# AIR DRAG EFFECT ON A SATELLITE ORBIT DESCRIBED BY DIFFERENCE EQUATIONS IN THE REVOLUTION NUMBER*

BY

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**Abstract.** For a satellite on an orbit which is affected by air drag, difference equations are derived whose solutions express changes in orbital size and shape and give the satellite's time behavior as functions of revolution number. The results are obtained from a first order perturbation theory using a small air drag parameter.

**Introduction.** The behavior of a satellite orbit under the effects of air drag have been discussed by Petersen [1][1], the present author [2], and others. Using the method of variation of parameters, I derived some approximate expressions for the decay of eccentricity with radius, the decay of radius with revolution number, and the growth of time with revolution number. Those results appear to be especially useful for eccentricities which are quite small, say $\epsilon < 0.01$, although a complete investigation of their range of validity has not been investigated.

An alternative approach is available which may be preferable for larger eccentricities. It consists of a perturbation method in a small parameter, carried only to the first approximation, followed by the derivation of an exact relationship between values at the beginning and the end of any orbital revolution. Thus one obtains a set of difference equations in the values of the problem's dependent variables which occur at each full orbital revolution.

Because the approach leads to difference equations, it appears well suited to numerical treatment. It offers a significant advantage even when a high speed digital computer is available, for it obviates the numerical integration around each orbital revolution which can impose a significant burden on machine storage capacity and result in a prohibitively long solution time. It is clear that in a conventional numerical solution in which the satellite is followed step-wise around each revolution, even the finest practicable integration interval and the greatest care in handling errors may not avoid a significant accumulation of error over tens of thousands of revolutions. On the other hand, the difference equation method developed here allows the errors for any complete revolution to be made as small as desired without excessive difficulty.

**Outline of the method.** Like [2], this treatment begins with the drag law and equations of motion discussed by Petersen [1][2]. The following notation is used:

- $r$ radial distance from center of earth to satellite
- $\beta$ angular advance of satellite from arbitrary interially-fixed reference line in plane of orbit
- $K$ product of earth mass and constant of gravitation

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[1]Numbers in square brackets refer to the bibliography at the end of the paper.

[2]These apply only to the case where the aerodynamic force acts only as a drag along the line of the velocity vector. Lift forces are assumed absent. Other assumptions are given with the notation below.

$C_D$ drag coefficient, assumed constant
$A$ satellite projected area on plane normal to velocity vector, assumed constant[3]
$m$ satellite mass
$a$ equatorial radius of earth
$h$ altitude, $r - a$ (neglecting oblateness of earth)
$\rho(h)$ standard stable atmospheric density function

In these terms, the equations of motion are

$$r^{\cdot\cdot} - r\beta^{\cdot 2} = -\frac{K}{r^2} - \frac{C_D A}{2m}\,\rho(h) r^{\cdot}(r^{\cdot 2} + r^2\beta^{\cdot 2})^{1/2}, \tag{1}$$

$$r\beta^{\cdot\cdot} + 2r^{\cdot}\beta^{\cdot} = -\frac{C_D A}{2m}\,\rho(h) r\beta^{\cdot}(r^{\cdot 2} + r^2\beta^{\cdot 2})^{1/2}. \tag{2}$$

It is convenient to introduce auxiliary dimensionless variables

$$\xi = a/r, \tag{3}$$

$$\eta = Ka/(r^2\beta^{\cdot})^2, \tag{4}$$

and to regard them as functions of $\beta$. Primes henceforth denote differentiation with respect to $\beta$ as the new independent variable. Also define the parameter.

$$\nu = C_D A a/m. \tag{5}$$

The equations which are basic in the sequal then become

$$\xi'' + \xi = \eta, \tag{6}$$

$$\eta'/\eta = \nu\,\frac{\rho(a\xi^{-1} - a)}{\xi}\,(1 + (\xi'/\xi)^2)^{1/2}. \tag{7}$$

A key assumption is made, as in [2], that the density function is locally exponential. That is, in a restricted neighborhood of any particular $\xi$-value, the function $\rho(h)$ can be approximated by

$$\rho(a\xi^{-1} - a) = e^{\alpha\xi+\delta} \tag{8}$$

where, in general, $\alpha$ and $\delta$ vary slowly with the $\xi$ (altitude) level at which $\rho$ is evaluated.

The following procedure is used.

1.$\xi(\beta)$ and $\eta(\beta)$ of Eqs. 6 and 7 are expressed as power series in $\nu$ (second and higher order terms of which are subsequently neglected).

2. The zeroth approximation, elliptic motion, is found.

3. The first approximations to $\xi(2\pi)$, $\xi'(2\pi)$ and $\eta(2\pi)$ are found in terms of $\xi(0)$. $\xi'(0)$, and $\eta(0)$ as series in increasing powers of eccentricity, under the single assumption that $\rho(\xi)$ can be approximated by an exponential function over the range of $\xi$ represented in one revolution, as described above.

4. These values of $\xi(2\pi)$, $\xi'(2\pi)$, $\eta(2\pi)$ are used as initial conditions for the next revolution, and the same procedure is followed.

[3]This is the case if the satellite is spherical, and is closely true for other vehicles whose attitude relative to the earth is kept nearly constant by a suitable control system.

5. The results are precisely the same for connecting the initial conditions for the $(j + 1)$st revolution to the initial conditions for the $j$th revolution.

6. The difference equations are returned to the original physical variables of interest.

**Perturbation solution.** Let $\xi_j$, $\xi'_j$, $\eta_j$ $(j = 1, 2, \ldots)$ represent values of $\xi$, $d\xi/d\beta$ and $\eta$ at the beginning of the $j$th revolution and $\xi_j(\beta)$ and $\eta_j(\beta)$ represent the functions $\xi(\beta)$ and $\eta(\beta)$ during the $j$th revolution. Let $\xi_1 = a/r_1$, $\xi'_1 = -aV_V/r_1V_H$ and $\eta_1 = Ka/r_1^2V_H^2$ be given initial conditions, in terms of initial radius, horizontal velocity, and vertical velocity. However, there is no loss in generality in supposing that $\beta$ is measured from a point at which $\xi'_1 = 0$.

Assume that $\xi$ and $\eta$ can be expanded in power series in a small parameter $(\nu\rho_0)$, where $\rho_0$ is any value of density such that $\rho(h)/\rho_0$ is of the order of unity in the altitude region of interest. In particular, it may be very convenient to choose $\rho_0 = \rho(a\xi_1^{-1} - a)$. This implies the representation

$$\xi_j(\beta) = \xi_j^{(0)}(\beta) + (\nu\rho_0)\xi_j^{(1)}(\beta) + (\nu\rho_0)^2\xi_j^{(2)}(\beta) + \cdots, \tag{9}$$

$$\eta_j(\beta) = \eta_j^{(0)}(\beta) + (\nu\rho_0)\eta_j^{(1)}(\beta) + (\nu\rho_0)^2\eta_j^{(2)}(\beta) + \cdots. \tag{10}$$

The zeroth approximation during the $j$th revolution satisfies

$$\xi_j^{(0)\prime\prime}(\beta) + \xi_j^{(0)}(\beta) = \eta_j^{(0)}(\beta), \tag{11}$$

$$\eta_j^{(0)\prime}(\beta) = 0. \tag{12}$$

The first approximation satisfies

$$\xi_j^{(1)\prime\prime}(\beta) + \xi_j^{(1)}(\beta) = \eta_j^{(1)}(\beta), \tag{13}$$

$$\eta_j^{(1)\prime}(\beta) = \frac{\eta_j^{(0)}(\beta)\rho[\xi_j^{(0)}(\beta)]}{\rho_0\xi_j^{(0)}(\beta)}\left\{1 + \left[\frac{\xi_j^{(0)\prime}(\beta)}{\xi_j^{(0)}(\beta)}\right]^2\right\}^{1/2}. \tag{14}$$

As initial conditions, take

$$\left.\begin{aligned} \xi_j &= \xi_{j-1}^{(0)}(2\pi) + (\nu\rho_0)\xi_{j-1}^{(1)}(2\pi) + \cdots \\ \xi'_j &= \xi_{j-1}^{(0)\prime}(2\pi) + (\nu\rho_0)\xi_{j-1}^{(1)\prime}(2\pi) + \cdots \\ \eta_j &= \eta_{j-1}^{(0)}(2\pi) + (\nu\rho_0)\eta_{j-1}^{(1)}(2\pi) + \cdots \end{aligned}\right\}(j = 2, 3, \cdots) \tag{15}$$

and

$$\xi_1(0) = \xi_1 + (\nu\rho_0)0 + \cdots, \; \xi'_1(0) = 0, \; \eta_1(0) = \eta_1 + (\nu\rho_0)0 + \cdots. \tag{16}$$

We begin with the first revolution, solving Eqs. 11 and 12 to obtain

$$\eta_1^{(0)}(\beta) = \eta_1, \tag{17}$$

$$\xi_1^{(0)}(\beta) = \eta_1(1 + \epsilon_1 \cos\beta), \tag{18}$$

where

$$\epsilon_1 = (\xi_1/\eta_1) - 1. \tag{19}$$

Using these results in Eq. 14,

$$\eta_1^{(1)}(\beta) = \frac{1}{\rho_0}\int_0^\beta \rho[\eta_1(1 + \epsilon_1\cos\beta)]\frac{(1 + 2\epsilon_1\cos\beta + \epsilon_1^2)^{1/2}}{(1 + \epsilon_1\cos\beta)^2}\,d\beta. \tag{20}$$

Now the assumption discussed in connection with Eq. 8 enables us to simplify the the integrand of Eq. 20 by

$$\rho[\eta_1(1 + \epsilon_1 \cos \beta)] = \rho(\eta_1) \exp (\eta_1\alpha_1\epsilon_1 \cos \beta). \tag{21}$$

The remainder of the integrand can be expanded easily in a power series in $\epsilon$, the resulting coefficients involving the Legendre polynomials. These, in turn, can be developed in Fourier cosine series, whence

$$\frac{(1 + 2\epsilon \cos \beta + \epsilon^2)^{1/2}}{(1 + \epsilon \cos \beta)^2} = \sum_{n=0}^{\infty} (-\epsilon)^n \Gamma_n(\epsilon) \cos n\beta, \tag{22}$$

where

$$\left.\begin{aligned} \Gamma_0 &= 1 + \frac{3}{4}\epsilon^2 + \frac{13}{64}\epsilon^4 + \cdots \\ \Gamma_1 &= 1 + \frac{9}{8}\epsilon^2 + \frac{3}{32}\epsilon^4 + \cdots \\ \Gamma_2 &\ \ \frac{1}{4} + \frac{9}{16}\epsilon^2 + \cdots \\ &\cdots \end{aligned}\right\}. \tag{23}$$

An explicit general expression for $\Gamma_n$ can be given, but only in a rather complicated form. For $n \geq 3$, the constant term (independent of $\epsilon$) in $\Gamma_n$ is

$$\frac{1}{2^{n-1}} - \sum_{k=0}^{n-2} \frac{(2n - 2k - 3)!(k + 1)}{2^{2n-k-3}(n - k)!(n - k - 2)!}.$$

Combining the results of Eqs. 21 and 22, Eq. 20 becomes

$$\eta_1^{(1)}(\beta) = \frac{\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1) \int_0^{\beta} \exp (\eta_1\alpha_1\epsilon_1 \cos \beta) \cos n\beta \, d\beta. \tag{24}$$

The integral in Eq. 24 does not have to be evaluated because we need only $\eta_1^{(1)}(2\pi)$, and by [3], p. 181,

$$\int_0^{2\pi} \exp (x \cos \beta) \cos n\beta \, d\beta = 2\pi I_n(x), \tag{25}$$

$I_n(x)$ being the usual notation for the Bessel function of the first kind for purely imaginary argument. Thus we have

$$\eta_1^{(1)}(2\pi) = \frac{2\pi\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1) I_n(\eta_1\alpha_1\epsilon_1). \tag{26}$$

Next we must solve Eq. 13. The solution can be written

$$\xi_1^{(1)}(\beta) = \int_0^{\beta} \sin (\beta - x)\eta_1^{(1)}(x) \, dx \tag{27}$$

which, in view of Eq. 24, becomes

$$\begin{aligned} \xi_1^{(1)}(\beta) &= \frac{\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1) \int_0^{\beta} \sin (\beta - x) \int_0^{x} \exp (\eta_1\alpha_1\epsilon_1 \cos y) \cos ny \, dy \, dx, \\ &= \frac{\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1) \int_0^{\beta} \exp (\alpha_1\eta_1\epsilon_1 \cos y) \cos ny \, [\cos (\beta - y) - 1] \, dy. \end{aligned} \tag{28}$$

From this it follows that

$$\xi_1^{(1)}(2\pi) = \frac{2\pi\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1)[I_{n+1}(\eta_1\alpha_1\epsilon_1) + (n/\eta_1\alpha_1\epsilon_1 - 1)I_n(\eta_1\alpha_1\epsilon_1)]. \quad (29)$$

Similarly,

$$\xi_1^{(1)\prime}(\beta) = -\frac{\rho(\eta_1)}{\rho_0} \sum_{n=0}^{\infty} (-\epsilon_1)^n \Gamma_n(\epsilon_1) \int_0^{\beta} \exp(\eta_1\alpha_1\epsilon_1 \cos y) \cos ny \sin(\beta - y)\, dy. \quad (30)$$

When $\beta = 2\pi$, the integrand of Eq. 30 becomes an odd function of $y$ with period $2\pi$. Therefore, $\xi_1^{(1)\prime}(2\pi) = 0$ and one concludes that, to first order in $\nu$, the second cycle begins with the same kind of initial conditions as the first cycle, i.e., conditions appropriate to apogee or perigee. By induction, $\xi_j'$ vanishes identically in $j$.

Equations 26 and 29 relate conditions at the end of the first cycle to those at the beginning of that cycle. Moreover, because $\xi_j'$ is zero for all $j$, exactly the same argument can be used for any cycle provided the eccentricity and $\alpha$-value appropriate to that cycle are used.

A set of interim results complete through first order terms in the perturbation parameter is

$$\xi_{j+1} = \xi_j + 2\pi\nu\rho(\eta_j) \sum_{n=0}^{\infty} (-\epsilon_j)^n \Gamma_n(\epsilon_j)[I_{n+1}(\eta_j\alpha_j\epsilon_j) + (n/\eta_j\alpha_j\epsilon_j - 1)I_n(\eta_j\alpha_j\epsilon_j)], \quad (31)$$

$$\eta_{j+1} = \eta_j + 2\pi\nu\rho(\eta_j) \sum_{n=0}^{\infty} (-\epsilon_j)^n \Gamma_n(\epsilon_j) I_n(\eta_j\alpha_j\epsilon_j), \quad (32)$$

$$\xi_{j+1} = \epsilon_j + \frac{2\pi\nu\rho(\eta_j)}{\eta_j} \sum_{n=0}^{\infty} (-\epsilon_j)^n \Gamma_n(\epsilon_j)[I_{n+1}(\eta_j\alpha_j\epsilon_j) + (n/\eta_j\alpha_j\epsilon_j - 2 - \epsilon_j)I_n(\eta_j\alpha_j\epsilon_j)]. \quad (33)$$

Equation 33 follows from the fact that $\epsilon_j$ is defined as $\xi_j/\eta_j - 1$ for all $j$. Also, for $j = 1$ we have the initial conditions given at the beginning of this section.

**Some remarks on accuracy.** The accuracy with which one can find $\xi$, $\eta$ and $\epsilon$ from Eqs. 31–33 depends principally upon the accuracy with which $\nu$ and $\rho$ are known, and is therefore a physical problem. However, these equations do involve power series in $\epsilon$, so it is of some interest to see how the errors arising from truncating these series are related to the number of terms required.

In analyzing the orbit of artificial satellites, it should be noted that $\epsilon < 0.2$ is probably almost certain, even $\epsilon = 0.1$ being rather large. Also, one might typically be interested in altitudes to within one mile ($\xi$ about one part in 4000) at the end of 20,000 revolutions. (The latter is suggested by some of the numerical work in Ref. [2].) Because errors in the right-hand terms of Eq. 31 are additive, this means that for each $j$ the term must be correct to about one part in a million. With $\epsilon < 0.2$, noting that all of the leading coefficients of the $\Gamma_n$ are of the order of unity, it follows that one generally must carry $\epsilon^{10}$ terms in the $\Gamma_n$. This accuracy normally will be adequate in Eqs. 32 and 33 as well.

Not all the $\Gamma_n$ terms, of course, are equally important in contributing to the error, for the coefficients of $\Gamma_n$ in the infinite series (which converge) eventually become small. Unfortunately, it is not easy to establish just how fast these series converge, where they may be truncated, or how the accuracy requirements on the $\Gamma_n$ may be decreased with increasing $n$. However, some indication can be obtained by noting that the product $\eta\alpha\epsilon$ may be about 25 for $\epsilon = 0.2$. (A value of about 12 for $\epsilon = 0.1$ is given in [2]). The asymptotic expansions of $I_n$ and $I_{n+1}$ can be used in Eq. 31, and it is found that about twenty-five terms of the series are required before the residual drops below $10^{-8}$.

This argument must be done separately for each equation in view of the accuracy requirements one may wish to impose on the basic dependent variables. I believe, though, that this last result may be considered typical.

Thus the number of terms required to give high accuracy, in effect, to make the method adequate for a much greater eccentricity than that permitted by earlier methods, is substantial. However, it generally will be considerably easier to carry this number of terms in a numerical computation than to do a piecewise numerical integration separately around each orbital revolution, and to continue this through 20,000 or more revolutions.

**Radial and time behavior.** It is desired to return to the original physical variables of interest, $r$ and $t$. From $r = a/\xi$ it follows that to within terms of the order of $\nu^2$,

$$r_{j+1} = r_j - \frac{2\pi\nu}{a} r_j^2 \rho(\eta_j) \sum_{n=0}^{\infty} (-\epsilon_j)^n \Gamma_n(\epsilon_j)[I_{n+1}(\eta_j \alpha_j \epsilon_j) + (n/\alpha_j \epsilon_j - 1) I_n(\eta_j \alpha_j \epsilon_j)]. \tag{34}$$

Also

$$\frac{d\beta}{dt} = \left(\frac{K}{a^3}\right)^{1/2} \frac{\xi^2(\beta)}{\eta^{1/2}(\beta)}. \tag{35}$$

Integrating $dt/d\beta$ over one revolution as was done in finding $\eta_1^{(1)}(\beta)$,

$$t_2 - t_1 = \left(\frac{r_1^3(1 + \epsilon_1)^3}{K}\right)^{1/2} \left\{\frac{2\pi}{(1 - \epsilon_1^2)^{3/2}} + \nu \int_0^{2\pi} \left[\frac{\eta_1 \eta_1^{(1)}(\beta)}{2\xi_1^{(0)2}(\beta)} - \frac{2\eta_1^2 \xi_1^{(1)}(\beta)}{\xi_1^{(0)3}(\beta)}\right] d\beta\right\}. \tag{36}$$

Each of these integrals can be evaluated, albeit tediously. Doing this, and generalizing the argument to the $j$th revolution,

$$\begin{aligned} t_{j+1} = t_j + 2\pi\left(\frac{r_j^3}{K(1 - \epsilon_j)^3}\right)^{1/2} \Bigg[1 + \frac{\nu}{\eta_j} \Bigg\{ & \frac{5 - 4\epsilon_j^2}{2} \eta_1^{(1)}(2\pi) \\ & + \frac{\epsilon_j(1 + \epsilon_j^2)^{3/2}}{2} \sum_{n=0}^{\infty} (-\epsilon_j)^n \Gamma_n(\epsilon_j)[I_{n+1}(\eta_j \alpha_j \epsilon_j) + I_{n-1}(\eta_j \alpha_j \epsilon_j)]\Bigg\}\Bigg]. \end{aligned} \tag{37}$$

Equations 31, 32, 33, 36 and 37 form a complete set of difference equations from which the change of radius, shape and time with revolution number can be inferred. The $\eta_j$ enter this set only as auxiliary variables, and are not concerned in the final geometry of the orbit.

## References


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