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Alfven Wave Resonances in a Realistic Magnetospheric Magnetic Field Geometry

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The notion of magnetic field line resonance has been very effective in explaining many features of long-period geomagnetic pulsations. To date the decoupled transverse wave equations have been solved in a magnetic dipole field, whereas only WKB approximate solutions have been used in more general geometries. We have developed a solution of the decoupled equations that includes both a general magnetic field geometry and the effects of density and mass composition. The aim of this paper is to isolate and examine the effect on eigenfrequencies of only the field geometry by keeping density constant along all field lines. We review the diurnal variations in wave period predicted on the ground and in space by using the recent Olson-Pfitzer magnetospheric magnetic field model in our solution. For example, on the ground at 67° magnetic latitude the diurnal variation in period caused by field geometry is larger than a factor of 2. At 6.6 R_E, where the dipole field line from 67° crosses the magnetospheric equator, there is negligible diurnal variation in period. Significant diurnal variations in period ($\gtrsim$10%) at fixed radial distance in the equatorial plane in space occur only at distances $\gtrsim$10 R_E. Knowledge of the field geometry is shown to be important for the determination of mass density in space from ground pulsation observations. We discuss the impact of our results in interpretation of experimental data.

INTRODUCTION AND THEORY

In this paper we present a calculation of long-period, ultra-low-frequency (ULF) magnetospheric pulsations that uses a realistic earth's magnetic field geometry rather than a simple dipole field. The calculation pertains to transverse standing wave oscillations, whether they are Pc 3, 4, or 5 continuous pulsations with periods from 10 to 600 s or the irregular Pi pulsations in the same period range. The presence of standing Alfven wave resonances of the earth's magnetic field lines has been clearly demonstrated by conjugate point and spacecraft observations [see, e.g., *Nagata et al.*, 1963; *Sugiura and Wilson*, 1964; *Van Chi et al.*, 1968; *Lanzerotti et al.*, 1972, *Lanzerotti and Fukunishi*, 1974; *Kokubun et al.*, 1976; *Singer and Kivelson*, 1979]. Several mechanisms have been suggested to generate these waves, but regardless of the method of generation, intrinsic interest in this fundamental magnetohydrodynamic plasma process and the possibility of using the waves to diagnose magnetospheric properties make it worthwhile to model these standing wave oscillations.

In the following presentation the basic MHD equations are used to derive an equation for the period and amplitude of low-frequency transverse waves in an arbitrary field geometry. Next, the equation is numerically solved by using a field model that takes into consideration external current systems established because of the earth's interaction with the solar wind. Finally, the model results are compared to those calculated for oscillations in a dipole magnetic field.

In hydromagnetics, the wave equation for low-frequency waves in an infinitely conducting, stationary, magnetized plasma with zero pressure can be derived from the following linearized equations:

Faraday's law:

$$\nabla \times \mathbf{E} = \partial \mathbf{b}/\partial t \tag{1}$$

Ohm's law with infinite conductivity (or the frozen in flux condition):

$$\mathbf{E} = \frac{-\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \tag{2}$$

Ampere's law:

$$\nabla \times \mathbf{b} = \mu_0 \mathbf{j} \tag{3}$$

and the momentum equation:

$$\rho(\partial^2 \boldsymbol{\xi}/\partial t^2) = \mathbf{j} \times \mathbf{B}_0 \tag{4}$$

where $\mathbf{B}_0$ is the unperturbed background magnetic field; $\mathbf{E}$ and $\mathbf{b}$ are the wave perturbation electric and magnetic fields, respectively; $\boldsymbol{\xi}$ is the plasma (or field) displacement; $\mathbf{j}$ is the current; and ρ is the plasma mass density. Equation (4) is an approximate equation. It does ignore several effects of there being a background current in the plasma. The appendix discusses the validity of (4). The wave equation satisfied by the field displacement $\boldsymbol{\xi}$ is

$$[\partial^2(\mathbf{B}_0 \times \boldsymbol{\xi})/\partial t^2] = \mathbf{v}_A \times \mathbf{v}_A \times (\nabla \times \nabla \times (\mathbf{B}_0 \times \boldsymbol{\xi})) \tag{5}$$

where $\mathbf{v}_A = \mathbf{B}_0/(\mu_0\rho)^{1/2}$. This equation, or one equivalent to it for the wave electric or magnetic field, has been used by numerous authors as a starting point for examining long-period ULF waves in the earth's magnetic field [*Westphal and Jacobs*, 1962; *Dungey*, 1963; *Radoski and Carovillano*, 1966; *Cummings et al.*, 1969; *Orr and Matthew*, 1971; *Radoski*, 1974].

In a uniform plasma and magnetic field, (1)–(4) can be used to describe two magnetohydrodynamic wave modes, called the shear Alfven or transverse mode, and the fast mode. An important feature distinguishing these two wave modes is that the perturbation magnetic field is strictly transverse to the ambient field for the transverse mode, while the fast mode has a component along the ambient field. In other words the fast mode can compress the field, whereas the transverse mode only bends the field. Derivations are given, and further characteristics of the wave modes are discussed in detail in *Dungey* [1967, 1968]. In particular, in the transverse mode wave, en-

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ergy is guided along **B**, whereas in the fast mode, energy moves in the direction of the wave vector **k**, which can make any angle with **B**.

The introduction of a nonuniform magnetic field, such as a dipole field, makes it much more difficult to find solutions to the wave equation (5). Additional terms involving the spatial derivatives of **B** appear in the equations. The result is that the fast and transverse modes are coupled. The coupled equations have not been solved, even in a dipole background field [*Lanzerotti and Southwood,* 1979].

Since it has not been possible to solve the coupled equations, approximations have been made to find the eigenfrequencies for the waves in special situations that decoupled the equations [*Radoski and Carovillano,* 1966; *Dungey,* 1967, 1968; *Cummings et al.,* 1969; *Orr and Matthew,* 1971; *Orr,* 1973, *Lanzerotti and Fukunishi,* 1974]. Three of the cases for a dipole background field have been summarized by *Orr* [1973], who considers hydromagnetic waves with a longitudinal variation given by $e^{im\phi}$, where ϕ is longitude and m is the azimuthal wave number. Assuming an axisymmetric disturbance (putting $m = 0$), one finds the equations decouple to give signals with strictly toroidal and poloidal magnetic perturbations. The axially symmetric toroidal case is a transverse Alfven mode with torsional oscillations (b_ϕ) of an entire magnetic shell. The Poynting flux is aligned with $\mathbf{B}_0$, so this case is like a pure shear Alfven mode. The wave equation depends only on how the signal varies along $\mathbf{B}_0$, and as a result, eigenmodes with wavelengths comparable to flux tube length have eigenfrequencies that are different for different shells. The axially symmetric poloidal case corresponds to the fast mode with magnetic oscillations in a meridian plane (b_r and b_θ) and represents alternate symmetric compressions and expansions of the entire magnetosphere. The Poynting flux is directed across field lines. The third case where decoupling is possible is the highly asymmetric poloidal mode. If m is taken to be very large ($m \rightarrow \infty$), one again finds that magnetic oscillations confined to the meridian are possible. This case has also been called the guided poloidal mode [*Radoski,* 1967], because the Poynting flux is field aligned and again, this is like a pure shear Alfven mode.

Using a dipole magnetic field and an orthogonal dipole coordinate system, *Cummings et al.* [1969] wrote (5) in component form and decoupled the resulting equations for toroidal and poloidal fields by assuming that the wave magnetic field was strictly transverse ($\mathbf{B}_0 \cdot \mathbf{b} = 0$). The equations they solved were precisely those for the 'symmetric toroidal mode' and the 'highly asymmetric poloidal mode' we referred to earlier. Cummings et al. also assumed that the earth is a perfect conductor and that the wave electric field had a node at the earth. A more accurate statement is that the electric field (and so the plasma displacement $\boldsymbol{\xi}$) should be very small at the top of the ionosphere [see, e.g., *Hughes,* 1974; *Hughes and Southwood,* 1976]. The net result is that standing wave solutions are obtained along the field direction.

Cummings et al. [1969] numerically determined the eigenfrequencies for the uncoupled toroidal and poloidal modes on a field line extending to 6.6 R_E at the equator for the fundamental through sixth harmonic of the oscillation and for a plasma distribution along the field line given by $n = n_0(r_0/r)^\gamma$, where the density index γ varies from 0 to 6, n_0 is the proton number density at r_0, the geocentric distance to the equatorial crossing point of the field line considered, and r is the geocentric distance to the position of interest on the field line. *Orr and Matthew* [1971] also numerically solved the equations for toroidal and poloidal oscillations. Unlike Cummings et al., they presented the eigenfrequency solutions in a form that could be used at L values other than $L = 6.6$.

The assumption of $\mathbf{B}_0 \cdot \mathbf{b} = 0$ requires some discussion. If $\mathbf{B}_0 \cdot \mathbf{b}$ is zero, magnetic tension is the only stress in the wave, and local field line oscillations cannot couple to adjacent flux tubes. This decoupling means solutions along isolated flux tubes or magnetic shells are obtained as we have already mentioned. Cummings et al.'s assumptions can be justified as a reasonable starting point on observational grounds; in their paper they also report on purely transverse waves seen in synchronous orbit. There are also theoretical reasons given in works such as *Southwood* [1974] or *Chen and Hasegawa* [1974] for believing such an approach is reasonable. Both Southwood and Chen and Hasegawa, using simple models of field inhomogeneity with particular assumptions as to the energy source, showed, in the fully coupled situation, that the wave displacement $\boldsymbol{\xi}$ became very large when $\mathbf{B}_0 \cdot \mathbf{b} = 0$. These regions where $\mathbf{B}_0 \cdot \mathbf{b} = 0$ are called field line resonance regions. This feature, along with success of the experimental tests of the theories of Southwood, and Chen and Hasegawa, and the regular observation of quasitransverse signals, argue that the decoupling procedure adopted by *Cummings et al.* [1969] and *Orr and Matthew* [1971] is a worthwhile, if only approximate, means of estimating wave properties expected at particular points in space.

The earth's field is distorted from a dipole field geometry because of its strong interaction with the solar wind. In particular, on high-latitude magnetic field lines, any model of the earth's magnetic field must allow for the field induced by the interaction with the solar wind. In an early attempt to consider the effect of solar wind compression of the earth's dipole field, *Westphal and Jacobs* [1962] solved the toroidal mode wave equation in a cylindrical geometry by using a compressed dipole field. Their model predicted that wave periods observed from a fixed geomagnetic latitude would decrease as the dipole was compressed. Of course, the solar wind does not simply compress the earth's field uniformly at all local times, and recently, more accurate representations of the earth's magnetic field have become available. *Warner and Orr* [1979] used the *Mead and Fairfield* [1975] magnetic field model and solved for wave periods by using the WKB approximation to the toroidal wave equation. Their results included local time variations, but the WKB approximation is not valid when the wavelength of the oscillation is comparable to the scale size of the system, and is particularly poor for the fundamental mode oscillation of a field line.

Rather than use an approximate WKB solution or the uncoupled toroidal and poloidal mode wave equations, we derived a single exact linear wave equation which can be solved in a dipole field for both toroidal and poloidal mode oscillations. In addition, unlike the *Cummings et al.* [1969] and *Orr and Matthew* [1971] equations, which were for a strictly dipolar field geometry, our wave equation can also be solved in a nondipolar field geometry. Again, we have the constraint that the waves must be strictly transverse, without any perturbation along the ambient field direction. However, in our formulation the solution to the wave equation can be found for any linear polarization direction perpendicular to the ambient field. It is also worth noting that none of the models we have been discussing, including ours, solves the self-consistent problem which includes the effect of a field line perturbation on all field lines in its vicinity.

To examine the oscillation of isolated field lines in a more general field, first consider two adjacent field lines separated at some point along the normal to one by distance δ_α. At any other point along the field line, we can define h_α by requiring the normal separation to be

$$h_\alpha \delta_\alpha$$

If we then write the normal unit vector between the field lines as $\hat{\alpha}$, we can put

$$\nabla\alpha = \hat{\alpha}/h_\alpha$$

Now consider a small displacement in the $\hat{\alpha}$ direction, ξ_α. From (1) and (2) the displacement produces a magnetic perturbation

$$\mathbf{b} = \nabla \times (\xi_\alpha \hat{\alpha} \times \mathbf{B}_0)$$

Now

$$\mathbf{b}\cdot\nabla\alpha = \nabla\alpha\cdot\nabla \times (\xi_\alpha h_\alpha \nabla\alpha \times \mathbf{B}_0)$$
$$= \nabla\cdot(\xi_\alpha h_\alpha |\nabla\alpha|^2 \mathbf{B}_0)$$

because $\nabla\alpha$ has been defined as perpendicular to $\mathbf{B}_0$. Now also

$$|\nabla\alpha|^2 = h_\alpha^{-2}$$

and so

$$\mathbf{b}\cdot\nabla\alpha = (b_\alpha/h_\alpha) = \mathbf{B}_0\cdot\nabla(\xi_\alpha/h_\alpha) \qquad (6)$$

Now in the same manner as *Cummings et al.* [1969], assume that the displacement is made with negligible field compression. The magnetic perturbation b_α, using (3) and (4), produces a force in the momentum equation

$$\mu_0\rho(\partial^2\xi_\alpha\hat{\alpha}/\partial t^2) = (\nabla \times b_\alpha\hat{\alpha}) \times \mathbf{B}_0$$
$$= (\nabla \times b_\alpha h_\alpha \nabla\alpha) \times \mathbf{B}_0$$
$$= (\nabla b_\alpha h_\alpha \times \nabla\alpha) \times \mathbf{B}_0$$
$$= (\mathbf{B}_0\cdot\nabla b_\alpha h_\alpha)\nabla\alpha$$
$$\mu_0\rho(\partial^2\xi_\alpha/\partial t^2) = (1/h_\alpha)(\mathbf{B}_0\cdot\nabla b_\alpha h_\alpha) \qquad (7)$$

Taking this with (6) gives a wave equation

$$\mu_0\rho\frac{\partial^2(\xi_\alpha/h_\alpha)}{\partial t^2} = \frac{1}{h_\alpha^2}\mathbf{B}_0\cdot\nabla\{h_\alpha^2[\mathbf{B}_0\cdot\nabla(\xi_\alpha/h_\alpha)]\} \qquad (8)$$

To solve (8) numerically, we assume a time dependence of the form $e^{i\omega t}$. Writing ds for the increment of length along the magnetic field direction at any point, we can rewrite (8) as a second-order differential equation

$$\frac{\partial^2}{\partial s^2}\left(\frac{\xi_\alpha}{h_\alpha}\right) + \frac{\partial}{\partial s}\left[\ln(h_\alpha^2 B_0)\frac{\partial}{\partial s}\left(\frac{\xi_\alpha}{h_\alpha}\right)\right] + \frac{\mu_0\rho\omega^2}{B_0^2}\left(\frac{\xi_\alpha}{h_\alpha}\right) = 0 \qquad (9)$$

Once ξ_α is determined, we can find b_α from (6), which we rewrite as

$$b_\alpha = h_\alpha B_0\frac{\partial}{\partial s}\left(\frac{\xi_\alpha}{h_\alpha}\right) \qquad (10)$$

The wave electric field is given by

$$E_\beta = -i\omega\xi_\alpha B_0$$

where

$$\hat{\beta} = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}\times\hat{\alpha} \qquad (11)$$

and the plasma velocity by

$$u_\alpha = i\omega\xi_\alpha \qquad (12)$$

Any initial polarization or perturbation direction $\hat{\alpha}$ perpendicular to $\mathbf{B}_0$ can be chosen. In practice the geometrical factors h_α are determined by first taking a perturbation direction at a particular point, e.g., the equator, and then by field line tracing in whatever field model is being used. The factor h_α is proportional to field line separation and varies along $\mathbf{B}_0$ (i.e., varies with s).

ANALYSIS

With the formulation developed above, we can solve for Alfven eigenfrequencies in an arbitrary field geometry, and in particular in one that accurately represents the earth's magnetic field. First, we can compare (9) with the decoupled equations used by *Cummings et al.* [1969] if we solve (9) in a dipole field and use the same assumptions outlined by Cummings et al. The initial step in applying our method is to calculate the scale factors h_α. In Cummings et al. the scale factors are uniquely specified by the dipole field, whereas in our method the h_α's must be determined and depend on the field geometry used. In a dipole field the separation between two field lines in a meridian is proportional to $(rB \sin\theta)^{-1}$, and the separation between two lines on the same magnetic shell is proportional to $r \sin\theta$, where θ is colatitude and r is radial distance from the dipole. Thus with $h_\alpha = (rB \sin\theta)^{-1}$ in (9) we obtain the guided poloidal mode equation ($\mathbf{b}$, ξ, in the meridian), and with $h_\alpha = r \sin\theta$ we obtain the toroidal equation ($\mathbf{b}$, ξ, out of the meridian). Using these forms, we tested our numerical procedure and rederived the solutions for a dipole field, obtaining agreement with Cummings et al.'s results.

With confidence established in the application of wave equation (9) we introduced a more realistic magnetic field model in place of the dipole. The field from magnetospheric sources was calculated by using the Olson-Pfitzer model [W. P. Olson and K. A. Pfitzer, unpublished manuscript (preprint), 1977; *Walker*, 1979]. A dipole was used to model the earth's intrinsic field.

The procedure to calculate the standing wave periods is identical to that outlined above, except that through any point in space the field line in the Olson-Pfitzer model is substituted for the dipole field line. It is important to point out that the procedure used here for calculating the h_α's limits us to examining transverse waves polarized in the direction radially outward from the earth and perpendicular to that direction. In the case of a dipole field these directions are considered poloidal and toroidal, respectively. Another way of looking at it is that surrounding the Olson-Pfitzer field line we consider a flux tube that intersects the equatorial plane in a rectangle with sides parallel and perpendicular to the radial direction. The h_α's are then calculated by keeping flux constant in the tube as we progress up the field line in the calculation. There is a further implicit assumption in our method. Namely that locally all field lines are swept back out of the meridian to the same degree. If this is not so, the direction of polarization of the signal might be expected to vary as one moved along the field. We are currently developing procedures to probe this effect.

The Olson-Pfitzer model includes magnetic field contributions from the distributed quiet-time ring current, magnetopause current, and tail current systems. The model is designed to represent the observations of *Sugiura and Poros* [1973] and is limited to field lines that cross the equator inside of 15 R_E. Although the Olson-Pfitzer model can be used for all dipole

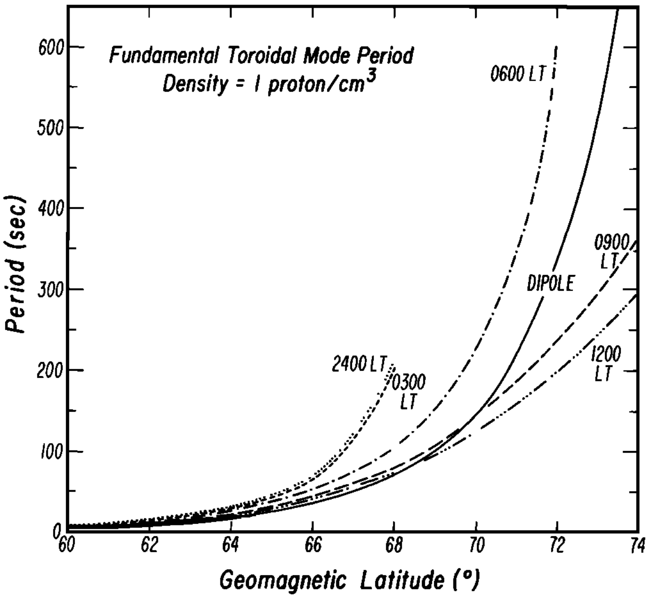

Fig. 1. Fundamental toroidal mode period for geomagnetic latitudes from 60° to 74°, using the model of W. P. Olson and K. A. Pfitzer (unpublished manuscript, 1977). Periods are given for several local times from midnight to noon and compared to dipole model periods. The results are symmetric about the noon-midnight meridian, and the mass density is 1 amu/cm³ along the entire field line.

tilt angles, we consider only zero tilt in the present analysis. As a result, the field lines are symmetric about the magnetic equatorial plane. We have tested the field line tracing procedures in our calculation by comparing our results with plots given in Olson and Pfitzer (unpublished manuscript, 1977) of ΔB (B model-B dipole) in the noon-midnight meridian, $|\mathbf{B}|$ in the equatorial plane, the equatorial intercept of field lines from various magnetic latitudes on the earth's surface, and the earth surface field line intercepts from synchronous orbit as a function of local time. In all cases, agreement with the calculations given by Olson and Pfitzer was excellent.

We formulated our solution to (9), using either a dipole field or the Olson-Pfitzer model, to allow for variation of several input parameters. These parameters include the position of the field line, which can be specified by a single point, either on the ground or in space; the mass density at the equatorial crossing point of the field line; and the density index. In addition, one may choose to solve for any harmonic for polarizations in the equatorial plane either parallel or perpendicular to the radial direction, which we continue to refer to as poloidal and toroidal, respectively.

RESULTS

In this section the effect of magnetic field geometry on eigenperiods of standing Alfven waves is illustrated for the fundamental mode toroidal oscillation with a density n_0 equal to 1 proton/cm³ everywhere along the field line. Periods for other mass densities can be determined by multiplying the result by $\sqrt{n}$, where n is the mass density of the plasma in amu/cm³. The terms toroidal and poloidal in this model indicate signals polarized east-west and radially at the equator, respectively. As a boundary condition we take the field displacement to be zero at the ionosphere, which is equivalent to assuming the ionosphere has infinite conductivity.

Figure 1 shows the period expected as a function of magnetic latitude for several local times at the ground position of the field line. The dipole model periods, for which there is no local time variation, are shown for purposes of comparison. Since the chosen nondipolar magnetic field model is symmetric about the noon-midnight meridian, only local times from midnight to noon are used. There are substantial deviations between the model and dipole periods at high latitudes, and periods can be larger or smaller than the dipole value, depending on latitude and local time. At any particular latitude the model period increases from noon to midnight. The steepest increase in period with latitude occurs at midnight.

Figure 2 illustrates these same results in a format that emphasizes the model period deviations from the dipole period. The ordinate of this figure is the percent increase (or decrease) of the model period from the dipole period. For example, at 67°, the magnetic latitude that corresponds to $L = 6.6$ and maps out to synchronous orbit in a dipole, the model fundamental period is ~130% larger than the dipole period at midnight. On the other hand, at noon, at high latitudes, the model period can be more than 50% less than the dipole period.

The previous two figures demonstrated local time effects on period expected for selected observation positions on the ground. In Figure 3 we examine the situation in space. The percentage deviation of the model periods from dipole values is shown for different local times and different radial distances in the equatorial plane. The table provides the resonant periods calculated for different L values by using a dipole field. The results here contrast strongly with those given for observing locations on the ground. Inside of ~9 R_E, deviations from the dipole periods are less than 10%, and more importantly, there is little local time variation of period. Whereas on the ground at 67° magnetic latitude the diurnal variation in period caused by field geometry is larger than a factor of 2; at 6.6 R_E, where the dipole field line from 67° crosses the magnetospheric equator, the computations that use the Olson-Pfitzer model show little period dependence on local time.

Another effect of a nondipolar magnetic field model is that field lines do not necessarily remain in meridian planes. The local time at which field lines leave the earth's surface can be different from the local time where the field lines intersect the

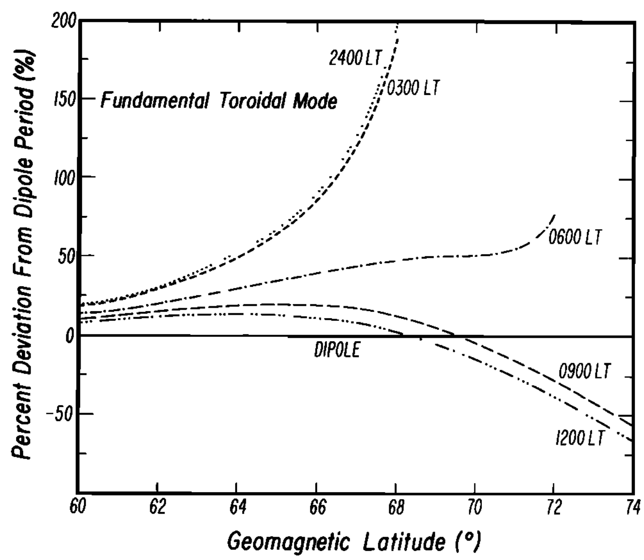

Fig. 2. Same as Figure 1 except the percent deviation from the dipole model results is shown.

equatorial plane, especially at high latitudes near dawn and dusk, where the field lines are swept tailward. Figure 4 shows the location in the equatorial plane of field lines that leave the earth at 1.5-hour steps from midnight to noon. From each of those local times the intercepts are given in 1° steps, starting with 64° geomagnetic latitude. The highest latitude positions vary as a function of local time and were selected to remain inside the region of validity of the Olson-Pfitzer model. The equatorial crossing point of a field line leaving the earth at a given latitude extends to larger radial distances as one proceeds from noon to midnight. The equatorial crossing point of a field line leaving the earth at a given local time intersects the equatorial plane at progressively earlier local times as the latitude increases, except for midnight and noon local time. The variations of field line intercept coordinates from those expected in a dipole field are important for the interpretation of space-ground conjugate studies.

DISCUSSION

The development of the model in this paper was partially prompted by the need to account for the effect on resonant periods of the difference in field line length between a dipole and a more realistic magnetic field model [*Singer and Kivelson*, 1979]. Nevertheless, it is clear from the approximate solution for the period of standing Alfven wave resonances given by the WKB method ($T \sim \oint ds/v_A$) that period depends on both the length of the field line and the Alfven velocity along the field line. For any reasonable density model the greatest contribution to the integral for the period comes from the region of smallest magnetic field strength along the field line, and differences in field line length from a dipole are often not as important as the magnitude of the field near the geomagnetic equator. For this reason we can qualitatively explain our results, as seen for example in Figure 2, by examining the field model in the equatorial region.

Figure 5 shows the ΔB contours in the noon-midnight meridian, using the Olson-Pfitzer magnetospheric model. (The figure is from Olson and Pfitzer (unpublished manuscript, 1977) and is also shown in *Walker* [1979]). The high-latitude field

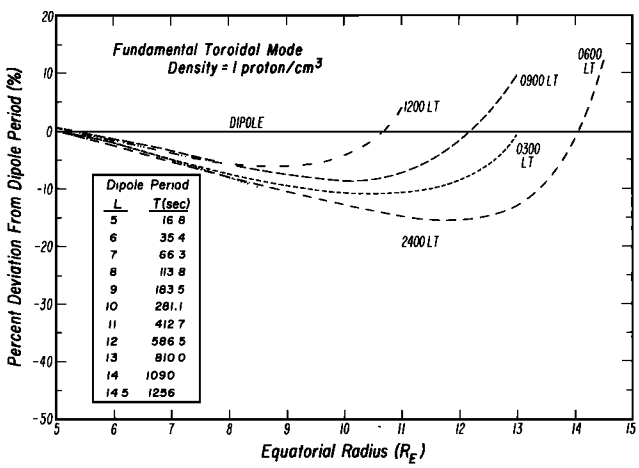

Fig. 3. Percentage deviation of period calculated by using the Olson-Pfitzer model from that which would be determined by using the dipole model for observations from different equatorial radial distances in the magnetosphere. Density and mode of oscillation are the same as Figure 1. The insert gives the dipole model period for field lines that cross the equator at several equatorial radial distances or L values.

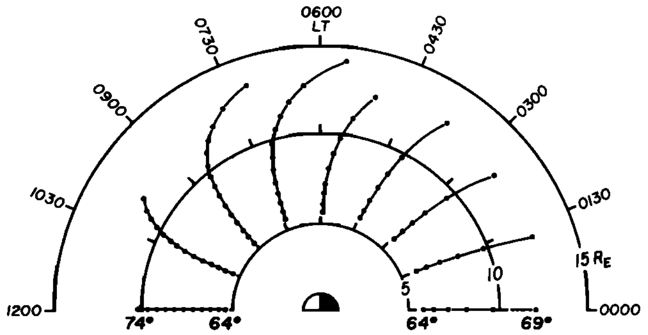

Fig. 4. Location in the equatorial plane of field lines leaving the earth at 1.5-hour steps from midnight to noon from the Olson-Pfitzer model. From each local time the intercepts are given in 1° steps, starting with 64° geomagnetic latitude.

lines at noon that cross the magnetospheric equator near the magnetopause have equatorial field strength larger than the dipole value because of magnetopause currents; therefore, as is clear from the WKB expression, the resonant period should be less than the dipole period. This effect was observed in Figure 2 as the decrease in the period from the dipole period at high latitudes at noon local time. Figure 5 shows that closer to the earth at noon there is a depression of the equatorial field strength as a result of the ring current. Consequently, the period should be larger than the dipole period, and this effect is also seen in Figure 2. At midnight the depression in the field strength near the equatorial plane because of the ring and tail currents increases the period from the dipole value. The effects at other local times can be qualitatively accounted for in a similar manner.

The insignificant variation of period with local time from constant radial distances in space less than ~9 R_E can be explained by examining Figure 6 (also from Olson and Pfitzer), which shows contours of constant field magnitude $|\mathbf{B}|$ in the equatorial plane. The local time variation in the field magnitude (the field strength at the equator becomes larger as one moves at a constant radial distance from midnight to noon) is only beginning to become significant at the ~100-γ contour shown in the figure. However, the field line length is also becoming larger as one moves at a constant radial distance from midnight to noon, and the two effects tend to counteract one another in determining the resonant period.

To determine the effect of the magnetic field model on the periods of standing Alfven waves, a constant plasma mass density of 1 amu/cm³ has been used everywhere in the magnetosphere. There are, of course, spatial, temporal, and compositional variations in the magnetospheric plasma that could substantially alter the local time and radial or latitudinal patterns that were shown to develop as a result of the realistic magnetic field model. In fact, using typical ion density observations from the OGO 5 satellite, for different magnetospheric regions, *Warner and Orr* [1979] have considered the effect of density on local time and latitudinal variations of standing wave periods. Their results point up the importance of considering density as well as field line configuration for determining standing wave periods. In our calculations, periods scale as the square root of equatorial mass density, and our plots can easily be recalibrated for different densities. We did not allow for density variation along the field.

Figures 1, 2, and 3 compare toroidal mode (azimuthally polarized) wave periods that are derived by using the Olson-Pfit-

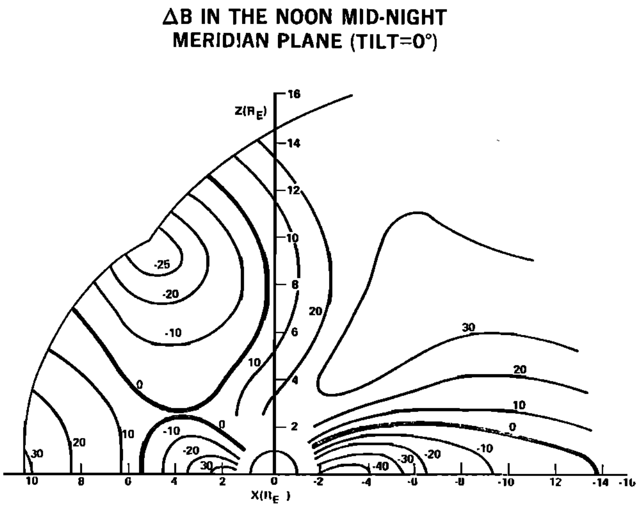

Fig. 5. ΔB contours (B Olson-Pfitzer—B dipole) in the noon-midnight meridian plane for the Olson-Pfitzer model with zero tilt. ΔB is given in gamma. (Figure is from Olson and Pfitzer, unpublished manuscript, 1977; and *Walker*, [1979]).

zer model field lines with periods that are derived by using dipole field lines. In a dipole field in the equatorial plane the azimuthal direction corresponds to the direction toward field lines with the same equatorial field magnitude. As can be seen in Figures 4 and 6 in the Olson-Pfitzer model, in the equatorial plane close to the earth the azimuthal direction and the direction of constant B are nearly identical. The two directions are identical at all radial distances at noon and midnight. However, at other local times, for example at dawn, at large radial distances the azimuthal direction and the direction of constant field magnitude do not coincide. Consequently, although consistent in our selection of polarization with respect to an Earth-based coordinate (azimuthal), the direction of polarization chosen does not align with any particular field parameter contour at the equator. For example, an alternative would be to compare periods at different radial distances for waves polarized in the direction of constant field magnitude. We are at present modifying our calculations to allow for the examination of perturbations in any direction.

The determination of magnetospheric plasma density from ground-based observations would be extremely useful as comparable coverage, using in situ measurements by spacecraft, is unlikely. In the past, VLF measurements from ground stations have been used to determine electron number densities in the magnetosphere [*Helliwell*, 1965; *Carpenter and Smith*, 1964]. More recently, using the notion of standing wave oscillations, densities have been determined by ULF techniques [e.g., *Lanzerotti et al.*, 1975; *Cummings et al.*, 1978] and compared to VLF density measurements [*Webb et al.*, 1977]. For ULF pulsations, usually the period and location of the pulsation can be well established, but assumptions have to be made about the field geometry and density distribution along the field line in order to infer equatorial plasma densities. *Webb et al.* [1977] assumed a dipole field geometry. Such an assumption is good at ≲60° magnetic latitude during quiet geomagnetic conditions; however, as we have shown at higher magnetic latitudes, the earth's field is often substantially distorted from the dipole geometry. *Walker et al.* [1979], using the STARE (Scandinavian Twin Auroral Radar Experiment) radar system at high magnetic latitudes (~70°), have estimated magnetospheric plasma density in the vicinity of pulsations by assuming fundamental, toroidal mode oscillations in a dipole field. They suggest the need for a more realistic field model for more accurate determination of the densities. The model developed in this paper would be useful for this purpose, as can be seen in Figure 7. Figure 7 shows the percentage deviation of inferred plasma mass density from what would be determined through the use of a dipole model as a function of magnetic latitude for several local times. The toroidal fundamental mode has been used here. Predictions from the more realistic magnetic field geometry show that there could be substantial error in the mass density determination at high latitudes if a dipole model is used.

To the best of our knowledge, the only other model of standing Alfven waves that uses a realistic magnetic field model for the earth is that of *Warner and Orr* [1979]. Using the WKB approximation and the *Mead and Fairfield* [1975] magnetic field model, they clearly demonstrate the importance of considering field model and density variations for calculating eigenperiods on field lines. As was discussed earlier, the WKB approximation is particularly poor for the fundamental period, which is the most probable mode, as reported by many observers [*Lanzerotti et al.*, 1972; *Cummings et al.*, 1975; *Singer and Kivelson*, 1979]. We preferred to use the Olson-Pfitzer model since it has been shown to fit the magnetic field strength at the equator in the noon-midnight meridian much better than the Mead-Fairfield model [*Walker*, 1976; W. P. Olson and A. Pfitzer, unpublished manuscript, 1977]. We have shown the effect of the field model on the local time variation of periods from ground observations as did Warner and Orr. We are unable to make direct comparisons with Warner and Orr's results because, in the published results, tilt, *Kp*, and density are varied simultaneously. We have isolated the effects of the magnetic field model and also have demonstrated the local time variation in period from a vantage point in space. Unlike the WKB approximation, the solution to the wave equation developed in this paper permits accurate determination of not only the period but also the perturbation fields along a field line.

A worthwhile direction for the further investigation of standing wave models is to combine the model which uses re-

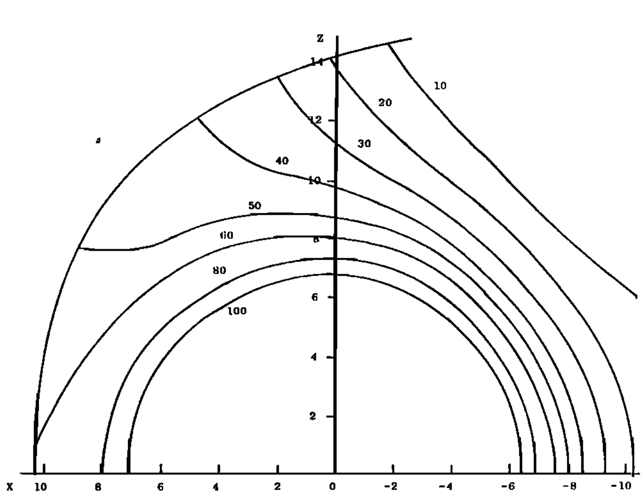

Fig. 6. Calculated from the model of Olson and Pfitzer (as described in W. P. Olsen and A. Pfitzer, unpublished manuscript, 1977) Contours of constant field magnitude $|\mathbf{B}|$ in the equatorial plane. B is given in gamma.

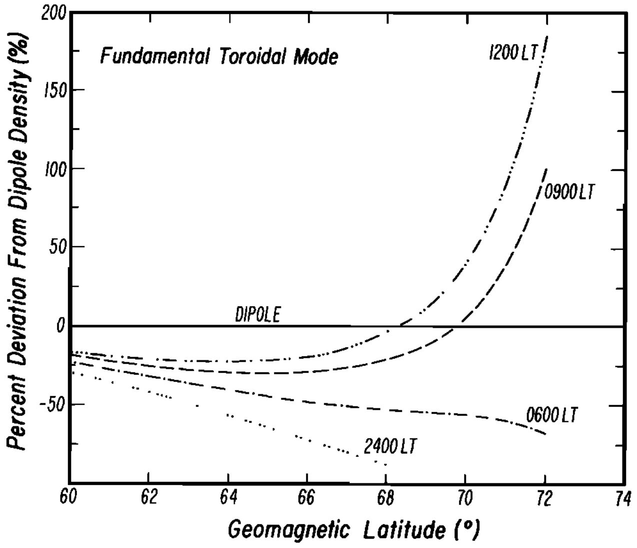

Fig. 7. Using the same model as described in Figure 1, the percentage deviation of the inferred plasma mass density from that which would be determined by using a dipole model is shown for different local times at different geomagnetic latitudes.

alistic field geometry in this paper with different density models in space and different ionospheric boundary conditions [*Newton et al.*, 1978]. The capability to adjust the ionospheric boundary conditions at the two ends of the field line to account for differing degrees of illumination should be included [*Allan and Knox*, 1979*a*, *b*]. The effect of a tilted magnetic field, such as considered by *Warner and Orr* [1979], should allow determination of the latitude of nodes of resonant oscillations and the amplitude structure along a field line for different harmonics. These results would be important for understanding seasonal variations of pulsation observations.

CONCLUSIONS

The linearized transverse wave equation for low-frequency propagation in a cold, collisionless, magnetized plasma has been solved in an arbitrary magnetic field geometry. We obtained solutions for the symmetric toroidal and highly asymmetric poloidal mode standing wave oscillations in both dipole and Olson-Pfitzer field models. The realistic description of the earth's field given by the Olson-Pfitzer model leads to substantial differences in eigenfrequencies from those determined by using dipole models. Therefore, our model should improve our ability to analyze pulsation observations.

The use of the model described in this paper is most important for studying oscillations on field lines which extend to large radial distances where large deviations from a dipole configuration occur. It has been shown that from a fixed latitude on the ground, the period of oscillation varies with local time and can be larger or smaller than the period calculated by using a dipole model. Local time variations of period from a fixed radial distance in space differ from observations from a fixed magnetic latitude. Due to field geometry, the local time variations of period from a fixed radial distance at the magnetospheric equator are not significant until large radial distances are reached. In particular, there are insignificant variations of period with local time at synchronous orbit.

The model developed is important for ground-satellite conjugate studies, since field lines going through a particular position in space can intersect the ground at a different local time and latitude than would be found by using a dipole. In addition, it has been shown that the diagnostic technique that uses ULF waves to determine mass density in space from ground observations requires a realistic field model, since the use of a dipole model can introduce errors, especially at high latitudes.

Finally, since the equation for standing Alfven waves derived in this paper can be used easily with any magnetic field model, we could replace the earth's field with that of Jupiter or Saturn. Prediction of pulsation periods in planetary magnetospheres will be one technique for determining if observed magnetic perturbations are standing Alfven waves.

APPENDIX: VALIDITY OF THE LINEARIZED MOMENTUM EQUATION (4).

Equation (4) takes the form

$$\rho(\partial^2\xi/\partial t^2) = \mathbf{j} \times \mathbf{B}_0$$

It explicitly ignores a force $\mathbf{j}_0 \times \mathbf{b}$, which must be present in a plasma carrying a zero-order current across the background field. The Olson-Pfitzer field model contains such currents (the ring current), and the force must be present. On order of magnitude grounds, however, it is unlikely to be significant, as the current density in the Olson-Pfitzer model does not exceed 1.5×10^{-10} A/m^2. Using a crude estimate, $j \sim b/l_{\|}$, where $l_{\|}$ = scale length parallel to $\mathbf{B}$, one finds $\mathbf{j} \times \mathbf{B}_0$ should exceed the neglected term by over an order of magnitude at synchronous orbit. Inside this orbit the neglect seems fully justified. Out to 12 R_E, the neglected term is everywhere less than ~25% of the term we have used.

Some further comments can be made. The force $\mathbf{j}_0 \times \mathbf{b}$ is parallel to $\mathbf{B}_0$ if the wave perturbation is transverse and thus its precise effect depends on the conditions governing ion and electron dynamics along the field. Unless conditions were just such that strong coupling could be established with an acoustic type of mode standing along the field line, the force only produces small net displacements of plasma back and forth along the field at the wave frequency. Equation (4) also ignores an implicit feature of the Olson-Pfitzer field. There must be hot plasma gradients present to provide the zero-order momentum balance;

$$\nabla P = \mathbf{j}_0 \times \mathbf{B}_0$$

if the pressure is isotropic. The plasma must be displaced by the wave, and even if the wave motion does not compress the plasma, changes in pressure will be produced by the convection of gradients by the wave. *Southwood* [1977] pointed out that this could lead to compressional magnetic field changes in the wave, as is often seen. We have made no effort to examine this further, but it will need consideration if this line of approach is continued.

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