinatorial* R*-matrix. Then*

$$H(\tilde{e}_i(b_2 \otimes b_1)) - H(b_2 \otimes b_1) = \begin{cases} -1 & \text{if } i = 0,\ \tilde{e}_0(b_2 \otimes b_1) = \tilde{e}_0(b_2) \otimes b_1 \text{ and} \\ & \qquad \tilde{e}_0(b_1' \otimes b_2') = \tilde{e}_0(b_1') \otimes b_2' \\ 1 & \text{if } i = 0,\ \tilde{e}_0(b_2 \otimes b_1) = b_2 \otimes \tilde{e}_0(b_1) \text{ and} \\ & \qquad \tilde{e}_0(b_1' \otimes b_2') = b_1' \otimes \tilde{e}_0(b_2') \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

H is called the local energy function. Its existence follows from the properties of the algebraic R-matrix. Its (essential) uniqueness follows from its defining properties and Theorem 3.17.

We normalize H as follows. Let $y_i = \mathrm{Key}(R_i)$ be classical highest weight vectors for $i = 1, 2$. Then

$$H(y_2 \otimes y_1) = \min(\mu_1, \mu_2)\min(\eta_1, \eta_2). \tag{3.14}$$

By definition it follows that

$$H_{R_2,R_1} = H_{R_1,R_2} \circ \sigma_{R_2,R_1}. \tag{3.15}$$

Now let $R = (R_1, R_2, \ldots, R_t)$ be a sequence of rectangles. Again by abuse of notation let H_i denote the energy function acting on the i-th and $(i+1)$-th tensor positions in some tensor product of crystals of the form $B^{k,l}$.

Following [24], define the global energy function $E_R : B^R \to \mathbb{Z}$ by

$$E_R(b) = \sum_{1 \le i < j \le t} H_i(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1}b). \tag{3.16}$$

Define the polynomials $E_{\lambda;R}(q)$ by

$$\mathrm{ch}_q(B^R) := \sum_{b \in B^R} e^{\mathrm{wt}(b)} q^{E_R(b)} = \sum_{\lambda \in \bar{P}^+_{cl}} \mathrm{ch}(V^\lambda) E_{\lambda;R}(q) \tag{3.17}$$

where ch is the formal character.

Theorem 3.23 *For all* $b \in B^R$,

$$E_R(b) = \mathrm{charge}_R(\mathbb{Q}(b)) \tag{3.18}$$

Corollary 3.24

$$E_{\lambda;R}(q) = \mathrm{LRT}_{\lambda;R}(q). \tag{3.19}$$

Theorem 3.23 was proved in [24] when all R_j are single rows and charge_R is replaced by charge.

The following Lemma is a higher-level generalization of [14, Lemma 5.1]. Compare this with [30, Proposition 24].

Lemma 3.25 *For all $b \in B^R$, $E_R(\tilde{e}_0(b)) - E_R(b) + 1$ is the number of indices $1 \le j \le t$ such that $\tilde{e}_0$ acts on the rightmost tensor factor of $\sigma_1\sigma_2\cdots\sigma_{j-1}(b)$. In particular, if $\varepsilon_0(b) > \mu_j$ for all j then $E_R(b) = E_R(\tilde{e}_0(b)) + 1$.*

Proof: Fix $b \in B^R$ such that $\tilde{e}_0(b)$ is defined. By Theorem 3.19, for every composition σ of combinatorial R-matrices acting on B^R, $\sigma(b)$ admits $\tilde{e}_0$ and $\tilde{e}_0(\sigma(b)) = \sigma(\tilde{e}_0(b))$. Let $p(\sigma)$ denote the index of the tensor position that is changed by $\tilde{e}_0$ acting on $\sigma(b)$. Write $k = p(id)$. Clearly

$$p(\sigma_i\sigma) = p(\sigma) \quad \text{unless } p(\sigma) \in \{i, i+1\}. \tag{3.20}$$

Write $\sigma_{i,j} = \sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1}$ for $0 \le i < j \le t$. By definition $\sigma_{i-1,j} = \sigma_i\sigma_{i,j}$ for $i \ge 1$. Fix $2 \le j \le t$. By (3.13),

$$\Delta_{i,j}(b) := H_i(\tilde{e}_0(\sigma_{i,j}b)) - H_i(\sigma_{i,j}(b)) = \begin{cases} -1 & \text{if } p(\sigma_{i,j}) = p(\sigma_{i-1,j}) = i+1 \\ 1 & \text{if } p(\sigma_{i,j}) = p(\sigma_{i-1,j}) = i \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that

$$\Delta_j(b) := \sum_{i=1}^{j-1} \Delta_{i,j}(b) = \begin{cases} 1 & \text{if } j \neq k \quad \text{and } p(\sigma_{0,j}) = 1 \\ -1 & \text{if } j = k \quad \text{and } p(\sigma_{0,k}) \neq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

There are three main cases. In each case it is shown that (3.21) is satisfied.

(1) $k > j$: $p(\sigma_{i,j}) = k$ for all $1 \le i < j$ and $\Delta_j = 0$.

(2) $k < j$: $p(\sigma_{i,j}) = k$ for $i \ge k$. This means $\Delta_{i,j} = 0$ for $j > i > k$. There are two subcases.

   (a) $p(\sigma_{k-1,j}) = k+1$. Then $p(\sigma_{i,j}) = k+1$ for $0 \le i \le k-1$, $\Delta_{i,j} = 0$ for $1 \le i \le k$ and $\Delta_j = 0$.

   (b) $p(\sigma_{k-1,j}) = k$. There is a minimum index q_j such that $p(\sigma_{k-1,j}) = k$, $p(\sigma_{k-2,j}) = k-1$, $p(\sigma_{k-3,j}) = k-2$, down to $p(\sigma_{q_j-1,j}) = q_j$, and thereafter $p(\sigma_{i,j}) = q_j$ for $0 \le i < q_j - 1$. Then $\Delta_{k,j} = 1$, $\Delta_{i,j} = 0$ for $k > i \ge q_j$, $\Delta_{q_j-1,j} = -1$, and $\Delta_{i,j} = 0$ for $i < q_j - 1$. Summing, $\Delta_j = 1$ if $q_j = 1$ (that is, $p(\sigma_{0,j}) = 1$) and $\Delta_j = 0$ otherwise (that is, $p(\sigma_{0,j} \neq 1)$.

(3) $k = j$: By definition $p(\sigma_{k-1,k}) = p(id) = k$. There is an index q_k such that $p(\sigma_{k-2,k}) = k-1$, $p(\sigma_{k-3,k}) = k-2$, down to $p(\sigma_{q_k-1,k}) = q_k$, and thereafter $p(\sigma_{i,k}) = q_k$ for $0 \le i < q_k - 1$. Then $\Delta_{i,k} = 0$ for $i \neq q_k - 1$ and $\Delta_{q_k-1,k} = -1$. Therefore $\Delta_k = 0$ if $q_k = 1$ (that is, $p(\sigma_{0,k}) = 1$) and $\Delta_k = -1$ if $q_k \neq 1$ (that is, $p(\sigma_{0,k}) \neq 1$).

Finally, suppose $\varepsilon_0(b) > \mu_j$ for all j. Since a tableau of shape R_j can have at most μ_j letters n, $\varphi(b_j) \leq \mu_j$ for all $b_j \in B^{R,j}$. By Theorem 3.19 $\varepsilon_0(\sigma(b)) > \mu_j$ for all j and all compositions σ of R-matrices. Therefore $p(\sigma) \neq 1$ for all σ. □

4. Affine crystals and *R*-cocyclage

In this section a connection is established between $\tilde{e}_0$ on B^R and the R-cocyclage poset structure on $\mathrm{LRT}(R)$.

4.1. Promotion and the RSK tableau pair

To give the effect of $\tilde{e}_0$ on the RSK tableau pair, in light of (3.8) the effect of pr on the tableau pair must be given first. Let $b \in B^R$. We give an algorithm to compute $(\mathbb{P}(\mathrm{pr}(b)), \mathbb{Q}(\mathrm{pr}(b)))$ in terms of $(\mathbb{P}(b), \mathbb{Q}(b))$ without using b directly. This algorithm can be reversed in order to compute the effect of pr^{-1} on the tableau pair.

Recall the notation of Section 2.2. Let w_0^R denote the automorphism of conjugation acting on words, corresponding to the longest permutation of the Young subgroup $S_{A_1} \times S_{A_2} \times \cdots \times S_{A_t}$ of the symmetric group of the alphabet $A_1 \cup \cdots \cup A_t$.

Proposition 4.1 *Let $b \in B^R$, $P = \mathbb{P}(b)$ and $Q = \mathbb{Q}(b)$. Let H be the horizontal strip* $\mathrm{shape}(P)/\mathrm{shape}(P|_{[n-1]})$, *and v be the row word and $Q^\wedge$ the tableau such that* $\mathrm{shape}(Q^\wedge) = \mathrm{shape}(P|_{[n-1]})$ *and $Q = P(vQ^\wedge)$ (see Lemma 7.2). Then*

$$\mathbb{Q}(\mathrm{pr}(b)) = P\big(\big(w_0^R Q^\wedge\big)\big(w_0^R v\big)\big). \tag{4.1}$$

Let H_1 be the horizontal strip $\mathrm{shape}(\mathbb{Q}(\mathrm{pr}(b)))/\mathrm{shape}(Q^\wedge)$, *$P_1$ the tableau given by adjoining to $P|_{[n-1]}$ the letters n at the cells of H_1. Then*

$$\mathbb{P}(\mathrm{pr}(b)) = \mathrm{pr}(P_1). \tag{4.2}$$

Example 4.2 Continuing the previous example, consider the biword whose lower word is $\mathrm{word}(b)$ and upper word is given by the row indices of letters in b viewed as a skew tableau.

7776665554443332211
3572463452341132211

Let $P = \mathbb{P}(b)$ and $Q = \mathbb{Q}(b)$. Then

$$P = \begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & \\ 5 & 5 & & \\ 6 & & & \\ 7 & & & \end{matrix} \qquad Q = \begin{matrix} 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 4 & 5 & 7 \\ 4 & 5 & 6 & \\ 5 & 7 & & \\ 6 & & & \\ 7 & & & \end{matrix}\ .$$

Then H is the skew shape given by the single cell (7, 1), $v = 7$, $w_0^R v = 6$,

$$Q^\wedge = \begin{matrix} 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 4 & 5 & 7 \\ 4 & 5 & 6 & \\ 5 & 7 & & \\ 6 & & & \end{matrix} \qquad w_0^R Q^\wedge = \begin{matrix} 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 4 & 5 & 7 \\ 4 & 5 & 6 & \\ 5 & 7 & & \\ 7 & & & \end{matrix},$$

and

$$\mathbb{Q}(\mathrm{pr}(b)) = \begin{matrix} 1 & 1 & 3 & 3 & 6 \\ 2 & 2 & 4 & 6 & \\ 3 & 4 & 5 & 7 & \\ 4 & 5 & 6 & & \\ 5 & 7 & & & \\ 7 & & & & \end{matrix}.$$

So H_1 consists of the single cell (1, 5) and

$$P_1 = \begin{matrix} 1 & 1 & 1 & 1 & 7 \\ 2 & 2 & 2 & 2 & \\ 3 & 3 & 3 & 3 & \\ 4 & 4 & 4 & & \\ 5 & 5 & & & \\ 6 & & & & \end{matrix} \qquad \mathbb{P}(\mathrm{pr}(b)) = \mathrm{pr}(P_1) = \begin{matrix} 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & \\ 4 & 4 & 4 & 4 & \\ 5 & 5 & 5 & & \\ 6 & 6 & & & \\ 7 & & & & \end{matrix}.$$

Proof of Proposition 4.1: Write $b = \cdots b_2 \otimes b_1$, $c_j = b_j|_{[n-1]}$, for all j, and $c = \cdots c_2 \otimes c_1$. By Lemmas 7.2 and 7.3 $Q^\wedge = \mathbb{Q}(c)$. By content considerations $v = \max(A_1)^{m_n(b_1)} \max(A_2)^{m_n(b_2)} \cdots$. Let $c'_j = \mathrm{P}_{\searrow}(c_j)$ for all j and $c' = \cdots c'_2 \otimes c'_1$. Applying Lemma 7.10 for each tensor factor, we have $\mathbb{Q}(c') = w_0^R(\mathbb{Q}(c)) = w_0^R(Q^\wedge)$.

By Lemmas 7.2, 7.3 and 3.14, there is a row word u such that

$$\mathbb{Q}(\mathrm{pr}(b)) = P\big(\mathbb{Q}\big(\mathrm{pr}(b)|_{[2,n]}\big)u\big) = P(\mathbb{Q}(c'+1)u) = P(\mathbb{Q}(c')u) = P\big(\big(w_0^R Q^\wedge\big)u\big).$$

By considering contents, $u = \min(A_1)^{m_n(b_1)} \min(A_2)^{m_n(b_2)} \cdots$. Since u and v are row words, it follows just by checking contents that $u = w_0^R v$. Therefore $\mathbb{Q}(\mathrm{pr}(b)) = P((w_0^R U)(w_0^R v))$. By Lemma 3.14 and Proposition 7.1,

$$\mathbb{P}\big(\mathrm{pr}(b)|_{[2,n]}\big) = \mathbb{P}(c'+1) = \mathbb{P}(c) + 1 = \mathbb{P}(b)|_{[n-1]} + 1.$$

By Lemma 3.13 applied to P_1, $\mathbb{P}(\mathrm{pr}(b)) = \mathrm{pr}(P_1)$. $\square$

4.2. $\tilde{e}_0$ and R-cocyclage

Any covering relation in the R-cocyclage is induced by an application of $\tilde{e}_0$ in B^R, assuming that $n > \sum_j \eta_j$. This assumption implies that $\mathrm{LRT}_{\le n}(R) = \mathrm{LRT}(R)$ since the maximum number of rows in an element of $\mathrm{LRT}(R)$ is $\sum_j \eta_j$, achieved by the tableau $P(\cdots Y_2 Y_1)$.

Theorem 4.3 *Suppose $n > \sum_j \eta_j$ and $S \leftarrow_R T$ for $S, T \in \mathrm{LRT}(R)$.*
(1) *There is an element $b \in B^R$ such that $\mathbb{Q}(b) = T$ and $\mathbb{Q}(\tilde{e}_0(b)) = S$.*
(2) *If in addition $S \lessdot_R T$ then $E_R(\tilde{e}_0(b)) = E_R(b) - 1$.*

Proof: Let $\mathrm{shape}(T) = \lambda$ and $S \leftarrow_{R,\nu} T$ with corner cell $s = \lambda/\nu = (m, \lambda_m)$. S is uniquely determined by T and ν by [30, Remark 18]. Let

$$K = \mathrm{Key}(\lambda_m, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \lambda_{m+1}, \lambda_{m+2}, \ldots).$$

By Proposition 3.4 there is a $b \in B^R$ such that $\mathbb{P}(b) = K$ and $\mathbb{Q}(b) = T$. By the uniqueness of S with respect to T and ν, for part (1) it suffices to show that $\mathbb{Q}(\tilde{e}_0(b)) \leftarrow_{R,\nu} T$.

Since $T \in \mathrm{LRT}(\lambda; R)$ and $n > \sum_j \eta_j$, $\lambda_n = 0$. It follows that $m < n$ and $m_n(b) = m_n(K) = 0$. Also $m_1(b) = m_1(K) = \lambda_m$, so $\varphi_0(b) = \lambda_m$. By the definition of $\lessdot_R$, $\lambda_m > \max_j \mu_j$. By Lemma 3.25, $E_R(\tilde{e}_0(b)) = E_R(b) - 1$. This proves part (2).

We shall compute $\mathbb{Q}(\tilde{e}_0(b))$ using (3.8). It will be shown first that

$$\begin{aligned} \mathbb{P}(\tilde{e}_1 \mathrm{pr}(b)) &= \tilde{e}_1(K+1) \\ \mathbb{Q}(\tilde{e}_1 \mathrm{pr}(b)) &= T. \end{aligned} \tag{4.3}$$

that is, H is empty. By Proposition 4.1, $\mathbb{P}(\mathrm{pr}(b)) = \mathrm{pr}(K)$ and $\mathbb{Q}(\mathrm{pr}(b)) = T$. Since $m_n(K) = 0$, $\mathrm{pr}(K) = K + 1$. Equation (4.3) follows by (3.2) for $\tilde{e}_1$.

Since $m_1(K+1) = 0$ and $m_2(K+1) = m_1(K) = \lambda_m > 0$, $K+1$ admits $\tilde{e}_1$. $\tilde{e}_1(K+1)$ is obtained from $K+1$ by changing the 2 in the northwest corner to a 1.

In order to obtain $\mathbb{Q}(\tilde{e}_0(b))$, by (3.8) the reverse of the procedure in Proposition 4.1 must be applied to the tableau pair in (4.3). The first ingredient is H_1. For this it is enough to give the shape of $\mathbb{P}(\tilde{e}_1(\mathrm{pr}(b)))|_{[n-1]}$. We have

$$\begin{aligned} \mathbb{P}(\tilde{e}_1(\mathrm{pr}(b)))|_{[n-1]} &= \mathrm{pr}^{-1}(\tilde{e}_1(K+1))|_{[n-1]} = P\big(\tilde{e}_1(K+1)|_{[2,n]}\big) - 1 \\ &= P\big(\tilde{e}_1(\cdots 3^{\lambda_1} 2^{\lambda_m}|_{[2,n]}\big) - 1 = P\big(\cdots 3^{\lambda_1} 1 2^{\lambda_m - 1}|_{[2,n]}\big) \\ &= P(\cdots 3^{\lambda_1} 2^{\lambda_m - 1}) = \mathrm{Key}(\lambda_m - 1, \lambda_1, \lambda_2, \ldots) =: K' \end{aligned}$$

by Lemma 3.13 and the definition of K. But $\mathrm{shape}(K') = \nu = \lambda - \{s\}$. Therefore $H_1 = \{s\}$. To compute $Q^\wedge$ and v, let u'' be the tableau of shape ν and x'' the letter, such that $T = P(u''x'')$ (see Lemma 7.2). Then $Q^\wedge = w_0^R u''$ and $v = w_0^R x''$ and $\mathbb{Q}(\mathrm{pr}^{-1}\tilde{e}_1 \mathrm{pr}(b)) = P(vQ^\wedge)$. Therefore $\mathbb{Q}(\tilde{e}_0(b)) \leftarrow_{R,\nu} T$. □

4.3. *Energy depends only on $\mathbb{Q}$ tableau*

Let $b \in B^R$. By definition $E_R(b)$ does not vary on $U_q(A_{n-1})$-components of B^R. Neither does $\mathbb{Q}(b)$, which uniquely specifies the $U_q(A_{n-1})$-component of $b \in B^R$. So there should be a way to compute $E_R(b)$ only in terms of the tableau $\mathbb{Q}(b)$. And indeed there is. Consequently, the parameter n that is always present when dealing with the $U_q'(A_{n-1}^{(1)})$-crystal B^R, has been rendered irrelevant, since one may choose to work with $\mathrm{LRT}(R)$, the parametrizing set for the "universal multiplicity space". The proof uses the connection between $\tilde{e}_0$ and R-cocyclage.

The following crucial Lemma has a subtle proof which involves very detailed properties of $\mathrm{LRT}(R)$ in the two-rectangle case $R = (R_1, R_2)$ given in [30, Section 5.2]. This proof is deferred to the Appendix.

Lemma 4.4 *Suppose $R = (R_1, R_2)$.*
(1) $\mathrm{LRT}_{\le n}(R_1, R_2)$ *has a unique $\le_R$-minimum element $T_{\min}$.*
(2) *If $T \in \mathrm{LRT}(R)$ is not $\le_R$-minimal, then there is an element $b \in B^R$ such that $\mathbb{Q}(\tilde{e}_0(b)) <_R T = \mathbb{Q}(b)$ and $H_{R_2,R_1}(\tilde{e}_0(b)) = H_{R_2,R_1}(b) - 1$.*

As a consequence, one obtains the two-rectangle special case of Theorem 3.23.

Proposition 4.5 *Let $R = (R_1, R_2)$. Then*

$$H_{R_2,R_1}(b) = d_{R_2,R_1}(\mathbb{Q}(b)) \tag{4.4}$$

for all $b \in B^R$.

Proof: By Theorem 3.22 and Proposition 3.3, both sides of (4.4) are constant on $U_q(A_{n-1})$-connected components. The subposet $\mathrm{LRT}_{\le n}(R) \subset \mathrm{LRT}(R)$ is connected by Lemma 4.4, which, with Theorem 2.13(C2), implies that the two sides of (4.4) differ by a global constant. But it is easy to check that they agree on the element $y = y_2 \otimes y_1$ where $y_j = \mathrm{Key}(R_j)$. □

By Proposition 4.5 and (3.12).

$$\begin{aligned} E_R(b) &= \sum_{1\le i<j\le t} H_i(\sigma_{i+1}\cdots\sigma_{j-1}b) \\ &= \sum_{1\le i<j\le t} d_i(\mathbb{Q}(\sigma_{i+1}\cdots\sigma_{j-1}b) \\ &= \sum_{1\le i<j\le t} d_i(\tau_{i+1}\cdots\tau_{j-1}\mathbb{Q}(b)) \end{aligned} \tag{4.5}$$

It now makes sense to define $E_R : \mathrm{LRT}(R) \to \mathbb{Z}_+$ by

$$E_R(Q) = \sum_{1\le i<j\le t} d_i(\tau_{i+1}\cdots\tau_{j-1}Q). \tag{4.6}$$

4.4. Proof of Theorem 3.23

It suffices to show

$$E_R(Q) = \mathrm{charge}_R(Q) \quad \text{for all } Q \in \mathrm{LRT}(R). \tag{4.7}$$

Since n is entirely absent from (4.7), we may assume that $n > \sum_j \eta_j$ and use Theorem 4.3, which only applies in this special case.

Let $Q \in \mathrm{LRT}(\lambda; R)$. Note that $\lambda_n = 0$ since the maximum number of rows in an element of $\mathrm{LRT}(R)$ is $\sum_j \eta_j$. It follows from the definition of R-LR and Theorem 2.7 that $R_j \subset \lambda$ for all j. In particular, $\lambda_1 \geq \max_j \mu_j$.

The proof now proceeds as in the cases (C1) through (C4) of Theorem 2.13.

Suppose first that Q is not $\leq_R$-minimal. By Theorems 4.3 and 2.13(C2), this case is finished by induction on charge_R.

Otherwise let Q be $\leq_R$-minimal, or equivalently, $\lambda_1 = \max_j \mu_j$ by Theorem 2.12(2). Suppose that $\lambda_1 = \mu_1$. By Theorem 2.12(3), $Q = Q^\wedge Y_1$ where $Q^\wedge \in \mathrm{LRT}(R^\wedge)$ where $R^\wedge = (R_2, R_3, \ldots)$. By Theorem 2.13(C3) and induction on the number t of rectangles in R, it suffices to show that $E_{R^\wedge}(Q^\wedge) = E_R(Q)$. By definition

$$E_R(Q) - E_{R^\wedge}(Q^\wedge) = \sum_{1<j\leq t} d_1(\tau_2 \cdots \tau_{j-1} Q).$$

Each summand vanishes by Remark 2.10.

Finally, suppose $\lambda_1 = \max_j \mu_j$ and the maximum is attained by μ_{p+1} but not by μ_j for $j \leq p$. By Theorem 2.13(C4) and induction on p (eventually reducing to the case $\lambda_1 = \mu_1 = \max_j \mu_j$) it suffices to show that $E_{r_p R}(\tau_p Q) = E_R(Q)$. For $1 \leq i < j \leq t$, write

$$w_{i,j} := \tau_{i+1}\tau_{i+2} \cdots \tau_{j-1} \tag{4.8}$$

and

$$d'_{i,j} := d_i(w_{i,j}\tau_p Q)$$
$$d_{i,j} := d_i(w_{i,j} Q).$$

By definition

$$E_R(Q) - E_{\tau R}(\tau Q) = \sum_{1\leq i<j\leq t} (d_{i,j} - d'_{i,j}).$$

The value $d'_{i,j}$ is computed using a case by case analysis.

(1) $i < p+1$. In this case it is clear that $d'_{i,j} = d_{i,j}$.
(2) $i = p+1$. Then $w_{p+1,j}\tau_p = \tau_p w_{p+1,j}$ and

$$d'_{p+1,j} = d_{p+1}(\tau_p w_{p+1,j} Q).$$

(3) $p = i$ and $p + 1 < j$. Then

$$w_{p,j}\tau_p = \tau_{p+1}w_{p+1,j}\tau_p = \tau_{p+1}\tau_p w_{p+1,j}$$

so that

$$d'_{p,j} = d_p(\tau_{p+1}\tau_p w_{p+1,j}Q).$$

(4) $p = i$ and $p + 1 = j$. Here $w_{i,j}$ is the identity, and

$$d'_{p,p+1} = d_p(\tau_p Q) = d_p(Q) = d_{p,p+1}.$$

(5) $i < p$ and $p + 1 < j$. Then $w_{i,j}\tau_p = \tau_{p+1}w_{i,j}$ so that

$$d'_{i,j} = d_i(\tau_{p+1}w_{i,j}Q) = d_i(w_{i,j}Q) = d_{i,j}.$$

since the restriction of $w_{i,j}Q$ to the i-th and $i + 1$-st subalphabets is not affected by τ_{p+1}.

(6) $i < p$ and $j = p + 1$. Then $w_{i,p+1}\tau_p = w_{i,p}$ and

$$d'_{i,p+1} = d_i(w_{i,p}Q) = d_{i,p}.$$

(7) $i < p$ and $j = p$. Then $w_{i,p}\tau_p = w_{i,p+1}$ and

$$d'_{i,p} = d_i(w_{i,p+1}Q) = d_{i,p+1}.$$

(8) $j < p$. In this case it is clear that $d_{i,j} = d'_{i,j}$.

Consider the contributions of such terms to $E_R(Q) - E_{\tau R}(\tau Q) = \sum_{1\le i<j\le t}(d_{i,j} - d'_{i,j})$. In cases 1, 4, 5, and 8, $d'_{i,j} = d_{i,j}$ and the terms cancel. The sum of the terms in cases 6 and 7 cancel. So it is enough to show that the sum of terms in 2 and 3 cancel, that is,

$$\begin{aligned} 0 = &\sum_{j>p+1}(d_{p+1}(\tau_p w_{p+1,j}Q) - d_{p+1}(w_{p+1,j}Q)) \\ &+ \sum_{j>p+1}(d_p(\tau_{p+1}\tau_p w_{p+1,j}Q) - d_p(w_{p,j}Q)). \end{aligned}$$

Rewriting $d_p(w_{p,j}Q) = d_p(\tau_{p+1}w_{p+1,j}Q)$, observe that without loss of generality it may be assumed that $t = 3$ and it is enough to show that

$$0 = d_2(\tau_1 Q) - d_2(Q) + d_1(\tau_2\tau_1 Q) - d_1(\tau_2 Q). \tag{4.9}$$

Using the previous cases one may assume that $\lambda_1 = \mu_{p+1}$ and $\lambda_1 > \mu_r$ for $r \le p$ where $p \in \{1, 2\}$. If $p = 2$ then by Remark 2.10 all terms vanish. If $p = 1$ Remark 2.10 implies that $d_2(Q) = d_1(\tau_2\tau_1 Q) = 0$. It is enough to show that $d_2(\tau_1 Q) = d_1(\tau_2 Q)$. Since λ has μ_2

columns, any τ_j that moves the wide rectangle R_2, merely causes the corresponding subtableau Y_2 to exchange vertically within its column in Q. It follows that $d_2(\tau_1 Q) = d_1(\tau_2 Q)$ since the shapes of the tableaux used to compute both quantities coincide.

This completes the proof.

Remark 4.6 Equation (4.9) also follows from the fact that the R-matrix acting on threefold tensor products of affinizations of modules of the form $W^{k,l}$, satisfies the Yang-Baxter equation.

5. Tensor product structure on Demazure crystals

5.1. Perfect crystals

In [6] the notion of a perfect crystal is defined. In [7] $B^{k,l}$ is shown to be perfect of level l for type $A^{(1)}_{n-1}$. We recall some consequences of this theory.

The level of a crystal B is defined by $\mathrm{lev}(B) = \min_{b\in B}\langle c, \varepsilon(b)\rangle$. For the root system $A^{(1)}_{n-1}$ the quantity being minimized is $\sum_{i\in I}\varepsilon_i(b)$. The set of minimal elements in B is given by $B_{\min} = \{b \in B \mid \langle c, \varepsilon(b)\rangle = \mathrm{lev}(B)\}$. Define $\varepsilon, \varphi : B \to P_{cl}$ by $\varepsilon(b) = \sum_{i\in I}\varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum_{i\in I}\varphi_i(b)\Lambda_i$ and $(P^+_{cl})_l = \{\lambda \in P^+_{cl} \mid \langle c, \lambda\rangle = l\}$.

Theorem 5.1 [6, 7] *Let $B = B^{k,l}$.*
(1) *The maps ε and φ restrict to bijections $B_{\min} \to (P^+_{cl})_l$.*
(2) *Given any $\lambda \in (P^+_{cl})_l$, there is an isomorphism of P_{cl}-weighted I-crystals*

$$B^{k,l} \otimes \mathcal{B}(\lambda) \cong \mathcal{B}(\lambda') \tag{5.1}$$

where $\lambda' = \varphi(\varepsilon^{-1}(\lambda))$ and $\varepsilon^{-1}(\lambda) \otimes u_\lambda \mapsto u_{\lambda'}$.

For the rest of Section 5, assume that μ_j is a constant, say l. Let $\lambda \in (P^+_{cl})_l$. Iterating Theorem 5.1, there is an $U'_q(A^{(1)}_{n-1})$-crystal isomorphism

$$B^R \otimes \mathcal{B}(\lambda) \cong \mathcal{B}(\lambda') \tag{5.2}$$

for a suitable λ'.

5.2. Tensor products as Demazure crystals

In [15] it was observed that under suitable conditions, tensor products of finite crystals are isomorphic to crystal graphs of Demazure submodules of irreducible integrable highest weight modules.

The tensor product structure for the Demazure crystals is a consequence of an inhomogeneous version of [15, Theorem 1] that uses Lemma 3.25.

Given a sequence $\eta = (\eta_1, \dots, \eta_t)$ with $0 < \eta_j < n$, let η^t be the partition with less than n parts, given by transposing the partition η^+ obtained by sorting the parts of η into decreasing order.

Let $w_0 \in \bar{W}$ be the longest element. Let $w_\eta = w_{\eta,0} \in W$ (not related to w_0 even for $\eta = 0$) be the shortest element such that

$$w_\eta \Lambda_{|\eta|} = \Lambda_0 + w_0 \mathrm{wt}_{\mathrm{sl}}(\eta^t). \tag{5.3}$$

Note that $l w_0 \eta^t$ is the most antidominant weight in B^R viewed as a $U_q(A_{n-1})$-crystal. If $|\eta| = n$ then w_η acts on $X = \Lambda_0 + \bar{P}_{cl}$ by the translation $t_{\mathrm{wt}_{\mathrm{sl}}(w_0(\eta^t))}$. Let $\chi = r_1 r_2 \cdots r_{n-1} \in \bar{W}$ and $\psi = t_{\bar{\Lambda}_1} \chi \in \tilde{W}$, where $\tilde{W} = \bar{P}_{cl} \rtimes \bar{W}$ is the extended affine Weyl group. Note that $\psi(\Lambda_i) = \Lambda_{i+1}$ and $\psi(\bar{\alpha}_i) = \bar{\alpha}_{i+1}$ for all $i \in I$. Define $t_{\eta,s} = \psi^s t_\eta \psi^{-s} \in W$.

Theorem 5.2 *Let $\eta = (\eta_1, \dots, \eta_t)$ with $0 < \eta_j < n$, $l > 0$, and $R_j = (l^{\eta_j})$ for all j. The isomorphism $\mathcal{B}(l\Lambda_{s+|\eta|}) \cong B^R \otimes \mathcal{B}(l\Lambda_s)$ of P_{cl}-weighted I-crystals given by iterating* (5.2), *restricts to a bijection*

$$\mathcal{B}_{w_{\eta,s}}\big(l\Lambda_{s+|\eta|}\big) \cong B^R \otimes u_{l\Lambda_s} \tag{5.4}$$

as full subgraphs. Moreover, suppose $s = 0$ and $v \mapsto b \otimes u_{l\Lambda_0}$. Let $\mathrm{wt} : B^R \otimes \mathcal{B}(l\Lambda_0) \to P$ *be given by* $\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) - \delta E_R(b) + \mathrm{wt}_P(b')$. *Then the isomorphism* (5.4) *preserves P-weights. That is,*

$$\big\langle d, l\Lambda_{|\eta|} - \mathrm{wt}(v)\big\rangle = E_R(b) \tag{5.5}$$

where the left hand side is the distance along the null root δ of v from the affine highest weight vector $u_{l\Lambda_{|\eta|}} \in \mathcal{V}(l\Lambda_{|\eta|})$.

If $s \neq 0$ then E_R must be modified to make the isomorphism preserve P-weights. Only in the $s = 0$ case is the above Demazure module closed under $U_q(A_{n-1})$-crystal operators.

Writing $q = e^{-\delta}$, we have the equality of characters in $\mathbb{Z}[P]$:

Corollary 5.3 *With hypotheses as in Theorem* 5.2 *with $s = 0$,*

$$\mathrm{ch}\big(\mathcal{V}_{w_\eta}(l\Lambda_0)\big) = e^{l\Lambda_0} \mathrm{ch}_q(B^R). \tag{5.6}$$

One obtains Theorem 1.2 by putting together Theorem 2.11 and Corollaries 3.24 and 5.3.

5.3. Generating Demazure crystals

Given an I-crystal B, a subset B', and $i \in I$, define $L_i(B') = \{\tilde{f}_i^m(b) \mid m \geq 0, b \in B'\}$. Given a sequence $\mathbf{i} = (i_p, \dots, i_1)$ define $L_{\mathbf{i}} = L_{i_p} \cdots L_{i_1}$.

Theorem 5.4 [8] *Let $\lambda \in P_{cl}^+$ and $w \in W$ such that w is shortest among elements of W sending λ to $w\lambda$. $\mathcal{B}_{id}(\lambda) = \{u_\lambda\}$. Suppose $w = r_i v$. Then*

$$L_i(\mathcal{B}_v(\lambda)) = \begin{cases} \mathcal{B}_w(\lambda) & \text{if } w > v \\ \mathcal{B}_v(\lambda) & \text{otherwise.} \end{cases}$$

In particular if $w = r_{i_p} \cdots r_{i_1}$ is a reduced decomposition then $L_{\mathbf{i}}(u_\lambda) = \mathcal{B}_w(\lambda)$.

5.4. Proof of Theorem 5.2

First (5.4) is proved. Following [14, Section 4] (but using different indexing), a reduced decomposition $\mathcal{R}(w_{\eta,s})$ of $w_{\eta,s} \in W$ is given as follows. All subscripts are taken modulo n. Define

$$\mathcal{R}\big(w_{(k)}\big) = (r_{n-k} r_{n-k+1} \cdots r_{n-1}) \cdots (r_2 r_3 \cdots r_{k+1})(r_1 r_2 \cdots r_k) \tag{5.7}$$

Let $\mathcal{R}(w_{(k),s})$ be obtained from $\mathcal{R}(w_{(k)})$ by adding s to each subscript. This takes care of the case that $\eta = (k)$ has a single part. In general define

$$\mathcal{R}(w_{\eta,s}) = \mathcal{R}\big(w_{(\eta_1),s}\big)\mathcal{R}\big(w_{(\eta_2),s+\eta_1}\big)\mathcal{R}\big(w_{(\eta_3),s+\eta_1+\eta_2}\big) \cdots \mathcal{R}\big(w_{(\eta_t),s+|\eta|-\eta_t}\big).$$

One may check that these are all reduced words for the elements $w_{\eta,s}$ defined as in (5.3).

Let $B_j = B^{\eta_j,l}$ for $1 \le j \le t$. Define $b_0 = u_{l\Lambda_s}$ and $b_j \in B_j$ by $\varepsilon(b_j) = \varphi(b_{j-1})$ for all $j \ge 1$. Iterating (5.2), one has

$$\begin{aligned} B\big(l\Lambda_{s+|\eta|}\big) &\cong B_t \otimes \cdots \otimes B_1 \otimes \mathcal{B}(l\Lambda_s) \\ u_{l\Lambda_{s+|\eta|}} &\mapsto b_t \otimes \cdots \otimes b_1 \otimes u_{l\Lambda_s}. \end{aligned} \tag{5.8}$$

In light of Theorem 5.4 it must be shown that

$$L_{\mathcal{R}(w_{\eta,s})}\big(b_t \otimes \cdots \otimes b_1 \otimes u_{l\Lambda_s}\big) = B_t \otimes \cdots \otimes B_1 \otimes u_{l\Lambda_s}. \tag{5.9}$$

For convenience in the proof of (5.4), using the rotation automorphism of the Dynkin diagram $A_{n-1}^{(1)}$, it may be assumed that $s = 0$. The proof proceeds by induction on the number of tensor factors. Write $\eta = (k, \eta^\wedge)$. Then $\mathcal{R}(w_\eta) = \mathcal{R}(w_{(k)})\mathcal{R}(w_{\eta^\wedge,k})$ and by induction

$$L_{\mathcal{R}(w_{\eta^\wedge,k})}\big(b_t \otimes \cdots \otimes b_2 \otimes u_{l\Lambda_k}\big) = B_t \otimes \cdots \otimes B_2 \otimes u_{l\Lambda_k}. \tag{5.10}$$

By (5.2) $\mathcal{B}(l\Lambda_k) \cong B_1 \otimes \mathcal{B}(l\Lambda_0)$ with $u_{l\Lambda_k} \mapsto b_1 \otimes u_{l\Lambda_0}$. It follows that

$$L_{\mathcal{R}(w_{\eta^\wedge,k})}\big(b_t \otimes \cdots \otimes b_2 \otimes b_1 \otimes u_{l\Lambda_0}\big) = B_t \otimes \cdots \otimes B_2 \otimes b_1 \otimes u_{l\Lambda_0}. \tag{5.11}$$

To prove (5.10) it suffices to show that

$$L_{\mathcal{R}(w_{(k),k})}\big(B_t \otimes \cdots \otimes B_2 \otimes b_1 \otimes u_{l\Lambda_0}\big) = B_t \otimes \cdots \otimes B_2 \otimes B_1 \otimes u_{l\Lambda_0}. \tag{5.12}$$

By Theorem 5.4 for type A_{n-1}, $L_{\mathcal{R}(w_{(k)})}(b_1) = B_1$. Since $w_{(k)}$ has no r_0 in it, $L_{\mathcal{R}(w_{(k)})}(b_1 \otimes u_{l\Lambda_0}) = B_1 \otimes u_{l\Lambda_0}$. Equation (5.12) follows from the following Lemma, applied with B_1 and $B_t \otimes \cdots \otimes B_2$.

Lemma 5.5 [15, Lemma 1] *Let $n \geq 0$, $i \in I$ and $b_j \in B_j$ for $j = 1, 2$. Then there exist $p, q \geq 0$ such that*

$$b_2 \otimes \tilde{f}_i^n b_1 = \tilde{f}_i^p \big(\tilde{e}_i^q (b_2) \otimes b_1\big).$$

It remains to show (5.5). Since $\langle d, \alpha_i \rangle = -\delta_{i0}$, the left hand side is also equal to the number of times $\tilde{f}_0$ has been applied in passing from $u_{l\Lambda_{|\eta|}}$ to v under $L_{\mathcal{R}(w_\eta)}$. It suffices to check that the energy increases by 1 upon the application of any of these operators $\tilde{f}_0$. Consider such an application of $\tilde{f}_0$, to $b' \otimes u_{l\Lambda_0} = b'_t \otimes \cdots \otimes b'_1 \otimes u_{l\Lambda_0} \in B^R \otimes \mathcal{B}(l\Lambda_0)$. Recall that in the computation of (5.11), the tensor factor $b_1 \in B_1$ is never disturbed. Also since $L_{w_{(k)}}$ has no $\tilde{f}_0$, it follows that $b'_1 = b_1$. But by definition $\varepsilon_0(b_1) = \varphi_0(u_{l\Lambda_0}) = l$. It follows that $\varepsilon_0(\tilde{f}_0(b')) > \varepsilon_0(b') \geq l = \mu_j$ for all $1 \leq j \leq t$. By Lemma 3.25 it follows that $E_R(\tilde{f}_0(b')) = E_R(b') + 1$.

This completes the proof of Theorem 5.2.

6. Generalization of Han's monotonicity for Kostka-Foulkes polynomials

The following monotonicity property for the Kostka-Foulkes polynomials was proved by G.-N. Han [4]:

$$K_{\lambda,\mu}(q) \leq K_{\lambda\cup\{a\},\mu\cup\{a\}}(q)$$

where $\lambda \cup \{a\}$ denotes the partition obtained by adding a row of length a to λ.

Here is the generalization of this result for the polynomials $K_{\lambda;R}(q)$ that was conjectured by A.N. Kirillov.

Theorem 6.1 *Let R be a dominant sequence of rectangles and $R_0 = (l^k)$ another rectangle. Then*

$$K_{\lambda;R}(q) \leq K_{\lambda\cup R_0;R\cup R_0}(q)$$

where $\lambda \cup R_0$ is the partition obtained by adding k rows of size l to λ and $R \cup R_0$ is any dominant sequence of rectangles obtained by adding the rectangle R_0 to R.

Proof: Let $R^+ = (R_0, R_1, \ldots, R_t)$. By Theorem 2.11, (4.7), and Theorem 2.13(C4), it suffices to define an embedding $i_R : \mathrm{LRT}(\lambda; R) \to \mathrm{LRT}(\lambda \cup R_0; R^+)$ such that $E_{R^+}(i_R(Q)) = E_R(i_R(Q))$. The elements of $W(R)$ will be regarded as being in an alphabet A, whose letters are larger than the zeroth alphabet A_0, which has size k.

Define the embedding $W(R) \to W(R^+)$ by $u \mapsto uY_0$ where $Y_0 = \mathrm{Key}((l^k))$. For any λ this embedding induces the desired embedding, which is defined by $i_R(Q) = P(QY_0)$. Since the letters of Y_0 are smaller than those of Q, it follows that $\mathrm{shape}(i_R(Q)) = \mathrm{shape}(Q) \cup (l^k)$. Thus i_R is well-defined.

Fix $1 \le j \le t$ and B be the union of the subalphabets A_0 and A'_j for R_0 and R_j in $R' = r_1 r_2 \cdots r_j R^+ = (R_0, R_j, R_1, R_2 \ldots)$. Let $(Y_0, Y', \ldots)$ be the sequence of tableaux as in the definition of $W(R')$. With $w_{i,j}$ as in (4.8) and writing $d_0 = d_{R_j,R_0}$,

$$\begin{aligned} d_0(w_{0,j} i_R(Q)) &= d_0(w_{0,j} Q Y_0) = d_0((w_{0,j} Q) Y_0) = d_0(((w_{0,j} Q) Y_0)|_B) \\ &= d_0\big((w_{0,j} Q)|_{A'_j} Y_0\big) = d_0\big(P\big((w_{0,j} Q)|_{A'_j}\big) Y_0\big) = d_0(Y'_j Y_0) = 0 \end{aligned} \tag{6.1}$$

by the Knuth invariance of d_0, the fact that $w_{0,j}$ doesn't touch letters in A_0, the definition of d_0, the fact that $w_{0,j} Q \in \mathrm{LRT}(w_{0,j} R)$, and Remark 2.10. If $i > 0$ then

$$\begin{aligned} d_i(w_{i,j} i_R(Q)) &= d_i((w_{i,j} i_R(Q))|_A \\ &= d_i(w_{i,j}(i_R(Q)|_A)) \\ &= d_i(w_{i,j} Q). \end{aligned} \tag{6.2}$$

From (6.1) and (6.2) it follows that $E_{R^+}(i_R(Q)) = E_R(Q)$. □

Appendix A

A.1. RSK miscellany

Proposition 7.1 [19] *If A is an interval and $u \equiv v$ then $u|_A \equiv v|_A$.*

Proof: Follows from (2.1). □

Lemma 7.2 [26]
(1) *Let v be a row word of length k, $S \in \mathrm{T}(\mu)$ and $\lambda = \mathrm{shape}(P(Sv))$. Then λ/μ is a horizontal strip of size k.*
(2) *Let $T \in \mathrm{T}(\lambda)$ and $\mu \subset \lambda$ such that λ/μ is a horizontal strip of size k. Then there is a unique pair (S, v) where v is a row word of length k and S is a tableau of shape μ, such that $T = P(Sv)$.*
The same is true if everywhere Sv is replaced by vS.

Lemma 7.3 *Let $w = \cdots w_2 w_1$ be a sequence of row words in the alphabet $[n]$.*
(1) *Let (v, Q_1) be the unique pair where v is a row word and $\mathrm{shape}(Q_1) = \mathrm{shape}(\mathbb{P}(w|_{[n-1]}))$, such that $P(vQ_1) = \mathbb{Q}(w)$. Then $Q_1 = \mathbb{Q}(w|_{[n-1]})$ and $m_j(v) = m_n(w_j)$ for all $j \ge 1$.*
(2) *Let (u, Q'_1) be the unique pair where u is a row word and $\mathrm{shape}(Q'_1) = \mathrm{shape}(\mathbb{P}(w|_{[2,n]}))$, such that $P(Q'_1 u) = \mathbb{Q}(w)$. Then $Q'_1 = \mathbb{Q}(w|_{[2,n]})$ and $m_j(u) = m_1(w_j)$ for all $j \ge 1$.*

A.2. Littlewood-Richardson rule

Given a word u and a letter i, define $p_i(u)$ to be the number of pairs of matched parentheses in the word obtained from u by replacing each letter i (resp. $i + 1$) by a right (resp. left) parenthesis, and ignoring other letters.

Given a row index i and a skew shape D, let $\mathrm{ov}_i(D)$ be the overlap of the i-th and $(i+1)$-th rows of D, that is, the number of columns of D containing cells in both the i-th and $(i+1)$-th rows.

Say that a word u is D-compatible if

(1) $m_i(u)$ is the number of cells in the i-th row of D for all i.
(2) $p_i(u) \geq \mathrm{ov}_i(D)$ for all i.

Let $C(\lambda; D)$ be the set of D-compatible tableaux of shape λ.

Remark 7.4 Let $D = \lambda/\mu$. Then u is D-compatible if and only if $\mathrm{content}(u) = \lambda - \mu$ and $u \cdots 2^{\mu_2} 1^{\mu_1}$ is Yamanouchi.

The following theorem is a reformulation of a theorem of Dennis White [36], which is a version of the Littlewood-Richardson rule [20].

Theorem 7.5 *$b \in \mathrm{T}(D)$ if and only if $\mathbb{Q}(b)$ is D-compatible. In particular, the RSK map $v \mapsto (\mathbb{P}(v), \mathbb{Q}(v))$ restricts to a bijection*

$$\mathrm{T}(D) \cong \bigcup_{\lambda} \mathrm{T}(\lambda) \times C(\lambda; D) \tag{7.1}$$

Corollary 7.6 *Let D and E be skew shapes and let A and B be the intervals of row indices of the subshapes D and E inside $D \otimes E$ respectively.*
(1) *A tableau Q is $D \otimes E$-compatible if and only if $Q|_A$ is D-compatible and $Q|_B$ is E-compatible.*
(2) *Suppose $c \in B^E$ and $b = \cdots b_2 b_1$ is a sequence of row words b_j such that the length of b_j is the size of the j-th row of D for all j. Then $b \in B^D$ if and only if $\mathbb{Q}(b \otimes c)|_A$ is D-compatible.*

Proof: Since the last row of E and the first row of D have zero overlap, the first part follows from the definition of compatibility. For the second part, Theorem 7.5 applied to c and E imply that $\mathbb{Q}(c)$ is E-compatible. The following are equivalent:

(1) $b \in B^D$.
(2) $b \otimes c \in B^{D \otimes E}$.
(3) $\mathbb{Q}(b \otimes c)$ is $D \otimes E$-compatible.
(4) $\mathbb{Q}(b \otimes c)|_A$ is D-compatible and $\mathbb{Q}(b \otimes c)|_B$ is E-compatible.
(5) $\mathbb{Q}(b \otimes c)|_A$ is D-compatible.

The first and second items are equivalent by the definition of a tableau, since it is assumed that $c \in B^E$. The second and third items are equivalent by Theorem 7.5 applied to $b \otimes c$ and $D \otimes E$. The third and fourth are equivalent by part 1. The fourth and fifth are equivalent because $\mathbb{Q}(b \otimes c)|_B = \mathbb{Q}(c)$ by definition of $\mathbb{Q}$ and the aforementioned fact that $\mathbb{Q}(c)$ is E-compatible. $\square$

Lemma 7.7 *The following are equivalent*:
(1) *u is λ-compatible.*
(2) *u is Yamanouchi of content λ.*
(3) $P(u) = \mathrm{Key}(\lambda)$.

Lemma 7.8 *Let $\lambda \subset (l^k)$ and $\lambda_{\searrow}$ the antinormal shape obtained from λ by a 180 degree rotation inside (l^k).*
(1) $\mathrm{Key}(\lambda_k, \ldots, \lambda_2, \lambda_1)$ *is the unique $\lambda_{\searrow}$-compatible tableau of partition shape.*
(2) *If $b \in \mathrm{T}(\lambda)$ then $\mathrm{P}_{\searrow}(b) \in \mathrm{T}(\lambda_{\searrow})$.*

Proof: Part 1 holds by direct computation. For part 2, suppose $\mathrm{shape}(\mathrm{P}_{\searrow}(b)) = \mu_{\searrow}$. Applying Theorem 7.5 to $\mathrm{P}_{\searrow}(b)$ and using part 1, we have $\mathbb{Q}(\mathrm{P}_{\searrow}(b)) = \mathrm{Key}(\mu_k, \ldots, \mu_1)$. On the other hand, $\mathbb{P}(\mathrm{P}_{\searrow}(b)) = b$ has shape λ. But these tableaux must have the same shape, so $\lambda = \mu$. □

Lemma 7.9 *u is R-LR if and only if it is $(R_t \otimes \cdots \otimes R_1)$-compatible.*

Proof: Follows from Corollary 7.6 part 1 and Lemma 7.7. □

Lemma 7.10 *Let $\lambda \subset (l^k)$ and $b \in B^{\lambda}$. Let $\mathrm{P}_{\searrow}(b)$ be regarded as a skew tableau of the antinormal shape $\lambda_{\searrow}$ defined by the 180-degree rotation of λ inside the rectangle (l^k). Let $b \in B^{\lambda}$, $c \in B^D$ for a skew shape D, A and B the sets of row indices for the subshapes λ and D of $\lambda \otimes D$, and w_0^A the automorphism of conjugation for the longest element of the symmetric group on A. Then*

$$\mathbb{Q}(\mathrm{P}_{\searrow}(b) \otimes c) = w_0^A(\mathbb{Q}(b \otimes c)).$$

Proof: Let us view b as an element of $B^{(\lambda_k)} \otimes \cdots \otimes B^{(\lambda_1)}$. Write $b = b_k \otimes \cdots \otimes b_2 \otimes b_1$ where b_j is the j-th row of b. Let σ^A be the composition of combinatorial R-matrices that reverses these k tensor factors. Let $b' = \sigma^A(b) = b'_k \otimes \cdots \otimes b'_1$. Applying Remark 3.20 repeatedly, one has $\mathbb{Q}(b' \otimes c) = w_0^A \mathbb{Q}(b \otimes c)$. It remains to show that $b' = \mathrm{P}_{\searrow}(b)$. We show that b' and $\mathrm{P}_{\searrow}(b)$, viewed as tableaux of the skew shape $(\lambda_1) \otimes \cdots \otimes (\lambda_k)$, have the same $\mathbb{P}$ and $\mathbb{Q}$ tableaux. By the definition of σ^A, $\mathbb{P}(b') = \mathbb{P}(b) = b = \mathbb{P}(\mathrm{P}_{\searrow}(b))$. In particular $\mathrm{shape}(\mathbb{Q}(b')) = \lambda$. Since $\mathrm{content}(\mathbb{Q}(b')) = (\lambda_k, \ldots, \lambda_1)$, $\mathbb{Q}(b') = \mathrm{Key}(\lambda_k, \ldots, \lambda_1)$. By Lemma 7.8 $\mathbb{Q}(\mathrm{P}_{\searrow}(b)) = \mathbb{Q}(b')$. □

A.3. Proof of Lemma 4.4

Recall that $R = (R_1, R_2)$. Applying the combinatorial R-matrix σ to exchange R_1 and R_2 if necessary, it may be assumed that either $\mu_1 > \mu_2$, or $\mu_1 = \mu_2$ and $\eta_1 \geq \eta_2$. This is justified by [30, Theorem 21], (3.15), and (3.12).

Suppose first that $\eta_1 + \eta_2 < n$. In this case $\mathrm{LRT}_{\leq n}(R) = \mathrm{LRT}(R)$. By Theorem 2.12 it follows that the unique $\leq_R$-minimum element is given by the tableau $T_{\min} = Y_2 Y_1$. Part 2 follows from Theorem 4.3(2).

Otherwise let $\eta_1 + \eta_2 \geq n$. Clearly, an element T of the subposet $\mathrm{LRT}_{\leq n}(R) \subset \mathrm{LRT}(R)$ is minimal if and only if, for every relation $S \lessdot_R T$ with $S \in \mathrm{LRT}(R)$, $S \notin \mathrm{LRT}_{\leq n}(R)$ (that is, S has more than n rows).

Let $T \in \mathrm{LRT}(R)$, with $\mathrm{shape}(T) = \lambda$, say. In [30, Section 5.2] the following explicit description of T is given. Write $A_1 = [\eta_1]$ and $A_2 = [\eta_1 + 1, \eta_1 + \eta_2]$ for the subalphabets of R. Then $T|_{A_1} = Y_1 = \mathrm{Key}(R_1)$. Let T_e and T_w be the east and west parts obtained by slicing $T|_{A_2}$ vertically between the μ_1-th and $(\mu_1 + 1)$-th columns. Let $\alpha = \mathrm{shape}(T_e)$; it has at most $\min(\eta_1, \eta_2)$ nonzero parts. Then $T_e = \mathrm{Key}(0^{\eta_1}, \alpha)$. Define $\beta_j = \mu_2 - \alpha_j$ for $1 \leq j \leq \eta_2$. Then $T_w = \mathrm{Key}(0^{\eta_2}, \beta)$. Moreover $\beta_j = \lambda_{\eta_1+\eta_2+1-j}$ for $1 \leq j \leq \eta_2$. Let $\alpha(T)$ be the partition α corresponding to an element $T \in \mathrm{LRT}(R)$.

It follows that $T \in \mathrm{LRT}_{\leq n}(R)$, that is, λ has at most n rows, if and only if $\alpha_j = \mu_2$ for $1 \leq j \leq \eta_1 + \eta_2 - n$, if and only if $(\mu_2^{\eta_1+\eta_2-n}) \subset \alpha(T)$.

By [30, Lemma 37], if $S \lessdot_{R,\nu} T$ with $\lambda = \mathrm{shape}(T)$ and $\lambda/\nu = s = (m, \lambda_m)$ then S is uniquely defined by the property that $\alpha(S)$ is obtained from $\alpha(T)$ by removing the corner cell in row m. Thus $T_{\min}$ exists and is given by the property that $\alpha(T_{\min}) = (\mu_2^{\eta_1+\eta_2-n})$.

For part 2, let $S, T \in \mathrm{LRT}_{\leq n}(R)$ as above. By the previous paragraph, $\alpha(S)$ is obtained from $\alpha(T)$ by removing a cell in the m-th row, so that $m > \eta_1 + \eta_2 - n$ and $\alpha_m > 0$.

Define $y \in B^R$ by $\mathbb{P}(y) = \mathrm{Key}(\lambda)$ and $\mathbb{Q}(y) = T$. By the above explicit form of T, together with Remark 3.7, y has the following explicit form. Let $y = y_2 \otimes y_1$ with $y_j \in B^{R_j}$. Then $y_1 = \mathrm{Key}(R_1)$ and $y_2 \in B^{R_2}$ is defined by

$$\begin{aligned} y_2|_{[\eta_1]} &= \mathrm{Key}(\alpha) \\ y_2|_{[\eta_1,n]} &= \mathrm{P}_{\searrow}\big(\mathrm{Key}\big(0^{\eta_1}, \lambda_{\eta_1+1}, \lambda_{\eta_1+2}, \ldots, \lambda_n\big)\big) \end{aligned} \tag{7.2}$$

Let K and b be as in the proof of Theorem 4.3. Note that

$$K = \tilde{r}_1 \cdots \tilde{r}_{m-1} \mathrm{Key}(\lambda) = \tilde{f}_1^{\lambda_1-\lambda_m} \cdots \tilde{f}_{m-1}^{\lambda_{m-1}-\lambda_m} \mathrm{Key}(\lambda).$$

By Proposition 3.3 and the fact that $\varphi_i(y_1) = 0$ for $1 \leq i \leq m-1$,

$$b = b_2 \otimes b_1 = \tilde{f}_1^{\lambda_1-\lambda_m} \cdots \tilde{f}_{m-1}^{\lambda_{m-1}-\lambda_m} (y_2 \otimes y_1) = \big(\tilde{f}_1^{\lambda_1-\lambda_m} \cdots \tilde{f}_{m-1}^{\lambda_{m-1}-\lambda_m} y_2\big) \otimes y_1.$$

By (7.2) and $m \leq \eta_1$,

$$b_2|_{[\eta_1+1,n]} = y_2|_{[\eta_1+1,n]} \tag{7.3}$$

and

$$\begin{aligned} b_2|_{[\eta_1]} &= \tilde{f}_1^{\lambda_1-\lambda_m} \cdots \tilde{f}_{m-1}^{\lambda_{m-1}-\lambda_m} \mathrm{Key}(\alpha) \\ &= \tilde{f}_1^{\alpha_1-\alpha_m} \cdots \tilde{f}_{m-1}^{\alpha_{m-1}-\alpha_m} \mathrm{Key}(\alpha) \\ &= \tilde{r}_1 \cdots \tilde{r}_{m-1} \mathrm{Key}(\alpha) \\ &= \mathrm{Key}(\alpha_m, \alpha_1, \alpha_2 \ldots, \alpha_{m-1}, \alpha_{m+1}, \ldots) =: K'. \end{aligned} \tag{7.4}$$

We wish to show

$$\varepsilon_0(b_2) = m_1(b_2). \tag{7.5}$$

Suppose this holds. Then since $\varphi_0(b_1) = 0$ and $\varepsilon_0(b_1) = \mu_1$, it follows that $\varepsilon_0(b) = m_1(b_2) + \mu_1 = \alpha_m + \mu_1 = \lambda_m > \max_j \mu_j$. By Lemma 3.25 $E_R(\tilde{e}_0(b)) = E_R(b) - 1$ for $R = (R_1, R_2)$, which proves the assertion on H.

Note that $m_n(b_2) = \lambda_n$ may be nonzero, so (7.5) (or equivalently, $\varepsilon_0(b) = \lambda_m$) is not trivial, as opposed to the situation in Theorem 4.3.

By (3.7), $\varepsilon_0(b_2) = \varepsilon_1(\mathrm{pr}(b_2)) = \varepsilon_1(\mathrm{pr}(b_2)|_{[2]})$. Equation (7.5) holds if and only if $\mathrm{pr}(b_2)|_{[2]}$ has only one row. It suffices to compute the shape of the complementary tableau $\mathrm{pr}(b_2)|_{[3,n]}$. Note that $\mathrm{pr}(b_2)|_{[3,n]} = \mathrm{P}_\searrow(b_2|_{[2,n-1]}) + 1$ using Lemma 3.14, restriction to $[3, n]$, and Theorem 2.1(2). We compute $\mathrm{P}_\searrow$ in two stages. First,

$$\begin{aligned} \mathrm{pr}(b_2)|_{[\eta_1+2,n]} &= \mathrm{P}_\searrow\big(b_2|_{[\eta_1+1,n-1]}\big) + 1 \\ &= \mathrm{P}_\searrow\big(\mathrm{Key}\big(0^{\eta_1}, \lambda_{\eta_1+1}, \lambda_{\eta_1+2}, \ldots, \lambda_n\big)|_{[\eta_1+1,n-1]}\big) + 1 \\ &= \mathrm{P}_\searrow\big(\mathrm{Key}\big(0^{\eta_1}, \lambda_{\eta_1+1}, \lambda_{\eta_1+2}, \ldots, \lambda_{n-1}, 0\big)\big) + 1 \end{aligned} \tag{7.6}$$

by (7.2), (7.3), and the definition of Key. By Lemma 7.7(2), the shape of the tableau (7.6) is the 180 degree rotation of the partition $(\lambda_{\eta_1+1}, \ldots, \lambda_{n-1})$. Using a jeu-de-taquin sliding algorithm to compute $\mathrm{P}_\searrow$ of $b_2|_{[2,n-1]}$, we may imagine that there are λ_n "holes" where the letters n were in b_2, and the holes are moving to the northwest in order from left to right. Equation (7.6) implies that when exchanging past the entries in the subalphabet $[\eta_1 + 1, n - 1]$, the holes moved directly vertically and now reside in the last λ_n columns of the p-th row, where $p = \eta_1 + \eta_2 - n + 1$. Call their current position the horizontal strip h. It remains to see how the holes enter the first row upon sliding them past $K' = b_2|_{[\eta_1]}$, which is a tableau of shape α. Cut K' vertically between the α_m-th and $(\alpha_m + 1)$-columns, yielding west and east subtableaux K_w and K_e. Then K_w is Yamanouchi for its shape and $K_e - 1$ is Yamanouchi for its shape. It follows that the holes exchange vertically within K_e on their way to the northwest. In particular, they enter the first row in columns strictly east of the α_m-th. It follows that $\mathrm{pr}(b_2)|_{[2]}$ is contained in the first row. This proves (7.5).

It remains to show that $\mathbb{Q}(\tilde{e}_0(b)) \lessdot_{R,\nu} T$. $\tilde{e}_0(b) = \tilde{e}_0(b_2) \otimes b_1$, for $\varepsilon_0(b_2) = \alpha_m > 0$ by (7.5) and (7.4). The computation of $\tilde{e}_0(b_2)$ consists of applying pr, then $\tilde{e}_1$, and then pr^{-1}. By our detailed description of the computation of $\mathrm{pr}(b_2)|_{[2,n]}$ and the fact that $\mathrm{pr}(b_2)|_{[2]}$ is the single row tableau $1^{\lambda_n} 2^{\alpha_m}$, it follows that

$$\mathrm{pr}^{-1}(\tilde{e}_1(\mathrm{pr}(b)))|_{[\eta_1]} = \mathrm{Key}(\alpha_m - 1, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_{m+1}, \ldots,) =: K''.$$

Consequently $P(\tilde{e}_0(b)|_{[\eta_1]}) = P(K'' y_1)$, which has the same shape as the first η_1 rows of λ but with the cell $s = (m, \lambda_m)$ removed. This means that $\alpha(\mathbb{Q}(\tilde{e}_0(b)))$ is obtained from $\alpha = \alpha(T)$ by removing a cell in the m-th row, or that $\alpha(S) = \alpha(\mathbb{Q}(\tilde{e}_0(b)))$. But both $S, \mathbb{Q}(\tilde{e}_0(b)) \in \mathrm{LRT}(R)$. This implies $S = \mathbb{Q}(\tilde{e}_0(b))$ and $\mathbb{Q}(\tilde{e}_0(b)) \lessdot_R T$ as desired.

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