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ABSTRACT

One of the main findings of the principal-agent literature has been that incentive schemes should be sensitive to all information that bears on the agent's actions. As a manifestation of this principle, incentive schemes tend to take quite complex (non-linear) forms. In contrast, real world schemes are often based on aggregate information with a rather simple structure.

This paper considers the optimality of linear schemes that use only aggregated information. The hypothesis is that linear schemes are to be expected in situations where the agent has a rich set of actions to choose from, because richness in action choice allows the agent to circumvent highly nonlinear schemes. We show that optimal compensation schemes are indeed linear functions of appropriate accounting aggregates in a multi-period model where the agent can observe and respond to his own performance over time. Furthermore, when profits evolve according to a controlled Brownian motion (with the agent at the controls) the optimal compensation scheme is linear in profits. The optimal scheme can be computed as if the principal could only choose among linear rules in a corresponding static problem. Applications of this *ad hoc* principle appear quite promising and are briefly illustrated.

AGGREGATION AND LINEARITY IN THE PROVISION OF INTERTEMPORAL INCENTIVES

by

Bengt Holmstrom and Paul Milgrom*

1. Introduction

Interest in the economics of information has surged in response to shortcomings in received micro-theory. A remarkable intellectual achievement, the Walrasian theory of general equilibrium is surprisingly superficial in some economic dimensions. Practitioners as well as welfare analysts have long recognized that the Walrasian theory fails to support realistic welfare judgments, because it does not bring out the inevitable trade-off between equity and efficiency, which society is faced with in the real world. On the positive side, Hayek was among the first to express strong discontent with general equilibrium theory, because it did not address what he saw as the fundamental economic question: how economic systems perform the task of aggregating efficiently widely dispersed information in society. Closely related is the critique that the Walrasian model does nothing to explain the real world abundance of nonmarket transactions and institutions, which certainly is a central part of the information processing marvel of a competitive system. Since nonmarket

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institutions emerge spontaneously in response to problems with price-mediated exchange, one is well justified in suspecting that their presence is of allocational significance and likely to alter the conclusions of a theory that views them as exogenous. Indeed, one of the main objectives of a more careful study of organizations is to see to what extent the interaction between market and nonmarket activities may be of help in explaining "anomalies" in the performance of a competitive system (e.g. strikes and underemployment), which cannot be accounted for in the traditional theory.

The aforementioned shortcomings of received micro-theory are largely associated with the implicit assumption that individuals have identical information and that only resource constraints matter. In contrast, the economics of information is based on the postulate that informational constraints are equally important for the allocation of goods. Private information may alter feasible trades and efficient transaction modes in fundamental ways. This observation underlies the extensive research on communication, information processing and incentives in various organizational contexts. Much of the incentive research has focused on the simplest organizational structure: the principal-agent relationship. Progress has not always been easy, however. The principal-agent literature has had to contend with analytical difficulties as well as weak economic implications.

Consider for instance the standard moral hazard model in which the agent controls a one-parameter family of return distributions through his choice of effort. Even for the most regular looking distributions a solution may fail to exist. As an example, Mirrlees [1974] showed that if the agent controls the mean of a normal distribution with fixed variance, then a first-best solution can be approximated arbitrarily closely by using schemes that punish the agent very severely with low probability although the first-best cannot actually be attained. On the other hand, when sufficient assumptions are made to assure existence, there is no guarantee yet that one can take much advantage of the one-dimensional action space, because the agent's incentive constraint cannot always be represented by a first-order condition. Some advances have recently been made in placing restrictions on the problem structure such that the common first-order characterization is valid (Grossman and Hart [1983], Rogerson [1984], and Holmstrom [1984]). Even so the resulting optima rarely take a simple form. One finds that they can be sensitive to what seems to be small details of the information structure of the problem. By changing the information structure almost any scheme can be made an optimum of some moral hazard problem. The complex forms optimal incentive schemes can take makes it hard to tell much about the agent's choice at equilibrium, although that should be a significant part of the model prediction. Besides the rather obvious observation that moral hazard leads to a second-best solution, one of the few general conclusions to emerge from the one-dimensional model is that an optimal incentive scheme uses all the available information about the agent's choice of effort (Holmstrom [1979] and Shavell [1979]), which is another reflection of the fact that optimal schemes respond to all changes in the information content of the output signals.

Thus, from a methodological point of view, it would be desirable to find models that yield simpler and sharper answers in such rudimentary organizations as the principal-agent set-up, in order to be able to go forward with the analysis and investigate more interesting aggregate questions. Such an effort is also supported by the fact that in reality incentive schemes tend to take rather simpler forms than the finely tuned rules predicted by the theory; for instance, real world schemes make much more use of aggregate information than what present theory might suggest. One could in part explain such simplicity by the costs of writing intricate contracts, but that is hardly the whole story. Our purpose is to show that simple schemes--particularly linear schemes based on aggregates--have other appealing features that provide more likely explanations for their extensive use.[1]

The main idea can best be grasped by considering what is "wrong" with the step-function punishment schemes that come close to first-best in the normal distribution example studied by Mirrlees. The practical problem with such schemes is that their performance is so sensitive to the particular assumptions of the model. The step-function schemes encourage effort by punishing certain rare "disasters," so their effectiveness relies both on an accurate specification of the probabilities of rare events and on the model's assumption that the agent's efforts at disaster prevention will increase the expected profits of the principal. Small probabilities are notoriously hard to estimate. Moreover, if the agent has the benefit of private information before choosing an action, it will rarely be true that disaster prevention enhances profits. For example, suppose that the agent can vary his effort over time in response to observations of how well he is doing and that the incentive scheme is a reward for performance at the end of some finite period of time (a year, say). In that case a step-function punishment scheme might induce the agent to bide his time early on and to start working hard only if that is necessary to avoid a disaster. In contrast, a linear scheme, which applies the same incentive pressure on the agent (income effects aside) no matter what the outcome history is, will lead to a more uniform choice of effort over time.

We show in the paper that these intuitive arguments can be made precise.

We consider a continuous time model in which the agent controls the drift of a Brownian motion over the unit time interval. Assuming that the agent has an exponential utility function (to remove income effects) and the cost of control is monetary, the optimal incentive scheme will indeed be linear in output. The agent will choose a constant drift rate independently of the path of output. This means that the optimal incentive scheme can be computed as if the agent were choosing the mean of a normal distribution only once and the principal were restricted a priori to using a linear rule. Thus, the dynamic perspective leads not only to a natural resolution of the nonexistence problem that the Mirrlees' example posed, but also to a remarkably simple and easily computed optimal incentive scheme.

While the linearity results that we will develop in this paper clearly depend on a particular intertemporal structure, there is a more general lesson to be learned from the analysis we present. The lesson is that finely tuned schemes may perform rather poorly (and simple schemes may become optimal or nearly so) when the agent's action space is enriched sufficiently. This point is best understood by reinterpreting the dynamic Brownian model as a static model in which the agent, instead of choosing a simple effort level, chooses a path-contingent strategy at the outset. As in standard static models, the agent's strategy induces a distribution over end-of-period outcomes. But instead of being constrained to a one-parameter family of outcome distributions, the rich set of contingent strategies now permits a vastly wider choice. The enormous expansion in the agent's opportunity set limits the principal's options dramatically; in fact, for each strategy that the principal wishes to implement, there is essentially a unique incentive scheme that he must use, which stands in sharp contrast to the usual flexibility in choice of schemes that the principal has in one-dimensional models.

We turn to a brief outline of the analysis we will pursue. Section 2 contains an analysis of a single-period model of moral hazard with the agent controlling a multinomial distribution. The purpose of this section is twofold: to isolate the key implications of the agent's exponential utility function and to illustrate how "richness" in the agent's opportunity set limits the principal's choice of incentive schemes.

In Section 3 we look at a finitely repeated version of the single-period multinomial model. The main result is that even though the principal can observe the outcomes of each period separately, the optimal scheme depends only on aggregates, and it depends linearly on them. The aggregates are the numbers of times that each particular outcome occurs during the pay period. For instance, in a three-period problem it is enough to know that the agent made $200 twice and $100 once; the principal gains nothing from further information about the precise sequence of these events. While it is not generally true that an optimal scheme is also linear in profits (i.e. one $300 outcome and one $100 outcome is not equivalent to two $200 outcomes), this additional feature does hold for the Bernoulli case in which there are only two possible outcomes in each period. For the Bernoulli case the linearity in end-of-period profits implies that optimal compensation depends only on accumulated profits; it does not matter when or how a dollar is made.

In Section 4 we go on to study a continuous time approximation of the discrete time model. The agent controls the instantaneous drift (but not the covariance) of a multi-dimensional Brownian motion. This model serves as a good approximation of a frequently repeated multinomial model provided that four conditions are met: (i) each outcome is expected to occur very often, (ii) the costs and profits in each individual period are negligible, (iii) the principal can observe the exact number of times that each outcome occurs, and (iv) the agent exercises very limited control over the outcome in any single period but can substantially affect profits by systematic effort over many periods. The optimal scheme for the multi-dimensional Brownian model is a linear function of the end-of-period levels of the different dimensions of the process (which we interpret as different account balances). In analogy with the discrete time model, the optimal scheme uses only account balances aggregated over time, but it generally requires more information than just the accumulated profits. For the one-dimensional case, which serves to approximate a frequently repeated binomial process, the scheme is of course linear in the end-of-period profits. Thus, the one-dimensional Brownian model is an appropriate approximation only for discrete time models in which the agent controls a process with two possible outcomes in each period.

This may seem to make the one-dimensional Brownian "linearity in profits" result very special. However, it has an important corollary implication for the multi-dimensional Brownian model, in which the manager controls a more interesting and complex process. If, in the latter model, the compensation paid must be a function of profits alone (because detailed reliable accounts are unavailable) or if the manager has sufficient discretion in how to account for revenues and expenses, then the optimal compensation scheme will be a linear function of profits. This is a central result, because it explains the use of schemes which are linear in profits even when the agent controls a complex multi-dimensional process.

We devote Section 5 to illustrations of the ease with which one can apply the continuous time model to obtain simple and explicit solutions to various kinds of agency problems. The examples we go through should be viewed as merely suggestive; we do not attempt any serious analysis of possible applications.

In Section 6 we address the robustness of our results, particularly with regard to the Brownian model. We show that the results survive some useful extensions; for instance, one can allow the principal to be risk averse (with exponential utility). But there are also some important restrictions to note for those who wish to make use of linearity. The most significant one is the necessity to assume that any control by the agent of the variance of the Brownian process is perfectly observable.

In the concluding section we provide a brief summary of our main findings.

2. A Single-Period Model

We begin by studying a fairly general single-period model of moral hazard.

There is a principal and an agent. The agent chooses an action privately, which results in a realization of a stochastic state θ . The set of possible states, Θ , is finite with $N+1$ elements $\{\theta_0, \ldots, \theta_N\}$. Each state θ_i is associated with a monetary payoff π_i that belongs to the principal and a publicly observable information outcome $x_i \in X$. We will write π for the random payoff and refer to it as "profits." The random public information is denoted x and called the "public outcome," or simply the "outcome," of the agent's action. Note that if $x_i = x_j$ implies $\pi_i = \pi_j$, then the outcome x contains at least as much information as π (i.e. profits can be observed). Notice, too, that it is possible that some of the information states θ_i are never observed.

The probability of state i is denoted p_i . The agent's action determines the probability vector $p = (p_0, \ldots, p_N)$. It is convenient to view p itself as the agent's action. The feasible set of actions is then a subset P of the N-dimensional simplex. P is assumed compact.

The principal is risk neutral and the agent is risk averse. In particular, the agent has a constant coefficient of absolute risk aversion, r , and we will take his utility function as $u(y) = -\exp(-ry)$, where y is money. Given an outcome x , the principal pays the agent $s(x)$. The rule $s(x)$ is called a sharing rule or incentive scheme. The agent's final income is $s(x)$ minus the cost of taking the action p . We assume the cost of action is monetary or that it has a monetary equivalent independently of other income. The cost is allowed to be stochastic and we write it $c(p; \theta_i)$. The cost function is assumed to be continuously differentiable on P .

The principal's problem is to select a sharing rule s and instructions p for the agent under the two standard constraints that (i) the agent can maximize his expected utility by following instructions and (ii) the agent can attain a certain minimum level of expected utility from the contract. We will measure this minimum expected utility level in terms of the agent's certain equivalent w ; thus, his expected utility has to be at least $u(w)$. The <u>Principal's Problem</u> can then be formally stated as:

(1) $$\max_{p,s} \Sigma(\pi_i - s(x_i))p_i \text{ , subject to:}$$

(2) $$p \text{ maximizes } \Sigma u(s(x_i) - c(p'; \theta_i))p'_i \text{ on } P \text{ ,}$$

(3) $$\Sigma u(s(x_i) - c(p; \theta_i))p_i \geq u(w) \text{ .}$$

Before going on to analyze this problem it may be helpful to suggest some interpretations that indicate the scope of the formulation.

In the simplest situation and the one most often studied, the agent controls a one-dimensional action variable, usually interpreted as effort. The cost of effort is deterministic and the agent chooses his effort with no more information about the production possibilities than the principal has. Commonly, output is the only observable variable. In our formulation this standard set-up would correspond to letting P be a one-dimensional manifold (a curve) in the (N+1)-simplex, letting $c(p; \theta_i) = c(p)$ for all θ_i and letting $x_i = \pi_i$ for all i.

The standard model can be enriched by having the agent observe a signal about the production technology or the cost function before choosing his effort level but after entering into a binding contract with the principal. Since the agent is assumed to have no private information at the time of contracting, this is not a model of adverse selection in the usual sense; we refer to it as the "Informed Agent Model". In the Informed Agent Model, the agent's action can be thought of as a strategy that maps his observed signal values into effort levels. Equivalently and more simply, the agent's strategy can be represented as a distribution over the set of states Θ. In that case the choice of distribution has to be constrained to be consistent with the composition of the distribution of the signals that the agent observes and the distributions he can choose from conditional on these signals (c.f. Milgrom and Weber [1984]). Thus, the Informed Agent Model is subsumed in our general formulation. In view of our subsequent interests it is worth noting that even if the agent's effort is a one-dimensional choice variable, a contingent strategy usually permits control of p in more than one dimension. In other words, a natural way of increasing the dimensionality of P, and hence enriching the agent's action space, is to let the agent act on the basis of private information.

Our formulation is even more general than the simple Informed Agent situation described above. The agent could be choosing a sequence of actions over time. Information of relevance for future decisions could be entering along the way. The agent's actions could influence this information stream as well as future costs, payoffs and opportunity sets. At each stage actions could be multi-dimensional (e.g. include effort choice, production decisions, project selections, etc.). The cost of action could be stochastic and the observable information essentially anything. In short, we could permit rather arbitrary production and information technologies and still have the reduced form map into the simple structure in (1)-(3). Our principal restrictions are that (i) the agent evaluates wealth at a single point in time, after all actions have been taken, (ii) the cost of actions can be expressed in monetary units, (iii) the utility function is exponential, and (iv) neither party has private information at the time of contracting.

We proceed to the analysis of the principal's problem. As is well known, existence of a solution to the program (1)-(3) cannot be taken for granted. The following result gives sufficient conditions for existence; it is a variant of an existence result by Grossman and Hart [1982], adapted to our framework. Here, p' is defined to be any solution to (2) when s is identically zero.

<u>Theorem 1</u>. If P is compact, $c(\cdot, \theta_i)$ is continuous, and if all p that assign zero probability to any information state are sufficiently expensive, that is, $c(p, \theta_i) > c(p', \theta_i) + \max_{ij}(\pi_i - \pi_j)$, for some $p' \varepsilon P$ then a solution (p^*, s^*) to the Principal's Problem exists, and p^* assigns positive probability to every information state.

Given any sharing rule s, the agent's problem (2) has a solution p because P is compact and the objective function is continuous. If the sharing rule s and the optimal response p results in an expected utility level with certain equivalent w, we say that s <u>implements</u> p <u>with certain equivalent</u> w. The set of sharing rules that implement p with certain equivalent w is denoted $S(p,w)$. This set may be empty for some (p,w). Therefore, define $P^0(w) = \{p|S(p,w) \text{ is not empty}\}$ and $P^*(w) = \{p|\text{for some } s, (s,p) \text{ solves the Principal's Problem}\}$. The key implications of assuming that the agent's utility is exponential can now be stated as follows:

<u>Theorem 2</u>. For any s, w and $p \in P^0(w)$:

(i) $s \in S(p,w)$ if and only if $s-w \in S(p,0)$,

(ii) $P^0(w) = P^0$ for all w,

(iii) $P^*(w) = P^*$ for all w.

<u>Proof</u>. Because utility is exponential,

$$\Sigma u(s(x_i) - w - c(p;\ \theta_i))p_i = -u(-w)\Sigma u(s(x_i) - c(p;\ \theta_i))p_i \ . \tag{4}$$

Since $-u(-w) > 0$, any p that is best for the agent against $s(x) - w$ is also best against $s(x)$, and conversely. Also, $-u(-w)u(w) = u(0)$.

This proves (i) and (ii). Part (iii) then follows from the form of the principal's objective function in (1).

Q.E.D.

Theorem 2, part (iii) asserts that the optimal choice of an instruction p^* given to the agent does not depend on the required minimum certain equivalent w. Also, the optimal incentive scheme s^* adjusts to changes in w by a simple shift, that is, $s^* - w$ is a constant. Computationally, this means that the principal can deal with the two constraints (2) and (3) separately. He governs the agent's incentives by the choice of the differences $s(x_i) - s(x_0)$, $i = 1, \ldots, N$, and he assures a sufficient expected utility level by adjusting $s(x_0)$. This separation result will play a key simplifying role in the subsequent multi-period analysis.

For notational convenience, we will henceforth write $S(p)$ for $S(p,0)$. From $S(p)$ we can recover schemes in $S(p,w)$ by adding w .

Before moving on to the multi-period case, we wish to make a brief digression into the relation between the dimensionality of the agent's feasible set of actions P and the principal's freedom in choosing a sharing rule to implement any particular $p \in P$ (for a more extensive discussion, see Holmstrom [1984]). Notice that the principal chooses a sharing rule, which is a point in an $N+1$ dimensional space, to control the agent's choice of action and to provide a particular equivalent. When the agent chooses a point in a one-dimensional space, as has been commonly assumed, and when $N > 1$, the principal normally has many sharing rules that will coax any implementable action from the agent. As the agent's action space grows in dimension, the principal's ability to control the agent becomes correspondingly more limited. We show below that in a simple version of our model where the agent's action space is of full dimension, the rule that implements any particular action is in fact unique. This requires an additional assumption.

Assumption A.

(i) P has a nonempty interior in the N-dimensional simplex;

(ii) $c(p;\theta_i) = c(p)$ for all θ_i ;

(iii) $c(p)$ is continuously differentiable on P ;

(iv) $c(p) - c(p') \geq \max_{ij}(\pi_i - \pi_j)$ for p on the boundary of P .

Theorem 3. For any p in the interior of P , the set $S(p)$ is either empty or a singleton under Assumption A; i.e. if an interior p can be implemented, then the implementing scheme (with any certain equivalent w) is unique. Specifically, the sharing rule that implements an optimal action p^* is unique.

Remark: It could still be the case that P^* has more than one element and therefore that there are many optimal incentive schemes.

Proof. Assume for the moment that $x_i = \pi_i$ for all i . Fix a p in the interior of P . Let c_j be the partial derivative of $c(p)$ with respect to p_j after substituting $p_0 = 1 - \sum_{i=1}^{N} p_i$ into the cost function. If $S(p)$ is empty we are done, so assume there exists an $s \in S(p)$. Since p is in the interior of P which is of full dimension, the first-order conditions for the agent's optimization problem (2), imply:

$$(5)\quad -\sum_{i=1}^{N} u'(s(x_i) - c(p))c_j p_i + u(s(x_j) - c(p)) - u(s(x_0) - c(p)) = 0 , \quad j = 1, \ldots, N$$

If we define $z_i = u(s(x_i))/u(s(x_0)) = -u(s(x_i) - s(x_0))$, then (5) can be written (using the exponential form of utility) as:

$$\sum_{i=0}^{N} r z_i p_i c_j + z_j - 1 = 0 \, , \quad j = 1, \ldots, N \, . \tag{6}$$

Let $K = \sum_{i=0}^{N} r z_i p_i$. Then $z_j = 1 - c_j K$ and

$$K = \sum_{i=0}^{N} r(1 - c_i K) p_i \quad \text{or}$$

$$K = r/(1 + \sum_{i=0}^{N} r c_i p_i) \, ,$$

which implies unique values for the z_j's . Note that if $\sum_{i=0}^{N} r c_i p_i = -1$, then (6) has no solution contradicting the assumption that $S(p)$ is non-empty. Consequently, $s(x_i) - s(x_0)$ is uniquely determined for all i ; specifying $s(x_0)$ determines the agent's certain equivalent.

If we do not have $x_i = \theta_i$ as assumed, but instead have that x provides coarser information than the state, then uniqueness is implied a fortiori, because system (6) will have added constraints of the form $z_i = z_j$ (in case $x_i = x_j$).

The last statement of the theorem follows from the first part and the obvious fact that part (iv) of Assumption A implies that P^* is in the interior of P .

Q.E.D.

Theorem 3 contrasts sharply with the conclusions of standard one-dimensional moral hazard models. When P is one-dimensional (i.e., a curve), $S(p,w)$ normally contains infinitely many schemes (unless $N = 1$) .

The analysis then centers on the characterization of the best scheme in $S(p,w)$. Under some rather restrictive assumptions (see Grossman and Hart [1983], Rogerson [1984] and Holmstrom [1984]) one can use variational techniques to provide an intuitive and useful characterization of the best scheme. However, little can in general be said about the optimal choice of p . Also, once the agent's action space expands, the corresponding characterization result becomes much less informative. It may then be both realistic and analytically tractable to go to the opposite extreme and let P be of full dimension so that $s(p)$ can be obtained uniquely from (6). This route has the potential of offering more information about the optimal p to be implemented and in addition provide a useful characterization of the best scheme. In fact, this point is illustrated by earlier Informed Agent models such as Mirrlees' [1971] model of optimal taxation (as well as by adverse selection and non-linear pricing models). Also--in the spirit of our main theme--it may be possible to study this one-period model more abstractly and find out what kind of richness in the agent's choice and what kind of cost functions will lead to simple incentive schemes. We do not pursue that route here, but instead specialize the model by focusing on a particular dynamic structure. (Note that the dynamic model we will study can, by our previous discussion, be mapped into the one-period model described here.)

3. A Multi-Period Model

Consider a T-period version of the previous model. In each period $t = 1, \ldots, T$, the agent picks a $p^t \in P$, incurring a periodic cost $c(p^t; \theta_i)$. We denote the outcome x^t , the resulting state θ^t and the profit level π^t . We call $X^t = (x^1, \ldots, x^t)$ the <u>history</u> of the stochastic outcome process up to time t. A key assumption is that the agent can observe X^{t-1} before deciding p^t. Thus, a strategy for the agent is a stochastic process $\{p^t(X^{t-1})\}$.

We assume that the principal pays the agent at the end of the last period based on the entire realized path X^T of the outcome process. The incentive scheme is denoted $s(X^T)$. The agent is assumed to be concerned about his final wealth, which will equal $s(X^T) - \sum_{t=1}^{T} c(p^t; \theta^t)$. He values this wealth according to the exponential utility function defined earlier. The principal's final wealth is $\sum_{t=1}^{T} \pi^t - s(X^T)$, over which he is risk neutral.

The principal's problem is to select a sharing rule and a strategy for the agent (interpreted as a set of instructions) such that it maximizes his expected end of period wealth, subject to the instructions being incentive compatible and the agent being assured a minimum certain equivalent, which we henceforth normalize to zero. Formally, the problem can be stated as:

(7) $$\max_{\{p^t\},s} E[\sum_{t=1}^{T} \pi^t - s(X^T)], \text{ subject to}$$

(8) $$E\{u(s(X^T) - \sum_{t=1}^{T} c(p^t(X^{T-1}); \theta^t)\} \geq u(0)$$

(9) $$\{p^t\} \text{ maximizes } E\{u(s(X^T) - \sum_{t=1}^{T} c(p^t(X^{T-1}); \theta^t\} \geq u(0).$$

The expectations are taken with respect to the distribution over states induced by the agent's strategy $\{p^t\}$.

We start by considering the agent's problem using dynamic programming.

Fix a compensation rule $s(X^T)$ and let $\{p^t(\cdot)\}$ be an optimal strategy for the agent given that rule. Define $V_t = V_t(X^t)$ by

$$V_t = E[u(s(X^T) - \sum_{s=t+1}^{T} c(p^s;\ \theta^s))|X^t] .$$

Since u is exponential, V_t differs from the standard dynamic programming value function only by the positive multiplicative factor $-u(\sum_{s=1}^{t} c(p^s;\ \theta^s))$, which is a constant from the perspective of time t. Thus, we may use V_t for purposes of dynamic programming, interpreting it as the maximal expected utility to the agent of continuing after time t given the history up to and including the outcome at time t, but excluding the accumulated sunk costs. Let $w_t = w_t(X^t)$ be the corresponding certain equivalent; i.e. $u(w_t) = V_t$. We wish to examine w_t as a function of x^t, holding the history X^{t-1} constant. For this purpose we will write $w_t(X^t) = w_t(X^{t-1},\ x^t)$. The dynamic programming equation for the agent's problem requires that $p^t(X^{t-1})$ solve

$$\text{(10)} \qquad \max_{p} \sum_{i=0}^{N} u(w_t(X^{t-1},\ x_i) - c(p;\ \theta_i))p_i .$$

Note that the sunk cost term $\sum_{s=1}^{t-1} c(p^s;\ \theta^s)$ has been dropped as it gives rise to a positive constant that can be factored out.

The problem in (10) is of the same form as the single-period problem (2). Thus, Theorem 2 applies. It follows that $p^t(X^{t-1})$ is optimal in (10) and makes the certain equivalent of the maximum value of (10) equal to $w_{t-1}(X^{t-1})$ if and only if:

$$(11) \qquad w_t(X^t) = s_t(x_t;\ p^t(X^{t-1})) + w_{t-1}(X^{t-1}) ,$$

where $s_t(\cdot;p)$ denotes a scheme in $S(p)$, that is, a scheme which in the single-period problem implements p with certain equivalent zero. Summing (11) over t (from 1 to T) and noting that, by definition, $w_T(X^T) = s(X^T)$ gives the following:

Theorem 4. A strategy $\{p^t(X^{t-1})\}$ can be implemented if and only if for every date and history, $p^t(X^{t-1}) \in P^0$ (that is, if and only if each $p^t(X^{t-1})$ can be implemented in the single period problem). A sharing rule $s(X^T)$ implements $\{p^t(X^{t-1})\}$ with certain equivalent w_0 if and only if it can be written in the form:

$$(12) \qquad s(X^T) = \sum_{t=1}^{T} s_t(x^t;\ p^t(X^{t-1})) + w_0 ,$$

where each $s_t(\cdot;p)$ is a sharing rule that implements p with certain equivalent zero in the single-period problem.

It is instructive to think of each possible outcome as being recorded in a different account. There may be fewer than N+1 such outcomes, since two different states may correspond to the same outcome $(x_i = x_j)$. If there are M possible outcomes, then there are M accounts. Let A_i^t be the number of times in the first t periods that the i^{th} outcome occurs and let A^t be the vector $(A_1^t, \ldots, A_M^t)$. Also, it is convenient to represent the sharing rule $s_t(\cdot;p)$ by the M-vector $s_t(p)$ whose i^{th} component is the compensation payable when the i^{th} outcome occurs. Then we can write (12) as:

$$(13) \qquad s(X^T) = \sum_{t=1}^{T} s_t(p^t(X^{t-1}))\cdot(A^t - A^{t-1}) + w_0 \ .$$

Written this way, the sharing rule can be thought of as a "stochastic integral" of the account process $\{A^t\}$ and it is this form that is suitable for an extension to continuous time models.

Theorem 4 recovers a sharing rule from the strategy that is to be implemented. Note that there may be many sharing rules that implement the same strategy, because we may not have uniqueness in the single-period model. Of course, in view of Theorem 3, if each p^t always lies in the interior of P , we do have a unique implementation.

We wish to stress that while (12) has an additive form over time, this does not imply that every sharing rule $s(x^T)$ is additively separable in the x^t's . Indeed no sharing rule that implements a history-contingent strategy has this separability property because, for history-contingent strategies, x^t affects the actions and hence the summands in periods after period t .

Without assuming exponential utility, we could have derived a formula similar to (12), namely:

$$(14) \qquad s(X^T) = \sum_{t=1}^{T} [(s_t(x^t;\ p^t(X^{t-1}),\ w_{t-1}(X^{t-1})) - w_{t-1}(X^{t-1})]$$

where $s_t(\cdot;p,w)$ is a scheme that implements p with certain equivalent w in the single-period problem. What is special about exponential utility is that $s_t(\ ;p,w) = w + s_t(\cdot;p)$ as we saw in Theorem 2. We will see shortly the ramifications of this separability.

Turning to the principal's problem, we see that with (12) the principal will get a payoff:

(15) $$\sum_{t=1}^{T} E\{\pi^t - s_t(x^t;\ p^t(X^{t-1}))\} - w_0 .$$

In view of the time-separable structure of (15) and the stochastic independence of periods, we have by inspection that one solution to the principal's problem is given by the following theorem:

Theorem 5. An optimal strategy for the principal to implement is $p^t(X^{t-1}) = p^*$, where p^* is any single period optimum. An optimal compensation rule to use is:

(16) $$s(X^T) = \sum_{t=1}^{T} s(x^t;\ p^*) = s(p^*)\cdot A^T ,$$

where $s(\cdot;\ p^*)$ is an optimal single-period scheme that implements p^* , and $s(p^*)$ is the corresponding M-vector.

Again we note that the scheme in (16) is generally not the unique scheme that implements p^* (unless Theorem 3 applies). And even if $s(x^T)$ is unique in implementing p^* , there will be several optima if P^* has more than one element. Any string of actions from P^* with accompanying one-period optimal schemes would solve the principal's problem. However, Theorem 5 tells us that there is no need to do anything more complex than apply the same scheme in each period separately.

The sharing rule in (16) has a ready interpretation in terms of aggregation and linearity. The agent's optimal compensation is a linear function of the account balances recorded in A^T . These balances represent time-aggregated information of the action outcome path. Thus, events in account x_i have equal value independently of when they occurred and the principal need not know the timing details of any of the events. In view of previous sufficient statistics results (Holmstrom [1979] and Shavell [1979]), it is rather notable that an optimal scheme can be based on the aggregated information A^T even though it is not a sufficient statistic for the agent's strategy. This means that the earlier sufficient statistics results, which were derived for the one-dimensional models, need to be reformulated when the agent's action space is of higher dimension.[2] In particular, one has to account for the possibility that in some dimensions of choice there is no conflict of interest between the principal and the agent. This is true here for the timing of actions, which explains the time-aggregation result.

Evidently, common accounting information is heavily time-aggregated. Aggregation saves costs, but it may also involve losses because of reduced control. We have identified a particular class of environments in which no loss of control occurs from aggregation over time, primarily because income effects are missing in the agent's attitude towards risk.

In the same vein, one could ask when aggregation across accounts is costless from an incentive point of view. For instance, suppose each x^t is a monetary payoff, say profits, of the agent's activity in period t, so that A_i^t is the number of periods the agent has made a profit of $\$x_i$. When will the optimal incentive scheme be linear in the total profits $\sum_{i=0}^{N} A_i^T x_i$? Theorem 5, it should be stressed, does not tell us that the optimal scheme is linear in profits: Two periods with a profit of \$100K each are not always compensated the same as one of \$50K and one of \$150K. Nevertheless, there is a special case for which the optimal scheme in (16) is linear in money. That is the case when there are only two outcomes yielding two different profit levels; in other words, if the agent controls a binomial process. In that case, the two account balances A_1^T and A_2^T are both linear in total profits, so the compensation rule (16) is linear in profits. We now turn to the Brownian model as a vehicle for investigating linearity in profits in more detail.

4. The Brownian Model

Suppose that the environment in which the agent functions is one in which he takes almost inconsequential actions frequently in time. In terms of the model in the preceding section, suppose the time between repetitions is small, the number of events in each account summed over all the periods is large, and the cost of action $c(p^t)$ as well as the money value of the outcome x^t in any single period is negligible to both principal and agent. In that case it is reasonable to model the problem as one of controlled vector Brownian motion, where each component of the vector represents the accumulated activity in one of the N accounts of the outcome process. (Account 0 is a residual that can be suppressed.)

Henceforth, we will think of the accounts as measured in dollars. The firm's profit is computed by adding the revenue accounts and subtracting the cost accounts. Because of the normalizations involved in making the Brownian approximation, it is best to view the Brownian process as the vector of differences between realized and planned outcomes or, equivalently, as the excess of the account balances over budget: These excesses are what accountants call "accounting variances." In particular, accounting variances can be positive or negative.

We measure time t on the unit interval. We let $Z(t)$ denote the vector of excesses over budget in each account as of time t; the vector may be positive or negative in each component. We assume that

$\{Z(t);\ 0 \leq t \leq 1\}$ is a controlled N-dimensional Brownian motion with a fixed covariance matrix denoted by $Var(Z)$. The agent can control the drift, but not the variance, of the process. In economic terms, the assumption that the agent cannot control the variance means that over short periods of time the developments in the accounts are almost entirely random; only over longer periods do the agent's systematic efforts accumulate to a quantity that is of the same order of magnitude as the accumulated random fluctuations.

Formally, we specify that Z is the solution to the following Ito stochastic differential equation:

$$dZ = \mu(t)dt + \Sigma dB ,$$

where B is a standard N-dimensional Brownian motion (with zero drift and the identity matrix as its covariance matrix), $\mu(t)$ is the drift vector, occasionally referred to as the agent's effort choice, and Σ is a non-singular matrix. Thus, the variance-covariance matrix for Z is $Var(Z) = \Sigma'\Sigma$.

For ease of presentation, we focus on the one-dimensional case for which the control equation is:

$$dZ = \mu(t)dt + \sigma dB , \tag{17}$$

where σ is a positive constant and $Z(t)$ is the excess of profit over budget at time t. The one-dimensional case corresponds to a particular limit of discrete time binomial processes. As we discussed in the previous section, the binomial case implies (i) that the sharing rule which implements any "interior" strategy for the agent is unique and (ii) that there is an optimal linear scheme. As we shall see, these results have precise analogues in the Brownian model.

Let $Z^t = \{Z(s);\ 0 \leq s \leq t\}$ be the history of profits up to time t. Then, since the agent can base his choice of $\mu(t)$ on Z^t, $\mu(t)$ is itself a stochastic process. Thus, the instructions given to the agent also take the form of a stochastic process. The compensation paid to the agent is allowed to be any function of the complete history of profits over the whole unit time interval, that is, any random variable $S(Z^1)$. Henceforth, we will often write $\mu(t)$ as $\mu(Z^t)$ to emphasize that the instructions governing the agent's behavior at time t can depend on the history of the process Z up to time t.

We assume that the agent can at each point in time choose an action μ from the finite open interval $(\underline{\mu},\overline{\mu})$. The total cost incurred by the agent over the unit time interval is $\int_0^1 c(\mu(Z^t))dt$. We assume that c is twice continuously differentiable and strictly convex on the closed interval $[\underline{\mu},\overline{\mu}]$ and $c'(\underline{\mu}) \leq 0$ and $c'(\overline{\mu}) > 1$.

We formulate the principal's problem as one of choosing a compensation scheme $S(Z^1)$ and instructions $M = \{\mu(t);\ 0 \leq t \leq 1\}$ such that (i) M is the agent's best strategy given S and (ii) the agent's expected utility, if he follows instructions, is at least $u(0)$.

As in the discrete time case, we begin by studying the agent's dynamic programming problem. To that end, fix a compensation scheme S and a set of instructions μ which are optimal for the agent, given S. Suppose that the agent has used the strategy $M' = \{\mu'(t)\}$ up to time t. Then the value to the agent of continuing optimally after time t is:

$$V(Z^t;\,M') = \max_{\mu''} E[u(S - \int_0^t c(\mu'(s))ds - \int_t^1 c(\mu''(s))ds)|Z^t]$$

$$(18)\qquad = \max_{\mu''} E[u(S - \int_t^1 c(\mu''(s))ds)|Z^t]\exp[r\int_0^t c(\mu'(s))ds]$$

$$= E[u(S - \int_t^1 c(\mu(s))ds)|Z^t]\exp[r\int_0^t c(\mu'(s))ds$$

$$= V(Z^t;\,M)C(M',\,t)$$

where $C(M',\,t) = \exp[r\int_0^t [c(\mu'(s)) - c(\mu(s))]ds]$.

Notice, too, that if M is a best strategy for the agent in response to the sharing rule S , then:

$$V(Z^t;\,M) = E[u(S - \int_0^1 c(\mu(s))ds)|Z^t] .$$

Since $V(Z^t;\,M)$ has the general form of $V(Z^t;\,M) = E[Y|Z^t]$, it is immediate that $\{V(Z^t;\,M),\,Z^t\}$ is a martingale. By construction, for any fixed M , the information (σ-field) corresponding to Z^t is the same as the information corresponding to B^t generated by the Brownian Motion. Hence, by the Martingale Representation Property of Brownian Motion (Jacod [1977]), we may write:

$$(19)\qquad dV(Z^t;\,M) = \gamma(Z^t)dB$$

for some stochastic process γ for which

$$P\{\int_0^1 \gamma^2(Z^t)dt] < \infty\} = 1 .$$

Recall that, at the agent's optimum, $dZ = \mu(Z^t)dt + \sigma dB$, and write:

$$(20) \qquad dV(Z^t; M) = [\gamma(Z^t)/\sigma]dZ - \mu(Z^t)[\gamma(Z^t))/\sigma]dt \ .$$

Equation (20) lets us study $V(Z^t; M)$ as a functional of the path Z^t without regard to whether the agent actually controls the path using M. Equation (19) holds only on the assumption that the agent actually uses the control strategy M. If instead he uses M', then since $dZ = \mu' dt + \sigma dB$, we have:

$$(21) \qquad dV(Z^t; M) = (\mu' - \mu(t))\frac{\gamma}{\sigma}dt + \gamma dB \ .$$

In economic terms, $V(Z^t; M)$ is the utility that the agent appears to be getting from the principal's point of view, when the principal observes the history Z^t and assumes that the agent has been following the instructions M. The actual probabilistic evolution of this quantity depends on the actual choices M' that the agent makes.

Now suppose that the agent has followed the optimal strategy M up to time t, and considers deviating by setting the drift to μ' over the interval $(t, t+dt)$. Call this modified strategy for the agent M'. Then in view of (17), (18), (21) and the definition of C, we must have:

$$\begin{aligned} dV(Z^t; M') &= d[V(Z^t; M)C(M', t)] \\ &= C(M', t)\{dV(Z^t; M) + V(Z^t; M)rdt[c(\mu') - c(\mu(t))]\} \\ &= C(M', t)\{(\mu' - \mu(t))\gamma(Z^t)/\sigma + V(Z^t; M)r[c(\mu') - c(\mu(t))]\}dt \\ &\quad + C(M', t)\gamma(Z^t)dB \ . \end{aligned}$$

According to Bellman's Principle of Optimality, the optimal strategy for the agent (M) must call for the agent to select the drift μ' in all states and at all times to maximize the drift of the value function, i.e., the coefficient of dt. This leads to the smooth concave maximization problem:

$$\max_{\mu'} [\gamma(Z^t)/\sigma]\mu' + V(Z^t; M) rc(\mu') .$$

The first-order necessary conditions for optimality are:

$$(23) \qquad -r\sigma c'(\mu(Z^t)) = \gamma(Z^t)/V(Z^t; M) .$$

Substituting (23) into (21) to eliminate γ would lead to a stochastic differential equation that determines V up to a boundary condition in terms of the instructions μ and the parameters of the problem. As in the discrete time case, the next main step is to invert the value function to obtain its certain equivalent, $W(Z^t; M) = u^{-1}(V(Z^t; M))$. Once we have $W(Z^1; M)$, it will be easy to recover the compensation scheme S in a form similar to (8).

To recover $W(Z^1; M)$, we apply Ito's lemma using the relation $W(Z^t; M) = -(1/r)\ln(-V(Z^t; M))$. Then, suppressing the arguments $(Z^t; M)$, and using (20):

$$(24) \qquad -r dW = [\gamma/V] dB - (1/2)[\gamma/V]^2 dt .$$

Recalling that $dZ = \mu(Z^t) dt + \sigma dB$, and substituting from (23), we get:

$$(25) \qquad dW = c'(\mu) dZ - [c'(\mu)\mu - (r/2)\sigma^2 c'^2(\mu)] dt .$$

Now, $W = W(Z^0; M)$ is the agent's certain equivalent from this problem. Given W and a stochastic process M, equation (25) can be integrated to determine the agent's certain equivalent at time 1.

$$W(Z^1;\, M) = W + \int_0^1 c'(\mu(Z^t))dZ - \int_0^1 [c'(\mu(Z^t))\mu(Z^t) - (r/2)\sigma^2 c'^2(\mu(Z^t))]dt \ .$$

The agent's certain equivalent at time 1 can also be expressed as the excess of the agent's compensation over his cost of effort. Equating those two representations of the certain equivalent quantities leads to the following result.

Theorem 6. The stochastic process M can be implemented with certain equivalent W by the sharing rule S only if S is given by:

$$(26)\quad W + \int_0^1 c(\mu(Z^t))dt + \{\int_0^1 c'(\mu(Z^t))dZ - \int_0^1 c'(\mu(Z^t))\mu(Z^t)dt\}$$
$$+ (r\sigma^2/2)\int_0^1 c'^2(\mu(Z^t))dt \ .$$

One can actually show under our hypotheses that a stochastic process M taking values in $(\underline{\mu},\overline{\mu})$ can always be implemented with any certain equivalent W by the sharing rule (26), however we shall need this result only for the special case where μ is a constant over time, and that case is covered by Theorem 4.1 of Chapter VI of Fleming and Rishel [1975].

Sharing rule (26) is analogous to (8) in the discrete time case with $N = 1$. We therefore conclude that our model is an appropriate approximation to the discrete time binomial formulation.

The sharing rule specified by (26) has a simple interpretation. The first terms provide the agent with the desired certain equivalent W plus direct compensation for his effort expenditures. The dZ term in the first line provides the incentive for effort. From the point of view of the agent's local optimization problem, only this term and the marginal cost are controlled, and the agent chooses $\mu(t)$ to set the marginal benefit equal to the marginal cost. From that incentive we subtract its expectation, so that the component of compensation in the bracketed terms in (26) has mean zero. Finally, to compensate the agent for the risk he must bear, a risk premium is paid, represented by the last term. The risk premium is proportional to the sum of the squares of the coefficients of dZ , a familiar form in exponential-normal models.

We have a representation of the principal's problem as one of picking a certain equivalent W and instructions M for the agent. Since the expected output is $E[Z(1)] = E[\int \mu(Z^t)dt]$ and the middle line of (26) has mean zero, the problem is to:

$$\max_{W,M} E[Z(1) - S] = \max \int_0^1 E[\mu(Z^t) - c(\mu(Z^t)) - (r\sigma^2/2)c'^2(\mu(Z^t))]dt - W$$

subject to $W \geq 0$ and M implementable.

We now have it by inspection that an optimum is to set $W = 0$ and to set $\mu(Z^t) = \mu^*$, where μ^* is any solution to

$$\max_{\mu}\{\mu - c(\mu) - (r\sigma^2/2)(c'(\mu))^2\}. \tag{27}$$

Our assumptions about the derivatives of c at the boundaries of the feasible region ensure that (27) has an interior optimum.

<u>Theorem 7</u>. An optimal solution to the Principal's Problem is to instruct the agent to set a constant drift μ^* that solves (27) and to pay the agent the following linear function of the end of period profits $Z(1)$:

$$c(\mu^*) + c'(\mu^*)[Z(1) - \mu^*] + (r\sigma^2/2)c'^2(\mu^*) \ . \tag{28}$$

Evidently, the solution in (28) corresponds to the solution to a static incentive problem in which the agent chooses the mean (μ) of a Normal distribution with fixed variance (σ^2) and the principal is constrained to offer a linear rule. Thus, we can analyze the dynamic problem as a static problem with the "ad hoc" constraint that only linear rules are permitted, knowing that the constraint is not really ad hoc, but justified by a full-blown intertemporal incentive problem. In view of the fact that the static problem has no solution at all without the linearity restriction, this is a rather striking demonstration of how enrichment of the action space can make matters much simpler. In the next section we will give some examples that illustrate the computational convenience that results from being able to solve the problem with the linearity restriction.

Another advantage of linearity is that one can compare the agent's effort level in second-best with that of first-best. In common moral hazard analyses this is often difficult. We have the following result directly from (27):

Theorem 8. The agent's choice of drift is lower in second-best than first-best. It is also lower than what the principal would desire given the optimal incentive scheme.

Remark: Note that this partial equilibrium result extends to a model of market equilibrium as well, because the agent's certain equivalent, which will be determined in equilibrium, has no impact on the choice of action.

The analysis of the one-dimensional case can be generalized to controlled vector Brownian motion. We will not provide any details, but merely state the conclusion. Letting $\pi = (\pi_0, \ldots, \pi_N)$ be the vector of profit weights for the accounts so that total profit is $\pi \cdot Z(1)$ we have:

Theorem 9. An optimal solution to the principal's problem is to instruct the agent to set a constant vector of drift μ^*, which solves:

$$\max_{\mu}(\pi \cdot \mu - c(\mu) - (r/2)c'(\mu)^T \mathrm{Var}(Z)c(\mu)) .$$

where $c'(\mu)$ is the gradient of the cost function. The corresponding optimal payment scheme is

$$c(\mu^*) + c'(\mu^*) \circ [Z(1) - \mu^*] + (r/2)c'(\mu^*)^T \mathrm{Var}(Z)c'(\mu^*) . \tag{29}$$

For the multi-dimensional case, we do not have a result corresponding to Theorem 8, unless, of course, the N dimensions of the Brownian process are independent, and the cost function separable.

As in the general discrete time model, the optimal scheme in (29) is linear in the final account balances, but not necessarily linear in profits. There is aggregation over time, but not across accounts. Evidently, the N-dimensional Brownian model is the right limit of the multinomial discrete time case discussed earlier, provided that the principal can distinguish events in different accounts even when actions are very frequent and individual outcomes of very minor importance, because the N-dimensional model preserves the profit levels generated by any outcome path and the associated frequency information in the accounts.

What does this mean for the interpretation of the one-dimensional Brownian model? It means that the model only serves as a good approximation for a discrete time binomial process (that is, a process with exactly two possible outcomes at each stage) if indeed the principal is assumed to be able to observe the detailed composition of profits. This appears very restrictive. However, the scope of the one-dimensional Brownian model --

particularly the result that the optimal scheme is linear in profits--is vastly expanded if we make assumptions that limit the principal's ability to observe the detailed composition of profits.

Specifically, let us assume that we begin with a discrete time model in which all that the principal can observe is the final level of profits $X(T)$. Now one might expect, based on the preceding discussion, that the one-dimensional Brownian model becomes an appropriate approximation, because the loss of information that occurs in the passage to the limit is inconsequential as the principal does not have that information at any point in the discrete time model either. It seems reasonable therefore to conjecture that the optimal solution to a frequently repeated multinomial process is approximately linear since it is linear for the one-dimensional Brownian model. Undoubtedly, one could prove an approximation theorem of this kind, but not without some additional assumptions. To give an indication of what can go wrong in general, consider a trinomial case with outcomes $x_0 = 0$, $x_1 = 1$, $x_2 = q$. Suppose there are T repetitions. Total profits are then $\pi(T) = n_1 + qn_2$, where n_0 , n_1 and n_2 are the number of times each individual outcome occurred; of course, $n_0 + n_1 + n_2 = T$. If q is an irrational number, then for any possible $\pi(T)$, one can recover uniquely n_0 , n_1 , and n_2 . In other words, observing $\pi(T)$ is as good as observing the account balances n_0 , n_1 , and n_2 . Consequently, the restriction on observability has no force and the one-dimensional Brownian model will not serve as a good approximation no matter how large T is, because it will not preserve the information content of the process.

The problem identified above depends on an implausible trick; the principal is able to decode a single one-dimensional statistic into N such statistics. Evidently, this is a problem of modeling--not one of economics. Rather than exploring the problem systematically, we finesse it by proceeding with a formulation in which the principal cannot untangle the N components of profits from the profit account alone.

Theorem 10. Let the agent control the vector of drift of an N-dimensional Brownian process $Z(t)$ with fixed covariance matrix. Assume accumulated profits at time t are a linear function of the balances in the N accounts, i.e. $\pi(t) = \pi \cdot Z(t)$, and that the principal is only able to observe the end-of-period level of profits $\pi \cdot Z(1)$. Then the optimal incentive scheme is linear in end-of-period profits.

Proof. Since the sum of any linear combination of normal variables is normally distributed, it is clear that the profit process $\pi \cdot Z(t)$ is a one-dimensional Brownian motion. The agent controls the drift (but not the variance) of this one-dimensional profit process by his choice of the vector $\mu(t)$. Each $\mu(t)$ implies a profit drift $\nu(t) = \pi \cdot \mu(t)$. Since only the choice of $\nu(t)$ matters for the final payoff to the agent (because payments are based on $\pi \cdot Z(1)$), it is clear that for any $\nu(t)$, $\mu(t)$ will be chosen so as to minimize the cost $c(\mu(t))$ subject to $\nu(t) = \pi \cdot \mu(t)$. If we replace $c(\mu(t))$ with the corresponding minimum cost function $c(\mu(t))$, we find that the principal's problem is one of providing incentives for the agent's control of the drift of a one-dimensional Brownian process at an instantaneous cost $c(\mu(t))$. The solution is linear in $\pi \cdot Z(1)$ according to Theorem 7.

Q.E.D.

In a similar vein, if the agent exercises some discretion over the accounting system, the principal may not be able to rely on the separate accounts. For example, suppose the agent has considerable latitude concerning which of several expense accounts to use for recording common items. Then, the relevant cost function is $c(\mu) = c(\mu_1 + \ldots + \mu_N)$, so a direct application of (29) establishes that the optimal scheme is linear in profits. Similarly, if the agent had enough latitude to control the allocation of expenses, but only within the expense categories, and the allocation of revenues, but only within the revenue categories, and the optimal scheme will be a linear function of revenues and expenses separately, but not of profits. Thus, our model is consistent with the idea that it may be desirable to use "cost centers" (and "revenue centers") rather than "profit centers" for evaluating managerial performance.

We have used the N-dimensional Brownian model as an approximation to the frequently repeated discrete time model with N+1 outcomes. In the standard theory of weak convergence of stochastic processes (c.f. Billingsley [1968]), processes like ours which are derived from summing multinomial random variables in which a single period outcome is always inconsequential are approximated by a <u>one</u>-dimensional Brownian motion. However, as Milgrom and Weber [1984] have shown, when the important economic issue is one of information, the topology of weak convergence is an inadequate notion of approximation. We have used an N-dimensional Brownian process here to approximate a multinomial with N+1 outcomes because this allows us to approximate both the profits and the relevant accounting information accurately.

We have found, however, that the N-dimensional controlled Brownian model can be adequately represented by a model of lower dimension when the agent has some control over the accounting process, or when the principal can observe only the total profits, and not the account balances that make up the profits. Evidently, this result considerably expands the applicability of the one-dimensional Brownian model; or put differently, it offers an alternative, much more encompassing interpretation of that model.

To conclude this section, we wish to point out that the Brownian models are not the only continuous time models one might consider as approximations of the model of Section 3. An alternative is a model in which the agent controls the rates of jump λ of N Poisson processes subject to a cost function $c(\lambda)$. Like the Brownian model, the Poisson model can be viewed as an approximation to a frequently repeated multinomial model, but the Poisson model replaces the assumption that each outcome is "very frequent" with the assumption that each outcome is far less frequent than outcome 0 (which might be interpreted as the outcome that "Nothing unusual is happening just now"). The results for the N-dimensional Poisson model resemble those for the Brownian model, with one major exception. Using the Poisson model, one can prove (i) a theorem like Theorem 6 that every implementable strategy has a unique implementing rule which can be written as a stochastic integral, (ii) theorems like Theorems 7 and 9 asserting the optimality of linear rules, and (iii) a theorem like Theorem 8 asserting that the agent puts forth too little effort. The one major difference between the two models lies in Theorem 10 and the subsequent discussion, which have no analogues for the Poisson case.

To better grasp the one key difference between the two continuous time models, consider the problem of compensating an industrial salesman whose efforts control (and are identified with) the arrival rates of large and small sales in a controlled Poisson process. Large sales are $5,000,000 and small ones are \$75,000. Assume that mark-ups are the same on all sales, so observing sales volume and observing profits amounts to the same thing. Only the total sales in a period can be observed. When the actual sales in some period are \$6.5 million, one can infer with certainty that the agent made one large sale and twenty small ones in the period. One can further infer from this that the agent did not devote his effort to large sales only or to small sales only and, indeed, that the maximum likelihood effort allocation is (1,20), where the components are the arrival rates of sales of the two kinds. Statistical inferences like that have no analogue in the Brownian model: In the Brownian model, all strategies for the agent that lead to the same instantaneous drift rate of total profits at all points in time also lead to the same distribution of end-of-period profits. Even strategies which assign zeros to some categories of revenue and expense cannot be distinguished from strategies with all nonzero drifts provided that the drift rates of total profits are identical.

5. Applications

The methodological value of our linearity results rest largely with the ease with which one can compute optimal solutions. One can solve the dynamic problem as a static one with the *ad hoc* restriction that the sharing rule is linear. Below we will offer some variations on an example that illustrates how simple the analysis becomes when the linearity restriction is appropriate. At the same time we wish to caution the readership against indiscriminate application of this *ad hoc* principle. It requires some fairly restrictive assumptions on the information and production technologies. The limitations will be discussed in more detail in the next section.

We begin by computing a closed form expression for the linear rule when the cost of varying the drift μ in the one-dimensional Brownian model is quadratic, for example, $c(\mu) = (a/2)\mu^2$. For the purpose of comparing the second-best solution with the first-best solution, let us first compute the latter. If the agent's action were observable and possible to control independently of the sharing rule, then the agent would be paid the constant amount $c(\mu)$ as compensation for his action choice μ and μ would be set so as to maximize $\mu - (a/2)\mu^2$. Thus, in the first-best solution, $\mu = a^{-1}$. This arrangement would yield a net return to the principal equal to $\pi = (2a)^{-1}$.

When there is a moral hazard problem, the constant payment rule would, of course, no longer be optimal. But some other linear rule $s(z) = \alpha z + \beta$ will be optimal according to Theorem 7. In order to find the best values for α and β , we first calculate the agent's best response against $s(z)$, given that z is normally distributed with mean μ and variance σ^2 . The agent's certain equivalent under $s(z)$ is given by:

$$(30) \qquad \alpha\mu + \beta - c(\mu) - (r/2)\alpha^2\sigma^2 .$$

The first-order condition is $\alpha = c'(\mu) = a\mu$, i.e. we set the slope of the incentive scheme equal to marginal cost at the desired level of action. Substituting this value for α into (30) and setting the resulting certain equivalent to zero, we get that

$$(31) \qquad \beta = -\tfrac{1}{2}a\mu^2 + (r/2)\sigma^2a^2\mu^2 .$$

This implies that the only linear sharing rule that implements μ with certain equivalent zero is:

(32) $$s(z) = \frac{1}{2}a\mu^2 + a\mu(z-\mu) + (r/2)a^2\mu^2\sigma^2 ,$$

which is the expression we have in (21). The principal's payoff with this arrangement is:

(33) $$\pi = \mu - \frac{1}{2}a\mu^2(1 + ra\sigma^2) .$$

Maximizing over μ gives:

(34) $$\mu^* = (1 + ra\sigma^2)^{-1} a^{-1} .$$

Inserting μ^* into (33) gives the principal's net return:

(35) $$\pi^* = (1 + ra\sigma^2)^{-1} (2a)^{-1} .$$

The optimal slope for the sharing rule is

(36) $$\alpha^* = (1 + ra\sigma^2)^{-1} .$$

Comparing the first-best values of effort and profit with μ^* and π^*, we see that the moral hazard problem causes a reduction in the action (as Theorem 8 stated) as well as a reduction in the principal's net payoff. Both reductions are proportional to $(1 + ra\sigma^2)^{-1}$. Thus, a small degree of risk aversion or uncertainty, or a low cost of effort will allow a solution close to the first-best.

The fact that expected profits increase with a reduction in uncertainty is consistent with the general result that additional information is valuable for the agency problem (Holmstrom [1979] and Shavell [1979]). A reduction in σ^2 can be viewed as stemming from additional monitoring of the agent's activities in the following way. Let $z = \mu + \varepsilon$ be a one-period representation of the dynamic problem with $\varepsilon \sim N(0, \sigma^2)$. Let y be another signal that the principal can observe which is correlated with z . For simplicity (though it is immaterial), assume y is an exogenous signal unaffected by μ --for instance, an observation of general economic conditions or of the performance of other agents. Let y be normally distributed with variance γ^2 and write γ for the correlation coefficient between z and y .

We may now think of (z,y) as end-of-period balances of a two-dimensional Brownian process and apply Theorem 9. The optimal sharing rule will be a linear function $s(z,y) = \alpha_1 z + \alpha_2 y + \beta$. It is then easy to calculate the optimal slopes in the same way as above. They are given by:

$$\alpha_1 = (1 + ra\sigma^2(1 - \gamma^2))^{-1} ,$$

$$\alpha_2 = -\sigma^{-1}(1 + ra\sigma_1^2))^{-1} = -\gamma\sigma^2(\gamma^2)^{-1}\alpha_1 .$$

The optimal action to induce and the principal's maximal net profits are:

$$\mu^* = a^{-1}(1 + ra\sigma^2(1 - \gamma^2))^{-1} ,$$

$$\pi^* = (2a)^{-1}(1 + ra\sigma^2(1 - \gamma^2))^{-1} .$$

From the formulas above we see that observing the additional signal y is equivalent to a reduction of the variance from σ^2 to $\sigma^2(1 - \gamma^2)$ as we claimed. The reduced variance is the conditional variance of ε given y as one could have guessed directly. If y and z are perfectly correlated, that is as good as observing ε and z , hence μ , and first-best can be achieved.

In accordance with earlier results on the value of relative performance evaluation (see e.g. Holmstrom [1982]) we find, of course, that the optimal compensation does depend on the uncontrolled variable y. But in contrast to general informativeness results, which state that there is <u>some</u> positive value to <u>costless</u> monitoring (or relative performance evaluation), it is here possible to compute the <u>net</u> value using the explicit formulas.

One can also extend the analysis to situations where the principal can monitor the agent at different levels of intensity. A natural way of doing it is to let him observe at a unit cost any number of signals $y_i = \varepsilon + \theta + \eta_i$, with η_i, ε and θ independent and normally distributed. The mean of the y_i's will then be a sufficient statistic for the additional information and one finds easily the implied reduction in the variance as before. It is also clear that the principal will engage in some additional monitoring if the unit cost of additional signals is sufficiently low. On the other hand, the optimal number of additional signals will always be finite because of diminishing returns (the incremental reduction in variance goes to zero). On its own, our suggested analysis of monitoring only confirms the obvious, but placed in a richer economic context it is likely to prove quite useful.

Proceeding along the same lines as above one can also study the value of diversification due to agency costs. In the simplest case, suppose the choice is between one project: $z = \mu + \varepsilon$ or two "half-sized" projects: $z_1 = \mu_1 + \frac{1}{2}\varepsilon_1$, and $z_2 = \mu_2 + \frac{1}{2}\varepsilon_2$, with ε_1 and ε_2 having the same variance as ε and μ_1 and μ_2 being the allocation of effort between the two. Even if the principal only can observe $z_1 + z_2$ it is clear that diversification pays as long as ε_1 and ε_2 are not perfectly correlated (positively) and the cost of effort is $c(\mu_1 + \mu_2)$. Diversification simply amounts to a reduction in the variance, as one would expect.

Another related question one might be interested in is the effect of letting the agent control a bigger project, e.g. have $z = n(\mu+\varepsilon)$; $n > 1$. The answer is easily obtained by writing $\mu_n = n\mu$ and $\varepsilon_n = n\varepsilon$. This transformation brings us back to the base case with a cost function $c(\mu_n) = (1/2)\mu_n^2/n^2$ and variance $\sigma_n^2 = n^2\sigma^2$. Substituting into (36) we see that the slope of the optimal incentive scheme remains unchanged. However, effort (and profits) will increase linearly in n . This suggests that the quadratic cost function is not very suitable for large values of μ . Obviously, other specifications of the cost function will result in different conclusions, but the point is that one is able to get precise answers of this kind.

As a final variation of the basic example, suppose the agent can allocate his effort between two activities $z_1 = \mu_1 + \varepsilon_1$ and $z_2 = \mu_2 + \varepsilon_2$ with a cost function $c(\mu_1, \mu_2) = \frac{1}{2}\mu_1^2 + \frac{1}{2}\mu_2^2$. Thus, there are diminishing returns to effort in each project. If the principal can only observe $z = z_1 + z_2$, then the solution will obviously be linear in z and the agent will choose $\mu_1 = \mu_2$ by symmetry. Somewhat surprisingly, the situation is rather different if z_1 and z_2 can be separately observed. By Theorem 9 the optimal scheme will be of the linear form $s(z_1, z_2) = \alpha_1 z_1 + \alpha_2 z_2 + \beta$, but α_1 need not be equal to α_2 . In fact, the only case for which the two are set equal is if ε_1 and ε_2 have the same variance. In general, $\alpha_1 > \alpha_2$ if $\sigma_1^2 < \sigma_2^2$. In the case of independent projects, the slopes are simply determined by (36) (with σ_1^2 or σ_2^2 in place of σ^2). It follows that there is generally value to observing the agent's different activities and also that incentive weights among different projects (or accounts) are not solely a function of the marginal cost or product of the activity, but also its variance. As an example, if the agent can allocate time between reducing costs or increasing revenues and is equally effective at both, and if revenues are subject to more exogenous variance than costs, then the incentive scheme should not be based on profits alone; it should reward cost reductions more highly than revenue increases.

6. Variations and Limitations of the Model

There are some significant restrictions on the applicability of our results. On the other hand, they also survive some interesting generalizations. Our purpose here is to discuss both extensions and limitations with the hope that they give a good idea of the overall relevance of the model.

Risk Averse Principal

If the principal is risk averse with a constant coefficient of absolute risk aversion (i.e. he has an exponential utility function) the main conclusions of the paper remain unchanged. This follows from the fact that Theorem 2 is still valid as can easily be checked. In the multi-period model the same dynamic programming arguments apply leading to the conclusion in Theorem 4. It is evident therefore that the aggregation and linearity results of the Brownian models also remain true, though the specific form of the optimal sharing rule and the optimal action to implement will change.

As an illustration of the impact of a risk averse principal, reconsider the earlier example (with $a = 1$, for simplicity) when the agent's coefficient of absolute risk aversion is r_A and the principal's is r_P. Given that the agent's sharing rule is linear it is easily found that the slope of the sharing rule is

$$\alpha = (1 + \sigma^2 r_P)/(1 + \sigma^2(r_A + r_P)) \quad . \tag{37}$$

Since $a = 1$, we have $\mu = \alpha$. Evidently, (37) generalizes the optimal sharing rule in the example (which has $\alpha = (1 + r_A\sigma^2)^{-1}$). The impact of a risk averse principal relative to a risk neutral one is that the agent's share increases and his effort μ does, too.

First-best in this situation is to choose $\mu = 1$ and set $\alpha = r_P/(r_P + r_A)$. This corresponds to optimal risk-sharing. Thus, second-best again dictates a lower action and a higher share for the agent than in the first-best. Note also that if the variance is very high, the second-best solution is close to the first-best solution in terms of risk sharing, while μ goes to zero. The reverse is true if variance is very low.

Cost/Production Shocks

In the discrete time model we could allow almost any kind of shocks to the cost or production function. The shocks could be observed privately or publicly, either before or after the agent takes an action (or they could be observable only partly or not at all). The only important restriction was that the shocks were independent over time.

When we took the discrete time model to a continuous time limit, we eliminated the stochastic element from costs in order to sidestep technical problems in defining a continuum of independent observed shocks over time. One way to deal with the agent's private information about costs and output is to allow the agent to observe processes $Y(t)$ and $Y'(t)$ which affect accumulated costs $C(t)$ and output $Z(t)$, as follows:

$$dZ(t) = \mu(t)dt + \sigma dB(t) + dY(t) \tag{38}$$

$$dC(t) = c(\mu(t))dt + dY'(t) \quad . \tag{39}$$

With these specifications of cost, the model retains its basic time-separable structure, and the main results (uniqueness of the implementing rule, optimality of linear rules) are unaffected.

A more interesting generalization is one in which the agent has private, advance knowledge about the timing of costs. Suppose $y(t)$ is a random disturbance term in the agent's cost function in the Brownian model. Further suppose that for each realized path $y(t)$, $\int_0^1 y(t)dt = 0$, i.e. there is no aggregate uncertainty. The instantaneous cost of effort is $c(\mu(t) + y(t))$. We make no further assumptions about the distribution of $y(t)$, and we allow that the agent may observe the entire path of $y(t)$ at time 0; we require only that the agent know $y(t)$ before acting at time t. Then, we claim, the solution remains linear in $Z(1)$.

The proof follows by a simple transformation of variables. Suppose the principal observes $y(t)$. Let $\mu'(t) = \mu(t) - y(t)$. Then the $Z'(t)$ process, constructed like $Z(t)$ with $\mu'(\cdot)$ in place of $\mu(\cdot)$, is:

$$dZ'(t) = \mu'(t)dt + \sigma dB(t) ,$$

where $Z(1) = Z'(1)$, because $\int_0^1 y(t)dt = 0$ for all realizations. We are back to our original model and the solution must therefore be a linear scheme with $\mu'(t) = \mu^*$. That means, it is optimal for the agent to choose:

$$\mu(t) = \mu^* - y(t) \quad \text{for every } t .$$

Notice that the solution would be unaltered even if the principal could observe the path of $y(t)$ at time 0. We conclude therefore that there is no cost to the principal in this case from ignoring transfers of resources across time by the agent, because the agent's interests in this matter do not conflict with those of the principal. Indeed, the principal does not need to know which night the agent "goes bowling" in order to fashion an optimal compensation scheme, which is another kind of aggregation result that reduces the principal's need for information. Note well, however, that this conclusion depends on the Brownian model, the specific way in which we have parameterized costs, and the assumption that $y(t)$ is only a timing effect, i.e., that it integrates to zero.

Wealth Restrictions

If the agent has finite wealth it may be argued that strictly speaking a linear scheme is infeasible because the range of a normal distribution is unbounded below. However, the following feasible scheme is approximately linear and approximately optimal: As long as the agent's certain equivalent stays above the minimum payment level, stick to the original payment plan; if the agent's certain equivalent with the linear scheme reaches the minimum payment level, instruct the agent to stop working and pay him the minimum at the end. From the stochastic integral representation of implementatable strategies, we see that this scheme will leave the agent's strategy unaltered as long as the minimum is not reached. If the wealth constraint is reached with only a small probability, the new scheme is approximately optimal (and linear). An alternative, approximately optimal scheme is to pay the agent a linear function which is truncated at the minimum payment level.

Thus, the linear rule features a very desirable robustness property, which is notably missing in the fine-tuned schemes that we discussed in the introduction. For the extreme punishment schemes, wealth restrictions have significant ramifications. They may perform very poorly and become far from optimal.

Quitting Option

One may wonder what happens if the agent can quit in the middle of the evaluation period. This depends on what is assumed about the agent's wealth and ability to bond himself. If the agent cannot be forced to pay anything to his present firm and he can switch costlessly to another firm at any point in time, he will quit with probability one in the Brownian model. His certain equivalent will dip below zero for sure in any initial interval. On the other hand, if the agent can be held to pay his certain equivalent (if it is negative) at the time of quitting, then there is no incentive to quit, because after having cleared his account with the present employer he can do no better on a new contract and he might as well stay. This applies equally in the case the agent contemplates to quit when he has accumulated a positive certain equivalent. There is a provision though. We are assuming implicitly that the next firm will offer a contract with zero certain equivalent. If firms are not making excess profits (i.e. they do not receive rents) that will be the case (having the certain equivalent zero was only a normalization, recall).

The importance of the agent's ability to quit obviously depends on mobility costs. When these costs are large then, even with wealth constraints, the agent will quit with a low probability, which makes the option insignificant.

Model Restrictions

Next we turn to the limitations of the model. Throughout both the discrete and continuous time models, the optimality of linear rules has rested on the absence of income effects and on a stochastic environment characterized by stationary independent increments. These assumptions are crucial. Without them, optimal schemes could not be based on time aggregates alone. Yet, we do not feel that these assumptions should be viewed as enormously restrictive. Whenever income effects can be deemed small we expect linear rules to be nearly optimal.

For the Brownian model, the conclusion that the optimal compensation scheme is only a function of final account balances depends on the specification that the agent does not control variances. If the agent did control variances, the principal would also want to keep track of a statistic that estimates the total variance over time, but the new problem retains its essential time stationarity. Despite the trickier accounting involved, the analysis involves no new principles.

The extra results obtained from the Brownian model--the explicit linear form of the sharing rule and the possibility of aggregation over accounts--depends on exact properties of the normal distribution. The possibility of aggregating over expense accounts or revenue accounts or both arises in this model when the agent has some control over the accounting system or the separate accounts are not reliably observable. Although the particular form of our aggregation results does depend on our normality assumption, the economic sense of the results is clear and can be extended to obtain related results in related (non-Brownian) models.

Our assumptions about the agent's information in all the models is restrictive and deserves further comment. First, the agent is assumed to have no private information at the time of contracting, that is, there is no problem of adverse selection. If there were a problem of adverse selection, then there would be moral hazard even if the agent were risk neutral. For example, suppose the principal were an inventor who contracts with an agent to market an invention. Being a marketing expert, the agent knows more than the principal about the product's market potential and the level of the agent's marketing effort will play a key role in determining the product's sales. It is standard in these combined moral hazard--adverse selection models that the principal should offer the agent a menu of contracts, and the question then becomes: When will the contracts in the menu all be linear functions of aggregates? We have not studied this question, though we expect that it is susceptible to a variant of our methods. (Likewise, it may be possible to find simpler and sharper results in the analysis of taxation, by exploiting richness of action choice along the lines we have pursued.)

Finally, we have assumed that the agent can monitor how the process is evolving over time. In the Brownian model, if the agent were to receive no information until the end of the compensation period, then the situation would be the same as in a one-period model where the agent chooses the mean, once and for all, of a normal distribution. We would be back to a static model with no solution (unless bounds on wealth are imposed). At the opposite extreme, if the agent could observe $B(1)$ at time zero, we would be in the Informed Agent model, and once again the solution would probably be non-linear. The Brownian model, which lies in between these two extremes, is likely to be the only one of the three in which a linear sharing rule is optimal.

7. Conclusion

There are two main ideas that motivate the kind of analysis we have pursued. The first is that one need not always use all of the information available for an optimal incentive contract. Accounting information which aggregates performance over time is sufficient for optimal compensation schemes in certain classes of environments and it is sometimes possible to aggregate further over the various accounts. The second idea is that optimal rules in a rich environment must work well in a range of circumstances and will therefore not be complicated functions of the outcome; indeed, in our model, linear functions are optimal.

Models that derive optimal rules in which small differences in outcomes lead to large differences in compensation are invariably based on an assumption that the agent finds it impossible, or very expensive, to cause small changes in individual outcomes. The optimal rule in such cases is usually inordinately sensitive to the distributional assumptions of the model. For example, in the model where the agent makes a one-shot choice that determines the mean of a normal distribution, by changing the distribution of outcomes for each action on a set of probability $\varepsilon > 0$, some of the near-optimal rules derived for that model can be made to perform worse than a flat compensation scheme, which provides no incentives at all for the agent to incur costs to increase production.

Linear rules, in contrast, are strikingly robust. For example, in the Brownian model, the agent's optimal response to a linear rule and the principal's expected payoff do not depend at all on the timing of the agent's information. Nor does this conclusion depend on normality: The agent's optimal response to a linear rule where the agent adds drift to any stochastic process is always the same.

It is probably the great robustness of linear rules based on aggregates that accounts for their popularity. That point is not made as effectively as we would like by our model; we suspect that it cannot be made effectively in any traditional Bayesian model. But issues of robustness lie at the heart of explaining any incentive scheme which is expected to work well in practical environments.

Finally, a brief remark on alternative dynamic models of agency is in order. Radner (1982) and Rubinstein (1982), among others, have developed multi-period models of moral hazard with distinctly different conclusions. They have shown that moral hazard models may yield close to first-best solutions when repetition permits very precise monitoring of the agent's strategy. We have deliberately designed our model so that repetition does not lead to first-best outcomes. The ingredients that assure that are: a finite horizon (rather than an infinite horizon with little or no discounting), a stochastic process in which uncertainty about the agent's actions remains significant[3] and evaluating the agent's utility at the end of the horizon rather than continuously, so as to make sure that the agent's marginal utility over income does not converge to a constant (i.e. to risk-neutrality) as time evolves.

We do not wish to argue that our modeling strategy is more realistic than that of Radner and Rubinstein; evidently, that depends on the context. Also, as we have tried to stress above, our main interest is not so much with dynamics *per se*, as with the issue of robustness. To us dynamics has mainly been a convenient (and we do not think an urealistic) vehicle for demonstrating that a rich action space may imply simple incentive schemes; for those who like it better, our model can be seen as a static one instead.

A major reason for our focus on simple schemes bears repeating, too. It seems clear that second-best analyses of moral hazard will be less helpful in reaching the ultimate objective of understanding the ramifications of organizational design on aggregate economic phenomena, unless the answers to the "small" design problems are simple and easy to work with in larger models. (Note that with this objective it is meaningless to set up models that yield first-best solutions.) It remains to be seen to what extent our linear incentive schemes will prove useful in this larger task and also whether there are more natural ways to translate robustness into simple and operative solutions of the small organizational building blocks that principal-agent models represent.

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FOOTNOTES

1. There is a trivial and uninteresting sense in which it is always possible to make the optimal compensation of the agent a linear function of a single numerical "aggregate," namely, his optimal compensation rule can be expressed as a function of half that rule, and the later is a single numerical aggregate. When we say that compensation can be based on aggregates, we have something more sensible in mind. First, the aggregate must be determined as a linear function of some separately observed variables, such as the profits earned in two different periods of time. Second, it must be a "natural aggregate" whose definition does not depend on such features of the problem as the agent's risk aversion or the costs of various actions the agent may take. For example, accounting systems are ideally based on aggregates of this sort. Account balances are accumulated sums over time, and they are defined in a way that depends only to a limited extent on the tastes of the manager.

2. "One-dimensional" refers to the dimensonality of P (see section 2) rather than the dimensionality of the agent's economic action. Thus, an effort model of standard type is not in general one-dimensional, because the convex hull of the distributions that effort can induce is not. This is why the "firt-order approach" often fails. See Holmstrom (1984).

3. We would get a result like Radner and Rubinstein if we let the binomial multi-period model converge to a Brownian model with zero variance. A linear scheme - one with unitary slope - would be optimal in that case.