Agency and fictional truth: a formal study on fiction-making

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Agency and Fictional Truth

A Formal Study on Fiction-making

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Abstract Fictional truth, or truth in fiction/pretense, has been the object of extended scrutiny among philosophers and logicians in recent decades. Comparatively little attention, however, has been paid to its inferential relationships with time and with certain deliberate and contingent human activities, namely, the creation of fictional works. The aim of the paper is to contribute to filling the gap. Toward this goal, a formal framework is outlined that is consistent with a variety of conceptions of fictional truth and based upon a specific formal treatment of time and agency, that of so-called *stit* logics. Moreover, a complete axiomatic theory of fiction-making **TFM** is defined, where fiction-making is understood as the exercise of agency and choice in time over what is fictionally true. The language $\mathcal{L}$ of **TFM** is an extension of the language of propositional logic, with the addition of temporal and modal operators. A distinctive feature of $\mathcal{L}$ with respect to other modal languages is a variety of operators having to do with fictional truth, including a 'fictionality' operator M (to be read as "it is a fictional truth that"). Some applications of **TFM** are outlined, and some interesting linguistic and inferential phenomena, which are not so easily dealt with in other frameworks, are accounted for.

Keywords Fiction · Agency · Tense logic · Stit logic · Branching time · Fiction-making

Introduction

We ordinarily regard works of fiction as *creations*, that is, as the outcomes of certain deliberate, self-conscious activities. We would likely balk, however, if asked to specify what these activities consist of. After all, fiction-making is seldom a linear process flowing from the work plan to the final editing—most often, it is a piecemeal collection of human miseries involving trials, errors, ephemeral inspirations, and strokes

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of luck. Furthermore, as you may guess, different types of fiction lend themselves to dramatically different processes of fiction-making; consider the production of a high-grossing movie compared to the telling of a goodnight tale.

Therefore, one might argue, fiction-making is too messy a phenomenon to be the topic of decently precise philosophical investigation. However, that might be a hasty conclusion. For instance, theory-revision is currently a well respected field of study, even though real-life theory-revision processes may often be highly unsystematic. This is possible insofar as we idealize away from certain details of the phenomena to be described, and we focus instead on their core features, or at least on those core features that are amenable to a fair treatment at a given stage of inquiry.

One of the most important and less disputed core features of fictional works is that they are endowed with (representational or propositional) content. When we say that certain propositions are true *in* a work of fiction—that they are *fictional truths* of that work—we are saying that they are part of the content of the work (see Ross 1997). And it is out of discussion that the creation of a work is a deliberate process that leads to the work, typically through a sequence of intermediate stages or drafts. Thus, it is only natural to think of fiction-making as a deliberate process leading to a certain content, typically through a series of intermediate contents. Let us call this view the *content-choice conception* of fiction-making.

This paper has three main aims. The first is to outline a general logical framework in which the content-choice conception can be made precise, and which paves the way for a more detailed study of the inferential relationships between fictional truth, time, and agency. Here, the content-choice conception is elaborated against the background of a specific view of time and agency, that of so-called *stit logics* (see Belnap et al. 2001 for a general overview). The second aim is to define a formal theory of fiction-making **TFM**, which is understood to encode (what is essential to) the content-choice conception. In spite of its limited expressive resources, **TFM** allows us to formulate and formally check a variety of distinctions concerning fiction and fiction-making. The third and final aim is to make a convincing case that a better understanding of the relationships between fictional truth, time, and agency comes with a better grasp of our common discourse about fiction and of its metaphysical underpinnings.

In recent decades, the content of fictional and representational artworks have been thoroughly investigated, from both a formal and an informal viewpoint (see, for instance, Blocker 1974; Woods 1974; Lewis 1978; Bertolet 1984a; Evans 1982, pp. 353–372; Walton 1990, pp. 138–187; Bonomi and Zucchi 2003; Nossum 2003; Woods and Alward 2004; Hill 2012). On the other hand, the concepts of action and performance have been assigned a key role in a variety of aesthetic and ontological theories of art, fiction, and their content (see Wolterstorff 1980; Walton 1990; Currie 1989, 1990, 2010; Kivy 2006, to mention but a few examples). To the best of my knowledge, however, **TFM** is the first axiomatic theory that explicitly concerns the relationship between action (or, perhaps more appropriately, agency) and the content of fictional works and other representational artifacts, performances or contexts.

TFM is a simple extension of a propositional tense-modal multi-agent *stit* logic. Like all standard *stit* logics, and in contrast to other proposals in the formal study of action (for instance so-called dynamic logics), **TFM** does not directly deal with actions. Its focus is, rather, on agents and the (possible) outcomes of their actions. This feature makes it especially well suited to cope with the content-choice conception, which is only concerned with authors and their achievements and not with the specific actions they perform. As an axiomatic *stit* theory, **TFM** may be classified in the same family as other intensional extensions of *stit* logic, such as the epistemic logics proposed by Broersen (see, e.g., Broersen 2008a,b). In the wider realm of *stit* accounts, the proposal that is closest in spirit to the one pursued here is probably Wansing's (2002, 2006) *stit*-theoretic approach to belief formation.

Now we shall proceed as follows. The first section is mainly devoted to an informal presentation of both the logical framework and **TFM**. It is also meant to introduce a bit of terminology and some basic notions from the philosophical study of fiction, time, and agency. In the second section, some interesting applications of both **TFM** and the underlying general framework are outlined and discussed. In the third and last section **TFM** is formally defined and shown to be complete.

1 Background notions and an outline of the proposal

1.1 Fictional truth and content: some background notions

> How easy is a bush supposed a bear!
>
> Shakespeare, *A Midsummer Night's Dream*

Since Kendall Walton's groundbreaking *Mimesis as Make-Believe*, the notion of fictional truth has been commonly tied to that of a *make-believe game*. A make-believe game is an imaginative activity that is based upon a number of bridge-rules or *principles of generation*. In a make-believe game, certain propositions are to be imagined as true. These are the *fictional truths* of the game. The fictional truths of a make-believe game are ideally determined by what is actually the case in the relevant context and by the principles of generation at work in the game. As an example, a Shakespeare-inspired, bush-bear make-believe game involves a principle of generation that we might formulate as "A bush counts as a bear." By virtue of this principle, whenever, say, a player is close to a bush, it is fictionally true that the player is close to a *bear*.

The fictional truths of a make-believe game are, intuitively, propositions that are true *in* the game. Analogously, we may speak of the fictional truths of a novel, of a movie, and more generally of a work of fiction to indicate the propositions that are true in the work. The same goes for other entities that we would not immediately classify either as make-believe games or works of fiction, for instance normative or communicative contexts and theatrical performances. In what follows, I shall adopt a broad conception of fictional truth and regard all these things as legitimate sources of fictional truths. Moreover, I shall use the expression *fictional setting* as an umbrella term that covers all of them. Thus, for instance, a bush-bear game, a performance of *A Midsummer Night's Dream*, a literary work like *War and Peace*, and your best game of *Angry Birds* all count as fictional settings in my sense. Like Walton, I assume that not just make-believe games but also fictional works and other fictional settings may be said to have principles of generation. For instance, the so-called say-so principle (roughly: whatever is said to be the case is fictionally the case) is naturally thought of as a standard principle of generation of literary fiction.

Throughout this paper, *fictional discourse* is used as a generic term to indicate any discourse that is about or is strictly related to works of fiction and other fictional settings. A key role in fictional discourse is played by so-called *internal sentences*[1] such as:

(1) Mickey Mouse wears gloves.
(2) Holmes met Watson in a lab.

Internal sentences are so called because they are accounts of what occurs *in* a novel, a movie or, more generally, in a fictional setting. If they are relative to a specific setting, we shall also say that they are internal *to* that setting.[2] For instance, (2) is naturally understood as internal to Conan Doyle's *A Study in Scarlet*. Note that a sentence may be internal without being internal to any specific setting. It seems perfectly possible, for example, that a sentence like (1) is uttered without having any specific fictional setting in mind.[3] Internal sentences may be contrasted with explicit reports of fictional settings, such as:

(3) In some piece of fiction, Mickey Mouse wears gloves.
(4) According to *A Study in Scarlet*, Holmes met Watson in a lab.

Statements like these have been called (internal) *metafictional sentences* (Kroon and Voltolini 2011). In the analysis of metafictional sentences, I borrow the 'fictionality' operator M_f (read: "It is a fictional truth of setting f that") and its generic counterpart M ("It is a fictional truth that") from the language of **TFM**. Thus, for instance, I write "M(Mickey Mouse wears gloves)" instead of (3).

Intuitively, metafictional sentences are constructions we use to describe a certain *content* (see Ross 1997, pp. 4–5). To assert that $M_f\mathcal{A}$ is to say that (the proposition expressed by) $\mathcal{A}$ is part of the content of f or, in other words, that the content of f is partly determined by $\mathcal{A}$'s being a fictional truth of f. For this reason, in what follows, we shall use "being a fictional truth of f" and "being part of the content of f" interchangeably. Arguably, the *content* of a work of fiction, so understood, is not to be identified with its *explicit* content, that is, with the content that the work explicitly represents or conveys. The reason is that the fictional truths of a work need not be part of its explicit content (see, e.g., Lewis 1978, p. 41). Conan Doyle, for instance, never bothered to tell us that Holmes had two nostrils, but that is nonetheless true in the Holmes stories. We shall set explicit content aside in what follows, and focus on fictional truth, for reasons of logical tractability and because the notion

[1] See, e.g., Reicher 2008. For the sake of simplicity, I follow common philosophical practice in classifying discourse about fiction into different sorts of *sentences*, although, strictly speaking, it would be better to draw the relevant distinctions at the level of language use.

[2] It is worth stressing that this is a clarification, not a definition of internal sentences. Moreover, internal sentences ought not to be confused with those sentences that directly occur in fictional stories.

[3] Of course, a sentence may also be internal to a plurality of fictional settings, but let us put this case aside for simplicity.

of explicit content is very difficult to characterize when non-purely-linguistic works such as movies or paintings are at stake (see Ross 1997, pp. 33–54 for a discussion).

Most philosophers think that, for many purposes, it is useful to replace internal sentences with suitably chosen metafictional sentences. According to this view, for instance, (1) is usefully paraphrased as (3) and (2), assuming that it is uttered during a discussion about *A Study in Scarlet*, as (4). We shall call this view *the paraphrase thesis*. The paraphrase thesis may be made precise in various ways, depending on the purpose the paraphrase is supposed to serve. I shall remain neutral on whether the paraphrase should reveal the meaning of the original sentence, or capture the speaker's communicative intentions. I assume, however, that an adequate paraphrase should be a better guide than the original sentence to what the speaker is committing her/himself to or, *at least*, not a worse one.[4] Given this assumption, it is natural to hold that the relation of paraphrasability at stake is transitive. If $\rightsquigarrow$ stands for such a relation, the paraphrase thesis may be roughly[5] expressed as follows:

(PT) If $\mathcal{A}$ is an internal sentence, then $\mathcal{A} \rightsquigarrow M(\mathcal{A})$. In certain circumstances, if it is clear that $\mathcal{A}$ is internal to a specific setting f, $\mathcal{A} \rightsquigarrow M_f(\mathcal{A})$.

1.2 Time and agency: some background notions

Now it is time to introduce some basic notions in the temporal logic of agency.[6] (Those familiar with the topic might want to skip all of this section but the last paragraph.) As is common in the formal study of agency, we assume as a working hypothesis that an indeterministic, *branching-time* conception is correct. In the branching-time conception, reality is represented as a *tree* of complete possible courses of affairs or *histories* (also called *possible worlds*). Intuitively, histories may be thought of as sequences of successive 'world-slices' or *moments*. We let $h, h', \ldots$ vary over histories and $m, m', \ldots, w, w', \ldots$ vary over moments, and we write $m \prec m'$ ($m \succ m'$) to indicate that m precedes (follows) m' in time.

A history is said to *pass through* a certain moment if, informally, that moment occurs on that history. In Figure 1, for instance, history h passes through moments m, m' and m''. Two histories are said to *share* a moment m if they both pass through m; they are said to *branch* or *divide* at m if m is the last moment they share; finally, they are said to be *undivided* at m if they share m and do not branch at m. In Figure 1, for

4 It is important to distinguish the paraphrase thesis from other, much stronger views, such as Lewis's (1978) idea that internal sentences are *abbreviations* of the corresponding metafictional sentences (see Bertolet 1984b and Predelli 1997 for critical discussions of this view).

5 This formulation is rough mainly because, as pointed out in note 1, to speak of internal *sentences* is not entirely appropriate. A more accurate formulation ought to take into the account the circumstances of use of the relevant sentences.

6 This section is only meant as a preliminary, informal presentation of these notions, which are officially introduced below, in § 3.2. For those interested in the formal aspects of the proposal, it is worth anticipating that **TFM** is based upon Ockhamist frames, not on Branching Time frames (see below, p. 8 and § 3). Thus, for instance, neither the notion of a moment nor that of a history are taken to be primitive notions, but are rather defined with reference to points and to accessibility relations between points.

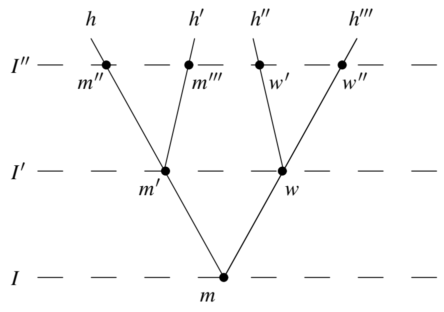

Fig. 1: A partial representation of a synchronized tree, where h–h''' are alternative histories, m–m''', w–w'' are moments such that $m \prec m' \prec m''$, $w \prec w' \prec w''$ (and so on), and I–I'' are successive instants.

instance, histories h and h' share moments m, m', divide at m', and are undivided at m (but not at m').

We assume that trees satisfy three basic conditions, namely, (i) trees are unbounded to the left and to the right; (ii) any two histories in a tree share some moment (*Tree* condition); (iii) if two histories in a tree share a moment, they share any previous moment (*No backward branching* condition).

Sometimes, intuitively, we speak about alternative *simultaneous* states of affairs (e.g., "Now I am awake but I could have been sleeping"). Ordinary trees, however, are 'blind' to such a relation of simultaneity. Since fictional works may obviously involve alternative simultaneous states of affair (say alternative versions of the Battle of Hastings), we need to extend a little bit the conceptual framework introduced thus far. We shall do that by adopting a novel temporal notion, that of an *instant*. We may think of an instant as an equivalence class of simultaneous moments (see, e.g., Belnap et al. 2001, pp. 194–196; below, the notion of an instant is formally defined in a slightly different way; see Def. 6). Intuitively, to say that moments m and m' are in the same instant is to say that they occur simultaneously, maybe on different histories. Moreover, we require trees to be *synchronized* (Di Maio and Zanardo 1994), that is, to contain only histories that share an isomorphic temporal ordering (this is ensured by condition SYNCHRONICITY below). Intuitively, this means that any moment is in some instant and that any instant intersects each history at precisely one moment. Figure 1 is a partial representation of a synchronized tree.

TFM is a modal logic, more precisely a multi-agent *stit* logic. As usual in modal logic, truth is relativized to certain parameters or *points* (of evaluation). A point corresponds to a pair whose elements are a history and a moment that passes through it. We write m/h to abbreviate (m,h) and, at the same time, indicate that m passes through h. These complicated points are needed to deal with sentences whose truth value may depend on future courses of events—and thus on the 'history' parameter. Atomic sentences (propositional letters) are assumed to "have no trace of futurity in them” (Prior 1967, p. 124), that is, to have the same truth value at all points that correspond to the same moment.

Stit logics revolve around a family of ‘agency’ operators *stit* (read: “sees to it that”). Intuitively, if α is an agent or a group of agents, and “$\mathcal{A}$” a sentence, to say that α *stit* $\mathcal{A}$ is to say that α ensures or guarantees that $\mathcal{A}$.

Standard *stit* logics represent actions and choices just in terms of the constraints that these impose on future courses of events. Moreover, they are idealized in many respects; for instance, they ignore the complications raised by vagueness, probability, the actual (non-null) duration of actions and choices, and the role of intentions and beliefs in the understanding of agency. **TFM** shares these limitations.

A central semantic notion in *stit* logics is that of a (possible) *choice*. A possible choice of an agent at a given moment *m* is a set of histories, which corresponds, intuitively, to a certain outcome that the agent is in a position to force at *m*. For instance, since I can now ensure that I am going to type a star (so: *), the class of histories on which I am going to type a star is presently a possible choice of mine. In standard multi-agent *stit* logics, possible choices obey two basic constraints. First, we have the *No choice between undivided history* (**Noch**) constraint: if two histories are undivided at moment *m*, then, for any agent α, they are precisely in the same possible choices of α at *m*. **Noch** makes it apparent that choices, as they are conceived in *stit* logics, are not vague intentions or wishful thoughts: to choose a possible outcome rather than another is to make an immediate difference in what is going to happen. The second constraint, *Independence of agents* (**Indag**), requires, intuitively, that the possible choices of distinct agents at a moment all be independent: it cannot be the case that a possible choice of an agent at a moment *m* is incompatible with some possible choice(s) of some other agent(s) at *m*. This requirement is reasonable: intuitively, an agent cannot be in a position to force a certain outcome if, at the same moment, one or more other agents can prevent that very outcome.

Many different *stit* operators are available in the philosophical market. A very common one is called *Chellas stit* (after Chellas 1969, 1992). A statement “α Chellas-sees to it that $\mathcal{A}$” is true at a point m/h iff $\mathcal{A}$ is true at *m* on all histories in the choice of α corresponding to m/h. Chellas *stit* operators are simple and logically well-behaving S5 boxes. For many purposes, however, it is useful to adopt a slightly more articulated *stit* notion, so-called *deliberative stit* (Von Kutschera 1986; Horty 1989). Informally, “α (deliberately) sees to it that $\mathcal{A}$” may be defined as “α Chellas-sees to it that $\mathcal{A}$ and *it is not inevitable that* $\mathcal{A}$” (see Def. 4 below). The negative clause encodes the intuition that we may only deliberate upon “what is future and capable of being otherwise” (Aristotle, *Nicomachean Ethics*, 1139b7). Deliberative *stit* will be our ‘official’ *stit* notion in all informal discussions.

1.3 Fictional truth, time, and agency: an outline of the proposal

The language $\mathcal{L}$ of **TFM** is an extension of the language of propositional logic, with the addition of the following tense-modal operators:

— Prior's tense operators P ("it has been sometimes the case that"), F ("it will be sometimes the case that"), with duals H, G, respectively.
— an operator of *historical possibility* or *non-inevitability* $\Diamond$ ("it is still not inevitable that") with dual $\Box$.
— an operator of *simultaneous possibility* $\Diamond_\sigma$ (roughly: "it is *now* possible that"[7]) with dual $\Box_\sigma$.
— a finite number of Chellas *stit* operators $[\alpha_1\, stit], \ldots, [\alpha_t\, stit]$, and their 'existential' version $[Stit]$ ("at least one agent (or group of agents) sees to it that"). The deliberative *stit* operators $[\alpha_1\, dstit], \ldots, [\alpha_t\, dstit], [Dstit]$ are defined in $\mathcal{L}$.
— a family of auxiliary operators $\Diamond_{f_1}, \ldots, \Diamond_{f_s}$ with duals $\Box_{f_1}, \ldots, \Box_{f_s}$. These have a role in the definition of the 'fictionality' operators $M_{f_1}, \ldots, M_{f_s}$.

Stit logics are traditionally based upon so-called Branching-Time (BT) frames (see Belnap et al. 2001), while the semantics of $\mathcal{L}$ is defined on **OTF** frames, which are Ockhamist frames (see Zanardo 1996). The choice of Ockhamist frames is made here essentially for reasons of mathematical tractability. BT and Ockhamist frames, albeit nonequivalent, are strictly related and, for our purposes, the latter may be taken as fair approximations of the former.

OTF frames are Kripke frames and are defined, as usual, as n-tuples whose elements are a domain, K, and some (classes of) accessibility relations on K. The elements of K are called *points* (of evaluation). In Ockhamist frames it is possible to define the notions of *history*, *moment*, *instant*, and *choice* we met in the previous section. The definitions of **OTF** models and of the corresponding semantic notions are standard. With the exception of 'fictionality' operators, the tense-modal operators of $\mathcal{L}$ are known from the literature, and semantically behave as expected.[8]

Given (i) some idealizations that are common in the formal study of content, (ii) an important restriction, and (iii) three background assumptions, these limited conceptual resources enable us to express a variety of theses concerning fictional truth, fictional settings, their modal properties, and their relationships with time and agency. Let us consider points (i)–(iii) in turn.

(i) The proposal is idealized in that it assumes that at each point any setting f has a single, well-determined content, and that such content reflects the common intuitions of clever and well-informed speakers on the fictional truths of f. As a consequence, we shall ignore such issues as ambiguity, obscurity, vagueness, and other potential sources of disagreement about fictional settings and their content among clever and well-informed speakers. Moreover, we shall assume that certain well-known problems involving fictional names (see, e.g., Kripke 1980, pp. 157–158) have been dealt with conveniently and possibly without invoking ad hoc, fictional entities.

[7] This informal reading is only adequate if $\Diamond_\sigma$ is not in the scope of a tense operator.

[8] See below, § 3.3. For the convenience of the reader, here is an informal characterization of the truth-conditions of sentences of $\mathcal{L}$ involving these operators (where 'true' means true in an **OTF** model):
– (a sentence of form) $P\mathcal{A}$ ($F\mathcal{A}$) is true at m/h iff $\mathcal{A}$ is true at some point m'/h, where $m' \prec m$ ($m' \succ m$).
– $\Diamond\mathcal{A}$ is true at m/h iff $\mathcal{A}$ is true at some point m/h'.
– $\Diamond_\sigma\mathcal{A}$ is true at m/h iff $\mathcal{A}$ is true at some point m'/h' in the same instant as m/h.
– $[\alpha_k\, dstit]\mathcal{A}$ ($[Dstit]\mathcal{A}$) is true at m/h iff (a) $\mathcal{A}$ is true at all points m/h' such that h' is in the same choice of α_k (of some agent) as h, and (b) $\mathcal{A}$ is false at some point m/h''.

(ii) The proposal is restricted in that it only applies to *possible* fiction, that is, to fictional settings endowed with possibly true content. The rationale behind this (simplifying) restriction is that the problems raised by impossible fiction, interesting as they may be on general grounds, are not immediately and specifically relevant to our discussion.[9]

(iii) According to our first background assumption, a fictional setting f has some fictional truth at a point only if f exists (or, if f is an event, a performance or the like, if f has occurred or is occurring; we shall skip this specification in what follows) at that point. For instance, if the 1929 Disney's movie *The Opry House* had not (yet) been produced, no proposition would be a fictional truth of *The Opry House*. We shall call this assumption *No fictional truth without fictional setting* (**Ntws**). Besides being independently plausible, **Ntws** is widely presupposed in our common discourse and reasoning about fiction and fiction-making. The second background assumption is the converse of **Ntws**. We shall call it *No fictional setting without fictional truth* (**Nswt**). **Nswt** says that having some fictional truth is essential to a fictional setting: no fictional setting may exist at a point and have no fictional truth at that point (see Ross 1997, p. 21 for a discussion). The third assumption introduces a strict connection between fiction-making and content. Let us indicate as a *fictional change* any change in the content of a setting (which may also consist of the creation of a setting). Now we may call the third assumption *No fiction-making without fictional change* (**Nfwc**). **Nfwc** says that no fiction-making performance occurs during a certain interval of time with respect to a setting f if the content of f remains the same throughout that interval. In other words, fiction-making as we shall understand it—that is, in accordance with the content-choice conception—is bound to produce some fictional change. Suppose, for instance, that you make a very small change in a literary work f (say you erase a comma), which does not affect either the identity or the content of f. In that case, your action does not count as a fiction-making performance according to **Nfwc**.

If no proposition is a fictional truth of f at a point m/h, we shall say that f has a *vacuos* content at m/h. A vacuous content may be safely identified with the empty set. By (i) and **Nswt**, at each point, each existing fictional setting corresponds to a well-determined, non-vacuous content. By restriction (ii), we may identify that content with a class of histories (intuitively, those compatible with all the fictional truths of the setting). As a consequence, fictional truth satisfies the following principle of closure:

Closure If (the proposition expressed by) $\mathcal{A}$ is a fictional truth of f at m/h and $\mathcal{B}$ is a logical consequence of $\mathcal{A}$, then $\mathcal{B}$ is a fictional truth of f at m/h.

The assumptions in (iii) are not themselves expressible in $\mathcal{L}$, but they allow the expression of a variety of claims about fictional settings and fiction-making. For instance, by **Ntws**, if a tautology $\top$ is a fictional truth of f, then f exists; conversely,

[9] Admittedly, the precise extent of this restriction is not entirely uncontroversial, for different philosophers may disagree on what fictional settings are possible in this sense. However, I would regard as possible most fictional works that are logically consistent and entail neither blatant metaphysical impossibilities nor disputable views about time, identity, the nature of things, and the like.

if a fictional setting f exists, then, by **Nswt**, some propositions are fictional truths of f, and by **Closure**, these include all tautologies. Then the formula $M_f\top$ (along with $\Diamond_f\top$, to which we shall return in a few lines) of $\mathcal{L}$ is true iff f exists, and can be thus regarded as a formal counterpart of "f exists."

According to **Ntws**, no proposition is a fictional truth of a setting f when f does not exist. On the other hand, the content-choice conception strongly suggests that f, like any other fictional setting, comes into being in time. Thus, there is some point at which no proposition, not even a tautology $\top$, is a fictional truth of f. To express this consequence using our 'fictionality' operator, we may say that there are points at which $\neg M_f\top$ is true. It is straightforward to conclude that M_f is not a normal modal box. Since there are reasons to take 'fictionality' operators as corresponding to universal quantifications from a semantic viewpoint (see, e.g., Lewis 1978, p. 39), we seem to have a problem. Luckily, this is a problem that admits a simple and very common solution. We may define an operator M_f with the proper behavior by means of normal modal operators. The expressions of $\mathcal{L}$ that play this role are the modal diamond $\Diamond_f$ (read: "it is compatible with the content of existing setting f that") and its dual $\Box_f$. Now $M_f\mathcal{A}$ may be defined as $\Diamond_f\top \wedge \Box_f\mathcal{A}$ (see Def. 2 below). As a result, $M_f\top$ is false when f does not exist, as desired.

TFM presupposes the general framework introduced thus far. In addition, it includes as axiom schemata four *basic postulates* concerning fictional truth and its relationships with time and agency (see below, § 3.4.2). These postulates are assumed to hold universally, for all fictional settings f (we may be interested in), relative to all moments, instants, and histories. Informally, they may be expressed as follows:

No Temporal Shift (**Nts**) Operator M_f does not shift the instant parameter.
Origin There is some moment before which f never existed.
No Endless Fiction-Making (**Nefm**) There is a moment after which no fiction-making performance ever occurs with respect to f.
Creation If f exists, then some agent deliberately saw to it that f would exist.

Standard epistemic operators do not shift the temporal parameter, and postulate **Nts** ensures us, plausibly enough, that 'fictionality' operators are not exceptional in this respect.[10]

Some philosophers, including most advocates of the branching-time conception, think of a tree as, quite literally, a representation of our physical universe, along with its possible causal developments. As a consequence, they are led to require that all

[10] It might appear that **Nts** is at odds with certain linguistic practices. In particular, we often use present-tense metafictional sentences to report fictional works set in the past or in the future. For instance, we can say:

(*) In *Julius Caesar*, Brutus commits suicide.

even though, in *Julius Caesar*, it is clearly false that Brutus's suicide occurs *now*. However, these uses may be explained by assuming that, in sentences like (*), the present tense is temporally idle ('eternal' or 'historical') and so without postulating genuine temporal shifts. Whether there are 'fictionality' operators that are prone to induce temporal shifts is a delicate issue, which we cannot tackle here (see Predelli 2008 for a discussion; see also Voltolini 2006 and references therein on the interplay between fictional discourse and contextual shifts). Be that as it may, **Nts** entails that *our* 'fictionality' operators are not among them.

histories in the tree are consistent with the laws of physics and are causally intertwined with one another. But then, by postulate **Nts**, all fictional works are bound to depict only states or events that are physically possible in a very strong sense: they must be among the possible causal developments of previous states or events of our physical universe. Arguably, this is too strong a constraint on the creativity of fiction-makers.[11]

Notice that this problem only arises if the tree induced by an **OTF** frame is assumed to literally represent our physical universe. This assumption is not forced upon us.[12] For those philosophers who are not prepared to drop it, however, my advice is to abandon **TFM** in favor of a weaker theory, say **TFM**$^{-}$, which does not encode the above *Tree* condition (p. 6). Intuitively, the underlying frames of **TFM**$^{-}$, say **OTF**$^{-}$ frames, may involve a *plurality* of tree-like structures. One of these 'trees' can be thought to represent our physical universe, the other ones, alternative universes, causally disconnected from the former and, possibly, nomologically inconsistent with it. The content of a fictional work is still represented as a class of histories in **TFM**$^{-}$, but these histories may be drawn from different 'trees'. As a result, one may represent our universe as a tree-like structure entirely made up of physically possible histories and, at the same time, deal with works that depict physical impossibilities. With the exception of few, mostly formal remarks, all I say about **TFM** and **OTF** frames in this paper may be restated with reference to **TFM**$^{-}$ and **OTF**$^{-}$ frames.[13]

Postulate **Origin** entails that fictional settings come into being in time (recall that, in our framework, time extends indefinitely in both directions). **Nefm** says, intuitively, that no fictional setting is eternally in progress. It is clearly empirical in nature and, *qua* empirical postulate, hardly controversial. By assumption **Nfwc**, we can say that **Nefm** holds if, for any setting f, there exists a moment after which the (possibly vacuous) content of f remains constant. Finally, given how deliberative *stit* is defined (see above, p. 7, and below, Defs. 4–5), **Creation** requires that, for any setting f, there is some moment at which, intuitively, (i) f will exist as a matter of a choice of some agent and (ii) it is not inevitable that f will exist. Among other things, **Creation** entails that all fictional settings are contingent entities: for any f, there is some history on which f never exists.

To express **Creation** in $\mathcal{L}$, the previously mentioned 'existential' *stit* operator [*Stit*] is needed. The reason that [*Stit*] is introduced as a primitive operator and not defined disjunctively in terms of $[\alpha_1\, stit], \dots, [\alpha_t\, stit]$ has to do with fiction-making *within* fiction. In *Hamlet*, for instance, we are told about a (non-actual) play entitled *The Murder of Gonzago*, but nothing is said about its author. **Creation** entails that, in *Hamlet*, some agent made up the play, but not that either α_1 or ... or α_t did—unless, of course, the number of possible agents is less than or equal to t.

Postulates **Origin** and **Creation** may be philosophically controversial. To begin with, both are false if some fictional setting is a Platonic entity. In addition, **Creation** might fail in some special circumstances, if, for instance, a work of fiction is allowed

[11] I am grateful to an anonymous referee for having brought this problem to my attention.

[12] Recall that here trees are taken to extend indefinitely toward both the past and a future (see above, p. 6), a feature that makes them at least questionable as representations of the physical universe.

[13] **TFM**$^{-}$ and **OTF**$^{-}$ frames are formally characterized below, in note 19.

to come into existence by chance or by a non-agentive process.[14] If you admit the existence of Platonic settings, the possibility of uncreated works, or the like, you may understand the role of these postulates as methodological in character. Since we are mainly interested in fiction-making, we are entitled to restrict our attention to fictional settings that are genuinely *made*, that is, deliberately brought into existence. Moreover, even if false for some (possibly very peculiar) choice of f, **Origin** and **Creation** would still retain their role as part of our ordinary conception of fiction and fiction-making.

We are now in a position to provide adequate truth-conditions for sentences of the form $M_{f_i}\mathcal{A}$. Namely, and informally, $M_{f_i}\mathcal{A}$ is true at point m/h iff (a) the content C of f_i at m/h is nonvacuous, and (b) $\mathcal{A}$ is true at any point m'/h' such that h' is in C and m' is in the same instant as m.[15]

In addition to **Nts**, **Origin**, **Nefm** and **Creation**, in our framework it is possible to impose further constraints on the relationships between fictional truth, time, and agency, which can be made to correspond to different sorts, or to different philosophical conceptions, of fictional settings and fictional truth. Outlining some of these constraints may be useful in gaining a better grasp of the proposal. (Below, in § 3.4.3, we shall see that the following constraints can be made to correspond to schemata of canonical formulae of $\mathcal{L}$, and so to axioms of suitable extensions of **TFM**; for the time being, we shall limit ourselves to an informal discussion.)

Fictional settings may be classified depending on whether their existence depends, at least sometimes, on future states of affairs—in other words, on whether there is any 'trace of futurity' in their existence conditions. To this goal, let us say that a setting is *objectual* if, whenever it exists (occurs), it is historically necessary that it exists (occurs) and *non-objectual* otherwise. Arguably, all object-like fictional settings such as novels or paintings (as opposed, for instance, to event-like settings such as theatrical or narrative performances) are objectual in this sense. On the other hand, if events are modally fragile (could not have been significantly different from what they actually are), as is plausible, then all or nearly all event-like fictional settings are non-objectual. To see why, consider a narrative performance f; arguably, there is always an initial stage at which it is still unsettled whether what actually takes place is f or some distinct performance that shares with f an initial temporal part.

Fictional settings may also be classified depending on what fictional changes they (can) undergo. Ideally, fictional changes come in a variety of sorts. Some are endowed with existential import, namely the *creation* and the *annihilation* of fictional settings; others are less dramatic in character. Such are, for instance, the changes by which a setting *acquires* or *loses* some fictional truths without either entering or passing out of existence. It is an interesting philosophical question whether all these fictional changes are—or at least can be—exemplified. With the exception of creations that,

[14] Walton (1990, p. 87) countenances the possibility that a story comes into existence as a result of purely natural, non-agentive processes; Currie (2010, § 1.4) disagrees. Anyway, we may easily imagine a mischievous author that, in order to falsify **Creation**, adopts a peculiar, 'random' method of composition that prevents the resulting work from being counted as a deliberate creation.

[15] Those readers who are especially at ease with formal definitions and semantic clauses might want to have a look below, to §§ 3.1–3.3, before reading the remainder of this section.

by **Origin**, must occur at least once in the life of any existing setting, **TFM** is neutral on this stance. For all that has been said so far, however, the most natural answer to the question is in the affirmative.

Be that as it may, we may recognize, among other things, a class of fictional settings that after their creation never acquire any novel fictional truth (call them *bounded*) and a class of fictional settings that never lose any fictional truth (*conservative*). We may also have settings that are both bounded and conservative, that is, that after their creation never undergo any fictional change (*temporally frozen*). Many philosophers think that literary works and other object-like settings are temporally frozen in this sense: make a change in the content of, say, a novel, and what you obtain is just another, numerically distinct work. It is worth noting that temporally frozen settings need not be *modally* frozen; that is, they need not have the same content at all points at which they exist.

Fictional settings may also be classified depending on their relationships with agency or the mode of their creation. Two examples should suffice. First, we shall say that a setting f is *fully controlled* if, for each fictional truth of f, some agent deliberately sees to it that it would be a fictional truth of f. Insofar as fictional truths need not be part of the explicit content of a setting (see above, p. 4), it is far from obvious that any ordinary work of fiction is fully controlled in this sense. (We shall return to fully controlled settings in the next section.) Second, we shall call a fictional setting *linear* if, after its creation, it is inevitable that it undergoes the fictional changes it actually undergoes. In slightly more formal terms, when f is linear, if f exists and it is historically possible that $\mathcal{A}$ and $\mathcal{B}$ will be, at successive moments, fictional truths of f, then it will be the case that, if f exists, $\mathcal{A}$ and $\mathcal{B}$ are fictional truths of f at successive moments. Frozen settings are trivially linear. But there are non-frozen settings, such as theatrical and narrative performances, that one might want to classify as linear settings. This is especially so if one regards the order in which certain fictional events are represented during a performance as essential to that performance (for instance, if one thinks that a *Hamlet* performance in which the second act precedes the first is essentially distinct from any conventionally ordered performance).

1.4 A short summary of the proposal

Before discussing some of the possible applications of the proposal outlined in the previous section, it may be useful to summarize the proposal and to address a general concern one might have about it.

A certain conception of fiction-making lies at the heart of the proposal, which I have called the content-choice conception. In the content-choice conception, a fiction-making performance is understood as a sequence of human actions that occurs during a finite interval of time and results in certain fictional changes. A fictional change is a change in the content of a work of fiction, a make-believe game, or, more generally, a fictional setting. There may be different types of fictional change. For instance, a fictional change may simply consist of the creation of a setting. Typically, an agent introduces a fictional change by deliberately seeing to it that some proposition is a fictional truth of some settings. All fictional settings come into existence (or take place) by virtue of deliberate actions. (However, it is not generally required that all fictional changes occur by virtue of deliberate actions.) In the content-choice conception, a fiction-making act is represented just in terms of the consequences it has on the content of some setting, that is, in terms of the fictional changes it results in. Given a certain level of idealization and some reasonable assumptions, it is possible to specify a formal framework in which fiction-making can be formally studied and to define an axiomatic theory, **TFM**, that encodes what is essential to the content-choice conception.

The above-mentioned general concern is this: In a conception that is traditionally associated with *stit*-logics, an action is thought of as the contribution of an agent to a change in the causal structure of the world; but is there not a contrast between this traditional conception and the content-choice conception, in which actions may also result in *fictional* changes? Should we regard fiction-making acts as strange, *sui generis* actions?

I think that the answer to these two questions is a qualified no. First, recall that a fictional change is a change in the content of some fictional setting. It is a consequence of very common views in the philosophy of language and mind that changes in content (globally) supervene on changes in the causal structure of the world. If so, then the content-choice conception, far from being incompatible with the traditional conception, actually entails it. Second, to adopt the content-choice conception we do not need to regard fiction-making acts as actions of an especially strange or peculiar sort. Think of speech acts, for instance. A promise, say, can also be understood as an action that may result in a change in content, that is, in the content of an agent's moral obligations (see Belnap et al. 2001, pp. 98–129 for a *stit* analysis of promising that goes in this direction).

This is not to deny that the role of content in an overall causal picture of the world is controversial and not well-understood. But this is an extremely general problem, which affects a variety of positions in the philosophy of language and mind, and by no means specific to the content-choice conception.

2 Applications, and beyond

The formal framework outlined thus far has some obvious applications in the logic of fiction, make-believe, and fiction-making. For instance, it is especially suited for modeling reasonings that are partly about the content of a setting and partly about the agents that operate over or in that setting. These include, for instance, inferences that lead from internal sentences to conclusions about a certain author's mind or attitude and vice versa. As another example, we may have reasonings that concern make-believe games and their principles of generation—including the ones Mark Richard (2000) would classify as *piggy backing*. The principles of generation of a setting and their inferential role may be modeled as well, for instance by letting them correspond to (non-logical) axioms of suitable extensions of **TFM**. Moreover, of course, the proposal is subject to a variety of limitations, the most apparent being those that directly or indirectly depend on the expressive poverty of $\mathcal{L}$.

There also are less obvious applications and limitations, though. This section provides a guided tour through some of these. It is not just a complement to the proposal, however, and is understood to have independent interest for all those concerned with the language, logic, and metaphysics of fiction-making.

2.1 Modeling philosophical debates about fiction

In the framework outlined in the previous section, it is possible to formulate different philosophical conceptions of fictional settings and fiction-making and to formally check their consequences, their mutual consistency, and so on.

For example, according to Amie Thomasson (1999, pp. 9–10), some or maybe even all existing works of fiction (object-like settings) could pass out of existence. This view may be expressed in $\mathcal{L}$ as a (canonical) schema, and the same goes for the contrary stance that all existing works of fiction will last forever. As another example, suppose you maintain, disputably, that object-like fictional settings are modally frozen. In **TFM**, we may prove that, if a setting is modally frozen, then it is also fully controlled. Thus, if your view is correct, we may conclude, somewhat surprisingly, that all object-like fictional settings are fully controlled. Finally, consider the view that fiction-making consists, at least typically, in deliberately changing the content of a setting f in a stepwise manner over a certain (non-null) amount of time. We might call this a *content-modeling* conception of fiction-making. (The content-modeling conception entails the content-choice conception, but is not to be confused with it.) In our framework, it is easy to show that, if a content-modeling conception is correct for some setting f, then f is non-linear.

It is interesting to note that, due to the generality of the notion of a fictional setting, a content-modeling conception of fiction-making may be correct even if all ordinary, object-like works are linear. Let us see why. Consider an object-like work of fiction, say Jane Austen's *Persuasion*, and the following sentence:

(5) In mid-1816, it was still unsettled what the final chapters of *Persuasion* would be like.

We may all agree that (5), suitably understood, is true. A way of accounting for the truth of (5) is simply to admit that the very novel, *Persuasion*, is non-linear. We are not *forced* to this view, however, for there are at least two alternative possibilities. We might either adopt a counterpart-theoretic approach or argue that, in (5), (more or less implicit) reference is made to whatever possible novel shares with *Persuasion* a certain initial trait of its history of production. Either way, what turns out to be relevant for the truth of (5) is not just a single novel, *Persuasion*, but, rather, an entity we might describe as a function from points (moment/history pairs) to possible novels. Let us call this entity a *non-linear expansion* of *Persuasion* (if you think *Persuasion* is itself non-linear, you can take it as a non-linear expansion of itself).[16] Intuitively, a non-linear expansion of a novel may be thought of as a tree whose 'branches' correspond to alternative choices of composition. Now, nothing in the notion of a fictional setting prevents us from regarding a non-linear expansion of a novel as a fictional setting itself. As a result, we might consistently adopt a content-modeling conception of novel-making and, at the same time, hold that *Persuasion* is linear. We can do so if we distinguish the result of Austen's novel-making performance, *Persuasion*, from an entity whose contents may be different at different moments and depend, in a stepwise manner, on Austen's choices of composition. The latter entity is, of course, an appropriate non-linear expansion of *Persuasion*.

2.2 Extending the paraphrase thesis

The paraphrase thesis (PT) (see above, § 1.1), suitably understood, is very plausible. However, it has limited scope, for internal sentences are but a small fragment of our common fictional discourse: a number of sentences that concern fictional settings and their content are not meant to express a fictional truth (a standard example, which we shall not deal with here, is "Holmes is a fictional character," but there are many others). Let us call these sentences *non-internal*. Providing an adequate representation of the commitments triggered by non-internal sentences is generally regarded as a major challenge for any theory of fictional discourse (see, e.g., Kroon and Voltolini 2011). If we let our 'fictionality' operator have plausible inferential relationships with *stit* operators and other sentential operators, however, it is possible to extend the paraphrase thesis to analyze a variety of non-internal sentences. Let us see how.

It has been observed that certain statements about fictional characters and events display a characteristic ambiguity, which is naturally understood to be structural in character (see, e.g., Currie 2003). Consider, for instance:

(6) Mickey Mouse will be wearing gloves.

We may take (6) as an internal sentence. But imagine it in Disney's mouth before the production of *The Opry House*, the movie in which Mickey first appeared with gloves on. If so, you are likely to understand (6) as saying, roughly speaking, that the proposition that Mickey wears gloves will be a fictional truth of some future work. This latter, non-internal reading of (6) may be made explicit as:

($6'$) It will be the case that (M(Mickey Mouse wears gloves)).

Intuitively, ($6'$) may be obtained from a natural analysis of (6); that is:

($6''$) It will be the case that Mickey Mouse wears gloves.

[16] This is obviously not a characterization of the notion of a non-linear expansion. I think it possible to provide some such characterization, at least given a certain level of idealization and a number of philosophical decisions, but this would require a richer formal framework than the one introduced here.

by replacing the occurrence of (1) (“Mickey Mouse wears gloves”) with a standard, ‘fictionality’ paraphrase. Furthermore, (1)—as it occurs in (6″)—clearly expresses a fictional truth, so it is reasonable to regard it as internal. These remarks strongly suggest that (PT) may be generalized to apply to occurrences of internal sentences in larger linguistic constructions. The following is an adequate, if rough (see note 5), formulation of this generalized version (where *Op* is a possibly empty sequence of monadic sentential operators and $\mathcal{B}$ is not already in the scope of a ‘fictionality’ operator):

(GPT) If $\mathcal{A} \rightsquigarrow Op(\mathcal{B})$ and $\mathcal{B}$ is an internal sentence, then $\mathcal{A} \rightsquigarrow Op(M(\mathcal{B}))$. In certain circumstances, when it is clear that $\mathcal{B}$ is internal to f, $\mathcal{A} \rightsquigarrow Op(M_f(\mathcal{B}))$.

It is apparent that (GPT) enables us to deal with statements to which the paraphrase thesis (PT) does not apply. An example may be, of course, (6). Other, more interesting examples shall be introduced in the next section.

2.3 Control statements and the creation problem

> If in the first act you have hung a pistol on the wall, then in the following one it should be fired. Otherwise don’t put it there.
>
> Anton Chechov

We often say or imply that a certain author *brought about* certain fictional events or *made* her/his characters act or look in such and such a way, or the like. For instance, we may claim that Conan Doyle made Holmes die or, in a somewhat more metaphorical fashion, that:

(7) Conan Doyle killed Holmes.

Statements of this sort might be called *control sentences*, for they seem to ascribe to authors of fiction a (sort of) causal control over certain characters and states of affairs portrayed in their works. Needless to say, control sentences have very puzzling consequences. For this reason, it is very plausible to hold that, when we assert them, we do not commit ourselves to their literal truth. (For instance, nobody would take the ‘killing’ of Holmes to be morally blameworthy.) If so, we have the problem of specifying as clearly as possible what they actually commit us to. This is a problem that, at least for a considerable number of control sentences, receives an immediate and elegant solution in our framework. Let us see how.

Nuances aside, many control sentences may be paraphrased as sentences of the following form (where $\mathcal{A}$ concerns purely fictional characters or states of affairs and is internal to some work authored by α):

(NF) α sees (saw, will see) to it that it will (would) be the case that $\mathcal{A}$.

If a statement is thus paraphrasable, we shall say that it is a *normal* control sentence. Now, the Generalized Paraphrase Thesis (GPT) enables us to paraphrase claims of form (NF) as:

α sees (saw, will see) to it that it will (would) be *fictionally* the case that $\mathcal{A}$.

As an example, consider again (7). Arguably, nothing of importance is lost if we paraphrase (7) as:

(7′) Conan Doyle saw to it that it would be the case that Holmes died.

Since "Holmes died" is apparently internal to a work by Conan Doyle, we may conclude that (7) is a normal control sentence. If so, it is usefully paraphrased as:

(7″) Conan Doyle saw to it that it would be fictionally the case that Holmes died.

The same strategy obviously applies to any other normal control sentence we may choose. It is worth noting that a paraphrase like (7″) is only sensible if authors have control over what is fictionally true; that is, if the relationships between agency and fictional truth are in accordance with the content-choice conception.

Now consider:

(8) Sherlock Holmes was created by Conan Doyle.

Let us label intuitively true statements like (8), in which an author α is said to create or bring into existence a fictional character of some work authored by α, as *creation sentences*. Some philosophers, so-called *creationists* about fictional characters, think that creation sentences are literally true (see, e.g., van Inwagen 1977; Salmon 1998; Thomasson 1999; Kripke 2011). Other philosophers disagree. A common view among the latter is *fictionalism* about creation sentences (see, e.g., Walton 1990, Brock 2002). Fictionalists hold that creation sentences are on a par with ordinary internal sentences like (2) ("Holmes met Watson in a lab"), with a single difference. Namely, creation sentences are not internal to standard fictional works but to more exotic fictional settings. For instance, according to Walton (1990, p. 410–411), creation statements are internal to "unofficial" make-believe games in which "to author a fiction about people and things of certain kinds is fictionally to create such."

Both creationism and fictionalism are questionable. The former relies on objectionable metaphysical assumptions (see, e.g., Yagisawa 2001; Everett 2005; Brock 2010); the latter is dubiously explicative (for instance, Thomasson (2003) observes that, by appeal to ad hoc make-believe games we could, in principle, 'explain away' any commonsense intuition we pleased; see also Stanley 2001 and, for a reply, Everett 2013, pp. 103–108). If the above treatment of control sentences goes in the right direction, however, a third, alternative approach to creation sentences immediately suggests itself. Clearly, (8) may be paraphrased as:

(8′) Conan Doyle saw to it that it would be the case that Holmes existed.

Since "Holmes existed" is naturally regarded as an internal statement—and internal to Conan Doyle's works—it is equally natural to understand (8) as a normal control sentence. If so, (8) is amenable to the same treatment as (7) (on the analogy between creation statements and statements like (7) see Kroon 2011, pp. 219–221 and Everett 2013, pp. 58–60).[17] This style of analysis involves no questionable metaphysical assumptions, or at least no more than fictionalism itself does, and deals uniformly—and

[17] It might be objected that Agilulf, the *nonexistent knight* of the eponym novel by Italo Calvino, provides a counter-example to this analysis. The reason is that it appears that Agilulf has been created by Calvino

without invoking ad hoc maneuvers—with a very large array of puzzling statements, over and above creation sentences. It is not clear whether any competing approach achieves as much.

2.4 Limits and prospects for future work

The present proposal is a formal study on human agency and fictional truth. As such, it imposes relatively mild constraints on fiction-making and it should not be regarded as an alternative to more substantial views on the nature of fictional works and related performances. Rather, it is thought to be compatible with (and possibly complementary to) very different takes on these matters.[18] A discussion of how precisely the proposal interacts with such different views would considerably clarify its limits and ambitions. However, this is a discussion that has to be left to another occasion. Here, three cursory remarks should suffice.

First, the content-choice conception shares an individualistic stance that is characteristic of multi-agent *stit* logics (and to a lesser extent of group *stit* logics, too). For instance, it abstracts away from those social and cultural facts, broadly conceived, that may affect the choices of an author and, therefore, the properties of the work. Undoubtedly, this is an idealization. But one may also regard it as a considerable limitation, especially if one's favorite conception of fiction, or one's favorite ontology of artworks, assigns a key role to such social and cultural backgrounds.

Second, as previously mentioned, standard *stit* logics assign no role to *intentions*, and **TFM** is no exception to the rule. Thus, in the picture of fiction-making I have drawn so far, we cannot make sense of an author intending to produce a fictional change (of course, unless the author succeeds in doing so). This restriction depends, ultimately, on limitations of the underlying *stit* framework, and there is no principled reason to suppose it cannot be overcome (see, e.g., Broersen 2011). But intentions come with considerable complications, and whether the price is worth the effort may depend, again, on one's overall conception of fictional or representational artworks.

Finally, what I have called fictional settings constitute a variegate plurality, which may include artifacts, performances, but also communicative contexts and systems of rules. Arguably, even those stories (myths, daydreams, and so on) we are told about *within* fictional works correspond to (actual) fictional settings. This variety may be startling from an ontological viewpoint, and one might feel that something substantial remains to be said on the nature of fictional settings, their identity, and their relationship with 'ordinary' fictional works. Actually, I think that such relative indeterminacy

even though in the story he is nonexistent. This objection has some initial plausibility, but it does not stand close scrutiny. Like any other character of a fictional work, in the work Agilulf is an agent, and as such has a number of existence-entailing properties (*being sentient*, *having causal efficacy*, and so on). Thus either the story is inconsistent, and Agilulf both exists and does not exist therein, or "nonexistent" is to be understood in a somewhat idiosyncratic way, for instance as a synonym of "immaterial." Either way, the counter-example is blocked.

[18] This is not to say that it is completely neutral, however. Among other things, it presupposes an indeterministic and possibilist framework that may or may not suit one's metaphysical tastes. (Albeit, of course, nothing forbids adopting an instrumentalist or fictionalist understanding of this presupposition.)

of the proposal *adds* to it in terms of generality, but I am happy to recognize that there is more work to be done in this area.

3 The Theory TFM

This section is mainly devoted to a formal definition of the theory **TFM** and to a proof of completeness for it. Informal comments are kept to a minimum, for the theory has been already outlined in Section 1. As stated above (p. 8), the semantics of **TFM** is based upon **OTF** frames, which are standard Kripke frames. Moments, histories, choices, and contents (of fictional settings) are defined in terms of points and accessibility relations between points. Frame properties are always expressed with reference to points and relations between them, but in many cases, a (frame-equivalent) reformulation in terms of moments, histories, instants, choices, or contents is provided for the sake of clarity and intuitiveness.[19]

3.1 Syntax

Definition 1 The language $\mathcal{L}$ of **TFM** may be formally specified as:

$$p \,|\, \neg\mathcal{A} \,|\, \mathcal{A}\wedge\mathcal{B} \,|\, P\mathcal{A} \,|\, F\mathcal{A} \,|\, \Diamond\mathcal{A} \,|\, \Diamond_\sigma\mathcal{A} \,|\, \Diamond_{f_i}\mathcal{A} \,|\, [\alpha_k\, stit]\mathcal{A} \,|\, [Stit]\mathcal{A}$$

where p is any propositional letter in $ATOM = \{p_0, p_1, \ldots\}$, $\Diamond_{f_i}$ a monadic operator in $\{\Diamond_{f_1} \ldots \Diamond_{f_s}\}$, and $[\alpha_k\, stit]$ a monadic operator in $\{[\alpha_1\, stit], \ldots, [\alpha_t\, stit]\}$.

Usual definitions hold for $\bot$, $\top$, $\rightarrow$, $\vee$ and the duals H, G, $\Box$, $\Box_\sigma$, $\Box_{f_i}$, $\langle\alpha_k\, stit\rangle$, $\langle Stit\rangle$ of P, F, $\Diamond$, $\Diamond_\sigma$, $\Diamond_{f_i}$, $[\alpha_k\, stit]$, $[Stit]$, respectively. For all $i \in \{1, \ldots, s\}$, the operator of fictional truth M_{f_i} is defined as:

Definition 2 $M_{f_i}\mathcal{A} =_{df} \Diamond_{f_i}\top \wedge \Box_{f_i}\mathcal{A}$

and its 'generic' counterpart M as:

Definition 3 $M\mathcal{A} =_{df} M_{f_1}\mathcal{A} \vee \cdots \vee M_{f_s}\mathcal{A}$

For all $k \in \{1, \ldots, t\}$ the deliberative *stit* operator $[\alpha_k\, dstit]$ is defined as:

Definition 4 $[\alpha_k\, dstit]\mathcal{A} =_{df} [\alpha_k\, stit]\mathcal{A} \wedge \Diamond\neg\mathcal{A}$

and its 'existential' counterpart $[Dstit]$ as:

Definition 5 $[Dstit]\mathcal{A} =_{df} [Stit]\mathcal{A} \wedge \Diamond\neg\mathcal{A}$

[19] The aforementioned theory **TFM**$^-$ (p. 11) and the corresponding **OTF**$^-$ frames are easily obtained from **TFM** and **OTF** frames by dropping the axiom **Tree** and the frame property TREE, respectively (see below, pp. 21–23). It is straightforward, and left to the reader, to extend the proofs of soundness and completeness for **TFM** to **TFM**$^-$.

3.2 Frame Definitions

Definition 6 An **OTF** (Ockhamist) frame is a tuple

$$\mathfrak{F} = (K, <, \approx, \sim, \mathcal{AG} = \{\widehat{\mathrm{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}\}, \mathcal{F} = \{\widehat{f_1}, \ldots, \widehat{f_s}\})$$

such that:

– K is a non-empty set of *points*. We let $x, y, \ldots$ vary over points.
– $<$ is a union of disjoint irreflexive linear orders on K that are unbounded to the left and to the right. We let $>$ be the converse relation of $<$ and $\leqslant$ be the reflexive closure of $<$. Maximal linearly ordered components of $<$ are called *histories*. We let h (possibly with indices) vary over histories.

– $\approx$ and $\sim$ are equivalence relations on K; equivalence classes modulo $\approx$ are called *moments*; equivalence classes modulo $\sim$ are called *instants*. We let m vary over moments. Moreover we have that:

(TREE) If $x \sim y$ then there exist points $x' \leqslant x$ and $y' \leqslant y$ such that $x' \approx y'$. Intuitively, this property corresponds to the above mentioned *Tree* condition (p. 6).
(IRR) If $x \sim y$ then $x \not> y$. Intuitively, no instant intersects any history at more than one point (or moment).
(NIN) If $x \approx y$ then $x \sim y$. Intuitively, if two points are in the same moment, they are in the same instant.
(MB) If $x \neq y$ and $x \approx y$, then there exists points $x' > x$ and $y' > x$ such that $x' \not\approx y'$. Intuitively, distinct histories are bound to branch.

Definition 7 (Instants and points: notation) By IRR and NIN, it immediately follows that, for each m, h, $m \cap h$ has at most one point x as its element. We write m/h only if $m \cap h$ is nonempty, to denote the unique point $x \in m \cap h$.

Definition 8 (Order of moments) We write $m \prec m'$ ("m precedes m'") iff, for some h, $m/h < m'/h$; $m \succ m'$ ("m follows m'") is defined in the obvious way. Once $\prec$ is so defined, histories may be also conceived as maximal $\prec$-connected sets of moments.

Definition 9 If R_1, R_2 are binary relations, $[R_1R_2]$ is a binary relation such that $x[R_1R_2]z$ iff $\exists y(xR_1y \wedge yR_2z)$. Similarly, $x[R_1R_2R_3]z$ iff $\exists yy'(xR_1y \wedge yR_2y' \wedge y'R_3z)$.

(SYNCHRONICITY) If $x[<\sim]y$ then $x[\sim<]z$ and if $x[>\sim]y$ then $x[\sim>]z$. Intuitively, instants preserve the order of moments toward both the past and the future.

– $\mathcal{AG}$ is a set of equivalence relations $\widehat{\mathrm{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}$ on K. Relations $\widehat{\alpha_1}, \ldots, \widehat{\alpha_t}$ correspond, intuitively, to distinct agents. For each $i \in \{1, \ldots, s\}$, $k \in \{1, \ldots, t\}$, and each point x (also indicated as m/h) we have that:

(NSTIT) For all y, if $x \widehat{\alpha_k} y$, then $x \approx y$ (that is, $\widehat{\alpha_k} \subseteq \approx$).

Definition 10 (Choices) The set $Choice^{\alpha_k}_{m/h} = \{h' : m/h \widehat{\alpha_k} m/h'\}$ is called *the choice of α_k at m/h*; the set $Choice^{\mathrm{A}}_{m/h} = \{h' : m/h \widehat{\mathrm{A}} m/h'\}$ is called *the choice of any agent* at m/h; for each h passing through m, each choice of α_k [any agent] at m/h is called *a possible choice of α_k [any agent] at m*. We let $Choice^{\alpha_k}_m$ [$Choice^{\mathrm{A}}_m$] vary over possible choices of α_k [any agent] at m.

(STITEX) For all y, if $x \frown_{\mathrm{A}} y$, then $x \frown_{\alpha_k} y$ (that is, $Choice^{\mathrm{A}}_{m/h} \subseteq Choice^{\alpha_k}_{m/h}$).
(NOCH) If $x \approx x'$ and $x > y$, then $y[\frown_{\mathrm{A}}<]x'$.
(INDAG) If $x \frown_{\alpha_1} y \approx \ldots \approx z \frown_{\alpha_t} w$, then $\exists x'(x' \frown_{\alpha_1} x \wedge \cdots \wedge x' \frown_{\alpha_t} w)$. In other words, for any moment m, any arbitrary sequence including exactly one possible choice $Choice^{\alpha_k}_{m}$ for each $k \in \{1, \ldots, t\}$ has a non-empty intersection.

STITEX is best understood in connection with the corresponding axiom **StitEx** (see below, p. 23). As for NOCH and INDAG, see above (p. 7). It is worth noting that the formal constraint corresponding to the above (p. 6) *No backward branching* condition is entailed by NSTIT, STITEX and NOCH.

–$\mathcal{F}$ is a set of relations $\frown_{f_1}, \ldots, \frown_{f_s}$ on K that, intuitively, correspond to fictional settings. As said above (in § 1.3), we assume that at any point m/h, each fictional setting f_i has a (possibly vacuous) content, which we represent as a (possibly empty) set of histories. The role of $\frown_{f_i}$ is, informally, that of individuating those contents. More precisely, for each $i \in \{1, \ldots, s\}$ and each m/h, we define *the content of f_i at m/h* as the set $\{h' : m/h \frown_{f_i} m'/h'$ for some $m'\}$. To indicate that the content of f_i is nonempty [empty] at a point, we say that f_i exists [does not exist] at that point. For each $k \in \{1, \ldots, t\}$, $i \in \{1, \ldots, s\}$, and each point x (or m/h), we have that:

(NTS) For all y, if $x \frown_{f_i} y$, then $x \sim y$ (that is, $\frown_{f_i} \subseteq \sim$).
(ORIGIN) For some $y < x$, f_i does not exist at any point $z < y$.
(NEFM) For some $y > x$, for all z, we have that (i) if $y[\frown_{f_i} <]z$, then $x[< \frown_{f_i}]z$; and (ii) if $y[< \frown_{f_i}]z$, then $x[< \frown_{f_i} <]z$. Intuitively, for any f_i, there is a future point after which the content of f_i remains constant.
(CREATION) If f_i exists at m/h, then some moment $m' \prec m$ is such that:
(i) for any h' in $Choice^{\mathrm{A}}_{m'/h}$, some $m'' \succ m'$ is such that f_i exists at m''/h', and
(ii) for some h'' passing through m', for any $m''' \succ m'$, f_i does not exist at m'''/h''.
In other words: $\exists y(x \frown_{f_i} y) \rightarrow \exists y < x((\forall z \frown_{\mathrm{A}} y(\exists x'[\frown_{f_i} >]z)) \wedge \exists y' \approx y(\neg \exists z'[\frown_{f_i} >]y'))$.

These properties correspond to the basic postulates outlined above (p. 10). Observe that, by constraint (NTS) and given how the notion of content is defined, we have that, for any $i \in \{1, \ldots, s\}$, relation $\frown_{f_i}$ holds between m/h and a point m'/h' iff (a) h' is in the content of f_i at m/h and (b) m' is in the same instant as m.

3.3 Models and Semantic Clauses

Definition 11 An **OTF** model (based) on an **OTF** frame $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathrm{V} \rangle$ and V is a valuation function such that, for all $p \in ATOM$, $V(p) \subseteq K$, and:

(PROP) if $x \approx y$ then $x \in V(p) \Leftrightarrow y \in V(p)$. This condition ensures that propositional atoms "have no trace of futurity in them" (see above, p. 7).

The semantic clauses are as follows ($\mathfrak{M}, x \vDash \mathcal{A}$ reads "$\mathcal{A}$ is true in $\mathfrak{M}$ at x"):

$\mathfrak{M}, x \vDash p$ iff $x \in V(p)$.
$\mathfrak{M}, x \vDash \neg\mathcal{A}$ iff $\mathfrak{M}, x \nvDash \mathcal{A}$.

$\mathfrak{M},x \models \mathcal{A} \wedge \mathcal{B}$ iff $\mathfrak{M},x \models \mathcal{A}$ and $\mathfrak{M},x \models \mathcal{B}$.
$\mathfrak{M},x \models P\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for some y such that $y < x$.
$\mathfrak{M},x \models F\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for some y such that $y > x$.
$\mathfrak{M},x \models \Diamond\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for some y such that $x \approx y$.
$\mathfrak{M},x \models \Diamond_\sigma\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for some y such that $x \sim y$.
for $k \in \{1,\dots,t\}$, $\mathfrak{M},x \models [\alpha_k\, stit]\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for all y such that $x \widehat{\alpha_k}\, y$.
$\mathfrak{M},x \models [Stit]\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for all y such that $x \widehat{\mathrm{A}}\, y$.
for $i \in \{1,\dots,s\}$, $\mathfrak{M},x \models \Diamond_{f_i}\mathcal{A}$ iff $\mathfrak{M},y \models \mathcal{A}$ for some y such that $x \widehat{f_i}\, y$.

3.4 Calculus

3.4.1 Rules

PC $\vdash \mathcal{A}$ if $\mathcal{A}$ is a tautology.
MP From $\mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{B}$ infer $\mathcal{B}$.
Gen If $\vdash \mathcal{A}$ then $\vdash H\mathcal{A}$, $\vdash G\mathcal{A}$, and $\vdash \Box_\sigma\mathcal{A}$.
Irr If $\vdash (\Box_\sigma p \wedge \Box_\sigma H\neg p) \rightarrow \mathcal{A}$, then $\vdash \mathcal{A}$, provided $p \in ATOM$ and does not occur in $\mathcal{A}$.

Irr is a Gabbay-style (1981) irreflexivity rule and corresponds to property IRR.

3.4.2 Axioms

For any $k \in \{1,\dots,t\}, i \in \{1,\dots,s\}$, we have the following axiom schemata:

1. The usual axiom schemata of tense logic with serial linear time orderings:
   $G(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (G\mathcal{A} \rightarrow G\mathcal{B})$, $\mathcal{A} \rightarrow HF\mathcal{A}$, $G\mathcal{A} \rightarrow GG\mathcal{A}$,
   $F\mathcal{A} \rightarrow G(F\mathcal{A} \vee \mathcal{A} \vee P\mathcal{A})$, $G\mathcal{A} \rightarrow F\mathcal{A}$
   and their mirror images.
2. S5-axioms for $\Box$, $\Box_\sigma$, $[Stit]$, and $[\alpha_k\, stit]$.
   K $\Box_{f_i}(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box_{f_i}\mathcal{A} \rightarrow \Box_{f_i}\mathcal{B})$
3. **Prop** $p \rightarrow \Box p$ for all $p \in ATOM$.
   Synchronicity $(P\Box_\sigma\mathcal{A} \rightarrow \Box_\sigma P\mathcal{A}) \wedge (F\Box_\sigma\mathcal{A} \rightarrow \Box_\sigma F\mathcal{A})$
   Nin $\Box_\sigma\mathcal{A} \rightarrow \Box\mathcal{A}$
   Tree $\Diamond_\sigma H\mathcal{A} \rightarrow P\Diamond\mathcal{A}$
4. **Nstit** $\Box\mathcal{A} \rightarrow [\alpha_k\, stit]\mathcal{A}$
   StitEx $[\alpha_k\, stit]\mathcal{A} \rightarrow [Stit]\mathcal{A}$
   Indag $\bigwedge_{1\leq k\leq t} (\Diamond[\alpha_k\, stit]\mathcal{A}_k) \rightarrow \Diamond(\bigwedge_{1\leq k\leq t} [\alpha_k\, stit]\mathcal{A}_k)$
   Noch $P[Stit]\mathcal{A} \rightarrow \Box P\mathcal{A}$
5. **Nts** $\Box_\sigma\mathcal{A} \rightarrow \Box_{f_i}\mathcal{A}$
   Origin $PH\neg\Diamond_{f_i}\top$
   Nefm $G\Box_{f_i}G\mathcal{A} \rightarrow (F\Box_{f_i}G\mathcal{A} \wedge FG\Box_{f_i}\mathcal{A})$
   Creation $\Diamond_{f_i}\top \rightarrow P[Dstit]F\Diamond_{f_i}\top$

No independency claim attaches to this axiomatization. Groups 1–2 need no presentation. As for the other groups, each axiom provided with a label is canonical for (a property equivalent to) the homonymous frame property, if any. So, for instance, **Origin** is canonical for the above frame property ORIGIN. **Prop** is an exception in that it corresponds to a condition on models (PROP). All the axioms in group 4 except **StitEx** are familiar from standard *stit* logic. **StitEx** is strictly tied to the intended reading of a sentence $[Stit]\mathcal{A}$, that is, "$[\alpha_k\, stit]\mathcal{A}$ for some $k \in \{1,\ldots,t\}$." The axioms in group 5 are nothing but (the formal counterparts of) the four basic postulates introduced in § 1.3 (p. 10).

3.4.3 Additional constraints

In § 1.3 we have discussed some additional constraints that can be imposed on a setting f to ensure that f satisfies certain properties (*being objectual*, *being bounded*, and so on). Each of these constraints corresponds to a schema of canonical formulae of $\mathcal{L}$ (and so also to a frame property), namely:

Objectuality $\Diamond_f \top \rightarrow \Box\Diamond_f \top$
Non-objectuality $\Diamond_\sigma(\Diamond_f \top \wedge \neg\Box\Diamond_f \top) \vee P\Diamond_\sigma(\Diamond_f \top \wedge \neg\Box\Diamond_f \top) \vee F\Diamond_\sigma(\Diamond_f \top \wedge \neg\Box\Diamond_f \top)$
Boundedness $\Diamond_f G\mathcal{A} \rightarrow G\Diamond_f \mathcal{A}$
Conservativity $M_f G\mathcal{A} \rightarrow G\Box_f \mathcal{A}$
Modal Frozenness f is bounded, conservative, and $M_f\mathcal{A} \rightarrow \Box_\sigma M_f\mathcal{A}$.
Full Control $M_f\mathcal{A} \rightarrow P[Stit]FM_f\mathcal{A}$
Linearity $(\Diamond_f \top \wedge \Diamond F(M_f\mathcal{A} \wedge FM_f\mathcal{B})) \rightarrow F(M_f\mathcal{A} \wedge FM_f\mathcal{B})$

3.5 Completeness

Throughout this section, completeness means completeness with respect to the **OTF** models. The proof that **TFM** is sound for **OTF**-validity is easy and is left to the reader. Aim of this section is to prove a completeness theorem for **TFM**. The proof is less than straightforward, as some complications are required to deal with the irreflexivity rule **Irr**. The strategy of proof is conceptually similar to that in (Zanardo 1991), but the terminology is mainly drawn from the classic (Blackburn et al. 2001, see esp. § 4.6). At its core, the proof consists of a step-by-step construction, by which an **OTF** model is shown to exist for each consistent formula of $\mathcal{L}$. Before the construction may begin, however, there is some preliminary work to be done.

3.5.1 Preliminary Definitions and Results

Let us suppose that the relation symbols $<, >, \approx, \sim, \widehat{\mathrm{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}, \widehat{f_1}, \ldots, \widehat{f_s}$ have been rewritten as $R_1,\ldots,R_{(s+t+5)}$. Let R_j be such that $1 \leqslant j \leqslant (t+s+5)$ and let $\langle j\rangle$ ($[j]$) indicate the diamond (box) of $\mathcal{L}$ having R_j as its accessibility relation. Let $\Delta, \Gamma, \Sigma, \ldots$ range over maximal consistent sets of formulas (MCSs). For each R_j, we define the following relation on the set of MCSs:

Definition 12 $\Delta\mathbf{R_j}\Gamma$ iff $[j]\mathcal{A} \in \Delta$ entails $\mathcal{A} \in \Gamma$.

We shall say that $\mathbf{R_j}$ is the *canonical correlate* of R_j ($\mathbf{<}, \mathbf{>}, \boldsymbol{\approx}, \ldots$, denote the canonical correlate of $<, >, \approx, \ldots$, respectively). It is straightforward to show that, for each $\mathbf{R_j}$:

Proposition 1 *$\Delta \mathbf{R_j} \Gamma$ iff $\mathcal{A} \in \Gamma$ entails $\langle j \rangle \mathcal{A} \in \Delta$.*

Furthermore, since **TFM** is a normal modal logic, we know by standard results (see, e.g., Blackburn et al. 2001, pp. 200–203) that the following *Existence lemma* holds:

Lemma 1 (Existence lemma) *If $\langle j \rangle \mathcal{A} \in \Delta$ then there exists some Γ such that $\Delta \mathbf{R_j} \Gamma$ and $\mathcal{A} \in \Gamma$.*

Given any formula $\mathcal{A}$ let $\mathcal{A}^+$ be the result of replacing p_i with p_{i+1} for every propositional variable p_i in $\mathcal{A}$. If X is a set of formulas, X^+ is $\{\mathcal{A}^+ : \mathcal{A} \in X\}$. $\mathcal{A}^-$ and X^- are defined in the obvious way, taking into account that $\mathcal{A}^-$ does not exist when $\mathcal{A}$ contains p_0. It is straightforward to show that (i) if Δ is a MCS, then Δ^- is a MCS as well; and (ii) if $\Delta \mathbf{R_j} \Gamma$, then $\Delta^- \mathbf{R_j} \Gamma^-$. Moreover:

Proposition 2 *If X is consistent, then X^+ is consistent.*

Proof Assume $\vdash \neg \mathcal{A}$ for some conjunction $\mathcal{A}$ of formulas in X^+. Then the proof of $\neg \mathcal{A}$ can be turned into a proof of $\neg \mathcal{A}^-$ (which is the negation of a conjunction in X) by replacing every propositional variable p_{i+1} with p_i and by replacing p_0 with a suitable new variable p_k. □

Let us denote by π_i the formula $\Box_\sigma p_i \wedge \Box_\sigma H \neg p_i$.

Proposition 3 *If X is consistent, then $X^+ \cup \{\pi_0\}$ is consistent.*

Proof Otherwise, there is a conjunction $\mathcal{A} = \mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_k$ of formulas in X^+ such that $\vdash \neg(\pi_0 \wedge \mathcal{A})$, which is equivalent to $\vdash \pi_0 \rightarrow \neg \mathcal{A}$. Since p_0 does not occur in $\mathcal{A}$, by **Irr** we have $\vdash \neg \mathcal{A}$, which contradicts the consistency of X^+ and hence, by Proposition 2, of X. □

Definition 13 A *rectangle* is a sequence $\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_n$ of MCSs such that, for $i, j = 1$ to n, $\Gamma_i \mathbf{<} \Delta_i$, $\Delta_i \sim \Delta_j$, $\Gamma_i \sim \Gamma_j$, and if $\Delta_i \approx \Delta_j$ then $\Gamma_i \widehat{\mathbf{A}} \Gamma_j$.

Definition 14 The rectangle $\Delta_1^*, \ldots, \Delta_n^*, \Gamma_1^*, \ldots, \Gamma_n^*$ is said to be *auxiliary* to the rectangle $\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_n$ if, for $i, j = 1$ to n, (a) $\Delta_i^+ \subseteq \Delta_i^*$, (b) $\Gamma_i^+ \cup \{\pi_0\} \subseteq \Gamma_i^*$, and (c) if $\Delta_i \approx \Delta_j$ then $\Delta_i^* \approx \Delta_j^*$.

Rectangles $\Delta_m, \ldots, \Delta_n, \Gamma_m, \ldots, \Gamma_n$ will be written as $[\Delta_i; \Gamma_i]$ $(m-n)$, possibly omitting the range $(m-n)$ of the index i if equal to $(1-n)$. The rectangle $[\Delta_i; \Gamma_i]$ $(0-n)$ will also be written as $[\Delta_0, \Delta_i; \Gamma_0, \Gamma_i]$ $(1-n)$.

Since we know that the axioms for $\Box_\sigma$ are complete with respect to S5 validities, and the axioms for P, F with respect to linear time validities, we shall freely use these validities as theorems. In particular, we have:

$$(*) \vdash \pi_i \leftrightarrow \Box_\sigma \pi_i \qquad (**) \vdash P(\pi_i \wedge \mathcal{A}) \rightarrow H(P\pi_i \rightarrow P(\pi_i \wedge \mathcal{A}))$$

Now we are in a position to prove the following:

$$
\begin{array}{ccccccccc}
\Delta_1 & \sim & \Delta_2 & \sim & \Delta_3 & \underset{\mathbf{A}}{\overset{\overset{\sim}{\approx}}{\frown}} & \Delta_4 & \overset{\sim}{\approx} & \Delta_5 \\
\vee & & \vee & & \vee & & \vee & & \vee \\
\Gamma_1 & \overset{\sim}{\approx} & \Gamma_2 & \sim & \Gamma_3 & \underset{\mathbf{A}}{\overset{\overset{\sim}{\approx}}{\frown}} & \Gamma_4 & \underset{\mathbf{A}}{\overset{\overset{\sim}{\approx}}{\frown}} & \Gamma_5
\end{array}
$$

Fig. 2: Example of a rectangle $[\Delta_i;\Gamma_i]$ $(1-5)$.

Lemma 2 *Assume that (a) the rectangle* $[\Delta_i;\Gamma_i]$ *has auxiliary rectangle* $[\Delta_i^*;\Gamma_i^*]$*, (b)* $P\mathcal{A} \in \Delta_1$*, and (c)* $\neg\mathcal{A} \wedge H\neg\mathcal{A} \in \Gamma_1$*. Then there exist MCSs* $\Sigma_1 \sim \ldots \sim \Sigma_n$ *such that (i)* $\mathcal{A} \in \Sigma_1$*, (ii)* $[\Delta_i;\Sigma_i]$ *and* $[\Sigma_i;\Gamma_i]$ *are rectangles, and (iii)* $[\Delta_i;\Sigma_i]$ *and* $[\Sigma_i;\Gamma_i]$ *have auxiliary rectangles.*

Proof Consider the sets Δ_1^* and Γ_1^*. By the assumptions, $P\mathcal{A}^+ \in \Delta_1^*$ and $\neg\mathcal{A}^+ \wedge H\neg\mathcal{A}^+ \in \Gamma_1^*$. Then, by Lemma 1 and Axioms 1, there exists a MCS Φ_1 such that $\mathcal{A}^+ \in \Phi_1$ and $\Gamma_1^* < \Phi_1 < \Delta_1^*$. Let Ψ_1 be any maximal consistent extension (MCE) of $\Phi_1^+ \cup \{\pi_0\}$. Then, for every conjunction δ^+ of elements of Δ_1^{*+} and every conjunction γ^+ of elements of Γ_1^{*+}, $F\delta^+ \wedge P\gamma^+ \in \Psi_1$. This implies that there exist MCEs $\Delta_1^{**}, \Gamma_1^{**}$ of $\Delta_1^{*+}, \Gamma_1^{*+}$, respectively, such that $\Gamma_1^{**} < \Psi_1 < \Delta_1^{**}$.

For every $i = 2$ to n, and every conjunction δ of elements of Δ_i^*, we have that either $\Diamond\delta^+$ or (just) $\Diamond_\sigma\delta^+$ is in Δ_1^{**}, depending on whether Δ_i^* is or is not in relation $\approx$ with Δ_1^*. Hence in the former case we can let a MCS $\Delta_i^{**} \supseteq \Delta_i^{*+}$ be in $\approx$ with Δ_1^{**}, otherwise we can let Δ_i^{**} be in $\sim$ with Δ_1^{**}. For all $i = 2$ to n such that Δ_i^{**} is in relation $\approx$ with Δ_1^{**}, we can let Ψ_i be a MCS such that $\Psi_1 \underset{\mathbf{A}}{\frown} \Psi_i < \Delta_i^{**}$ (by the canonicity of **Noch** for the frame property NOCH). After that, we pass to consider MCSs Δ_k^{**} such that $\Delta_k^{**} \overset{\sim}{\not\approx} \Delta_1^{**}$, if any, and let Ψ_k be a MCS such that $\Psi_1 \sim \Psi_k < \Delta_k^{**}$ (by the canonicity of **Synchronicity** for SYNCHRONICITY). Then for all $i = 2$ to n such that $\Delta_i^{**} \overset{\approx}{\neq} \Delta_k^{**}$, we can let Ψ_i be a MCS such that $\Psi_k \underset{\mathbf{A}}{\frown} \Psi_i < \Delta_i^{**}$. By repeated application of this procedure, we may obtain MCSs $\Psi_2, \ldots, \Psi_n$ such that $[\Delta_i^{**};\Psi_i]$ is a rectangle.

For $i = 2$ to n, consider an arbitrary formula $\gamma \in \Gamma_i^*$. Since $\pi_0 \leftrightarrow \Box_\sigma\pi_0$ is a theorem and $\pi_0 \in \Gamma_i^*$, the formula $P(\gamma \wedge \pi_0) \in \Delta_i^*$ and $P(\gamma^+ \wedge \pi_1) \in \Delta_i^{**}$. Observe that $P\pi_1 \in \Psi_i$ because $P\pi_1 \in \Psi_1$ and $P\pi_i \leftrightarrow \Box_\sigma P\pi_i$ is a theorem. Then, by $(**)$, $P(\gamma^+ \wedge \pi_1) \in \Psi_i$. This implies that there exists a MCE Γ_i^{**} of Γ_i^{*+} such that $\Gamma_1^{**} \sim \Gamma_i^{**} < \Psi_i$. For $i, j = 1$ to n, it is immediate to check that if $\Psi_i \approx \Psi_j$ then $\Gamma_i^{**} \underset{\mathbf{A}}{\frown} \Gamma_j^{**}$.

We now set $\Sigma_i = \Psi_i^{--}$ for $i = 1$ to n. The formula $\mathcal{A}$ is in Σ_1. To conclude the proof, we have to provide auxiliary rectangles for $[\Delta_i;\Sigma_i]$ and $[\Sigma_i;\Gamma_i]$. As for the latter, an auxiliary rectangle is given by the starting sets Γ_i^* with the sets $\Sigma_i^* = \Psi_i^-$. As for $[\Delta_i;\Sigma_i]$, consider the new sets Δ_i' and Σ_i' obtained by exchanging p_0 and p_1 in the sets Δ_i^{**} and Ψ_i. Since the exchange is simply a renaming of propositional variables, the sets Δ_i' and Σ_i' are in the same relation as the corresponding Δ_i^{**} and Ψ_i. Moreover, $\Delta_i = \Delta_i'^{--}$ and $\Sigma_i = \Sigma_i'^{--}$. Finally, the effect of the exchange is that every Σ_i' contains π_1. Then, $[\Delta_i'^{-};\Sigma_i'^{-}]$ is auxiliary to $[\Delta_i;\Sigma_i]$. □

3.5.2 Completeness Theorem

Now we shall proceed step-by-step (see Blackburn et al. 2001, § 4.6), as follows. First we shall pair up certain finite frames, which are thought to approximate **OTF** frames, with functions from points to MCSs and from rectangles to auxiliary rectangles, to obtain (what we shall label as) *networks*. Then we shall let networks correspond to special, *induced* models. Finally, we shall show that each consistent formula of $\mathcal{L}$ is true in some induced model.

Definition 15 A *network* $\mathcal{N}$ is a tuple:

$$(K, <, \approx, \sim, \mathcal{AG} = \{\widehat{\mathrm{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}\}, \mathcal{F} = \{\widehat{f_1}, \ldots, \widehat{f_s}\}, L, \rho)$$

where K is a (possibly finite) set of points, $<, \sim, \ldots, \widehat{f_s}$ are relations on K,[20] L is a labelling function from points in K to MCSs and ρ is a possibly partial function from rectangles to auxiliary rectangles (see Defs. 13–14). $\mathfrak{F}^{\mathcal{N}} = (K, <, \approx, \sim, \mathcal{AG}, \mathcal{F})$ is said to be the *underlying frame* of $\mathcal{N}$.

Definition 16 A network $\mathcal{N}$ is *coherent* if:

(C1) For all $x, y \in K$, if $xR_j y$ then $L(x)\mathbf{R_j}L(y)$.
(C2) $<$ is a union of disjoint irreflexive linear orders on K.
(C3) $\approx$ and $>$ satisfy property MB (see Def. 6).
(C4) $\approx, \sim$ and $\widehat{\mathrm{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}$ are equivalence relations on K, are disjoint from $<$ and are such that, for any $k \in \{1, \ldots, t\}$, $\widehat{\mathrm{A}} \subseteq \widehat{\alpha_k} \subseteq \approx \subseteq \sim$.
(C5) $\sim$ and $<$ satisfy SYNCHRONICITY (see Def. 6).
(C6) $\sim, \approx$ and $<$ satisfy property TREE (see Def. 6).
(C7) $\widehat{\mathrm{A}}, \approx$ and $<$ satisfy property NOCH (see Def. 6).
(C8) $\widehat{f_i} \subseteq \sim$ for all $i \in \{1, \ldots, s\}$ (property NTS, Def. 6).

Definition 17 A *row* $x_1 \sim \ldots \sim x_n$ in K is a sequence on K such that $y \in \{x_1, \ldots, x_n\}$ iff $y \sim x_1$. A row $x_1 \sim \ldots \sim x_n$ in K is said to *immediately follow* row $y_1 \sim \ldots \sim y_n$ in K if, for $i = 1$ to n, $x_i > y_i$ and no $z \in K$ is such that $x_i > z > y_i$.

(C9) If row $x_1 \sim \ldots \sim x_n$ immediately follows row $y_1 \sim \ldots \sim y_n$, then $[L(x_i); L(y_i)]$ is a rectangle and there exists rectangle $\rho([L(x_i); L(y_i)])$ auxiliary to it.

Definition 18 A network $\mathcal{N}$ is *saturated* if:

(S1) For all j, x if $\langle j \rangle \mathcal{A} \in L(x)$ then $\mathcal{A} \in L(y)$ for some y such that $xR_j y$.
(S2) The irreflexive linear orders in $<$ are unbounded to the left and to the right.
(S3) $\approx$ and $\widehat{\alpha_1}, \ldots, \widehat{\alpha_t}$ satisfy property INDAG (see Def. 6).
(S4) $<$ and $\widehat{f_1}, \ldots, \widehat{f_s}$ satisfy property ORIGIN (see Def. 6).
(S5) $<$ and $\widehat{f_1}, \ldots, \widehat{f_s}$ satisfy property NEFM (see Def. 6).
(S6) $\widehat{f_1}, \ldots, \widehat{f_s}, \widehat{\mathrm{A}}$ and $<$ satisfy property CREATION (see Def. 6).

[20] Note that here the symbols $<, \sim, \ldots, \widehat{f_s}$ are not understood to denote the same relations as in Def. 6 (unless $\mathcal{N}$ is perfect, see Def. 19). This (harmless) ambiguity is allowed too keep terminological complexity to a minimum.

Definition 19 A network is *perfect* if it is both coherent and saturated.

Definition 20 Let $\mathcal{N}$ be a network with underlying frame $\mathfrak{F}^{\mathcal{N}}$. The *induced valuation* on $\mathfrak{F}^{\mathcal{N}}$ is defined as $V(p) = \{x \in K : p \in L(x)\}$. The structure $\mathfrak{M}^{\mathcal{N}} = \{\mathfrak{F}^{\mathcal{N}}, V\}$ is the *induced model* of $\mathcal{N}$.

Lemma 3 (Truth lemma) *If $\mathcal{N}$ is a perfect network with induced model $\mathfrak{M}^{\mathcal{N}} = \{\mathfrak{F}^{\mathcal{N}}, V\}$, then for all $x \in K$ we have that $\mathfrak{M}^{\mathcal{N}}, x \vDash \mathcal{A}$ iff $\mathcal{A} \in L(x)$.*

Proof Straightforward, by induction on $\mathcal{A}$. □

Now it should be clear enough that all is needed to prove that **TFM** is complete is to show that, for any consistent $\mathcal{A}$, it is possible to construct a perfect network $\mathcal{N}$ such that $\mathcal{A} \in L(x)$ for some $x \in K$. Before starting the construction let us define six *defects* a network may have, which correspond to failures of the saturation conditions (S1)–(S6).

Definition 21 Let $\mathcal{N} = (K, <, \approx, \sim, \{\widehat{\text{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}\}, \{\widehat{f_1}, \ldots, \widehat{f_s}\}, L, \rho)$ be a network. We may have the following defects ($u, x, \ldots$ range on K):

(S1)-defect: A pair $(u, \langle j \rangle \mathcal{A})$, where $\langle j \rangle \mathcal{A} \in L(u)$, for which there is no x such that $uR_j x$ and $\mathcal{A} \in L(x)$.
(S2)-defect: A point u for which there is either no $x < u$ or no $x > u$.
(S3)-defect: A tuple $(u_1, \ldots, u_{(2t)}, \Diamond)$ such that $u_1 \widehat{\alpha_1} u_2 \approx \ldots \approx u_{(2t-1)} \widehat{\alpha_t} u_{(2t)}$, and there exists no x such that $x \widehat{\alpha_k} u_j$ for all $j \in \{1, \ldots, 2t\}$, $k \in \{1, \ldots, t\}$.
(S4)-defect: A pair $(u, \Diamond_{f_i})$ such that $H\neg\Diamond_{f_i}\top \notin L(x)$ for all $x < u$.
(S5)-defect: A triple $(u, \Diamond_{f_i}, P)$ such that for no $x > u$ the content of f_i remains the same at all $y > x$ (see Def. 6).
(S6)-defects: A triple $(u, \Diamond_{f_i}, [Dstit])$ such that $\Diamond_{f_i}\top \in L(u)$ and for all $x < u$ we have that $[Dstit]F\Diamond_{f_i}\top \notin L(x)$.

Now we need to define the notion of an *extension* of a network:

Definition 22 Let $\mathcal{N}$ and $\mathcal{N}'$ be networks (here and in what follows, we indicate the elements of $\mathcal{N}'$ as $K', <', \sim', \ldots$). $\mathcal{N}'$ is said to *extend* $\mathcal{N}$ if $\mathfrak{F}^{N}$ is a subframe of $\mathfrak{F}^{\mathcal{N}'}$, $L \subseteq L'$, and $\rho \subseteq \rho'$.

Finally we have the following:

Lemma 4 (Repair lemma) *For any defect of a finite, coherent network $\mathcal{N}$ there is a finite, coherent $\mathcal{N}'$ that extends $\mathcal{N}$ and lacks that defect.*

Proof Let $\mathcal{N} = (K, <, \approx, \sim, \{\widehat{\text{A}}, \widehat{\alpha_1}, \ldots, \widehat{\alpha_t}\}, \{\widehat{f_1}, \ldots, \widehat{f_s}\}, L, \rho)$ be a finite, coherent network having some defect. We have to prove that all these defects may be repaired. To this aim, we consider, in turn, defects of each sort (S1)–(S6) and show how to define a coherent extension $\mathcal{N}'$ of $\mathcal{N}$ lacking the defects.

(S1)-defects: (S1)-defects come in seven different sorts, depending on which $\mathcal{L}$-diamond, or family of $\mathcal{L}$-diamonds, they involve. We shall deal with them separately.

(P) Let $(u, P\mathcal{A})$ be an (S1)-defect, i.e., $P\mathcal{A} \in L(u)$ but $\mathcal{A} \notin L(x)$ for all $x < u$. Consider the unique point $u_1 \leqslant u$ such that $(u_1, P\mathcal{A})$ is an (S1)-defect, and for all $x < u_1, (x, P\mathcal{A})$ is *not* an (S1)-defect. Assume that u_1 is not $<$-minimal, that is, $x < u_1$ for some $x \in K$. Let w_1 be the immediate $<$-predecessor of u_1. Then the row $u_1 \sim \ldots \sim u_n$ immediately follows row $w_1 \sim \ldots \sim w_n$. By (C9), let $[\Delta_i;\Gamma_i]$ be the rectangle corresponding to these rows, and let $\rho([\Delta_i;\Gamma_i]) = [\Delta_i^*;\Gamma_i^*]$. We know that $P\mathcal{A} \in \Delta_1$ and $\neg\mathcal{A} \wedge H\neg\mathcal{A} \in \Gamma_1$. Thus, by Lemma 2, there exist MCSs $\Sigma_1 \sim \ldots \sim \Sigma_n$ such that $\mathcal{A} \in \Sigma_1$, and $[\Delta_i;\Sigma_i]$, $[\Sigma_i;\Gamma_i]$ are rectangles and have auxiliary rectangles $[\Delta_i'^*;\Sigma_i'^*]$, $[\Sigma_i^*;\Gamma_i^*]$. Given the *new* points $v_1,\ldots,v_n$ (new in that $v_1,\ldots,v_n \notin K$), the repaired network $\mathcal{N}'$ can be defined as follows:

$K' := K \cup \{v_1,\ldots,v_n\}$.
$<' := \bigcup_{1 \leq i \leq n} \{<, \{(x,v_i) : x \in K \wedge x < u_i\}, \{(v_i,x) : x \in K \wedge u_i \leqslant x\}\}$.
$\approx' := \approx \cup \{(v_i,v_j) : i,j \in \{1,\ldots,n\}$ and $(u_i,u_j) \in \approx\}$.
$\sim' := \sim \cup \{v_1,\ldots,v_n\}^2$.
$\widehat{\text{A}}' := \widehat{\text{A}} \cup \{(v_i,v_j) : i,j \in \{1,\ldots,n\}$ and $(u_i,u_j) \in \approx\}$.
For $k = 1$ to t, $\widehat{\alpha_k}' := \widehat{\alpha_k} \cup \{(v_i,v_j) : i,j \in \{1,\ldots,n\}$ and $(u_i,u_j) \in \approx\}$.
For $i = 1$ to s, $\widehat{f_i}' := \widehat{f_i}$.
$L' := L \cup \{(v_1,\Sigma_1),\ldots,(v_n,\Sigma_n)\}$.
$\rho' := \rho \cup \{([\Delta_i;\Sigma_i],[\Delta_i'^*;\Sigma_i'^*]),([\Sigma_i;\Gamma_i],[\Sigma_i^*;\Gamma_i^*])\}$.

It is immediate to check that $\mathcal{N}'$ satisfies conditions (C1)–(C9). So far, we have only discussed the case in which u_1 is not $<$-minimal. However, the other case is really simpler and is left as an exercise to the reader.

(F) By symmetry, these defects can be treated as the previous ones.

$(\Diamond_\sigma)$ Let $(u_1, \Diamond_\sigma\mathcal{A})$ be such that $\Diamond_\sigma\mathcal{A} \in L(u_1)$ but $\mathcal{A} \notin L(x)$ for all $x \sim u_1$. Let us consider row $u_1 \sim \ldots \sim u_n$, and suppose that it immediately follows some row $w_1 \sim \ldots \sim w_n$ (if there is no such row, we can add it, by Axioms 1, as in the above defect $(u,P\mathcal{A})$). For any $i \in \{1,\ldots,n\}$, let $L(u_i) = \Delta_i$, $L(w_i) = \Gamma_i$, and $\rho([\Delta_i;\Gamma_i]) = [\Delta_i^*;\Gamma_i^*]$. Since $\Diamond_\sigma\mathcal{A}^+ \in \Delta_1^*$, by Lemma 1 and the canonicity of **Synchronicity** for SYNCHRONICITY, there exists a rectangle $[\Phi,\Delta_i^*;\Upsilon,\Gamma_i^*]$ such that $\mathcal{A}^+ \in \Phi$. Now we add new points u,w and set $L'(u) = \Delta = \Phi^-$, $L'(w) = \Gamma = \Upsilon^-$, $\Delta^* = \Phi$, and $\Gamma^* = \Upsilon$. Hence we have suitable rectangle $[\Delta,\Delta_i;\Gamma,\Gamma_i]$ with auxiliary $[\Delta^*,\Delta_i^*;\Gamma^*,\Gamma_i^*]$. By a similar procedure we provide, for each rectangle $[X_i,Y_i]$ in the domain of ρ, an extended rectangle $[X,X_i;Y,Y_i]$ with auxiliary rectangle $[X^*,X_i^*;Y^*,Y_i^*]$ (this is required to ensure coherence). Finally, consider the bottom rectangle $[\Lambda_i;\Sigma_i]$, and let $[\Lambda,\Lambda_i;\Sigma,\Sigma_i]$ be the corresponding extended rectangle with auxiliary rectangle $[\Lambda^*,\Lambda_i^*;\Sigma^*,\Sigma_i^*]$. We know by (C1), (C6) that $\Sigma_1 \approx \Sigma_i$ for $i = 2$ to n. If $\Sigma \approx \Sigma_1$, we stop here. Otherwise, to restore coherence, we must add a new bottom row of points $z,z_1,\ldots,z_n$ such that $L'(z) \approx L'(z_1) \approx \ldots \approx L'(z_n)$, as in the above (S1)-$P$ defects. Again, this part of the proof is left to the reader, along with the definition of the repaired network.

$(\Diamond, \Diamond_{f_i}, \langle\alpha_k\, stit\rangle, \langle Stit\rangle)$ These defects can be treated in analogy with (S1)-$\Diamond_\sigma$ defects.

(S2)–(S6)-defects: Let u be an (S2)-defect; we can suppose without loss of generality that there exists no $y < u$. Since, by Axioms 1, $P\top \in L(u)$, our (S2)-defects boils down to the (S1)-defect $(u,P\top)$. By similar reasonings, it is straightforward to show that any (S3)–(S6)-defect boils down to one or more suitable (S1)-defects. □

Theorem 1 ***TFM*** *is strongly complete.*

Proof Let X be a consistent set of formulae. Consider any MCS $\Delta \supseteq X$. Given a point u, let $\mathcal{N}_0$ be a network such that $K_0 = \{u\}$, $< = \emptyset$, $\approx = \sim = \widehat{\mathrm{A}} = \widehat{\alpha_1} = \ldots = \widehat{\alpha_t} = \{(u,u)\}$, $\widehat{f_i} = \emptyset$ for all i, $L_0 = \{(u,\Delta)\}$, and $\rho_0 = \emptyset$. Trivially, $\mathcal{N}_0$ is finite and coherent. Our aim is to show that it is possible to extend it step-by-step up to a perfect network $\mathcal{N}'$. Let S be a countable set of points such that $u \in S$ (intuitively, these are the points we shall use in the construction). We start by enumerating the set of potential defects we might encounter in the process (where, for instance, the set of potential (S1)-P defects is $\{(x, P\mathcal{A}) : x \in S$ and $\mathcal{A}$ is a formula of $\mathcal{L}\}$). Then we let d_0 be the defect of $\mathcal{N}$ that is minimal in our enumeration. By Lemma 4, we can obtain a repaired network $\mathcal{N}_1$ lacking defect d_0. Then, for any network $\mathcal{N}_n$, we may consider the defect d of $\mathcal{N}_n$ that is minimal in our enumeration and obtain, by Lemma 4, a repaired network $\mathcal{N}_{n+1}$ lacking that defect. Observe that d does not affect any extension of $\mathcal{N}_{n+1}$. By standard results, we conclude that there exists a perfect network $\mathcal{N}'$ such thaxt $L'(x) = \Delta$ for some $x \in K'$. Therefore, by Lemma 3, $\mathfrak{M}', x \vDash \mathcal{A}$ for any $\mathcal{A} \in X$. □

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