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Akash Distribution and Its Applications

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Abstract A new one parameter lifetime distribution named “Akash distribution” for modeling lifetime data has been introduced. Some important mathematical properties of the proposed distribution including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability have been discussed. The condition under which Akash distribution is over-dispersed, equi-dispersed, and under-dispersed are presented along with the conditions under which exponential and Lindley distributions are over-dispersed, equi-dispersed and under-dispersed. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. The usefulness and the applicability of the proposed distribution have been discussed and illustrated with two real lifetime data sets from medical science and engineering.

Keywords Lifetime distribution, Moments, Hazard rate function, Mean residual life function, Mean deviations, Order statistics, Estimation of parameter, Goodness of fit

1. Introduction

The modeling and analyzing lifetime data are crucial in many applied sciences including medicine, engineering, insurance and finance, amongst others. There are a number of continuous distributions for modeling lifetime data such as exponential, Lindley, gamma, lognormal, and Weibull and their generalizations. The exponential, Lindley and the Weibull distributions are more popular than the gamma and the lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. Though each of exponential and Lindley distributions has one parameter, the Lindley distribution has one advantage over the exponential distribution that the exponential distribution has constant hazard rate whereas the Lindley distribution has monotonically decreasing hazard rate.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Lindley (1958) distribution are given by

$$f\left(x;\theta\right)=\frac{\theta^2}{\theta+1}\left(1+x\right)e^{-\theta x}\quad ;x>0,\,\theta>0 \qquad (1.1)$$

$$F\left(x;\theta\right)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\;;x>0,\theta>0 \qquad (1.2)$$

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The density (1.1) is a two-component mixture of an exponential distribution having scale parameter θ and a gamma distribution having shape parameter 2 and scale parameter θ with their mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. A detailed study about its various mathematical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by Ghitany *et al* (2008). The Lindley distribution has been generalized, extended, modified and its detailed applications in reliability and other fields of knowledge by different researchers including Hussain (2006), Zakerzadeh and Dolati (2009), Nadarajah *et al* (2011), Deniz and Ojeda (2011), Bakouch et al (2012), Shanker and Mishra (2013 a, 2013 b), Shanker *et al* (2013), Elbatal *et al* (2013), Ghitany *et al* (2013), Merovci (2013), Liyanage and Pararai (2014), Ashour and Eltehiwy (2014), Oluyede and Yang (2014), Singh *et al* (2014), Sharma *et al* (2015), Shanker *et al* (2015), Alkarni (2015), Pararai *et al* (2015), Abouammoh *et al* (2015) are some among others.

Although the Lindley distribution has been used to model lifetime data by many researchers and Hussain (2006) has shown that the Lindley distribution is important for studying stress-strength reliability modeling, there are many situations in the modeling of real lifetime data where the Lindley distribution may not be suitable from a theoretical or applied point of view. Therefore, to obtain a new distribution which is flexible than the Lindley distribution for modeling lifetime data in reliability and in terms of its hazard rate shapes, we introduced a new distribution by considering a two-considering a two- component mixture of an exponential distribution having scale parameter θ and a gamma distribution having shape parameter 3 and scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively.

The probability density function (p.d.f.) of a new one parameter lifetime distribution can be introduced as

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x} \quad ;x>0,\ \theta>0 \qquad (1.3)$$

We would call this distribution, “Akash distribution. This distribution can be easily expressed as a mixture of exponential (θ) and gamma $(3,\theta)$ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. We have

$$f(x,\theta)=p\,g_1(x)+(1-p)\,g_2(x)$$

where

$$p=\frac{\theta^2}{\theta^2+2},\ g_1(x)=\theta e^{-\theta x},\ \text{and}\ g_2(x)=\frac{\theta^3x^2e^{-\theta x}}{2}.$$

The corresponding cumulative distribution function (c.d.f.) of (1.3) is given by

$$F(x)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x};\ \ x>0,\theta>0 \qquad (1.4)$$

The graphs of the p.d.f. and the c.d.f. of Lindley and Akash distributions for different values of θ are shown in **figures 1 and 2.**

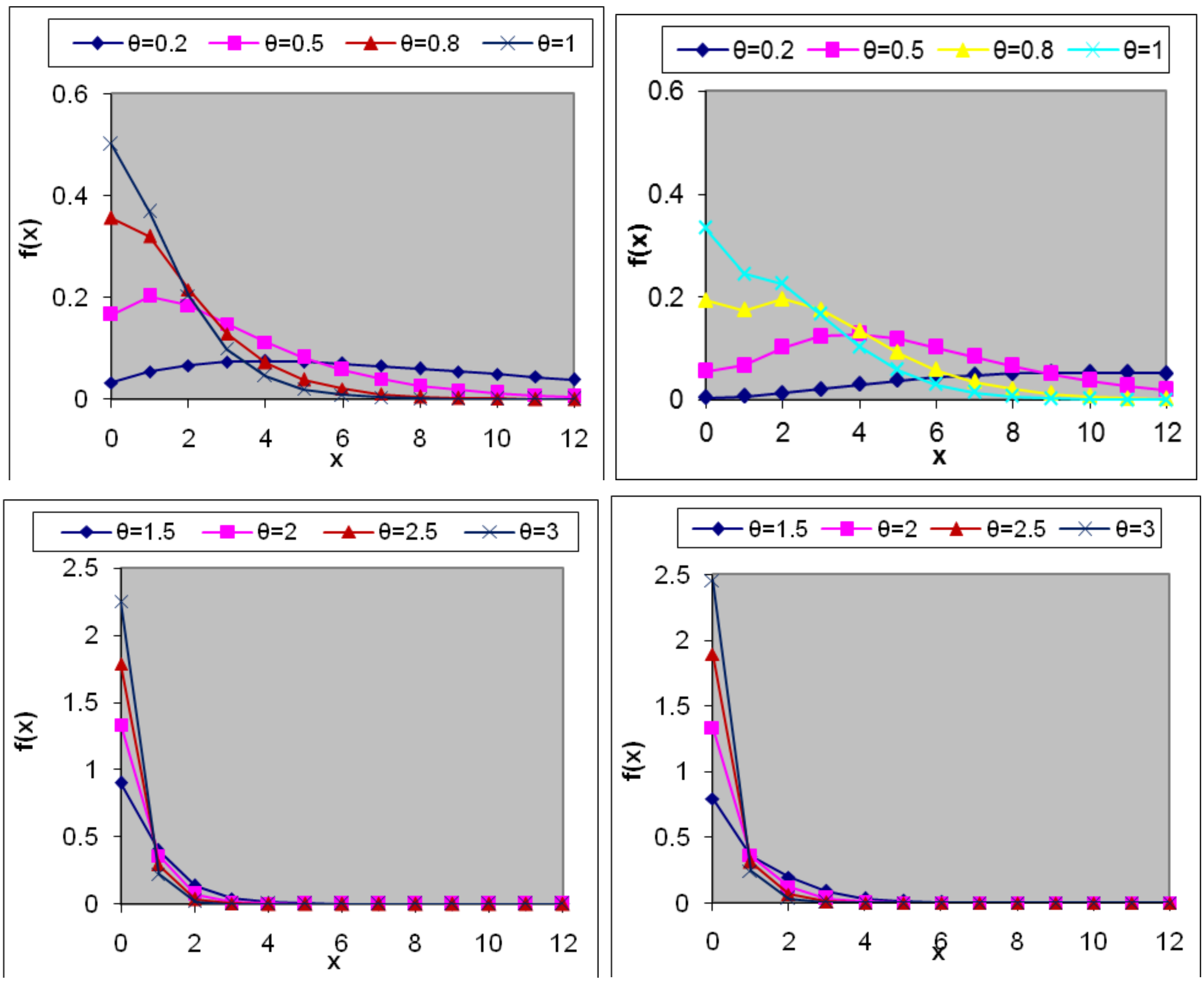

Figure 1. Graphs of the pdf of Lindley and Akash distributions for different values of parameter θ. Left hand side graphs are for Lindley distribution and right hand sides graph are for Akash distribution

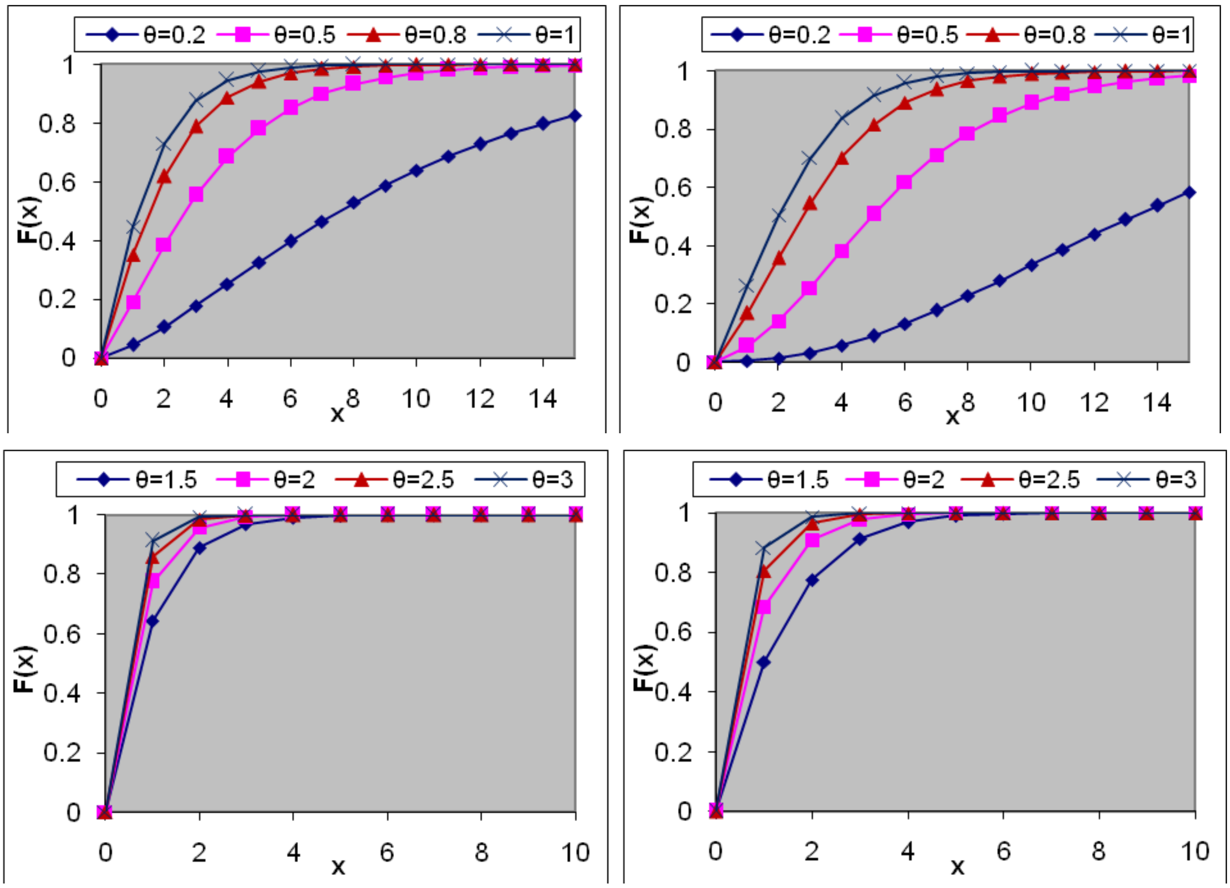

Figure 2. Graphs of the cdf of Lindley and Akash distribution for different values of parameter θ. Left hand side graphs are for Lindley distribution and right hand side graphs are for Akash distribution

2. Moments and Related Measures

The r the moment about origin of Akash distribution (1.3) has been obtained as

$$\mu_r' = \frac{r!\left[\theta^2 + (r+1)(r+2)\right]}{\theta^r\left(\theta^2+2\right)} \quad ; r = 1,2,3,...$$

and so the first four moments about origin as

$$\mu_1' = \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}, \quad \mu_2' = \frac{2\left(\theta^2+12\right)}{\theta^2\left(\theta^2+2\right)},$$

$$\mu_3' = \frac{6\left(\theta^2+20\right)}{\theta^3\left(\theta^2+2\right)}, \quad \mu_4' = \frac{24\left(\theta^2+30\right)}{\theta^4\left(\theta^2+2\right)}$$

Thus the moments about mean of Akash distribution are obtained as

$$\mu_2 = \frac{\theta^4+16\theta^2+12}{\theta^2\left(\theta^2+2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6+30\theta^4+36\theta^2+24\right)}{\theta^3\left(\theta^2+2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8+128\theta^6+408\theta^4+576\theta^2+240\right)}{\theta^4\left(\theta^2+2\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2) and index of dispersion (γ) of Akash distribution are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4+16\theta^2+12}}{\theta^2+6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^6+30\theta^4+36\theta^2+24\right)}{\left(\theta^4+16\theta^2+12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(3\theta^8+128\theta^6+408\theta^4+576\theta^2+240\right)}{\left(\theta^4+16\theta^2+12\right)^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\theta^4+16\theta^2+12}{\theta\left(\theta^2+2\right)\left(\theta^2+6\right)}$$

It can be easily shown that Akash distribution is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispers $\left(\mu>\sigma^2\right)$ for $\theta<(=)>\theta^*=1.515400063$. It would be recalled that Lindley distribution is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispers $\left(\mu>\sigma^2\right)$ for $\theta<(=)>\theta^*=1.170086487$ while exponential distribution is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispersed $\left(\mu>\sigma^2\right)$ for $\theta<(=)>\theta^*=1$.

3. Moment Generating Function

The moment generating function of Akash distribution (1.3) are obtained as

$$\begin{aligned}M_X(t)&=\frac{\theta^3}{\theta^2+2}\int_0^\infty e^{-(\theta-t)}\left(1+x^2\right)dx\\&=\frac{\theta^3}{\theta^2+2}\left[\frac{1}{\theta-t}+\frac{2}{(\theta-t)^3}\right]\\&=\frac{\theta^3}{\theta^2+2}\left[\frac{1}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^3}\sum_{k=0}^\infty\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k\right]\\&=\sum_{k=0}^\infty\frac{\theta^2+(k+1)(k+2)}{\theta^2+2}\left(\frac{t}{\theta}\right)^k\end{aligned}$$

4. Hazard Rate Function and Mean Residual Life Function

Let X be a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x\mid X>x\right)}{\Delta x}=\frac{f(x)}{1-F(x)} \tag{4.1}$$

and

$$m(x)=E\left[X-x\mid X>x\right]=\frac{1}{1-F(x)}\int_x^\infty\left[1-F(t)\right]dt \tag{4.2}$$

The hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of Lindley distribution are given by

$$h(x)=\frac{\theta^2(1+x)}{(\theta+1)+\theta x} \tag{4.3}$$

And

$$m(x)=\frac{\theta+2+\theta x}{\theta(\theta+1+\theta x)} \tag{4.4}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the Akash distribution are given by

$$h(x)=\frac{\theta^3\left(1+x^2\right)}{\theta x(\theta x+2)+\left(\theta^2+2\right)} \tag{4.5}$$

and

$$m(x)=\frac{\theta^2+2}{\left[\theta x(\theta x+2)+\left(\theta^2+2\right)\right]e^{-\theta x}}\int_x^\infty\left[\frac{\theta t(\theta t+2)+\left(\theta^2+2\right)}{\theta^2+2}\right]e^{-\theta t}dt=\frac{\theta^2x^2+4\theta x+\left(\theta^2+6\right)}{\theta\left[\theta x(\theta x+2)+\left(\theta^2+2\right)\right]} \tag{4.6}$$

It can be easily verified that $h(0)=\dfrac{\theta^3}{\theta^2+2}=f(0)$ and $m(0)=\dfrac{\theta^2+6}{\theta\left(\theta^2+2\right)}=\mu_1'$. It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is an increasing function of x, and θ, whereas $m(x)$ is a decreasing function of x, and θ. The hazard rate function and the mean residual life function of the Akash distribution show its flexibility over Lindley distribution and exponential distribution.

The graphs of the hazard rate function and mean residual life function of Lindley and Akash distributions are shown in figures 3 and 4.

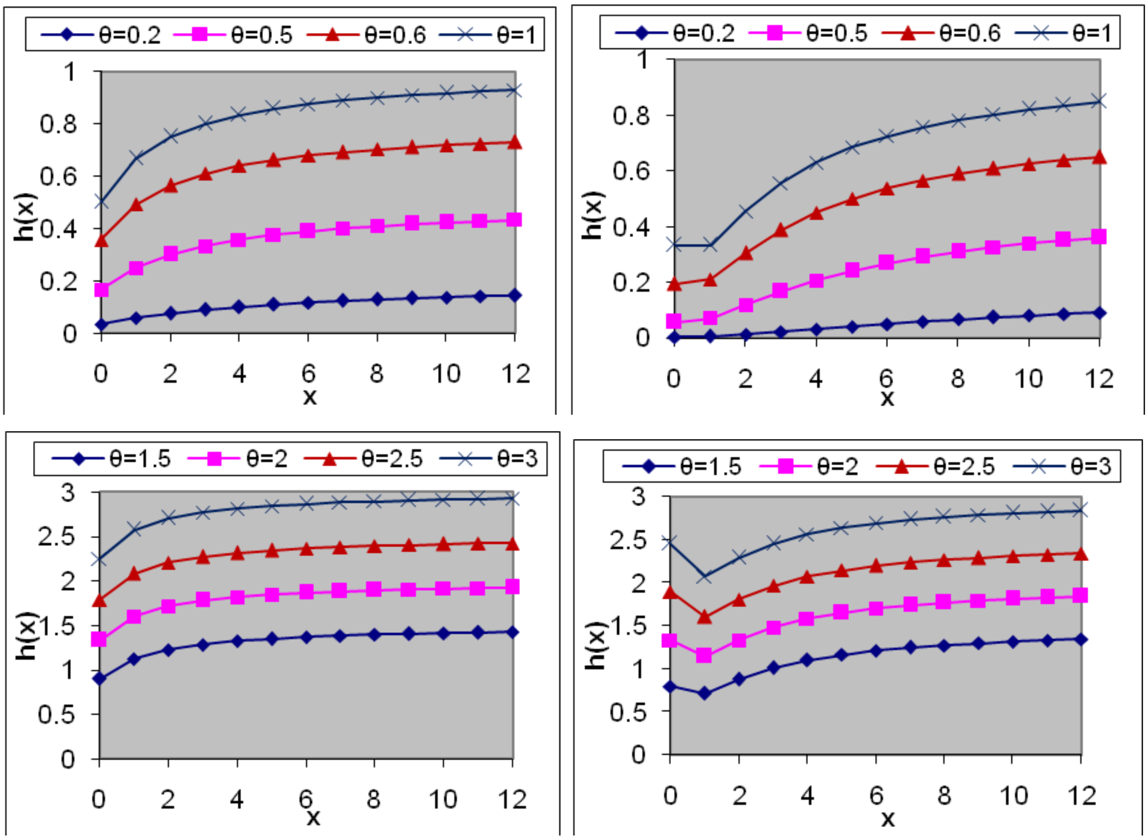

Figure 3. Graphs of hazard rate function of Lindley and Akash distributions for different values of parameter θ. Left hand side graphs are for Lindley distribution and right hand side graphs are for Akash distribution

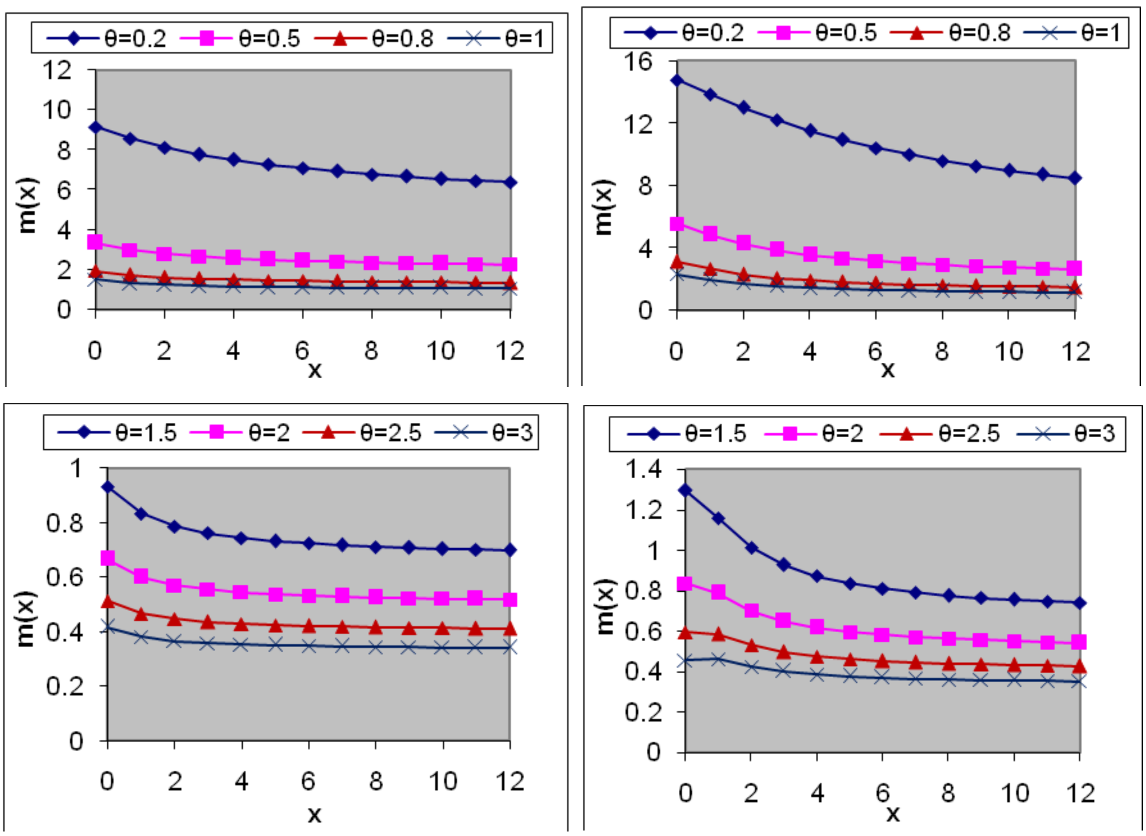

Figure 4. Graphs of mean residual life function of Lindley and Akash distributions for different values of parameter θ. Left hand side graphs are for Lindley distribution and right hand side graphs are for Akash distribution

5. Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X\left(x\right) \ge F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X\left(x\right) \ge h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X\left(x\right) \le m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\frac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \tag{5.1}$$

$$\Downarrow$$
$$X \le_{st} Y$$

The Akash distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ Akash distributon $\left(\theta_1\right)$ and Y ~ Akash distribution $\left(\theta_2\right)$. If $\theta_1 \ge \theta_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3\left(\theta_2^2+2\right)}{\theta_2^3\left(\theta_1^2+2\right)} e^{-(\theta_1-\theta_2)x} \quad ; \; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^3\left(\theta_2^2+2\right)}{\theta_2^3\left(\theta_1^2+2\right)}\right] - \left(\theta_1-\theta_2\right)x$$

This gives $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = -\left(\theta_1-\theta_2\right)$

Thus for $\theta_1 \ge \theta_2$, $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

6. Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations usually from mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined by

$$\delta_1\left(X\right) = \int_0^\infty \left|x-\mu\right| f\left(x\right)dx \text{ and } \delta_2\left(X\right) = \int_0^\infty \left|x-M\right| f\left(x\right)dx,$$

respectively, where $\mu = E\left(X\right)$ and $M = \text{Median}\left(X\right)$. The measures $\delta_1\left(X\right)$ and $\delta_2\left(X\right)$ can be calculated using the relationships

$$\begin{aligned}\delta_1\left(X\right) &= \int_0^\mu \left(\mu-x\right)f\left(x\right)dx + \int_\mu^\infty \left(x-\mu\right)f\left(x\right)dx \\ &= \mu F\left(\mu\right) - \int_0^\mu x\,f\left(x\right)dx - \mu\left[1-F\left(\mu\right)\right] + \int_\mu^\infty x\,f\left(x\right)dx \\ &= 2\mu F\left(\mu\right) - 2\mu + 2\int_\mu^\infty x\,f\left(x\right)dx \\ &= 2\mu F\left(\mu\right) - 2\int_0^\mu x\,f\left(x\right)dx \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\delta_2\left(X\right) &= \int_0^M \left(M-x\right)f\left(x\right)dx + \int_M^\infty \left(x-M\right)f\left(x\right)dx \\ &= M\,F\left(M\right) - \int_0^M x\,f\left(x\right)dx - M\left[1-F\left(M\right)\right] + \int_M^\infty x\,f\left(x\right)dx \\ &= -\mu + 2\int_M^\infty x\,f\left(x\right)dx \\ &= \mu - 2\int_0^M x\,f\left(x\right)dx \end{aligned} \tag{6.2}$$

Using p.d.f. (1.3) and expression for the mean of Akash distribution, we get

$$\int_0^\mu x\,f\left(x\right)dx = \mu - \frac{\left\{\theta^3\left(\mu^3+\mu\right)+\theta^2\left(3\mu^2+1\right)+6\left(\theta\mu+1\right)\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\right)} \tag{6.3}$$

$$\int_0^M x\,f\left(x\right)dx = \mu - \frac{\left\{\theta^3\left(M^3+M\right)+\theta^2\left(3M^2+1\right)+6\left(\theta M+1\right)\right\}e^{-\theta M}}{\theta\left(\theta^2+2\right)} \tag{6.4}$$

Using expressions from (6.1), (6.2), (6.3), and (6.4), the mean deviation about mean, $\delta_1\left(X\right)$ and the mean deviation about median, $\delta_2\left(X\right)$ of Akash distribution are obtained as

$$\delta_1\left(X\right) = \frac{2\left\{\theta^2\left(\mu^2+1\right)+2\left(2\theta\mu+3\right)\right\}e^{-\theta\mu}}{\theta\left(\theta^2+2\right)} \tag{6.5}$$

$$\delta_2(X) = \frac{2\left\{\theta^3\left(M^3+M\right)+\theta^2\left(3M^2+1\right)+6\left(\theta M+1\right)\right\}e^{-\theta M}}{\theta\left(\theta^2+2\right)} - \mu \tag{6.6}$$

7. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from Akash distribution (1.3). Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ denote the corresponding order statistics. The p.d.f. and the c.d.f. of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\left\{1-F(y)\right\}^{n-k} f(y)$$

$$= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l F^{k+l-1}(y) f(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\left\{1-F(y)\right\}^{n-j}$$

$$= \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j}\binom{n-j}{l} (-1)^l F^{j+l}(y),$$

respectively, for $k = 1, 2, 3, ..., n$.

Thus, the p.d.f. and the c.d.f of k th order statistics are given by

$$f_Y(y) = \frac{n!\theta^3\left(1+x^2\right)e^{-\theta x}}{\left(\theta^2+2\right)(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left[1 - \frac{\theta x(\theta x+2)+\left(\theta^2+2\right)}{\theta^2+2} e^{-\theta x}\right]^{k+l-1}$$

and

$$F_Y(y) = \sum_{j=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j}\binom{n-j}{l} (-1)^l \left[1 - \frac{\theta x(\theta x+2)+\left(\theta^2+2\right)}{\theta^2+2} e^{-\theta x}\right]^{j+l}$$

8. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx = \frac{1}{p\mu}\left[\int_0^\infty x f(x)dx - \int_q^\infty x f(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty x f(x)dx\right] \tag{8.1}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)dx = \frac{1}{\mu}\left[\int_0^\infty x f(x)dx - \int_q^\infty x f(x)dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x f(x)dx\right] \tag{8.2}$$

respectively or equivalently

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{8.3}$$

$$\text{and }\; L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{8.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{8.5}$$

$$\text{and }\; G=1-2\int_0^1 L(p)\,dp \tag{8.6}$$

respectively.

Using p.d.f. (1.3), we get

$$\int_q^\infty x\,f(x)\,dx=\frac{\left\{\theta^3\left(q^3+q\right)+\theta^2\left(3q^2+1\right)+6\left(\theta q+1\right)\right\}e^{-\theta q}}{\theta\left(\theta^2+2\right)} \tag{8.7}$$

Now using equation (8.7) in (8.1) and (8.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\theta^3\left(q^3+q\right)+\theta^2\left(3q^2+1\right)+6\left(\theta q+1\right)\right\}e^{-\theta q}}{\theta^2+6}\right] \tag{8.8}$$

$$\text{and }\; L(p)=1-\frac{\left\{\theta^3\left(q^3+q\right)+\theta^2\left(3q^2+1\right)+6\left(\theta q+1\right)\right\}e^{-\theta q}}{\theta^2+6} \tag{8.9}$$

Now using equations (8.8) and (8.9) in (8.5) and (8.6), the Bonferroni and Gini indices are obtained as

$$B=1-\frac{\left\{\theta^3\left(q^3+q\right)+\theta^2\left(3q^2+1\right)+6\left(\theta q+1\right)\right\}e^{-\theta q}}{\theta^2+6} \tag{8.10}$$

$$G=-1+\frac{2\left\{\theta^3\left(q^3+q\right)+\theta^2\left(3q^2+1\right)+6\left(\theta q+1\right)\right\}e^{-\theta q}}{\theta^2+6} \tag{8.11}$$

9. Renyi Entropy

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy (1961). If X is a continuous random variable having probability density function $f(.)$, then Renyi entropy is defined as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}$$

where $\gamma>0$ and $\gamma\neq 1$.

Thus, the Renyi entropy for the Akash distribution (1.3) is obtained as

$$T_R(\gamma)=\frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\theta^{3\gamma}}{\left(\theta^2+2\right)^{\gamma}}\left(1+x^2\right)^{\gamma}e^{-\theta\gamma x}\,dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\theta^{3\gamma}}{\left(\theta^2+2\right)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\left(x^2\right)^{j}e^{-\theta\gamma x}\,dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{\theta^{3\gamma}}{\left(\theta^2+2\right)^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}x^{2j}e^{-\theta\gamma x}\,dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{3\gamma}}{\left(\theta^2+2\right)^{\gamma}}\int_0^\infty e^{-\theta\gamma x}x^{2j+1-1}\,dx\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{3\gamma}}{\left(\theta^2+2\right)^{\gamma}}\frac{\Gamma\left(2j+1\right)}{\left(\theta\gamma\right)^{2j+1}}\right]$$

$$=\frac{1}{1-\gamma}\log\left[\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\theta^{3\gamma-2j-1}}{\left(\theta^2+2\right)^{\gamma}}\frac{\Gamma\left(2j+1\right)}{\left(\gamma\right)^{2j+1}}\right]$$

10. Stress-strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till $X>Y$. Therefore, $R=P\left(Y<X\right)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having Akash distribution (1.3) with parameter θ_1 and θ_2 respectively. Then the stress-strength reliability R is obtained as

$$R=P\left(Y<X\right)=\int_0^{\infty}P\left(Y<X\mid X=x\right)f_X\left(x\right)dx$$

$$=\int_0^{\infty}f\left(x;\theta_1\right)F\left(x;\theta_2\right)dx$$

$$=1-\frac{\theta_1^3\begin{bmatrix}\theta_2^6+4\theta_1\theta_2^5+2\left(3\theta_1^2+4\right)\theta_2^4\\+2\left(2\theta_1^2+11\right)\theta_1\theta_2^3+\left(\theta_1^4+22\theta_1^2+40\right)\theta_2^2\\+10\left(\theta_1^2+2\right)\theta_1\theta_2+2\left(\theta_1^2+2\right)\theta_1^2\end{bmatrix}}{\left(\theta_1^2+2\right)\left(\theta_2^2+2\right)\left(\theta_1+\theta_2\right)^5}.$$

11. Estimation of Parameter

11.1. Maximum Likelihood Estimation

Let $\left(x_1,x_2,x_3,\ldots,x_n\right)$ be a random sample from Akash distribution (1.3). The likelihood function, L of (1.3) is given by

$$L=\left(\frac{\theta^3}{\theta^2+2}\right)^n\prod_{i=1}^{n}\left(1+x_i^2\right)e^{-n\theta\overline{x}}$$

and so its log likelihood function is thus obtained as

$$\log L=n\log\left(\frac{\theta^3}{\theta^2+2}\right)+\sum_{i=1}^{n}\log\left(1+x_i^2\right)-n\theta\overline{x}$$

Now $$\frac{d\log L}{d\theta}=\frac{3n}{\theta}-\frac{2n\theta}{\theta^2+2}-n\overline{x}$$

where $\overline{x}$ is the sample mean.

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\frac{d\log L}{d\theta}=0$ and so it can be obtained by solving the following non-linear equation

$$\overline{x}\theta^3-\theta^2+2\overline{x}\theta-6=0 \qquad (11.1.1)$$

11.2. Method of Moment (MoM) Estimation

Equating the population mean of the Akash distribution to the corresponding sample mean, the method of moment (MOM) estimate, $\tilde{\theta}$, of θ is the same as given by equation (11.1.1).

12. Applications of Akash Distribution

The Akash distribution has been fitted to a number of data sets from medical science and engineering. In this section, we present the fitting of Akash distribution to two real data sets and compare its goodness of fit with the one parameter exponential and Lindley distributions.

Data set 1: The first data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105). The data are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Data set 2: The second data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994)

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381

In order to compare distributions, $-2\ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion), K-S Statistics (Kolmogorov-Smirnov Statistics) for two real data sets have been computed. The formulae for computing AIC, AICC, BIC, and K-S Statistics are as follows:

$$AIC=-2\ln L+2k,\quad AICC=AIC+\frac{2k\left(k+1\right)}{\left(n-k-1\right)},$$

$$BIC=-2\ln L+k\ln n \text{ and}$$

Table 1. MLE's, - 2ln L, AIC, AICC, BIC, and K-S Statistics of the fitted distributions of data sets 1 and 2

	Model	Parameter estimate	$-2\ln L$	AIC	AICC	BIC	K-S statistic
Data 1	Akash	1.1569	59.5	61.7	61.7	62.5	0.3205
	Lindley	0.8161	60.5	62.5	62.7	63.5	0.3410
	Exponential	0.5263	65.7	67.7	67.9	68.7	0.3895
Data 2	Akash	0.09706	240.7	242.7	242.8	244.1	0.2664
	Lindley	0.06299	254.0	256.0	256.1	257.4	0.3332
	Exponential	0.03246	274.5	276.7	276.7	277.9	0.4264

$D=\operatorname*{Sup}_{x}\left|F_n(x)-F_0(x)\right|$, where k = the number of parameters, n = the sample size and $F_n(x)$ is the empirical distribution function.

The best distribution corresponds to lower $-2\ln L$, AIC, AICC, BIC, and K-S statistics.

It can be easily seen from above table that the Akash distribution is better than the Lindley and exponential distributions for modeling life time data and thus Akash distribution should be preferred to exponential distribution and Lindley distributions for modeling lifetime data-sets.

13. Conclusions

A one parameter lifetime distribution named, "Akash distribution" has been proposed. Its mathematical properties including shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics have been discussed. The condition under which Akash distribution is over-dispersed, equi-dispersed, and under-dispersed are presented along with the conditions under which exponential and Lindley distributions are over-dispersed, equi-dispersed and under-dispersed. Further, expressions for Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability of the proposed distribution have been derived. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameter. Finally, the goodness of fit test using K-S Statistics (Kolmogorov-Smirnov Statistics) for two real lifetime data- sets have been presented to illustrate its applicability and superiority over exponential and Lindley distributions.

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