Czechoslovak Mathematical Journal

Mary Ellen Rudin

ℵ-Dowker spaces

Czechoslovak Mathematical Journal, Vol. 28 (1978), No. 2, 324–326

Persistent URL: http://dml.cz/dmlcz/101534

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Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

$\varkappa$-DOWKER SPACES

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(Received October 5, 1976)

In a written communication to the Prague Topology Symposium of 1976, K. MORITA proposed the following:

Conjecture 1. *If a Hausdorff space Y has the property that $X \times Y$ is normal for all normal Hausdorff spaces X, then Y is discrete.*

In an abstract and talk at this symposium M. ATSUJI pointed out that Morita's conjecture follows from:

Conjecture 2. *For each infinite cardinal $\varkappa$, there is a normal Hausdorff space $X_\varkappa$ which has a decreasing family $\{D_\alpha\}_{\alpha<\varkappa}$ of closed sets such that $\bigcap_{\alpha<\varkappa} D_\alpha = \emptyset$ and, if $\{U_\alpha\}_{\alpha<\varkappa}$ is a family of open sets with $D_\alpha \subset U_\alpha$ for each α, then $\bigcap_{\alpha<\varkappa} U_\alpha \neq \emptyset$.*

A space $X_\varkappa$ having the properties described in Atsuji's conjecture could be called a *$\varkappa$-Dowker space* since X_ω would be an ordinary Dowker space. The purpose of this note is to prove that there are $\varkappa$-Dowker spaces for all cardinals $\varkappa$, thus proving conjectures (1) and (2).

I. Assume that $\varkappa$ is an infinite cardinal; we construct $X_\varkappa$ by simply generalizing the construction given in [1] of an ordinary Dowker space.

We begin by choosing an increasing family $\{\lambda_\alpha\}_{\alpha<\varkappa}$ of regular cardinals such that $\lambda_0 < \varkappa^+$ and $\lambda_\alpha = \lambda_\alpha^\varkappa$. Let $\lambda = \sup\{\lambda_\alpha \mid {}_{\alpha<\varkappa}\}$.

Let $F = \{f : \varkappa \to \lambda \mid f(\alpha) \leqq \lambda_\alpha \text{ for all } \alpha < \varkappa\}$.

Let $G = \{g \in F \mid g(\alpha) < \lambda_\alpha \text{ for all } \alpha < \varkappa\}$.

Let $X = X_\varkappa = \{f \in F \mid \exists \beta < \varkappa \text{ such that } \varkappa^+ \leqq \operatorname{cf}(f(\alpha)) \leqq \lambda_\beta \text{ for all } \alpha < \varkappa\}$.

If f and g belong to F we say $g < f$ if $g(\alpha) < f(\alpha)$ for all α, and we say $g \leqq f$ if $g(\alpha) \leqq f(\alpha)$ for all α. If $g < f$, define $U_{gf} = \{h \in X \mid g < h \leqq f\}$. We topologize X by using $\{U_{gf} \mid g < f \text{ in } F\}$ as a basis.

To check that $X_\varkappa = X$ has the properties desired in Conjecture (2), the reader acquained with [1] will probably have no difficulty. Significant changes needed other than replacing ω by $\varkappa$ are indicated below.

II. For each $\alpha < \varkappa$, let $D_\alpha = \{f \in X \mid f(\beta) = \lambda_\beta \text{ for all } \beta \leqq \alpha\}$. Clearly $\bigcap_{\alpha<\varkappa} D_\alpha = \emptyset$ and each D_α is closed. Assume that for each $\alpha < \varkappa$, $D_\alpha \subset U_\alpha$ which is open. We want to prove:

Lemma (3). $\bigcap_{\alpha<\varkappa} U_\alpha \neq \emptyset$.

As in [1] it would suffice to prove:

Lemma (2). *Suppose that $\alpha < \varkappa$. There is a term f of G such that $\{g \in G \cap \cap X \mid f < g\} \subset U_\alpha$.*

Since $\{\beta < \alpha\}$ need not be finite if $\varkappa \neq \omega$, we need a different proof of Lemma (2) than that given in [1]. This is again relevant in the proof of Lemma (6). In both proofs we make use of $\lambda_\alpha^\varkappa = \lambda_\alpha$.

Proof of Lemma (2). For each $\delta < \lambda_\alpha$, we choose $h_\delta \in \prod_{\beta\leqq\alpha} \lambda_\beta$ in such a way that $\delta < \lambda_\alpha$ and $h \in \prod_{\beta\leqq\alpha} \lambda_\beta$ imply that there is a $\gamma < \lambda_\alpha$ with $\delta < \gamma$ and $h_\gamma = h$. This is possible since $\{\lambda_\beta\}_{\beta\leqq\alpha}$ are increasing and $\lambda_\alpha = \lambda_\alpha^\varkappa$ imply that $\lambda_\alpha^\varkappa$ and λ_α^α and $\prod_{\beta\leqq\alpha} \lambda_\beta$ all have cardinality λ_α.

Assuming there is no f satisfying Lemma (2), we define terms g_δ and f_δ of G for all $\delta < \lambda_\alpha$ by induction on δ. If f_γ has been defined for all $\gamma < \delta$, define g_δ by $g_\delta(\beta) = h_\delta(\beta)$ for $\beta \leqq \alpha$ and $g_\delta(\beta) = \sup \{f_\gamma(\beta) \mid \gamma < \delta\}$ for $\alpha < \beta < \varkappa$. Then choose $f_\delta \in (X \cap G) - U_\alpha$ with $g_\delta < f_\delta$ as guaranteed by assumption. Let f be the term of F with $f(\beta) = \lambda_\beta$ for $\beta \leqq \alpha$ and $f(\beta) = \sup \{f_\delta(\beta) \mid \delta < \lambda_\alpha\}$ for $\alpha < \beta < \varkappa$. Since $f \in D_\alpha, f \in U_\alpha$ and there is $g < f$ with $U_{gf} \subset U_\alpha$. For $\alpha < \beta < \varkappa$, $\{f_\delta(\beta)\}_{\delta<\lambda_\alpha}$ *is* strictly increasing. So there is a $\delta < \lambda_\alpha$ such that $f_\delta(\beta) > g(\beta)$ for all $\alpha < \beta < \varkappa$. Thus there is a $\gamma < \lambda_\alpha$ with $\delta < \gamma$ and $h_\gamma(\beta) = g(\beta)$ for all $\beta \leqq \alpha$. But then $f_\gamma \in U_{gf}$ contradicting $f_\gamma \notin U_\alpha$.

III. It remains to prove that X is normal. We might as well prove that X is collectionerise normal. So assume that $\mathscr{H}$ is a closed discrete family of closed sets. By exactly the same proof given in [1], we can find disjoint open sets separating the members of $\mathscr{H}$ provided we can prove:

Lemma (4). *The intersection of any family of less than $\varkappa$ open sets is open.*

Lemma (5). *Suppose that $t \in F$ and $\varkappa^+ \leqq \text{cf}(t(\alpha))$ for all $\alpha < \varkappa$. There is an $f \in F$ such that $f < t$ and $\{h \in X \mid f < h \leqq t\}$ intersects at most one member of $\mathscr{H}$.*

Lemma (4) is proved exactly as in [1]; and, as in [1], Lemma (5) follows from:

Lemma (6). *Suppose that* $\alpha < \varkappa$ *and define* $U_\alpha = \{h \in X \mid h \leqq t$ *and* $\mathrm{cf}(h(\beta)) \leqq \lambda_\alpha$ *for all* $\beta < \varkappa\}$. *There is an* $f \in F$ *such that* $f < t$ *and* $\{h \in U_\alpha \mid f < h\}$ *intersects at most one term of* $\mathscr{H}$.

Proof of Lemma (6). Let

$$N = \{\beta < \varkappa \mid \mathrm{cf}(t(\beta)) \leqq \lambda_\alpha\} \quad \text{and} \quad M = \{\beta < \varkappa \mid \mathrm{cf}(t(\beta)) > \lambda_\alpha\}.$$

As in the proof of Lemma (2), for each $\delta < \lambda_\alpha$ we choose $g_\delta \in \prod_{\beta \in N} t(\beta)$ in such a way that $\delta < \lambda_\alpha$ and $g \in \prod_{\beta \in N} t(\beta)$ imply there is a $\gamma < \lambda_\alpha$ with $\delta < \gamma$ and $g < g_\gamma$. The fact that the cardinality of N is at most $\varkappa$ and $\lambda_\alpha^\varkappa$ is λ_α makes this possible.

Assuming there is no f satisfying Lemma (6), we define $h_\delta \in U_\alpha$, $k_\delta \in U_\alpha$, and $f_\delta \in F$ for each $\delta < \lambda_\alpha$ by induction on δ. Assuming that h_γ and k_γ have been defined for all $\gamma < \delta$, define f_δ by $f_\delta(\beta) = g_\delta(\beta)$ for $\beta \in N$ and $f_\delta(\beta) = \sup(\{h_\gamma(\beta)\}_{\gamma<\delta} \cup \cup \{k_\gamma(\beta)\}_{\gamma<\delta})$ for $\beta \in M$. Then choose h_δ and k_δ to be terms U_α with $f_\delta < h_\delta$ and $f_\delta < k_\delta$ belonging to *different* terms of $\mathscr{H}$.

Let $f \in F$ be defined by $f(\beta) = t(\beta)$ for $\beta \in N$ and $f(\beta) = \sup\{f_\delta(\beta) \mid \delta < \lambda_\alpha\}$ for $\beta \in M$. There is $g < f$ such that U_{gf} intersects at most one term of $\mathscr{H}$. Also there is a $\delta < \lambda_\alpha$ with $f_\delta(\beta) > g(\beta)$ for all $\beta \in M$. So there is $\gamma < \lambda_\alpha$ with $\delta < \gamma$ and $f_\gamma(\beta) > g_\gamma(\beta)$ for all $\beta \in N$. Hence $f_\gamma \in U_{gf}$. But $f_\gamma < h_\gamma$ and $f_\gamma < k_\gamma$ and thus $h_\gamma \in U_{gf}$ and $k_\gamma \in U_{gf}$. However this contradicts U_{gf} intersecting only one term of $\mathscr{H}$.

Bibliography

[1] *M. E. Rudin:* A normal space X for which $X \times I$ is not normal, Fund. Math. *LXXIII* (1971), 179—186.

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