AFFINE WEYL GROUPS, DISCRETE DYNAMICAL SYSTEMS AND PAINLEVÉ EQUATIONS

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Abstract. A new class of representations of affine Weyl groups on rational functions are constructed, in order to formulate discrete dynamical systems associated with affine root systems. As an application, some examples of difference and differential systems of Painlevé type are discussed.

Introduction

In this paper, we propose a class of discrete dynamical systems associated with affine root systems, by constructing new representations of affine Weyl groups. This class of difference systems covers certain types of discrete Painlevé equations, and is expected also to provide a general framework to describe the structure of Bäcklund transformations of differential systems of Painlevé type.

By a series of works by K. Okamoto [12], it has been known since 80's that Painlevé equations $P_{\rm II}$, $P_{\rm III}$, $P_{\rm IV}$, $P_{\rm V}$ and $P_{\rm VI}$ admit the affine Weyl groups of type $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$, respectively, as groups of Bäcklund transformations. The relationship between the affine Weyl group symmetry and the structure of classical solutions has been clarified through the studies of irreducibility of Painlevé equations in the modern sense of H. Umemura (see [12],[6],[16],[8], for instance).

In a recent work [9], the authors introduced a new representation (5.12) of the fourth Painlevé equation $P_{\rm IV}$ from which the structures of Bäcklund transformations and of special solutions of $P_{\rm IV}$ are understood naturally. This sort of "symmetric forms" can be formulated for other Painlevé equations as well (see [11]). One important point of symmetric forms is that the structure of Bäcklund transformations of these Painlevé equations can be described in a unified manner, by introducing a class of representations of affine Weyl groups inside certain Cremona groups. Also, with the τ-functions appropriately defined, the dependent variables of the Painlevé equations allow certain "multiplicative formulas" in terms of τ-functions. One remarkable fact about our multiplicative formulas (2.2) is that the factors are completely determined by the Cartan matrix of the corresponding affine root system. Similar structures can be found commonly in various (discrete) integrable systems with Painlevé (singularity confinement) property ([13],[4],[3]).

The main purpose of this paper is to present a new class of representations of affine Weyl groups which provides a prototype of affine Weyl group symmetry in nonlinear differential and difference systems.

In Sections 1 and 2, we introduce a class of representations of the Coxeter groups of Kac-Moody type on certain fields of rational functions (on the levels of *f-variables* and *τ-functions*, respectively). This class of representations was found as a generalization of the structure of Bäcklund transformations in the symmetric forms of

Painlevé equations P_{IV}, P_{V} and P_{VI} which are the cases of $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$ respectively.

Our representation in the case of an affine root system provides naturally a discrete dynamical system from the lattice part of the affine Weyl group. We introduce in Section 3 the discrete dynamical systems associated with affine root systems in this sense. The case of $A_l^{(1)}$ is discussed in Section 4 in some detail as an example. One interesting aspect of our system is that *continued fractions* arise naturally in the discrete dynamical system, with variations depending on the affine root system.

In the final section, we explain how one can apply our discrete dynamical systems to the problem of symmetry of nonlinear differential (or difference) systems. In particular, we present a series of nonlinear ordinary differential systems which have symmetry under the affine Weyl groups of type $A_l^{(1)}$. This series of nonlinear equations gives a generalization of the Painlevé equations P_{IV} and P_{V} to higher orders.

1. A representation of the Coxeter group $W(A)$

We fix a *generalized Cartan matrix* (or a *root datum*) $A = (a_{ij})_{i,j\in I}$ with I being a finite indexing set. By definition, A is a square matrix with the properties

(C1) $a_{jj} = 2$ for all $j \in I$,
(C2) a_{ij} is a nonpositive integer if $i \neq j$,
(C3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0 \quad (i, j \in I)$.

(See Kac [2] for the basic properties of generalized Cartan matrices. Although we assume that I is finite, a considerable part of the following argument can be formulated under the assumption that A is *locally finite*, namely, for each $j \in I$, $a_{ij} = 0$ except for a finite number of i's.) We define the *root lattice* $Q = Q(A)$ and the *coroot lattice* $Q^\vee$ for A by

$$Q = \bigoplus_{j\in I} \mathbb{Z}\,\alpha_j \quad \text{and} \quad Q^\vee = \bigoplus_{j\in I} \mathbb{Z}\,\alpha_j^\vee \tag{1.1}$$

respectively, together with the pairing $\langle\ ,\ \rangle : Q^\vee \times Q \to \mathbb{Z}$ such that $\langle \alpha_i^\vee, \alpha_j\rangle = a_{ij}$ for $i, j \in I$. We denote by $W = W(A)$ the Coxeter group defined by the generators s_i $(i \in I)$ and defining relations

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1 \quad (i, j \in I, i \neq j), \tag{1.2}$$

where $m_{ij} = 2, 3, 4, 6$ or ∞ according as $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4. The generators s_i act naturally on Q by reflections

$$s_i(\alpha_j) = \alpha_j - \alpha_i\langle \alpha_i^\vee, \alpha_j\rangle = \alpha_j - \alpha_i a_{ij} \tag{1.3}$$

for $i, j \in I$. Note that the action of each s_i on Q induces an automorphism of the field $\mathbb{C}(\alpha) = \mathbb{C}(\alpha_i; i \in I)$ of rational functions in α_i $(i \in I)$ so that $\mathbb{C}(\alpha)$ becomes a left W-module.

Introducing a set of new "variables" f_j $(j \in I)$, we propose to extend the representation of W on $\mathbb{C}(\alpha)$ to the field $\mathbb{C}(\alpha; f) = \mathbb{C}(\alpha)(f_j; j \in I)$ of rational functions in α_j and f_j $(j \in I)$. In order to specify the action of s_i on f_j, we fix a matrix $U = (u_{ij})_{i,j\in I}$ with entries in $\mathbb{C}$ such that

$$\begin{array}{llll}
(0) & u_{ij}=0 & \text{if} & i=j \ \text{ or } \ a_{ij}=0,\\
(1) & u_{ij}=-u_{ji} & \text{if} & (a_{ij},a_{ji})=(-1,-1),\\
(2) & u_{ij}=-u_{ji} \text{ or } -2u_{ji} & \text{if} & (a_{ij},a_{ji})=(-2,-1),\\
(3) & u_{ij}=-u_{ji}, -\frac{3}{2}u_{ji}, -2u_{ji} \text{ or } -3u_{ji} & \text{if} & (a_{ij},a_{ji})=(-3,-1).
\end{array}$$

Theorem 1.1. *Let $A=(a_{ij})_{i,j\in I}$ be a generalized Cartan matrix and $U=(u_{ij})_{i,j\in I}$ a matrix satisfying the conditions above. For each $i\in I$, we extend the action of s_i on $\mathbb{C}(\alpha)$ to an automorphism of $\mathbb{C}(\alpha;f)$ such that*

$$s_i(f_j)=f_j+\frac{\alpha_i}{f_i}u_{ij} \quad (j\in I). \tag{1.4}$$

Then the actions of these s_i define a representation of the Coxeter group $W=W(A)$ (i.e. a left W-module structure) on the field $\mathbb{C}(\alpha;f)$ of rational functions.

We have only to check that the automorphisms s_i on $\mathbb{C}(\alpha;f)$ are involutions ($s_i^2=1$ for all $i\in I$) and that they satisfy the Coxeter relations $(s_is_j)^{m_{ij}}=1$ when $i\neq j$ and $m_{ij}=2,3,4,6$. This can be carried out by direct computations since, for any $i\in I$, the automorphism s_i stabilizes the subfield $\mathbb{C}(\alpha)(f_i,f_k)$ for each $k\in I$ and, for any $i,j\in I$, both s_i and s_j stabilize the subfield $\mathbb{C}(\alpha)(f_i,f_j,f_k)$ for each $k\in I$.

We remark that Theorem 1.1 provides a systematic method to realize the Coxeter groups of Kac-Moody type nontrivially inside *Cremona groups* (groups of the birational transformations of affine spaces).

Remark 1.2. An important class of generalized Cartan matrices is that of symmetrizable ones, which includes the matrices of finite type and of affine type. Our condition on $U=(u_{ij})_{ij\in I}$ described above requires that U should be "almost" skew-symmetrizable. The matrix U can be thought of as specifying a sort of *orientation* of the Coxeter graph of A. It is also related to *Poisson structures* of dynamical systems.

Remark 1.3. Practically, it is sometimes necessary to consider the extension $\widetilde{W}=W\rtimes\Omega$ of $W=W(A)$ by a group Ω of diagram automorphisms of A. Recall that a *diagram automorphism* ω is by definition a bijection on I such that $a_{\omega(i)\omega(j)}=a_{ij}$ for all $i,j\in I$; the commutation relations of each $\omega\in\Omega$ with elements of W are given by $\omega s_i=s_{\omega(i)}\omega$ for all $i\in I$. Suppose that the matrix U satisfies in addition the following compatibility condition with respect to Ω : $u_{\omega(i)\omega(j)}=u_{ij}$ for all $i,j\in I$, $\omega\in\Omega$. Then, together with the automorphisms ω of $\mathbb{C}(\alpha;f)$ such that $\omega(\alpha_j)=\alpha_{\omega(j)}$, $\omega(f_j)=f_{\omega(j)}$ $(j\in I)$, the representation of W in Theorem 1.1 lifts to a representation of the *extended* Coxeter group $\widetilde{W}=W\rtimes\Omega$ on $\mathbb{C}(\alpha;f)$.

2. τ-Functions – A further extension of the representation

We now introduce another set of variables τ_j $(j\in I)$, which we call the "τ-functions" for the f-variables f_j $(j\in I)$. Considering the field extension $\mathbb{C}(\alpha;f;\tau)$ $=\mathbb{C}(\alpha;f)(\tau_j;j\in I)$, we propose a way to extend the representation of W of Theorem 1.1 to $\mathbb{C}(\alpha;f;\tau)$.

Theorem 2.1. *Let A be a generalized Cartan matrix and $U=(u_{ij})_{i,j\in I}$ a matrix with entries in $\mathbb{C}$ satisfying the conditions*

$$\begin{array}{lll}
(0) & u_{jj}=0 & \textit{for all } j\in I,\\
(1) & u_{ij}=u_{ji}=0 & \textit{if} \quad a_{ij}=a_{ji}=0,\\
(2) & u_{ij}=-ku_{ji} & \textit{if} \quad (a_{ij},a_{ji})=(-k,-1) \textit{ with } k=1,2 \textit{ or } 3.
\end{array}$$

We extend the action of each generator s_i of W on $\mathbb{C}(\alpha; f)$ to an automorphism of $\mathbb{C}(\alpha; f; \tau)$ by the formulas

$$s_i(\tau_j) = \tau_j \quad (i \neq j), \quad s_i(\tau_i) = f_i\ \tau_i \prod_{k\in I} \tau_k^{-a_{ki}} = f_i \frac{\prod_{k\in I\setminus\{i\}} \tau_k^{|a_{ki}|}}{\tau_i}, \tag{2.1}$$

for all $i, j \in I$. Then these automorphisms define a representation of W on $\mathbb{C}(\alpha; f; \tau)$

The formulas (2.1) of Theorem 2.1 specify how the f-variables should be expressed in terms of the τ-functions:

$$f_j = \frac{\tau_j\ s_j(\tau_j)}{\prod_{i\in I\setminus\{j\}} \tau_i^{|a_{ij}|}} \tag{2.2}$$

for all $j \in I$. We remark that this type of *multiplicative formulas by τ-functions* is of a *universal* nature as can be found in various discretized integrable systems such as T-systems, discrete Toda equations and discrete Painlevé equations (see [4],[3],[13],...). In that context, the existence of multiplicative formulas is thought of as a reflection of *singularity confinement* which is a discrete analogue of the Painlevé property.

Remark 2.2. If the matrix U is invariant with respect to a group Ω of diagram automorphisms, then the action of the extended Coxeter group $\widetilde{W} = W \rtimes \Omega$ on $\mathbb{C}(\alpha; f)$ extends naturally to $\mathbb{C}(\alpha; f; \tau)$ by $\omega.\tau_j = \tau_{\omega(j)}$ for all $j \in I$.

Theorem 2.1 can be proved essentially by direct computation to verify the fundamental relations of the Coxeter group with respect to the action on the τ-functions τ_k $(k \in I)$. Instead of giving the detail of such a proof, we will explain some of the ideas behind these multiplicative formulas. We consider that the τ-functions should correspond to the *fundamental weights* Λ_j, while the f-variables do to *simple roots* α_i. Let us denote by $L = \mathrm{Hom}_{\mathbb{Z}}(Q^\vee, \mathbb{Z})$ the dual $\mathbb{Z}$-module of the coroot lattice $Q^\vee$, and take the dual basis $\{\Lambda_j\}_{j\in I}$ of $\{\alpha_i^\vee\}_{i\in I}$ so that $L = \bigoplus_{j\in I} \mathbb{Z}\Lambda_j$. Note that L, being the dual of Q, has a natural action of W and that there is a natural W-homomorphism $Q \to L$ such that

$$\alpha_j \mapsto \sum_{i\in I} \Lambda_i a_{ij} \quad (j \in I) \tag{2.3}$$

through the pairing $\langle\ ,\ \rangle$. (The lattice L is in fact the weight lattice modulo the null roots.) The action of W on L is then described as

$$s_i(\Lambda_j) = 0\ (i \neq j), \quad s_i(\Lambda_i) = \Lambda_i - \sum_{k\in I} \Lambda_k a_{ki} \tag{2.4}$$

for $i, j \in I$. We remark that formulas (2.1) in Theorem 2.1 are a multiplicative analogue of (2.4) *except for the factor f_j*.

Let us introduce the notation of formal exponentials for τ-functions:

$$\tau^\lambda = \prod_{i\in I} \tau_i^{\lambda_i} \quad \text{for each} \quad \lambda = \sum_{i\in I} \lambda_i \Lambda_i \in L, \tag{2.5}$$

where $\lambda_i = \langle \alpha_i^\vee, \lambda\rangle$. In order to clarify the meaning of Theorem 2.1, we consider the action of each element $w \in W$ on τ^λ for $\lambda \in L$. Suppose now that the action of W on $\mathbb{C}(\alpha; f)$ can be extended to $\mathbb{C}(\alpha; f; \tau)$ as described in Theorem 2.1. Since formulas (2.1) read as $s_i(\tau^{\Lambda_j}) = f_j^{\delta_{ij}}\, \tau^{s_i(\Lambda_j)}$ for $j \in I$, we have by linearity

$$s_i(\tau^{\lambda}) = f_i^{\lambda_i}\, \tau^{s_i(\lambda)} \tag{2.6}$$

for each $\lambda \in L$. Hence, for each $w \in W$, we should have rational functions $\phi_w(\lambda) \in \mathbb{C}(\alpha; f)$ indexed by $\lambda \in L$ such that

$$w(\tau^{\lambda}) = \phi_w(\lambda)\, \tau^{w.\lambda} \quad (w \in W, \lambda \in L). \tag{2.7}$$

Furthermore, these functions $\phi_w(\lambda)$ should satisfy the following *cocycle condition*:

$$\phi_{w_1 w_2}(\lambda) = w_1(\phi_{w_2}(\lambda))\, \phi_{w_1}(w_2.\lambda) \tag{2.8}$$

for all $w_1, w_2 \in W$ and $\lambda \in L$. Conversely, if one has a family $(\phi_w(\lambda))_{w\in W, \lambda\in L}$ of rational functions satisfying the cocycle condition (2.8), one can define a representation of W on $\mathbb{C}(\alpha; f; \tau)$ by means of (2.7). Theorem 2.1 is thus equivalent to the following proposition.

Proposition 2.3. *Under the same assumption of Theorem 2.1, there exists a unique cocycle $\phi = (\phi_w(\lambda))_{w\in W, \lambda\in L}$ such that*

$$\phi_1(\lambda) = 1, \quad \phi_{s_i}(\lambda) = f_i^{\langle \alpha_i^{\vee}, \lambda\rangle} \quad (\lambda \in L) \tag{2.9}$$

for each $i \in I$.

Remark 2.4. Any family $\{\phi_w(\lambda)\}_{w\in W, \lambda\in L}$ of rational functions in $\mathbb{C}(\alpha; f)$ can be identified with a mapping

$$\phi : W \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}(\alpha; f)^{\times}) : w \mapsto \phi_w, \tag{2.10}$$

where $\mathbb{C}(\alpha; f)^{\times}$ stands for the multiplicative group of $\mathbb{C}(\alpha; f)$ regarded as a $\mathbb{Z}$-module. The cocycle condition (2.8) is then equivalent to saying that the mapping ϕ of (2.10) is a *Hochschild 1-cocycle* of W with respect to the natural W-bimodule structure of $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}(\alpha; f)^{\times})$. Furthermore, formula (2.7) means that, this cocycle ϕ becomes the coboundary of the 0-cochain

$$\tau \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}(\alpha; f; \tau)^{\times}) : \lambda \mapsto \tau^{\lambda} \tag{2.11}$$

after the extension of the W-module $\mathbb{C}(\alpha; f)$ to $\mathbb{C}(\alpha; f; \tau)$. Thus one could say that: *The role of τ-functions is to trivialize the Hochschild 1-cocycle defined by the f-variables.* From the cocycle condition, it follows that the cocycle $\phi_w : L \to \mathbb{C}(\alpha; f)^{\times}$ of Proposition 2.3 can be expressed as

$$\phi_w(\lambda) = \prod_{r=1}^{p} s_{j_1} \cdots s_{j_{r-1}}(f_{j_r})^{\langle \alpha_{j_r}^{\vee},\, s_{j_{r+1}} \ldots s_{j_p}.\lambda\rangle} \quad (\lambda \in L) \tag{2.12}$$

for any expression $w = s_{j_1} \ldots s_{j_p}$ of w in terms of generators.

The cocycle $\phi = (\phi_w(\lambda))_{w\in W, \lambda\in L}$ defined above plays a crucial role in application of our representation to discrete dynamical systems. One remarkable thing about this cocycle is that ϕ seem to have a very strong *regularity* as described in the following conjecture.

Conjecture 2.5. *In addition to conditions* (1) *and* (2) *of Theorem 2.1, suppose that the matrix $U = (u_{ij})_{i,j\in I}$ satisfies the condition*

(3′) $\quad u_{ij}a_{ji} + a_{ij}u_{ji} = 0 \quad$ *for all $i, j \in I$.*

Then, for any $k \in I$, the rational functions $\phi_w(\Lambda_k)$ $(w \in W)$ of (2.7) *are polynomials in α_j, f_j and u_{ij} $(i, j \in I)$ with coefficients in $\mathbb{Z}$.*

Remark 2.6. In Sections 1 and 2, we presented a nontrivial class of representations of Coxeter groups $W(A)$ over the fields of f-variables and τ-functions, with A being a generalized Cartan matrix. This class of representations appears in fact as *Bäcklund transformations* (or the *Schlesinger transformations*) of the Painlevé equations $P_{\mathrm{IV}}, P_{\mathrm{V}}$ and P_{VI}, which correspond to the cases of the generalized Cartan matrices A of type $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$, respectively. As to these Painlevé equations, one can define appropriate f-variables and τ-functions for which the Bäcklund transformations are described as in Theorems 1.1 and 2.1 (see [9], [11]). (In the cases of P_{II} and P_{III}, which have symmetries of type $A_1^{(1)}$ and $C_2^{(1)}$, the corresponding representations of the affine Weyl groups on f-variables must be modified appropriately, while the multiplicative formulas in terms of τ-functions keep the same structure.) In the context of Bäcklund transformations of Painlevé equations, the functions $\phi_w(\Lambda_k)$, specialized to certain particular solutions, give rise to the special polynomials, called *Umemura polynomials* (see [17],[7]), which are defined to be the main factors of τ-functions for algebraic solutions of the Painlevé equations. For this reason, we expect that the functions $\phi_w(\Lambda_k)$ $(w \in W, k \in I)$ should supply an ample generalization of Umemura polynomials in terms of root systems.

3. Affine Weyl groups and discrete dynamical systems

In what follows, we assume that the generalized Cartan matrix A is indecomposable and is *of affine type*. We use the standard notation of the indexing set $I = \{0, 1, \dots, l\}$ so that $\alpha_1, \dots, \alpha_l$ form a basis for the corresponding finite root system. Recall that the null root δ is expressed as $\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots + a_l\alpha_l$ with certain positive integers $a_0, a_1, \dots, a_l$. The affine Weyl group $W = W(A)$ is generated by the fundamental reflections $s_0, s_1, \dots, s_l$ with respect to the simple roots $\alpha_0, \alpha_1, \dots, \alpha_l$:

$$W = W(A) = \langle s_0, \dots, s_l \rangle. \tag{3.1}$$

One important aspect of the affine case is that the affine Weyl group $W = W(A)$ has an alternative description as the semi-direct product of a free $\mathbb{Z}$-submodule M of rank l of $\mathring{\mathfrak{h}}_{\mathbb{R}} = \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i^\vee$, and the finite Weyl group W_0 acting on M:

$$W \ \stackrel{\sim}{\leftarrow}\ M \rtimes W_0 \quad \text{with} \quad W_0 = \langle s_1, \dots, s_l \rangle. \tag{3.2}$$

For each element $\mu \in M$, we denote by t_μ the corresponding element of the affine Weyl group W, so that $t_{\mu+\nu} = t_\mu t_\nu$ for all $\mu, \nu \in M$. Note that the structure of the *lattice part* M depends on the type of the affine root system and that, if A is nontwisted, i.e., of type $X_l^{(1)}$, then M is identified with the coroot lattice $\mathring{Q}^\vee$ of the finite root system with basis $\{\alpha_1, \dots, \alpha_l\}$. (There are descriptions analogous to (3.2) for certain *extended* affine Weyl groups $\widetilde{W} = W \rtimes \Omega$ as well.) As we already remarked, the field $\mathbb{C}(\alpha) = \mathbb{C}(\alpha_0, \alpha_1, \dots, \alpha_l)$ has a natural structure of W-module. The lattice part M in the decomposition (3.2) acts on $\mathbb{C}(\alpha)$ by

$$t_\mu(\alpha_j) = \alpha_j - \langle \mu, \alpha_j \rangle \delta \quad (j = 0, 1, \dots, l; \mu \in M) \tag{3.3}$$

as shift operators with respect to the simple affine roots. (The null root δ is a W-invariant element of $\mathbb{C}(\alpha)$. For this reason, it is sometimes more convenient to consider δ to be a nonzero constant which represents the scaling of the lattice M.)

Suppose now that one has extended the action of W from $\mathbb{C}(\alpha)$ to $\mathbb{C}(\alpha; f) = \mathbb{C}(\alpha)(f_0, f_1, \ldots, f_l)$. At this moment, we can consider an arbitrary extension $\mathbb{C}(\alpha; f)$ as a W-module, assuming that each element of W acts on the function field as an automorphism; the representation of W presented in Sections 1 and 2 provides a choice of such an extension. For each $\nu \in M$, we define a family of rational functions $F_{\nu j}(\alpha; f) \in \mathbb{C}(\alpha; f)$ by

$$t_\nu(f_j) = F_{\nu j}(\alpha; f) \quad (j = 0, 1, \ldots, l). \tag{3.4}$$

Then these formulas can already be considered as a *discrete dynamical system*, defined by a set of commuting discrete time evolutions. In other words, we obtain a commuting family of rational mappings on the affine space where α_j and f_j play the role of coordinates of the discrete time variables and the dependent variables, respectively.

To make clear the meaning of (3.4) as a difference system, we set

$$\alpha_j[\mu] = t_\mu(\alpha_j) = \alpha_j - \langle \mu, \alpha_j \rangle \delta, \quad f_j[\mu] = t_\mu(f_j) \quad (j = 0, \ldots, l) \tag{3.5}$$

for each $\mu \in M$, and consider them as representing functions on M with initial values $\alpha_j[0] = \alpha_j$, $f_j[0] = f_j$ $(j = 0, \ldots, l)$. Then formulas (3.4) implies that

$$f_j[\mu + \nu] = F_{\nu j}(\alpha[\mu]; f[\mu]) \quad (j = 0, 1, \ldots, l). \tag{3.6}$$

In this sense, the functions $F_{\nu j}(\alpha; f)$ defined above provide a difference dynamical system on the lattice M. Since $f_j[\mu]$ is a rational function in $f_0, \ldots, f_l$, for each $\mu \in M$, the general solution of the difference system (3.6) *a priori* depends rationally on initial values $f_0, f_1, \ldots, f_l$. Note also that the action of the affine Weyl group W on $f_j[\nu]$ is described as

$$(w.f_j)[\mu] = w(f_j[w^{-1}\mu]) \quad (j = 0, \ldots, l; \mu \in M) \tag{3.7}$$

for all $w \in W$. In this sense, our difference system admits the action of the affine Weyl group $W(A)$. Note that, if one take the representation of Theorem 1.1, one has

$$(s_i.f_j)[\mu] = f_j[\mu] + \frac{\alpha_i[\mu]}{f_i[\mu]} u_{ij} \tag{3.8}$$

for $i, j = 0, \ldots, l$.

Suppose that one can extend the action of W further to the τ-functions as in Theorem 2.1 and set $\tau_i[\mu] = t_\mu(\tau_i)$, regarding $\tau_i[0] = \tau_i$ as initial values of the τ-functions. Then from (2.2) we obtain the multiplicative formulas

$$f_j[\mu] = \frac{\tau_j[\mu]\ s_{\alpha_j[\mu]}(\tau_j[\mu])}{\prod_{i \in I \setminus \{j\}} \tau_i[\mu]^{|a_{ij}|}} \qquad (j = 0, \ldots, l) \tag{3.9}$$

for the f-variables in terms of τ-functions. In terms of the cocycle ϕ, these formulas are rewritten by (2.7) into

$$f_j[\mu] = \frac{\phi_{t_\mu}(\Lambda_j)\ \phi_{t_\mu s_j}(\Lambda_j)}{\prod_{i \in I \setminus \{j\}} \phi_{t_\mu}(\Lambda_i)^{|a_{ij}|}} \qquad (j = 0, \ldots, l), \tag{3.10}$$

which give a complete description of the general solution of the difference system (3.4) in terms of the initial values $f_0, \ldots, f_l$. In this sense, the cocycle ϕ *solves* our difference system (3.4). It should be noted that all these properties of the difference system (3.4), or (3.6) equivalently, are already guaranteed when we take the representation of the affine Weyl group $W(A)$ as in Theorem 2.1. Also, it is meaningful if one could find other types of representations of affine Weyl groups which have the properties of Theorem 2.1.

We now take the representation of the affine Weyl group $W(A)$ on $\mathbb{C}(\alpha;f;\tau)$ introduced in Theorem 2.1. One interesting feature of our representation is that *continued fractions* arise naturally in the description of discrete dynamical systems, and that the structure of continued fractions is determined by the affine root system. We assume for simplicity that the generalized Cartan matrix A is of type $X_l^{(1)}$. For a given element $w \in W(A)$, take a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_p}$ of w, and define the affine roots $\beta_1, \beta_2, \cdots, \beta_p$ by

$$\beta_1 = \alpha_1, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}). \tag{3.11}$$

Note that these β_r $(r = 1, \ldots, p)$ give precisely the set of all positive real roots whose reflection hyperplanes separate the fundamental alcove C and its image $w.C$ by w. Since the action of s_i on f_j is given by

$$s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij} \qquad (i, j = 0, \ldots, l), \tag{3.12}$$

we have inductively

$$w(f_j) = f_j + \frac{\alpha_{i_1}}{f_{i_1}} u_{i_1 j} + s_{i_1}\big(\frac{\alpha_{i_2}}{f_{i_2}}\big) u_{i_2 j} + \cdots + s_{i_1} \cdots s_{i_{p-1}}\big(\frac{\alpha_{i_p}}{f_{i_p}}\big) u_{i_p j}. \tag{3.13}$$

Each summand of this expression is given by the continued fraction

$$s_{i_1} \cdots s_{i_{r-1}}\big(\frac{\alpha_{i_r}}{f_{i_r}}\big) = \cfrac{\beta_r}{f_{i_r} + u_{i_{r-1} i_r} \cfrac{\beta_{r-1}}{f_{i_{r-1}} + \ddots \cfrac{\beta_2}{f_{i_2} + u_{i_1 i_2} \cfrac{\beta_1}{f_{i_1}}}}} \tag{3.14}$$

along the reduced decomposition $w = s_{i_1} \cdots s_{i_p}$. Note also that formula (3.13) for $w(f_j)$ has an alternative expression

$$w(f_j) = \frac{\phi_w(\Lambda_j)\ \phi_{ws_j}(\Lambda_j)}{\prod_{i \in I \setminus \{j\}} \phi_w(\Lambda_i)^{|a_{ij}|}} \tag{3.15}$$

in terms of the cocycle ϕ, which is implied by (2.2) and (2.7). If one take an element $\nu \in M = \mathring{Q}^\vee$ of the dual root lattice, the rational functions $t_\nu(f_j) = F_{\nu j}(\alpha; f)$ $(j = 0, \ldots, l)$ for the time evolution with respect to ν are determined in the form

$$F_{\nu j}(\alpha; f) = f_j + \sum_{r=1}^{p} s_{i_1} \cdots s_{i_{r-1}}\big(\frac{\alpha_{i_r}}{f_{i_r}}\big) u_{i_r j} \tag{3.16}$$

as a sum of continued fractions along the reduced decomposition of t_ν, with positive real roots separating the fundamental alcove C and its translation $C+\nu$. We remark that a similar description of the rational functions $F_{\nu j}(\alpha; f)$ can be given also for the cases of extended affine Weyl groups $\widetilde{W} = W \rtimes \Omega$. A series of such discrete dynamical systems will be given in the next section.

4. Discrete dynamical system of type $A_l^{(1)}$

As an example of our discrete dynamical systems associated with affine root systems, we will give an explicit description of the case of $A_l^{(1)}$ with $l \geq 2$. Consider the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{4.1}$$

of type $A_l^{(1)}$ ($l \geq 2$), and identify the indexing set $\{0, 1, \ldots, l\}$ with $\mathbb{Z}/(l+1)\mathbb{Z}$. We take following matrix of "orientation" to specify our representation of $W = W(A_l^{(1)})$:

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \tag{4.2}$$

Then the action of the affine Weyl group $W = \langle s_0, \ldots, s_l \rangle$ on the variables α_j, f_j and τ_j is given explicitly as follows:

$$\begin{array}{llll} s_i(\alpha_i) = -\alpha_i, & s_i(\alpha_j) = \alpha_j + \alpha_i & (j = i \pm 1), & s_i(\alpha_j) = \alpha_j \quad (j \neq i, i \pm 1) \\ s_i(f_i) = f_i, & s_i(f_j) = f_j \pm \dfrac{\alpha_i}{f_i} & (j = i \pm 1), & s_i(f_j) = f_j \quad (j \neq i, i \pm 1) \\ s_i(\tau_i) = f_i \dfrac{\tau_{i-1}\tau_{i+1}}{\tau_i}, & s_i(\tau_j) = \tau_j & (j \neq i) & \end{array} \tag{4.3}$$

Note that U is invariant with respect to the diagram rotation $\pi : i \to i + 1$. Hence this action of W extends to the extended affine Weyl group $\widetilde{W} = W \rtimes \{1, \pi, \ldots, \pi^l\}$ by

$$\pi(\alpha_j) = \alpha_{j+1}, \quad \pi(f_j) = f_{j+1}, \quad \pi(\tau_j) = \tau_{j+1}. \tag{4.4}$$

The group $\widetilde{W}$ is now isomorphic to $\mathring{P} \rtimes W_0$, where $\mathring{P}$ is the weight lattice of the finite root system of type A_l and $W_0 = \langle s_1, \ldots, s_l \rangle \simeq \mathfrak{S}_{l+1}$. Taking the first fundamental weight $\varpi_1 = (l\alpha_1 + (l-1)\alpha_2 + \cdots + \alpha_l)/(l+1)$ of the finite root system, we set

$$T_1 = t_{\varpi_1}, \quad T_i = \pi T_{i-1} \pi^{-1} \quad (i = 2, \ldots, l+1). \tag{4.5}$$

These *shift operators* are expressed as

$$T_1 = \pi s_l s_{l-1} \cdots s_1, \quad T_2 = s_1 \pi s_l \ldots s_2, \quad \ldots, \quad T_{l+1} = s_l \cdots s_1 \pi \tag{4.6}$$

in terms of the generators of $\widetilde{W}$. Note that $T_1 \cdots T_{l+1} = 1$ and that $T_1, \ldots, T_l$ form a basis for the lattice part of $\widetilde{W}$.

The simple affine roots $\alpha_0, \dots, \alpha_l$ are the dynamical variables for the shift operators $T_1, \dots, T_l$ such that

$$T_i(\alpha_{i-1}) = \alpha_{i-1} + \delta, \quad T_i(\alpha_i) = \alpha_i - \delta, \quad T_i(\alpha_j) = \alpha_j \ \ (j \neq i-1, i). \tag{4.7}$$

For each $k \in \mathbb{Z}/(l+1)\mathbb{Z}$ and $r = 0, 1, \dots, l-1$, we define $g_{k,r}$ to be the continued fraction

$$\begin{aligned} g_{k,r} &= s_{k+r} s_{k+r-1} \cdots s_{k+1} \big(\frac{\alpha_k}{f_k}\big) \\ &= \frac{\alpha_k + \dots + \alpha_{k+r}|}{| \qquad f_k} - \frac{\alpha_{k+1} + \dots + \alpha_{k+r}|}{| \qquad f_{k+1}} - \dots - \frac{\alpha_{k+r}|}{|f_{k+r}}. \end{aligned} \tag{4.8}$$

Then the discrete time evolution by T_1 is expressed as

$$\begin{aligned} T_1(f_0) &= f_1 - g_{2,l-1} + g_{0,0}, \\ T_1(f_1) &= f_2 - g_{3,l-2}, \\ T_1(f_2) &= f_3 - g_{4,l-3} + g_{2,l-1}, \\ &\cdots \\ T_1(f_{l-1}) &= f_l - g_{0,0} + g_{l-1,2}, \\ T_1(f_l) &= f_0 + g_{l,1}. \end{aligned} \tag{4.9}$$

The corresponding formulas for $T_2, \dots, T_l$ are obtained from these by applying the diagram rotation π.

5. Nonlinear systems with affine Weyl group symmetry

As we already remarked in Section 2, the representation of $W(A)$ introduced in Theorems 1.1 and 2.1, for the cases $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$, arises in nature as Bäcklund transformations of Painlevé equations $P_{\rm IV}, P_{\rm V}$ and $P_{\rm VI}$, respectively. Hence, the Painlevé equations $P_{\rm IV}, P_{\rm V}$ and $P_{\rm VI}$ have the structure of discrete dynamical systems on the lattice as described in Section 3 with respect to Bäcklund transformations. As to P_{IV}, this point has been discussed in detail in our previous paper [9]. "Symmetric forms" of all Painlevé equations $P_{\rm II}, \dots, P_{\rm VI}$ and their Bäcklund transformations will be discussed in our forthcoming paper [11].

From the viewpoint of nonlinear equations of Painlevé type, an important problem would be the following:

Problem 5.1. *For each affine root system* (*or for each generalized Cartan matrix A, in general*)*, find a system of differential* (*or difference*)*equations for which the Coxeter group $W = W(A)$ acts as Bäcklund transformations.*

We believe that such differential (or difference) systems with affine Weyl group symmetry should provide an intriguing class of dynamical systems with rich mathematical structures, to be compared to Painlevé equations. We also remark that, if one specifies the representation of $W = W(A)$ in advance as in Theorem 2.1, then the problem mentioned above is equivalent to finding such derivations (or shift operators) on $\mathbb{C}(\alpha; f)$ and $\mathbb{C}(\alpha; f; \tau)$ that commute with the action of $W(A)$.

In this section, we will introduce some examples of type $A_l^{(1)}$ of difference and differential systems with affine Weyl group symmetry, as well as remarks on the continuum limit from the difference to the differential systems.

We first explain a general idea to construct difference systems with affine Weyl group symmetry by means of our discrete dynamical systems associated with affine root systems. Consider the discrete dynamical system defined by an affine root system as in Section 3. If we take a sublattice $N \subset M$ of rank r, then the centralizer $Z_{W(A)}(N)$ of N in W gives rise to a group of Bäcklund transformations of the discrete system

$$t_\nu(f_j) = F_{\nu j}(\alpha; f) \quad (j = 0, \ldots, l) \tag{5.1}$$

on the sublattice N of rank r, with α_j $(j = 0, \ldots, l)$ regarded as functions on N such that $t_\nu(\alpha_j) = \alpha_j - \langle \nu, \alpha_j \rangle$. The centralizer $Z_{W(A)}(N)$ contains in fact subgroups generated by reflections acting on the quotient M/N. For instance, let $W_{M/N}$ be the group generated by the reflections s_α with respect to the affine roots α that are perpendicular to the lattice N. Then $W_{M/N}$ is contained in the group of Bäcklund transformations of the discrete system (5.1). (The group of symmetry thus obtained may have a different structure from that of our representations of Sections 1 and 2.)

For example, the difference system (4.9) with respect to the shift operator T_1 has symmetry under the affine Weyl group $W(A_{l-1}^{(1)}) = \langle r, s_2, \ldots, s_l \rangle$, where $r = s_0 s_1 s_0$. The corresponding simple affine roots are given by $\alpha_0 + \alpha_1, \alpha_2, \ldots, \alpha_l$. Note that the root $\alpha_0 + \alpha_1$ is invariant under T_1. The reflection r acts on the variables f_j as follows:

$$\begin{aligned} &r(f_1) = f_1 + \frac{\alpha_0 + \alpha_1}{s_1(f_0)}, \quad r(f_2) = f_2 + \frac{\alpha_0 + \alpha_1}{s_0(f_1)}, \\ &r(f_l) = f_l - \frac{\alpha_0 + \alpha_1}{s_1(f_0)}, \quad r(f_0) = f_0 - \frac{\alpha_0 + \alpha_1}{s_0(f_1)}, \\ &r(f_j) = f_j \qquad (j = 3, \ldots, l-1). \end{aligned} \tag{5.2}$$

We remark that the two element $s_1(f_0)$ and $s_0(f_1) = T_1 s_1(f_0)$ are invariant under the action of r as well as $f_3, \ldots, f_{l-1}$. Similarly, the difference system with respect to the commuting operators $T_1, \ldots, T_k$ has affine Weyl group symmetry under the subgroup $W(A_{l-k}^{(1)}) = \langle r, s_{k+1}, \ldots, s_l \rangle$ where $r = s_0 s_1 \cdots s_{k-1} s_k s_{k-1} \cdots s_0$.

If one can take an appropriate continuum limit of the sublattice N inside M, one would possibly obtain a differential system in r variables whose group of Bäcklund transformations contains a reasonable reflection group. We show an example in which the idea explained above works nicely, in some detail.

Consider the discrete dynamical system of type $A_2^{(1)}$ with extended affine Weyl group $\widetilde{W} = W \rtimes \Omega$ as in Section 4. We take the element $T = T_1 = \pi s_2 s_1 \in \widetilde{W}$ which represents to the translation t_{ϖ_1} with respect to the first fundamental weight $\varpi_1 = (2\alpha_1 + \alpha_2)/3$ of the finite root system, so that

$$T(\alpha_0) = \alpha_0 + \delta, \quad T(\alpha_1) = \alpha_1 - \delta, \quad T(\alpha_2) = \alpha_2. \tag{5.3}$$

Note that T can be considered as a shift operator with respect to the variable α_1. Our discrete dynamical system for this case is described as follows:

$$\begin{aligned} T(f_0) &= f_1 + \frac{\alpha_0}{f_0} - \frac{\alpha_0+\alpha_1|}{|\quad f_2} - \frac{\alpha_0|}{|f_0}, \\ T(f_1) &= f_2 - \frac{\alpha_0}{f_0}, \\ T(f_2) &= f_0 + \frac{\alpha_0+\alpha_1|}{|\quad f_2} - \frac{\alpha_0|}{|f_0} \end{aligned} \tag{5.4}$$

in terms of continued fractions. We remark that $T^{-1}(f_0)$ takes a simpler form than $T(f_0)$ above:

$$T^{-1}(f_0) = f_2 + \frac{\alpha_1}{f_1}. \tag{5.5}$$

Noticing that the element $f_0+f_1+f_2$ is invariant under the action of $\widetilde{W}$, we set $f_0+f_1+f_2 = c$. Then from (5.4) and (5.5) we obtain the following equivalent form of our difference system:

$$T^{-1}(f_0) + f_0 = c - f_1 + \frac{\alpha_1}{f_1}, \quad f_1 + T(f_1) = c - f_0 - \frac{\alpha_0}{f_0}. \tag{5.6}$$

With the notation $f_i[n] = T^n(f_i)$ for $n \in \mathbb{Z}$, this equation gives rise to a representation of the *second discrete Painlevé equation* dP_{II} ([15],[14]):

$$\begin{aligned} f_0[n-1] + f_0[n] &= c - f_1[n] + \frac{\alpha_1 - n\delta}{f_1[n]}, \\ f_1[n] + f_1[n+1] &= c - f_0[n] - \frac{\alpha_0 + n\delta}{f_0[n]} \quad (n \in \mathbb{Z}). \end{aligned} \tag{5.7}$$

Since the shift operator $T = \pi s_2 s_1$ commute with the two reflections $r_0 = s_0 s_1 s_0$ and $r_1 = s_2$, we see that the difference system (5.6) or (5.7) has symmetry of the affine Weyl group $W(A_1^{(1)}) = \langle r_0, r_1 \rangle$. (The corresponding simple roots are $\beta_0 = \alpha_0 + \alpha_1$ and $\beta_1 = \alpha_2$.)

The second Painlevé equation P_{II} arises as a continuum limit of the difference system (5.7), and that $A_1^{(1)}$-symmetry of (5.7) naturally passes to P_{II}. Introduce a small parameter ε such that $\delta = \varepsilon^3$, and set

$$\begin{aligned} &f_0[n] = 1 + \varepsilon\psi + \varepsilon^2\varphi_0, \quad f_1[n] = 1 - \varepsilon\psi + \varepsilon^2\varphi_1, \quad c = 2, \\ &\alpha_0 + n\delta = -1 + \varepsilon^2 x + \varepsilon^3 a_0, \quad \alpha_1 - n\delta = 1 - \varepsilon^2 x + \varepsilon^3 a_1, \quad \alpha_2 = \varepsilon^3 b_1. \end{aligned} \tag{5.8}$$

Then in the limit as $\varepsilon \to 0$, the difference equations (5.7) imply the following differential equation for $\varphi_0, \varphi_1, \psi$:

$$\begin{aligned} &\varphi_0' = 2\varphi_0\psi + a_0 - \frac{1}{2}, \ \varphi_1' = 2\varphi_1\psi + a_1 - \frac{1}{2}, \\ &\psi' = 2(\varphi_0 + \varphi_1) - \psi^2 + x. \end{aligned} \tag{5.9}$$

From this we get the second Painlevé equation for ψ

$$\psi'' = 2\psi^3 - 2x\psi - 2b_1 + 1, \tag{5.10}$$

and the other dependent variables φ_0, φ_1 are determined by quadrature from ψ. At the same time, we obtain the following Bäcklund transformations r_0 and r_1 for ψ:

$$r_0(\psi) = \psi - \frac{2b_0}{\psi' - \psi^2 + x}, \quad r_1(\psi) = \psi - \frac{2b_1}{\psi' + \psi^2 - x}, \tag{5.11}$$

where $b_0 = a_0 + a_1 = 1 - b_1$. The parameters b_0, b_1 are the simple roots for the $A_1^{(1)}$-symmetry of P_{II}.

Finally, we present a series of differential systems with $A_l^{(1)}$-symmetry ($l \geq 2$), which give a generalization of the Painlevé equations P_{IV} and P_{V}.

In our previous paper [9], we introduced the symmetric form of the fourth Painlevé equation:

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2. \end{aligned} \tag{5.12}$$

This system defines in fact a derivation $'$ of the field $\mathbb{C}(\alpha; f)$ which commute with the action of the extended affine Weyl group $\widetilde{W}$ of type $A_2^{(1)}$ as in (4.3) and (4.4). (Note that the convention of [9] corresponds to the transposition of U in (4.2).) We remark that the sum $f_0 + f_1 + f_2$ is invariant under $\widetilde{W}$, and satisfies the equation $(f_0 + f_1 + f_2)' = \alpha_0 + \alpha_1 + \alpha_2 = \delta$. Introduce the independent variable x so that $x' = 1$, and eliminate one of the three f-variables, noting that $f_0 + f_1 + f_2$ is a linear function of x. Then the differential system above is rewritten into a system of order 2, which is equivalent to the Painlevé equation P_{IV}.

Differential system (5.12) has a generalization to higher orders. For example, when $l = 4$, the differential system

$$\begin{aligned} f_0' &= f_0(f_1 - f_2 + f_3 - f_4) + \alpha_0, \\ f_1' &= f_1(f_2 - f_3 + f_4 - f_0) + \alpha_1, \\ f_2' &= f_2(f_3 - f_4 + f_0 - f_1) + \alpha_2, \\ f_3' &= f_3(f_4 - f_0 + f_1 - f_2) + \alpha_3, \\ f_4' &= f_4(f_0 - f_1 + f_2 - f_3) + \alpha_4 \end{aligned} \tag{5.13}$$

has $A_4^{(1)}$-symmetry. Note that the sum $f_0 + f_1 + f_2 + f_3 + f_4$ is a linear function of the independent variable x such that $x' = 1$ and that the system above is essentially of order 4. In general, when $l = 2n$, the following differential system (essentially of order $2n$) turns out to have $A_{2n}^{(1)}$-symmetry with the Bäcklund transformations defined as in Section 4:

$$f_j' = f_j \sum_{1 \leq r \leq n} \big(f_{j+2r-1} - f_{2r}\big) + \alpha_j \quad (j = 0, 1, \ldots, 2n). \tag{5.14}$$

We remark that this differential system is obtained as a continuum limit from the difference system with $A_{2n}^{(1)}$-symmetry which arises from the discrete dynamical system of type $A_{2n+1}^{(1)}$, in the manner as we explained above.

We also found a series of differential systems with $A_{2n+1}^{(1)}$-symmetry ($n = 1, 2, \ldots$) which generalize the fifth Painlevé equation P_{V}:

$$\begin{aligned} f_j' = f_j\big(&\sum_{1 \leq r \leq s \leq n} f_{j+2r-1} f_{j+2s} - \sum_{1 \leq r \leq s \leq n} f_{j+2r} f_{j+2s+1}\big) \\ &+ \big(\frac{\delta}{2} - \sum_{1 \leq r \leq n} \alpha_{j+2r}\big) f_j + \alpha_j \big(\sum_{1 \leq r \leq n} f_{j+2r}\big) \quad (j = 0, 1, \ldots, 2n+1), \end{aligned} \tag{5.15}$$

where $\alpha_0+\cdots+\alpha_{2n+1}=\delta$. We remark that differential system (5.15) is also essentially of order $2n$, since each of the sums $\sum_{r=0}^{n} f_{2r}$ and $\sum_{r=0}^{n} f_{2r+1}$ is determined elementarily. The Painlevé equation P_{V} is covered as the case $n=1$ (see [11]):

$$\begin{aligned}
f_0' &= f_0(f_1f_2-f_2f_3)+(\frac{\delta}{2}-\alpha_2)f_0+\alpha_0 f_2,\\
f_1' &= f_1(f_2f_3-f_3f_0)+(\frac{\delta}{2}-\alpha_3)f_1+\alpha_1 f_3,\\
f_2' &= f_2(f_3f_0-f_0f_1)+(\frac{\delta}{2}-\alpha_0)f_2+\alpha_2 f_0,\\
f_3' &= f_3(f_0f_1-f_1f_2)+(\frac{\delta}{2}-\alpha_1)f_3+\alpha_3 f_1,
\end{aligned} \tag{5.16}$$

where $\alpha_0+\alpha_1+\alpha_2+\alpha_3=\delta$.

These two series of differential systems with affine Weyl group symmetry can be considered as a variation of Lotka-Voltera equations and Bogoyavlensky lattices, including the parameters $\alpha_0,\ldots,\alpha_l$. Also, the structures of their Bäcklund transformations can be described completely in terms of the discrete dynamical systems we have introduced in this paper. (Details will be discussed elsewhere.) We expect that these systems of differential equations with affine Weyl group symmetry deserve to be studied individually from various aspects, since they already give a candidate for systematic generalization of Painlevé equations to higher orders.

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