Algebra of Dunkl Laplace–Runge–Lenz vector

Misha Feigin[1,2] · Tigran Hakobyan[3]

Abstract

We introduce the Dunkl version of the Laplace–Runge–Lenz vector associated with a finite Coxeter group W acting geometrically in $\mathbb{R}^N$ and with a multiplicity function g. This vector generalizes the usual Laplace–Runge–Lenz vector and its components commute with the Dunkl–Coulomb Hamiltonian given as the Dunkl Laplacian with an additional Coulomb potential γ/r. We study the resulting symmetry algebra $R_{g,\gamma}(W)$ and show that it has the Poincaré–Birkhoff–Witt property. In the absence of a Coulomb potential, this symmetry algebra $R_{g,0}(W)$ is a subalgebra of the rational Cherednik algebra $H_g(W)$. We show that a central quotient of the algebra $R_{g,\gamma}(W)$ is a quadratic algebra isomorphic to a central quotient of the corresponding Dunkl angular momenta algebra $H_g^{so(N+1)}(W)$. This gives an interpretation of the algebra $H_g^{so(N+1)}(W)$ as the hidden symmetry algebra of the Dunkl–Coulomb problem in $\mathbb{R}^N$. By specialising $R_{g,\gamma}(W)$ to $g = 0$, we recover a quotient of the universal enveloping algebra $U(so(N+1))$ as the hidden symmetry algebra of the Coulomb problem in $\mathbb{R}^N$. We also apply the Dunkl Laplace–Runge–Lenz vector to establish the maximal superintegrability of the generalised Calogero–Moser systems.

Keywords Calogero–Moser systems · Cherednik algebras · Dunkl operators · Superintegrability

Mathematics Subject Classification 81R12

✉ Misha Feigin
misha.feigin@glasgow.ac.uk

Tigran Hakobyan
tigran.hakobyan@ysu.am

1   School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, UK

2   Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia

3   Yerevan State University, 1 Alex Manoogian St., Yerevan 0025, Armenia

1 Introduction

The discovery of the classical version of the Laplace–Runge–Lenz (LRL) vector goes back to a letter of Johann Bernoulli written in 1710 and a letter by J. Hermann written earlier that year (see [1] and references there). The quantum version of the LRL vector was used by Pauli [2] for the derivation of the spectrum of the hydrogen atom. Pauli also found relations between the components of the LRL vector, the Hamiltonian of the hydrogen atom (equivalently, the Hamiltonian of the Coulomb problem), and the angular momentum operators. Hulthén (with an acknowledgment to O. Klein) pointed out a connection to the orthogonal (or rather the Lorentz) group in a four-dimensional space in 1933 [3]. Indeed, it is clear from these relations that the conserved quantities at a fixed energy level satisfy $so(4)$ relations. This algebra is also called the hidden symmetry algebra of the Coulomb problem as it extends the more straightforward $so(3)$ symmetry algebra generated by the angular momenta. In the case of the Coulomb problem in N-dimensional space, the hidden symmetry algebra is $so(N + 1)$ [4]. Another explanation of the hidden rotational symmetry of the Coulomb problem based on the momentum space representation of wave functions was presented in [5].

Dunkl operators arose in the theory of generalised harmonic polynomials associated with a root system $\mathcal{R}$ of a finite reflection group W and a W-invariant function $g : \mathcal{R} \to \mathbb{C}$ [6]. These operators give a deformation of partial derivatives with nonlocal additional terms which vanish if $g = 0$. Dunkl operators commute pairwise, and the sum of the squares of these operators corresponding to an orthonormal basis gives the Dunkl Laplacian which is a deformation of the usual Laplace operator [7, 8].

Dunkl operators are also the key ingredients of the rational Cherednik algebras [9], and they are closely related to the Calogero–Moser integrable systems. The Calogero–Moser system describes interacting particles on a line with a pairwise inverse square distance potential [10]. Liouville integrability of the classical system was established by the Lax method in [11]. The Calogero–Moser system has integrable generalisations related to the root systems $\mathcal{R}$ [12]. Integrability of these systems can be established with the help of Dunkl operators. Indeed, the restriction of the Dunkl Laplacian to the space of W-invariant functions gives a generalised Calogero–Moser Hamiltonian associated with $\mathcal{R}$. Moreover, integrals of motions can also be expressed as the restrictions of W-invariant combinations of Dunkl operators [13] (see also [14] for the usual Calogero–Moser system in the harmonic confinement).

In addition to Liouville integrability, the Calogero–Moser Hamiltonian has extra integrals of motion ensuring its maximal superintegrability. This was established for the classical Calogero–Moser system in [15]. The maximal superintegrability of a classical integrable system leads to the property that compact trajectories are closed. In the quantum case, superintegrability corresponds to a degeneration of the spectrum. The maximal superintegrability of the quantum Calogero–Moser system was shown in [16], where additional integrals were constructed with the help of Dunkl operators (see also [17] for another proof). Superintegrability for the quantum system was claimed in [18] for any root system, and additional integrals were given. Algebraic structures formed by the Calogero–Moser operators together with their integrals were investigated in [16, 18] (see also [19] where a closely related quadratic algebra was considered). Superintegrability of spin Calogero–Moser systems was investigated in

[20, 21]. Most recently, it was established for the generalised spin Calogero–Moser systems related to complex reflection groups [22].

A related algebraic structure is the Dunkl angular momenta algebra $H_g^{so(N)}(W)$ [23]. This algebra is generated by Dunkl angular momentum operators and the group W. It can be thought of as a flat deformation of the skew product of a quotient of the universal enveloping algebra $U(so(N))$ and W. The central element of this algebra acts naturally on the W-invariant functions as the angular part of the corresponding Calogero–Moser Hamiltonian. A central quotient of the algebra $H_g^{so(N)}(W)$, that is the quotient over the ideal generated by a central element, can be identified with the algebra of global sections of a sheaf of Cherednik algebras on a smooth quadric [37]. Dunkl angular momenta also lead to various symmetries of a Dirac operator in the Clifford extension of the rational Cherednik algebra studied in [24].

The Dunkl–Coulomb Hamiltonian $\mathcal{H}_\gamma$ is a nonlocal operator given as the Dunkl Laplacian with an additional Coulomb potential γ/r. It depends on the coupling parameter g and the Coulomb force parameter γ. Such a two-dimensional Hamiltonian was considered in [25] for the root system $A_1 \times A_1$, where a version of the LRL vector for the corresponding Hamiltonian was presented. In the N-dimensional case, for the root system A_{N-1} the Dunkl LRL vector was introduced in [26] (see also [27] for the initial attempt and for a discussion of superintegrability for classical generalised Calogero–Moser Hamiltonians with a Coulomb potential). It commutes with the Dunkl–Coulomb Hamiltonian $\mathcal{H}_\gamma$. If the coupling parameter $g = 0$, then the Dunkl LRL vector reduces to the usual LRL vector. On the other hand, if $\gamma = 0$, then we get a version of the LRL vector for the Dunkl Laplacian. The Dunkl LRL vector for the type A Dunkl–Coulomb problem on the sphere was considered in [28, 29]. For certain integrable perturbations, which break $H_g^{so(N)}(W)$ symmetry along a particular direction, the corresponding component of the LRL vector is preserved [30].

In the present paper, we introduce the Dunkl LRL vector for the Dunkl–Coulomb Hamiltonian $\mathcal{H}_\gamma$ associated with the root system $\mathcal{R}$ of an arbitrary finite Coxeter group W. Components of this vector commute with the Dunkl–Coulomb Hamiltonian. Another set of operators commuting with the Hamiltonian is given by Dunkl angular momenta. This gives a new way to prove the maximal superintegrability of the quantum generalised Calogero–Moser system related to a root system $\mathcal{R}$. Furthermore, it leads to additional quantum integrals of the generalised Calogero–Moser systems which do not have full Coxeter symmetry and are related to special representations of the rational Cherednik algebras [31–33]. The components of the Dunkl LRL vector, the Dunkl angular momenta, the Dunkl–Coulomb Hamiltonian and the elements of W generate an algebra $R_{g,\gamma}(W)$, which may be thought of as the symmetry algebra of the Hamiltonian $\mathcal{H}_\gamma$. It is the main object of this paper.

After reviewing the background information in Sect. 2, we establish relations involving the Dunkl LRL vector in Sect. 3. In Sect. 4, we establish the defining relations of the algebra $R_{g,\gamma}(W)$ and find its basis. Then, we consider a central quotient of $R_{g,\gamma}(W)$ and show that it is isomorphic to a central quotient of the Dunkl angular momenta algebra $H_g^{so(N+1)}(W)$ where the group W acts naturally on the first N components of vectors in $\mathbb{C}^{N+1}$. These central quotients are (non-homogeneous) quadratic PBW algebras in the sense of [34]. This isomorphism gives an interpretation of the

Dunkl angular momenta algebra $H_g^{so(N+1)}(W)$ as the hidden symmetry algebra of the Dunkl–Coulomb problem in $\mathbb{R}^N$; this is similar to (the quotient of the universal enveloping algebra of) $so(N + 1)$ being the hidden symmetry algebra of the Coulomb problem in $\mathbb{R}^N$. The latter property can also be deduced from our considerations by specialising the algebra $R_{g,\gamma}(W)$ at $g = 0$ so that the Dunkl–Coulomb Hamiltonian $\mathcal{H}_\gamma$ takes the form of the usual Coulomb Hamiltonian in $\mathbb{R}^N$. At this specialisation the central quotient of $R_{0,\gamma}(W)$ becomes isomorphic to the skew product of a quotient of the universal enveloping algebra $U(so(N + 1))$ and the Coxeter group W.

Of particular interest is the case $\gamma = 0$, when the Coulomb potential is absent since the algebra $R_{g,0}(W)$ is a subalgebra of the corresponding rational Cherednik algebra $H_g(W)$. Even though $H_g^{so(N+1)}(W)$ is naturally a subalgebra of the rational Cherednik algebra with group W acting in $(N + 1)$-dimensional vector space, its central quotient appears to be isomorphic to a central quotient of the subalgebra $R_{g,0}(W)$ in the rational Cherednik algebra $H_g(W)$ with the group W acting in the space $\mathbb{C}^N$.

In Sect. 5, we apply the Dunkl LRL vector and the algebra $R_{g,\gamma}(W)$ we have developed, and establish the maximal superintegrability of the Calogero–Moser systems for any root system $\mathcal{R}$ with an additional (possibly, vanishing) Coulomb potential. We also establish the maximal superintegrability of generalisations of such systems which do not have full Coxeter symmetry.

2 Dunkl operators and their properties

Let $\mathcal{R}$ be a Coxeter root system in the N-dimensional Euclidean space $\mathbb{R}^N$ [35]. Let $V \cong \mathbb{C}^N$ be the complexification of this vector space with the bilinear inner product denoted by $(\cdot, \cdot)$. The corresponding finite Coxeter group W is generated by the orthogonal reflections

$$s_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha, \quad \alpha \in \mathcal{R}, \; x \in V.$$

The root system $\mathcal{R}$ can be represented as a disjoint union $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$, where $\mathcal{R}_+$ is a system of non-collinear positive roots and $\mathcal{R}_- = -\mathcal{R}_+$. The root system $\mathcal{R}$ satisfies the invariance $s_\alpha \mathcal{R} = \mathcal{R} \; \forall \alpha \in \mathcal{R}$.

Consider a multiplicity function $g: \mathcal{R} \to \mathbb{C}$. Let $g_\alpha = g(\alpha)$ for $\alpha \in \mathcal{R}$. We assume that g is W-invariant, that is

$$g_{w(\alpha)} = g_\alpha$$

for any $w \in W, \alpha \in \mathcal{R}$.

The Dunkl operator ∇_ξ for any $\xi \in V$ is defined by [6]

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(\alpha, \xi)}{(\alpha, x)} s_\alpha,$$

where $\partial_\xi = (\xi, \partial)$ is the partial derivative in the direction ξ and the reflections s_α act on the functions $\psi(x)$ in a standard way,

$$s_\alpha \psi(x) = \psi(s_\alpha(x)), \quad x \in V.$$

The Dunkl operators satisfy commutativity [6] $[\nabla_\xi, \nabla_\eta] = 0$, and their commutators with linear functions produce nonlocal exchange operators given as follows:

$$\left[\nabla_\xi, (x, \eta)\right] = (\xi, \eta) + \sum_{\alpha \in \mathcal{R}_+} \frac{2g_\alpha(\alpha, \xi)(\alpha, \eta)}{(\alpha, \alpha)} s_\alpha, \tag{1}$$

where $\xi, \eta \in V$. These relations reduce to the Weyl algebra commutation relations in the zero coupling limit, $g_\alpha = 0$ for all $\alpha \in \mathcal{R}$. For general coupling values, they lead to the rational Cherednik algebra $H_g(W)$ which is a deformation of the skew product of the Weyl algebra with the Coxeter group W [9].

It is also convenient to consider N Dunkl operators corresponding to an orthonormal basis $e_1, \ldots, e_N$ in V. For $x \in V$ and $\alpha \in \mathcal{R}$, we let

$$x = \sum_{i=1}^N x_i e_i, \quad \alpha = \sum_{i=1}^N \alpha_i e_i,$$

where $x_i, \alpha_i \in \mathbb{C}$, and we denote $x_\alpha = (x, \alpha) = \sum_{i=1}^N \alpha_i x_i$. We have

$$\nabla_i = \nabla_{e_i} = \partial_i - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha \alpha_i}{x_\alpha} s_\alpha,$$

where $\partial_i = \frac{\partial}{\partial x_i}, i = 1, \ldots, N$. In this notation, the commutation relations (1) take the form:

$$[\nabla_i, x_j] = S_{ij}, \tag{2}$$

where

$$S_{ij} = \delta_{ij} + \sum_{\alpha \in \mathcal{R}_+} \frac{2g_\alpha \alpha_i \alpha_j}{(\alpha, \alpha)} s_\alpha, \tag{3}$$

$i, j = 1, \ldots, N$, and δ_{ij} is the Kronecker symbol. Note that relations (2) remain invariant under the formal Hermitian conjugation,

$$\nabla_i^+ = -\nabla_i, \quad x_i^+ = x_i, \quad S_{ij}^+ = S_{ij}. \tag{4}$$

Lemma 2.1 *The commutators of the elements S_{ij} with coordinates x_k satisfy the relations*

$$[S_{ij}, x_k] = [S_{kj}, x_i] \tag{5}$$

for all i, j, $k = 1, \ldots, N$.

Proof For any $\alpha \in \mathcal{R}$, we have

$$s_\alpha(x_k) = x_k - \frac{2x_\alpha \alpha_k}{(\alpha, \alpha)}, \tag{6}$$

and therefore

$$[x_k, s_\alpha] = \frac{2x_\alpha \alpha_k}{(\alpha, \alpha)} s_\alpha.$$

Hence, we obtain from (3) that

$$[x_k, S_{ij}] = \sum_{\alpha \in \mathcal{R}_+} \frac{4g_\alpha \alpha_i \alpha_j \alpha_k}{(\alpha, \alpha)^2} x_\alpha s_\alpha. \tag{7}$$

The right-hand side of the equality (7) is symmetric in the three indices i, j and k, which implies the required relation (5). $\qquad \square$

Next we consider the following element S of the Coxeter group algebra $\mathbb{C}W$:

$$S = - \sum_{\alpha \in \mathcal{R}_+} g_\alpha s_\alpha. \tag{8}$$

Lemma 2.2 *The element S satisfies the following relations:*

$$[S, s_\alpha] = 0 \tag{9}$$

for any $\alpha \in \mathcal{R}$, and

$$\sum_{i=1}^{N} S_{ii} = N - 2S. \tag{10}$$

Proof The relation (9) follows from the W-invariance of the multiplicity function (2) and the property

$$w s_\alpha w^{-1} = s_{w(\alpha)},$$

which is valid for any $w \in W$. The relation (10) follows directly from (3). $\qquad \square$

Relations (6) and (8) imply that

$$(x, \nabla) = \sum_{k=1}^{N} x_k \nabla_k = r\partial_r + S, \quad (\nabla, x) = \sum_{k=1}^{N} \nabla_k x_k = r\partial_r - S + N, \tag{11}$$

where $r = \left(\sum_{k=1}^{N} x_k^2 \right)^{1/2}$ and $\partial_r = \frac{1}{r} \sum_{k=1}^{N} x_k \partial_k$.

Lemma 2.3 *The following relations take place for any* $i = 1, \ldots, N$:

$$\sum_{j=1}^{N} x_j S_{ij} = x_i + [S, x_i], \qquad \sum_{j=1}^{N} S_{ij} x_j = x_i - [S, x_i]; \qquad (12a)$$

$$\sum_{j=1}^{N} \nabla_j S_{ij} = \nabla_i + [S, \nabla_i], \qquad \sum_{j=1}^{N} S_{ij} \nabla_j = \nabla_i - [S, \nabla_i]. \qquad (12b)$$

Proof It follows from (3) that

$$\sum_{j=1}^{N} x_j S_{ij} = x_i + \sum_{\alpha \in \mathcal{R}_+} \frac{2 g_\alpha x_\alpha \alpha_i}{(\alpha, \alpha)} s_\alpha.$$

We also have

$$[g_\alpha s_\alpha, x_i] = -\frac{2 g_\alpha x_\alpha \alpha_i}{(\alpha, \alpha)} s_\alpha,$$

which implies the first relation in (12a). The other relations can be checked similarly.
□

As a corollary of Lemma 2.3, we get the following statement.

Lemma 2.4 *The following (anti-)commutation relations take place for any* $i = 1, \ldots, N$:

$$\sum_{j=1}^{N} \{x_j, S_{ij}\} = 2x_i, \qquad \sum_{j=1}^{N} [x_j, S_{ij}] = 2[S, x_i]; \qquad (13a)$$

$$\sum_{j=1}^{N} \{\nabla_j, S_{ij}\} = 2\nabla_i, \qquad \sum_{j=1}^{N} [\nabla_j, S_{ij}] = 2[S, \nabla_i]. \qquad (13b)$$

3 The Dunkl–Coulomb model

3.1 Nonlocal Hamiltonian

We define the Dunkl–Coulomb Hamiltonian as:

$$\mathcal{H}_\gamma = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha (g_\alpha - s_\alpha)(\alpha, \alpha)}{x_\alpha^2} + \frac{2\gamma}{r}, \qquad (14)$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator and $\gamma \in \mathbb{C}$ is a parameter. Note that s_α commutes with x_α^2; hence, there is no ambiguity in the fractions inside the sum. In

the case $\mathcal{R} = \mathcal{A}_{N-1}$ the Hamiltonian $\mathcal{H}_\gamma$ was considered in [26, 27], and in the case $\mathcal{R} = \mathcal{A}_1 \times \mathcal{A}_1$ it was considered in [25] (here we rescale the Hamiltonian by a factor of -2). In the zero coupling limit $g_\alpha = 0$ for all $\alpha \in \mathcal{R}$, we get the Coulomb problem. On the other hand, in the zero charge limit $\gamma = 0$ the Hamiltonian (14) reduces to the nonlocal version of the generalised Calogero–Moser Hamiltonian associated with a root system $\mathcal{R}$. The latter Hamiltonian can also be referred to as the Dunkl Laplacian as it can be expressed as the sum of squares of Dunkl operators [13] (see also [14] for type $\mathcal{A}_{N-1}$) which allows us to represent the Hamiltonian (14) as:

$$\mathcal{H}_\gamma = \nabla^2 + \frac{2\gamma}{r},$$

where $\nabla^2 = \sum_{i=1}^{N} \nabla_i^2$. Being restricted to the wavefunctions $\psi(x)$ which are symmetric or antisymmetric with respect to the action of W

$$\psi(s_\alpha x) = \varepsilon \psi(x), \quad \alpha \in \mathcal{R}, \quad \varepsilon = \pm 1,$$

the nonlocal Hamiltonian reduces to the Calogero–Moser–Coulomb model [27]. It is obtained by replacing s_α in (14) with ε.

3.2 Dunkl angular momenta

Now we describe some symmetries of the Hamiltonian $\mathcal{H}_\gamma$. In the Coulomb limit $g_\alpha = 0$, the Calogero–Moser terms are absent, and the rotational $so(N)$ symmetry exists, whose generators are given by the angular momenta. The Calogero–Moser terms break the rotational symmetry so that the angular momentum is not conserved any more. Instead, we construct a version of angular momenta using the Dunkl operators. Consider the Dunkl angular momentum

$$L_{ij} = x_i \nabla_j - x_j \nabla_i, \tag{15}$$

where $1 \leq i, j \leq N$. These operators satisfy the commutation relation of the $so(N)$ Lie algebra with the Kronecker delta replaced by the elements $S_{ij} \in \mathbb{C}W$ [23] (see also [16] for $\mathcal{R} = \mathcal{A}_{N-1}$):

$$[L_{ij}, L_{kl}] = L_{il} S_{kj} + L_{jk} S_{li} - L_{ik} S_{lj} - L_{jl} S_{ki}. \tag{16}$$

Similarly, there are closed commutation relations between Dunkl angular momenta and Dunkl operators or coordinates. More precisely, the following lemma holds which can be checked directly.

Lemma 3.1 *Dunkl angular momenta L_{ij} satisfy*

$$[L_{ij}, x_k] = x_i S_{jk} - x_j S_{ik}, \quad [L_{ij}, \nabla_k] = \nabla_i S_{jk} - \nabla_j S_{ik}, \tag{17}$$

where $i, j, k = 1, \ldots, N$.

This allows us to establish some symmetries of the Hamiltonian $\mathcal{H}_\gamma$.

Proposition 3.2 *The Hamiltonian* (14) *preserves Dunkl angular momenta:*

$$[\mathcal{H}_\gamma, L_{ij}] = 0$$

for any $i, j = 1, \ldots, N$.

Proof By Lemma 3.1, we have

$$[L_{ij}, x_k^2] = \{[L_{ij}, x_k], x_k\} = x_i\{S_{jk}, x_k\} - x_j\{S_{ik}, x_k\}$$

for any $k = 1, \ldots, N$. It follows now from Lemma 2.4 that

$$[r^2, L_{ij}] = 0.$$

Similarly, one can check that

$$[\nabla^2, L_{ij}] = 0 \tag{18}$$

(alternatively, see [36]), and the statement follows. $\qquad\square$

The Dunkl angular momenta algebra $H_g^{so(N)}(W)$ was defined in [23] as the algebra generated by the Dunkl angular momenta L_{ij} and the Coxeter group algebra $\mathbb{C}W$. It has a second-order Casimir element $\mathcal{I} = \mathcal{I}_N$, which is an analogue of the angular momentum square:

$$\mathcal{I} = L_{(N)}^2 - S(S - N + 2), \tag{19}$$

where $L_{(N)}^2 = \sum_{i<j}^N L_{ij}^2$, and

$$[\mathcal{I}, L_{ij}] = 0$$

for any $i, j = 1, \ldots, N$. The element $\mathcal{I}$ represents the angular part of the nonlocal Hamiltonian $\mathcal{H}_\gamma$ [23]:

$$\mathcal{H}_\gamma = \nabla^2 + \frac{2\gamma}{r} = \partial_r^2 + \frac{N-1}{r}\partial_r + \frac{2\gamma}{r} + \frac{\mathcal{I}}{r^2}. \tag{20}$$

Remark 3.3 It is stated in [23] that the centre of $H_g^{so(N)}(W)$ is generated by $\mathcal{I}$ and $1 \in W$. If W contains element σ such that $\sigma(x) = -x$ for any $x \in V$, then it is easy to see that the centre is generated by $\mathcal{I}$ and σ. The generator σ is missing in [23].

Let us also recall that generators L_{ij} satisfy additional *crossing* relations [23]

$$L_{ij}(L_{kl} - S_{kl}) + L_{jk}(L_{il} - S_{il}) + L_{ki}(L_{jl} - S_{jl}) = 0 \tag{21}$$

for any $i, j, k, l = 1, \ldots, N$.

For any $\xi = (\xi_1, \ldots, \xi_N), \eta = (\eta_1, \ldots, \eta_N) \in V$ define $L_{\xi\eta} = \sum_{i,j=1}^{N} \xi_i \eta_j L_{ij}$. Then, one has

$$wL_{\xi\eta} = L_{w(\xi), w(\eta)} w \tag{22}$$

for any $w \in W$.

Relations (16), (21), (22) are defining relations of algebra $H_g^{so(N)}(W)$ [23].

We will also need the next lemma which generalises a well-known orthogonality relation between the coordinate vector and the angular momentum vector in a three-dimensional space.

Lemma 3.4 *Relations*

$$L_{ij}x_k + L_{jk}x_i + L_{ki}x_j = x_k L_{ij} + x_i L_{jk} + x_j L_{ki} = 0,$$
$$L_{ij}\nabla_k + L_{jk}\nabla_i + L_{ki}\nabla_j = \nabla_k L_{ij} + \nabla_i L_{jk} + \nabla_j L_{ki} = 0$$

hold for any $i, j, k = 1, \ldots, N$.

Proof It follows from the relation (2) that

$$L_{ij}x_k + L_{jk}x_i + L_{ki}x_j = x_i L_{kj} + x_k L_{ji} + x_j L_{ik}.$$

Therefore,

$$\{L_{ij}, x_k\} + \{L_{jk}, x_i\} + \{L_{ki}, x_j\} = 0,$$

and hence

$$x_k L_{ij} + x_j L_{ki} + x_i L_{jk} = -\frac{1}{2}([L_{ij}, x_k] + [L_{jk}, x_i] + [L_{ki}, x_j]) = 0$$

by Lemma 3.1.

The remaining relations can be established similarly. $\square$

3.3 Dunkl Laplace–Runge–Lenz vector

In addition to the angular momentum symmetry, the standard Coulomb model possesses a hidden symmetry known as the Runge–Lenz or Laplace vector. In the presence of an extra nonlocal Calogero–Moser potential term, this vector can be defined by making use of Dunkl operators. For the Calogero–Moser–Coulomb nonlocal (Dunkl–Coulomb) problem related to the root system $\mathcal{R} = \mathcal{A}_{N-1}$, this conserved quantity was introduced in [26], while in the case of $\mathcal{R} = \mathcal{A}_1 \times \mathcal{A}_1$, it was introduced in [25]. Now we extend this construction to an arbitrary root system $\mathcal{R}$. We define the components

A_i of a vector $A = (A_1, \ldots, A_N)$ by the formula

$$A_i = -\frac{1}{2} \sum_{j=1}^{N} \{L_{ij}, \nabla_j\} + \frac{1}{2}[\nabla_i, S] - \frac{\gamma x_i}{r}, \tag{23}$$

where $i = 1, \ldots, N$. This reduces to the usual Laplace–Runge–Lenz (LRL) vector in the zero coupling limit $g_\alpha = 0$. Below in this section we derive various relations which the components A_i satisfy. Most of these statements can be found in [25, 26] for the root systems $\mathcal{A}_{N-1}$ and $\mathcal{A}_1 \times \mathcal{A}_1$, respectively.

Proposition 3.5 *The components* (23) *of the Dunkl LRL vector A satisfy the relation*

$$[A_i, \mathcal{H}_\gamma] = 0$$

for any $i = 1, \ldots, N$.

Proof Note that for any $\alpha \in \mathcal{R}$

$$\left[s_\alpha, \sum_{k=1}^{N} \nabla_k^2 \right] = \sum_{k=1}^{N} \left(\nabla_k - \frac{2\alpha_k}{(\alpha, \alpha)}(\alpha, \nabla) \right)^2 s_\alpha - \sum_{k=1}^{N} \nabla_k^2 s_\alpha = 0. \tag{24}$$

Proposition 3.2 together with relation (24) implies that

$$[A_i, \mathcal{H}_\gamma] = \gamma \sum_{j=1}^{N} \left(\left\{ L_{ij}, \left[\frac{1}{r}, \nabla_j \right] \right\} - \left\{ \left[\frac{x_i}{r}, \nabla_j \right], \nabla_j \right\} \right) - \gamma \left[\left[\frac{1}{r}, \nabla_i \right], S \right]. \tag{25}$$

Notice that

$$\left[\frac{1}{r}, \nabla_j \right] = \frac{x_j}{r^3}. \tag{26}$$

Equalities (25), (26) together with (2) imply that

$$[A_i, \mathcal{H}_\gamma] = \frac{\gamma}{r^3} \sum_{j=1}^{N} \{L_{ij}, x_j\} - \gamma \sum_{j=1}^{N} \left\{ \frac{x_i x_j}{r^3} - \frac{S_{ij}}{r}, \nabla_j \right\} + \frac{\gamma}{r^3}[S, x_i]. \tag{27}$$

Notice that

$$\frac{1}{r^3} \sum_{j=1}^{N} \{L_{ij}, x_j\} = \sum_{j=1}^{N} \left((\nabla_j x_i - \nabla_i x_j)\frac{x_j}{r^3} + \frac{x_j}{r^3}(x_i \nabla_j - x_j \nabla_i) \right)$$

$$= \sum_{j=1}^{N} \left\{ \frac{x_i x_j}{r^3}, \nabla_j \right\} - \left\{ \frac{1}{r}, \nabla_i \right\}. \tag{28}$$

By substituting (28) in the right-hand side of (27), we get

$$[A_i, \mathcal{H}_\gamma] = \gamma \sum_{j=1}^{N} \left\{ \frac{S_{ij} - \delta_{ij}}{r}, \nabla_j \right\} + \frac{\gamma}{r^3}[S, x_i].$$

By making use of (26), we obtain

$$[A_i, \mathcal{H}_\gamma] = \frac{\gamma}{r} \sum_{j=1}^{N} \left\{ S_{ij} - \delta_{ij}, \nabla_j \right\} + \frac{\gamma}{r^3} \left(x_i - \sum_{j=1}^{N} x_j S_{ij} + [S, x_i] \right).$$

It follows from formulas (12a), (12b) that $[A_i, \mathcal{H}_\gamma] = 0$. $\square$

Other forms of the Dunkl LRL vector are given by the following statement.

Proposition 3.6 *The components* (23) *of the Dunkl LRL vector A can be represented in the following ways:*

$$A_i = -x_i \left(\nabla^2 + \frac{\gamma}{r} \right) + \nabla_i \left(r\partial_r + \frac{N-3}{2} \right), \tag{29}$$

$$A_i = - \left(\nabla^2 + \frac{\gamma}{r} \right) x_i + \left(r\partial_r + \frac{N+3}{2} \right) \nabla_i \tag{30}$$

for any $i = 1, \ldots, N$.

Proof By applying relations (17), (12b) and (10), we obtain the following formula:

$$\sum_{j=1}^{N} [\nabla_j, L_{ij}] = -(N-1)\nabla_i + \{S, \nabla_i\}.$$

This transforms expression (23) for the components of the Dunkl LRL vector to the form:

$$A_i = - \sum_{j=1}^{N} L_{ij} \nabla_j - \left(S - \frac{N-1}{2} \right) \nabla_i - \frac{\gamma x_i}{r}.$$

By substituting (15) in the above expression and using the first identity in (11) together with the commutator

$$[r\partial_r, \nabla_i] = -\nabla_i,$$

we arrive at the first required relation (29). Then, relation (30) follows since

$$[x_i, \nabla^2] = -2\nabla_i$$

due to (13b). Note that

$$(r\partial_r)^+ = -r\partial_r - N,$$

which implies formula (29) since the Dunkl LRL vector satisfies Hermitian property $A_i^+ = A_i$ for any i. ☐

Proposition 3.6 allows us to derive commutation relations between the components of the Dunkl LRL vector and Dunkl angular momenta.

Lemma 3.7 *The following relation holds*

$$[A_i, L_{kl}] = A_l S_{ki} - A_k S_{li}$$

for any $i, k, l = 1, \ldots, N$.

Lemma 3.7 follows immediately from Lemma 3.1, Proposition 3.6 and formula (18).

It appears that commutators of the components of the Dunkl LRL vector can be expressed in a compact form.

Lemma 3.8 *The following relation holds*

$$[A_i, A_j] = \mathcal{H}_\gamma L_{ij}$$

for any $i, j = 1, \ldots, N$.

Proof Consider the product $A_i A_j$, where the operator A_i has the form (30) and the operator A_j has the form (29). Subtract the same expression with the indexes swapped, $i \leftrightarrow j$, from this product. We obtain

$$[A_i, A_j] = \left(\left(r\partial_r + \frac{N+3}{2} \right) \left(\nabla^2 + \frac{\gamma}{r} \right) - \left(\nabla^2 + \frac{\gamma}{r} \right) \left(r\partial_r + \frac{N-3}{2} \right) \right) L_{ij}$$
$$(31)$$

as L_{ij} commutes with both r and ∇^2 by the proof of Proposition 3.2. Since

$$[r\partial_r, \nabla^2] = -2\nabla^2, \quad [r\partial_r, r^{-1}] = -r^{-1}, \tag{32}$$

we get from (31) that

$$[A_i, A_j] = \left(\frac{2\gamma}{r} + \nabla^2 \right) L_{ij} = \mathcal{H}_\gamma L_{ij}$$

as required. ☐

The next lemma gives a compact expression for the square of the length of the LRL vector.

Lemma 3.9 *Let $A^2 = \sum_{i=1}^{N} A_i^2$. Then, the following relation holds*

$$A^2 = \mathcal{H}_\gamma \left(\mathcal{I} + S - \frac{(N-1)^2}{4} \right) + \gamma^2, \tag{33}$$

where $\mathcal{I}$ is given by formula (19).

Proof Formulas (29), (30) together with the relation (11) imply that

$$A^2 = \left(\nabla^2 + \frac{\gamma}{r} \right) \left(r^2 \nabla^2 + \gamma r \right) + \left(r\partial_r + \frac{N+3}{2} \right) \nabla^2 \left(r\partial_r + \frac{N-3}{2} \right)$$
$$- \left(\nabla^2 + \frac{\gamma}{r} \right) (r\partial_r + S) \left(r\partial_r + \frac{N-3}{2} \right)$$
$$- \left(r\partial_r + \frac{N+3}{2} \right) (r\partial_r - S + N)) \left(\nabla^2 + \frac{\gamma}{r} \right). \tag{34}$$

Taking into account the first commutator in (32), we can simplify (34) to the form:

$$A^2 = \left(\nabla^2 + \frac{\gamma}{r} \right) \left(r^2 \nabla^2 + \gamma r \right) + \left(\frac{N-1}{2} \nabla^2 - \gamma \partial_r - \left(\nabla^2 + \frac{\gamma}{r} \right) S \right)$$
$$\left(r\partial_r + \frac{N-3}{2} \right) - \left(r\partial_r + \frac{N+3}{2} \right) (r\partial_r - S + N) \left(\nabla^2 + \frac{\gamma}{r} \right). \tag{35}$$

The right-hand side of the equality (35) is a second-order polynomial in γ. First, consider terms which do not contain γ. They have the form:

$$\nabla^2 \left(r^2 \nabla^2 + \left(\frac{N-1}{2} - S \right) \left(r\partial_r + \frac{N-3}{2} \right) \right)$$
$$- \left(r\partial_r + \frac{N+3}{2} \right) (r\partial_r - S + N)) \nabla^2$$
$$= \nabla^2 \left(r^2 \nabla^2 + \left(\frac{N-1}{2} - S \right) \left(r\partial_r + \frac{N-3}{2} \right) \right.$$
$$\left. - \left(r\partial_r + \frac{N-1}{2} \right) (r\partial_r - S + N - 2) \right) \tag{36}$$

due to (32). We rearrange (36) further to

$$\nabla^2 \left(r^2 \nabla^2 - (r\partial_r)^2 - (N-2) r\partial_r + S - \frac{(N-1)^2}{4} \right) = \nabla^2 \left(\mathcal{I} + S - \frac{(N-1)^2}{4} \right) \tag{37}$$

since $\mathcal{I} = r^2 \nabla^2 - r^2 \partial_r^2 - (N-1) r\partial_r$ by formula (20), and $(r\partial_r)^2 = r^2 \partial_r^2 + r\partial_r$.

Let us now consider terms in (35) containing γ in power one. We have

$$r\nabla^2 + \nabla^2 r - (\partial_r + r^{-1}S)\left(r\partial_r + \frac{N-3}{2}\right) - \left(r\partial_r + \frac{N+3}{2}\right)(r\partial_r + S + N)r^{-1}$$

$$= r^{-1}\left(r^2\nabla^2 + r\nabla^2 r - 2(r\partial_r)^2 - 2(N-1)r\partial_r + 2S - \frac{N^2-1}{2}\right), \tag{38}$$

where we used $[\partial_r, r^{-1}] = -r^{-2}$. It follows from the formula (20) that

$$[\nabla^2, r] = \left[\partial_r^2 + \frac{N-1}{r}\partial_r, r\right] = 2\partial_r + (N-1)r^{-1}.$$

Therefore,

$$r\nabla^2 r = r^2\nabla^2 + r[\nabla^2, r] = r^2\nabla^2 + 2r\partial_r + N - 1. \tag{39}$$

Substituting expression (39) in formula (38), we obtain the expression

$$r^{-1}\left(2r^2\nabla^2 - 2(r\partial_r)^2 - 2(N-2)r\partial_r + 2S - \frac{(N-1)^2}{2}\right)$$

$$= 2r^{-1}\left(\mathcal{I} + S - \frac{(N-1)^2}{4}\right)$$

by the relation (37). The statement follows. $\qquad\square$

The next statement generalises the orthogonality relation between angular momenta and components of the LRL vector in a three-dimensional space.

Lemma 3.10 *For any $i, j, k = 1, \ldots, N$ we have*

$$L_{ij}A_k + L_{jk}A_i + L_{ki}A_j = A_k L_{ij} + A_i L_{jk} + A_j L_{ki} = 0.$$

Proof The statement follows from Lemma 3.4 and the commutativity

$$[\nabla^2, L_{ij}] = [r^{-1}, L_{ij}] = [r\partial_r, L_{ij}] = 0.$$

$\qquad\square$

Remark 3.11 The commutativity property of the nonlocal Hamiltonian $\mathcal{H}_\gamma$ established in Propositions 3.2, 3.5 allows to obtain integrals of the local Hamiltonian [27]

$$H_{\gamma,g}^{\mathrm{loc}} = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(g_\alpha - 1)(\alpha, \alpha)}{x_\alpha^2} + \frac{2\gamma}{r}, \tag{40}$$

similarly to the case $\gamma = 0$ [13]. Namely, let $\mathrm{Res}(B)$ be the restriction of a W-invariant operator B to the W-invariant functions. Then, $H_{\gamma,g}^{\mathrm{loc}} = \mathrm{Res}(\mathcal{H})$. Furthermore, if

P is a polynomial in the non-commuting variables A_i, L_{kl} and in elements $w \in W$ such that P is W-invariant, then it follows that $[H_{\gamma,g}^{\mathrm{loc}}, \mathrm{Res}(P)] = 0$. Note that integrals corresponding to two W-invariant polynomials P_1, P_2 do not commute one with another in general.

Remark 3.12 One may also adopt approach of [13, 36] to construct the shift operators for the operator (40), namely the differential operators $\mathcal{D}$ such that

$$H_{\gamma,g+1}^{\mathrm{loc}} \circ \mathcal{D} = \mathcal{D} \circ H_{\gamma,g}^{\mathrm{loc}}. \tag{41}$$

These operators arise from the application of the W-anti-invariant operators P to W-invariants. In the case of P not dependent on variables A_i, such shift operators of degree 0 were constructed in [36] for $\gamma = 0$, and they satisfy the intertwining relation (41) for any γ.

4 Algebraic structure

Now we introduce the following associative algebra $R_{g,\gamma}(W)$ over $\mathbb{C}$ with identity $1 \in W$. It is generated by the elements $\mathcal{A}_i$, $\mathcal{L}_{kl}$, $\mathcal{H}$ and $\mathbb{C}W$, where $1 \leq i, k, l \leq N$. It is convenient to define

$$\mathcal{A}_\xi = \sum_{i=1}^N \xi_i \mathcal{A}_i, \quad \mathcal{L}_{\xi\eta} = \sum_{i,j=1}^N \xi_i \eta_j \mathcal{L}_{ij},$$

where $\xi = (\xi_1, \ldots, \xi_N), \eta = (\eta_1, \ldots, \eta_N) \in \mathbb{C}^N$. Then, the defining relations of the algebra $R_{g,\gamma}(W)$ are

$$\mathcal{L}_{ij} = -\mathcal{L}_{ji}, \tag{42}$$

$$w\mathcal{A}_\xi = \mathcal{A}_{w(\xi)}w, \quad w\mathcal{L}_{\xi\eta} = \mathcal{L}_{w(\xi)w(\eta)}w, \tag{43}$$

$$[\mathcal{H}, \mathcal{A}_i] = [\mathcal{H}, \mathcal{L}_{ij}] = 0, \tag{44}$$

$$\sum_{i=1}^N \mathcal{A}_i^2 = \mathcal{H}\left(\sum_{i<j}^N \mathcal{L}_{ij}^2 - S(S - N + 1) - \frac{(N-1)^2}{4}\right) + \gamma^2, \tag{45}$$

$$[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{H}\mathcal{L}_{ij}, \tag{46}$$

$$[\mathcal{A}_i, \mathcal{L}_{kl}] = \mathcal{A}_l S_{ki} - \mathcal{A}_k S_{li}, \tag{47}$$

$$[\mathcal{L}_{ij}, \mathcal{L}_{kl}] = \mathcal{L}_{il} S_{jk} + \mathcal{L}_{jk} S_{il} - \mathcal{L}_{ik} S_{jl} - \mathcal{L}_{jl} S_{ik}, \tag{48}$$

$$\mathcal{L}_{ij}\mathcal{L}_{kl} + \mathcal{L}_{jk}\mathcal{L}_{il} + \mathcal{L}_{ki}\mathcal{L}_{jl} = \mathcal{L}_{ij} S_{kl} + \mathcal{L}_{jk} S_{il} + \mathcal{L}_{ki} S_{jl}, \tag{49}$$

$$\mathcal{L}_{ij}\mathcal{A}_k + \mathcal{L}_{jk}\mathcal{A}_i + \mathcal{L}_{ki}\mathcal{A}_j = 0, \tag{50}$$

where $i, j, k, l = 1, \ldots, N$. Note that algebras $R_{g,\gamma}(W)$ are isomorphic for all $\gamma \neq 0$ as one can rescale generators $\mathcal{A}_i \to \gamma \mathcal{A}_i$, $\mathcal{H} \to \gamma^2 \mathcal{H}$.

Remark 4.1 Let $g = 0$. Then, $S_{ij} = \delta_{ij}$. Let us also set $\mathcal{H} = c \in \mathbb{C}^\times$. Consider the universal enveloping algebra $U(so(N+1))$ with the generators $K_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the matrix unit at (ij)-place, $1 \le i < j \le N+1$. For $\xi = (\xi_1, \ldots, \xi_{N+1})$, $\eta = (\eta_1, \ldots, \eta_{N+1})$ define $E_{\xi\eta} = \sum_{i,j=1}^{N+1} \xi_i \eta_j E_{ij}$. Consider the skew product $U(so(N+1)) \rtimes W$ where the group W acts naturally in $\mathbb{C}^N$ embedded into the subspace of $\mathbb{C}^{N+1}$ with the last coordinate of elements being 0, $w E_{\xi\eta} = E_{w(\xi),w(\eta)} w$. Due to relations (46)–(48), we have a surjective homomorphism $\psi \colon U(so(N+1)) \rtimes W \to R_{g,\gamma}(W)/(\mathcal{H} - c)$ given by $\psi(K_{ij}) = \mathcal{L}_{ij}$ for $1 \le i < j \le N$, $\psi(K_{i,N+1}) = (-c)^{1/2} \mathcal{A}_i$ $(1 \le i \le N)$, $\psi(w) = w$ $(w \in W)$. Relations (45), (49), (50) then define ideal $I = \operatorname{Ker} \psi \subset U(so(N+1)) \rtimes W$ so that $U(so(N+1)) \rtimes W/I \cong R_{g,\gamma}(W)/(\mathcal{H} - c)$.

Consider the algebra D of operators on functions which is generated by the differential operators with rational coefficients, the function r^{-1} and the group W.

Proposition 4.2 *There exists a homomorphism* $\rho \colon R_{g,\gamma}(W) \to D$ *given by*

$$\rho(\mathcal{L}_{ij}) = L_{ij}, \quad \rho(\mathcal{A}_i) = A_i, \quad \rho(\mathcal{H}) = \mathcal{H}_\gamma, \quad \rho(w) = w, \tag{51}$$

where $w \in W$, $i, j = 1, \ldots, N$.

Proof We have to establish relations (42)–(50) with $\mathcal{L}_{ij}$ replaced with L_{ij}, $\mathcal{A}_i$ replaced with A_i, and $\mathcal{H}$ replaced with $\mathcal{H}_\gamma$. The corresponding relations (42)–(47) follow from the definitions of A_i, L_{ij} and $\mathcal{H}_\gamma$, Propositions 3.2, 3.5 and Lemmas 3.7, 3.8, 3.9. The corresponding relations (48), (49) follow from (16), (21). Finally, the relation (50) follows from Lemma 3.10. $\qquad\square$

Following [23] we will call relations (49), (50) the *crossing* relations. This terminology has the following background. Suppose we consider an element from the algebra $R_{g,\gamma}(W)$ of the form:

$$\mathcal{L}_{i_1 j_1}^{n_1} \ldots \mathcal{L}_{i_k j_k}^{n_k}, \tag{52}$$

where $k \in \mathbb{N}$, $1 \le i_s < j_s \le N$ for $s = 1, \ldots, k$, $n_s \in \mathbb{N}$, and we assume that pairs of indices (i_s, j_s) are all different. Imagine the integral points from 1 to N plotted on the real line and represent element (52) geometrically by connecting the points i_s and j_s by n_s arcs in the upper half-plane. We say that element (52) has *no crossings* if the corresponding arcs do not intersect (cf. [23]). This property can also be restated as follows. Suppose that $i_s < i_{s'} < j_s$ for some $1 \le s, s' \le k$. Then, this implies that $j_{s'} \le j_s$. Now take four indices $1 \le i < j < k < l \le N$. Then, exactly two of the three elements $\mathcal{L}_{ij}\mathcal{L}_{kl}$, $\mathcal{L}_{jk}\mathcal{L}_{il}$, $\mathcal{L}_{ik}\mathcal{L}_{jl}$, namely elements $\mathcal{L}_{ij}\mathcal{L}_{kl}$, $\mathcal{L}_{jk}\mathcal{L}_{il}$ have no crossings.

Furthermore, similar to elements of the form (52) let us consider elements from $R_{g,\gamma}(W)$ of the form:

$$\mathcal{L}_{i_1 j_1}^{n_1} \ldots \mathcal{L}_{i_k j_k}^{n_k} \mathcal{A}_{r_1}^{m_1} \ldots \mathcal{A}_{r_l}^{m_l}, \tag{53}$$

where i_s, j_s, n_s are as in (52), $l \in \mathbb{N}$, $1 \leq r_t \leq N$ and $m_t \in \mathbb{N}$ for $t = 1, \ldots, l$ ($r_t \neq r_{t'}$ if $t \neq t'$). Let us represent this element geometrically as follows. Plot the integral points on the real line from 1 to $N + 1$. Connect pairs of points (i_s, j_s) by n_s arcs in the upper half-plane as we did for the element (52). Further, connect the pairs of points $(r_t, N + 1)$ by m_t arcs in the upper half-plane ($t = 1, \ldots, l$). We say that element (53) has *no crossings* if the corresponding arcs do not intersect. Then, for any triple of indices $1 \leq i < j < k \leq N$ exactly two of the three elements $\mathcal{L}_{ij}\mathcal{A}_k$, $\mathcal{L}_{jk}\mathcal{A}_i$, $\mathcal{L}_{ik}\mathcal{A}_j$ in (50), namely elements $\mathcal{L}_{ij}\mathcal{A}_k$, $\mathcal{L}_{jk}\mathcal{A}_i$ have no crossings.

More generally, we consider elements of $R_{g,\gamma}(W)$ of the form (53) multiplied (on the right) by $\mathcal{H}^q w$, where $q \geq 0$ and $w \in W$. We say that such an element has no crossings if the corresponding element (53) has no crossings, and we represent such an element geometrically in the same way as element (53) for any q, w.

Similarly, an element $L_{i_1 j_1}^{n_1} \ldots L_{i_k j_k}^{n_k} A_{r_1}^{m_1} \ldots A_{r_l}^{m_l} \mathcal{H}^q w \in D$ is said to have no crossings ($m_t, q \geq 0$) if the corresponding element

$$\mathcal{L}_{i_1 j_1}^{n_1} \ldots \mathcal{L}_{i_k j_k}^{n_k} \mathcal{A}_{r_1}^{m_1} \ldots \mathcal{A}_{r_l}^{m_l} \mathcal{H}^q w \tag{54}$$

in $R_{g,\gamma}(W)$ has no crossings.

It follows from the relations (49), (50) that any quadratic polynomial in $\mathcal{L}_{ij}$, $\mathcal{A}_r$ can be rearranged as a linear combination of elements from $R_{g,\gamma}(W)$ with no crossings. Furthermore, any element of $R_{g,\gamma}(W)$ can be represented as a linear combination of elements of the form (54) with no crossings.

We will also use similar terminology for the classical version of the above considerations. More exactly, consider classical angular momenta $M_{ij} = x_i p_j - x_j p_i$, where $1 \leq i, j \leq N$, and all the variables $x_1, p_1, \ldots, x_N, p_N$ commute. Consider the monomial

$$M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k}, \tag{55}$$

where, as in (52), $k \in \mathbb{N}$, $1 \leq i_s < j_s \leq N$ for $s = 1, \ldots, k$, $n_s \in \mathbb{N}$, and we assume that pairs of indices (i_s, j_s) are all different. We say that element (55) has no crossings if the corresponding element (52) has no crossings. Equivalently, the arcs constructed in the same way as for the monomial (52) do not intersect.

Lemma 4.3 *Any monomial* (55) *can be represented as a linear combination of monomials in the classical angular momenta which have no crossings.*

Proof Consider a monomial which contains a factor $M_{ij}M_{kl}$, where $i < k < j < l$. It is easy to see that by applying the crossing relation

$$M_{ij}M_{kl} = M_{ik}M_{jl} - M_{il}M_{jk}$$

we obtain two monomials such that each of them has less intersecting pairs of arcs than the original monomial. The statement follows by induction. $\square$

Lemma 4.4 *Different monomials* (55) *are linearly independent provided that they have no crossings.*

Proof Let $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_k)$, $n = (n_1, \ldots, n_k)$. Suppose that monomials (55) with no crossings admit a linear dependence

$$\sum_{i,j,n;k} a_{ijn}^{(k)} M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k} = 0, \tag{56}$$

where $a_{ijn}^{(k)} \in \mathbb{C}^\times$. We can assume that not all the monomials in (56) are divisible by M_{12} as otherwise we can divide this polynomial relation by a suitable power of M_{12}. Furthermore, note that each angular momentum $M_{i_r j_r}$ is homogeneous in any pair of variables (x_s, p_s). Therefore, we can assume that all monomials in the relation (56) have the same multidegree $(k_1, \ldots, k_N)$ in the pairs of variables (x_s, p_s) $(s = 1, \ldots, N)$. Consider now the homomorphism φ from the algebra generated by angular momenta in $N + 1$ variables to the algebra generated by angular momenta in the last N variables given by $\varphi(M_{1i}) = M_{2i}$ for any $i \geq 3$, $\varphi(M_{12}) = 0$, and $\varphi(M_{ij}) = M_{ij}$ if $i, j \geq 3$. The relation (56) implies that

$$\sum_{i,j,n;k} a_{ijn}^{(k)} \varphi(M_{i_1 j_1})^{n_1} \ldots \varphi(M_{i_k j_k})^{n_k} = 0, \tag{57}$$

and each nonzero monomial $\varphi(M_{i_1 j_1})^{n_1} \ldots \varphi(M_{i_k j_k})^{n_k}$ has no crossings. Furthermore, given a nonzero monomial $\overline{M} = \varphi(M_{i_1 j_1})^{n_1} \ldots \varphi(M_{i_k j_k})^{n_k}$ in (57), there exists a unique monomial M in (56) such that $\varphi(M) = \overline{M}$ (namely, $M = M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k}$). Indeed, the monomial M has no crossings and it has fixed degrees k_1, k_2. The latter condition means geometrically that the monomial M has k_1 arcs attached to the point 1 and k_2 arcs attached to the point 2, which allows us to find the monomial M uniquely for a given $\overline{M}$. It follows by induction in N that the coefficients at monomials not containing M_{12} in the sum (56) are all zero. This contradiction implies the statement. $\square$

For a given $m \in \mathbb{N}$, we introduce the following combination of classical angular momenta:

$$M_{(m)}^2 = \sum_{1 \leq i < j \leq m} M_{ij}^2 = \sum_{i=1}^m x_i^2 \sum_{i=1}^m p_i^2 - \left(\sum_{i=1}^m x_i p_i \right)^2.$$

Consider the algebra $B = B_{N+1} \subset \mathbb{C}[x_1, \ldots, x_{N+1}, p_1, \ldots, p_{N+1}]$ generated by the classical angular momenta M_{ij}, $1 \leq i < j \leq N + 1$. Let $I = I_{N+1}$ be the ideal in B generated by the element $M_{(N+1)}^2$.

Lemma 4.5 *A linear basis in the quotient B/I is given by the coset classes of different monomials $M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k}$ ($i_s < j_s$ for all $s = 1, \ldots, k$) such that they have no crossings and $M_{N,N+1}$ has power at most 1.*

Proof Let $\overline{M}_{ij}$ be the image of M_{ij} in B/I. Suppose there is a linear dependence between the specified monomials in the quotient, that is $Q(\overline{M}) = 0$, where $Q(\overline{M})$

is a linear combination of monomials $\overline{M}_{i_1 j_1}^{n_1} \ldots \overline{M}_{i_k j_k}^{n_k}$. Let $Q(M)$ be the same linear combination of the elements $M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k}$. We get the following polynomial relation in the algebra B:

$$Q(M) = M_{(N+1)}^2 q(M), \tag{58}$$

where $q(M)$ is a polynomial in variables M_{ij}. We can assume that monomials in the polynomial $q(M)$ have no crossings. Note that $q \neq 0$ by Lemma 4.4. Let us consider terms Q_0, q_0 in the polynomials $Q(M)$ and $q(M)$, respectively, which have the maximal total degree in the pairs of variables (x_N, p_N) and (x_{N+1}, p_{N+1}) (equivalently, the corresponding monomials have the maximal possible number of indexes N and $N+1$). Then, relation (58) implies the relation $Q_0 = M_{N,N+1}^2 q_0$. Monomials in Q_0 have no crossings and they contain $M_{N,N+1}$ in the power at most 1. Monomials in $M_{N,N+1}^2 q_0$ have no crossings too and they contain $M_{N,N+1}^2$ in the power at least 2. This is a contradiction with Lemma 4.4 which implies that monomials $M_{i_1 j_1}^{n_1} \ldots M_{i_k j_k}^{n_k}$ are linearly independent. $\qquad\square$

Lemma 4.6 *Let* $x = (x_1, \ldots, x_{N+1})$, $p = (p_1, \ldots, p_{N+1}) \in \mathbb{C}^{N+1}$ *be generic vectors satisfying relation* $M_{(N+1)}^2 = 0$. *Then, there exist* $\lambda, \mu \in \mathbb{C}$ *such that*

$$\hat{x} = (\hat{x}_1, \ldots, \hat{x}_{N+1}) = \lambda x, \quad \hat{p} = (\hat{p}_1, \ldots, \hat{p}_{N+1}) = \lambda^{-1} p + \mu x \tag{59}$$

satisfy

$$\sum_{k=1}^{N+1} \hat{x}_k \hat{p}_k = 0, \quad \sum_{k=1}^{N+1} \hat{p}_k^2 = 0. \tag{60}$$

Moreover, $M_{ij} = \hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i$ *for any* $i, j = 1, \ldots, N+1$.

Proof By substituting expressions (59) in (60), we get the conditions

$$\sum_{k=1}^{N+1} x_k p_k + \lambda \mu \sum_{k=1}^{N+1} x_k^2 = 0, \tag{61}$$

$$\lambda^{-2} \sum_{k=1}^{N+1} p_k^2 + 2\lambda^{-1} \mu \sum_{k=1}^{N+1} x_k p_k + \mu^2 \sum_{k=1}^{N+1} x_k^2 = 0. \tag{62}$$

The condition (62) can then be replaced with the relation

$$\lambda^2 \mu^2 \sum_{k=1}^{N+1} x_k^2 = \sum_{k=1}^{N+1} p_k^2. \tag{63}$$

Note that relation (63) follows from (61) since

$$M^2_{(N+1)} = \sum_{k=1}^{N+1} x_k^2 \sum_{k=1}^{N+1} p_k^2 - \left(\sum_{k=1}^{N+1} x_k p_k \right)^2 = 0.$$

We get that

$$\lambda\mu = -\frac{\sum_{k=1}^{N+1} x_k p_k}{\sum_{k=1}^{N+1} x_k^2},$$

which determines required λ and μ up to a scaling factor. $\qquad\square$

We are going to use the previous lemmas to study bases in the non-commutative algebras $R_{g,\gamma}(W)$, $\rho(R_{g,\gamma}(W))$. It is convenient to fix an ordering when writing out the elements of these algebras. It is clear that any element of the algebra $R_{g,\gamma}(W)$ can be represented as a linear combination of elements

$$\mathcal{L}_{12}^{i_{12}} \dots \mathcal{L}_{N-1,N}^{i_{N-1,N}} \mathcal{A}_1^{k_1} \dots \mathcal{A}_N^{k_N} \mathcal{H}^l w, \tag{64}$$

where we write $\mathcal{L}_{rs}$ to the left of $\mathcal{L}_{r's'}$ ($1 \le r < s \le N, 1 \le r' < s' \le N$) if $r < r'$ or $r = r'$ and $s < s'$, and all the powers in (64) are natural numbers or zeroes. We will use the same ordering for the elements of algebra $\rho(R_{g,\gamma}(W))$ in the next theorem.

Theorem 4.7 *Consider elements of the algebra D of the form:*

$$L_{12}^{i_{12}} \dots L_{N-1,N}^{i_{N-1,N}} A_1^{k_1} \dots A_N^{k_N} \mathcal{H}^l w, \tag{65}$$

with $i_{12}, \dots, i_{N-1,N}, k_1, \dots k_N, l \in \mathbb{N}\cup\{0\}$, $w \in W$, such that they have no crossings and $k_N \le 1$. Then, all such elements are linearly independent.

Proof Let us assume a linear dependence of the elements of the form (65). Then, there is a linear dependence between their highest symbols. To get the highest symbols replace L_{ij} with the classical angular momentum $M_{ij} = x_i p_j - x_j p_i$, replace A_i with

$$A_i^{cl} = -\sum_{j=1}^{N} p_j(x_i p_j - x_j p_i),$$

and replace $\mathcal{H}_\gamma$ with $H^{cl} = \sum_{i=1}^{N} p_i^2$. It is sufficient to consider the case $w = 1$. We can assume that the highest symbol is homogeneous if $\deg p_i = 1$ and $\deg x_i = -1$ for all i. Therefore, after dividing by a suitable power of $(-\sum p_i^2)^{1/2}$ we get a polynomial Q in $A_i^{cl}/(-H^{cl})^{1/2}$, M_{ij} ($1 \le i < j \le N$) which is equal to zero.

Let us introduce new variables

$$p_{N+1} = \left(-\sum_{j=1}^{N} p_j^2\right)^{1/2}, \quad x_{N+1} = -\frac{\sum_{j=1}^{N} x_j p_j}{p_{N+1}}. \tag{66}$$

Notice that

$$M_{i,N+1} = x_i p_{N+1} - x_{N+1} p_i = \frac{p_i \sum_{j=1}^{N} x_j p_j - x_i \sum_{j=1}^{N} p_j^2}{\left(-\sum_{j=1}^{N} p_j^2\right)^{1/2}}$$

$$= \frac{\sum_{j=1}^{N} p_j (x_j p_i - x_i p_j)}{\left(-\sum_{j=1}^{N} p_j^2\right)^{1/2}} = \frac{A_i^{cl}}{(-H^{cl})^{1/2}}.$$

We find that $Q(M) = 0$, where the polynomial $Q(M)$ is a polynomial in variables M_{ij} ($1 \leq i < j \leq N+1$) obtained by replacing variables $A_i^{cl}/(-H^{cl})^{1/2}$ with $M_{i,N+1}$ ($1 \leq i \leq N$). Note that $(x_1, \ldots, x_{N+1})$, $(p_1, \ldots, p_{N+1})$ satisfy the condition $M_{(N+1)}^2 = 0$. Furthermore, it follows by Lemma 4.6 that for any generic point on $M_{(N+1)}^2 = 0$ we can assume that x_{N+1}, p_{N+1} are given by formulas (66) without change of the angular momenta M_{ij}, $1 \leq i < j \leq N+1$. Therefore, we get that $Q(M) = 0$, where angular momenta satisfy $M_{(N+1)}^2 = 0$, and there are no other constraints on angular momenta M_{ij}. By Hilbert's Nullstellensatz, we get that $Q(M)^k \in I_{N+1}$ for some $k \in \mathbb{N}$. Since the scheme $M_{(N+1)}^2 = 0$ is integral (see, for example, [37]), it follows that $k = 1$ so that $Q(\overline{M}) = 0$ in the quotient algebra B_{N+1}/I_{N+1}, where $Q(\overline{M})$ stands for replacing each variable M_{ij} in the polynomial Q with its image $\overline{M_{ij}}$ in the quotient. By Lemma 4.5, we get that Q is the zero polynomial. □

Theorem 4.7 allows us to give a linear basis for the algebra $R_{g,\gamma}(W)$.

Theorem 4.8 *Consider the elements of the algebra $R_{g,\gamma}(W)$ of the form*

$$\mathcal{L}_{12}^{i_{12}} \ldots \mathcal{L}_{N-1,N}^{i_{N-1,N}} \mathcal{A}_1^{k_1} \ldots \mathcal{A}_N^{k_N} \mathcal{H}^l w, \tag{67}$$

where $w \in W$, and $i_{12}, \ldots, i_{N-1,N}, k_1, \ldots, k_N, l \in \mathbb{N} \cup \{0\}$ are such that $k_N \leq 1$ and there is no crossings. These elements form a basis in the algebra $R_{g,\gamma}(W)$.

Proof First, we note that the relations of the algebra $R_{g,\gamma}(W)$ imply that any element of the algebra can be represented as a linear combination of the elements of the form (67) which have no crossings and where $k_N \in \mathbb{N} \cup \{0\}$ is arbitrary. Suppose now that $k_N \geq 2$. We apply the relation (45) in the form:

$$\mathcal{A}_N^2 = -\sum_{i=1}^{N-1} \mathcal{A}_i^2 + \mathcal{H}\left(\sum_{i<j}^{N} \mathcal{L}_{ij}^2 - S(S-N+1) - \frac{(N-1)^2}{4}\right) + \gamma^2 \tag{68}$$

to the monomial (67), and let us rearrange the resulting terms as a linear combination of monomials with no crossings. For a given monomial and its geometrical realisation let P be the total number of arcs ending in the points N or $N+1$ with arcs connecting these points counted twice. Then, as we apply relation (68), the maximum of P across all the resulting monomials is reduced by at least 2 in comparison with the original monomial. By continuing the process we arrive at monomials with $k_N \leq 1$. Therefore elements (67) with no crossings and $k_N \leq 1$ span the algebra $R_{g,\gamma}(W)$. It follows from Proposition 4.2 and Theorem 4.7 that these elements are linearly independent. □

Theorem 4.8 can be interpreted as the Poincaré–Birkhoff–Witt theorem for the algebra $R_{g,\gamma}(W)$. Theorems 4.7, 4.8 also have the following implication.

Corollary 4.9 *The homomorphism of algebras $\rho \colon R_{g,\gamma}(W) \to D$ given by formulas (51) has trivial kernel.*

Consider the Dunkl angular momenta algebra $H_g^{so(N+1)}(W)$, where group W acts by its geometric representation on $V \cong \mathbb{C}^N$. Vector space V is embedded into $\mathbb{C}^{N+1}$ via the first N components:

$$V \ni (x_1, \ldots, x_N) \to (x_1, \ldots, x_N, 0) \in \mathbb{C}^{N+1},$$

and the group W acts trivially on the last component of vectors in $\mathbb{C}^{N+1}$. It appears that a central quotient of the algebra $R_{g,\gamma}(W)$ is isomorphic to a central quotient of the corresponding angular momenta algebra $H_g^{so(N+1)}(W)$.

Theorem 4.10 *For any $a \in \mathbb{C}^\times$, there is an isomorphism of algebras*

$$\varphi \colon R_{g,\gamma}(W)\big/(\mathcal{H} - a) \to H_g^{so(N+1)}(W)\big/(\mathcal{I}_{N+1} - b),$$

where $b = \frac{(N-1)^2}{4} - \frac{\gamma^2}{a}$. Under this isomorphism

$$\varphi(\mathcal{L}_{ij}) = L_{ij}, \quad \varphi(\mathcal{A}_i) = (-a)^{-1/2} L_{i,N+1}, \quad \varphi(w) = w,$$

where $1 \leq i, j \leq N$, $w \in W$.

Proof Note that elements

$$S_{ij} = \delta_{ij} + \sum_{\alpha \in \mathcal{R}_+} \frac{2g_\alpha \alpha_i \alpha_j}{(\alpha, \alpha)} s_\alpha \in \mathbb{C}W \subset H_g^{so(N+1)}(W),$$

$1 \leq i, j \leq N+1$, satisfy the properties

$$S_{N+1,N+1} = 1, \quad S_{i,N+1} = S_{N+1,i} = 0 \quad (1 \leq i \leq N).$$

Therefore, relations (46), (47), (48) correspond under the map ρ to all the cases of the relation (16) (with $1 \leq i, j, k, l \leq N+1$). Similarly, relations (49), (50) correspond

to all the cases of the relation (21). Note also that relation (45) at $\mathcal{H} = a$ corresponds under ρ to the condition $\mathcal{I}_{N+1} = \frac{(N-1)^2}{4} - \frac{\gamma^2}{a}$ (see (19)), and that all the remaining relations match as well. $\square$

Note that for $\gamma = 0$ the algebra $R_{g,0}(W)$ is isomorphic to the subalgebra $\widehat{R}_g(W)$ of the rational Cherednik algebra $H_g(W)$ given as $\widehat{R}_g(W) = \rho(R_{g,0}(W))$. Furthermore, the central quotient $R_{g,0}(W)/(\mathcal{H}-a)$ does not depend on the nonzero value of a and it is a flat deformation in g of the non-homogeneous quadratic Poincaré–Birkhoff–Witt algebras in the sense of [34]. Indeed, let us consider the central quotient

$$\varphi(R_{g,0}(W)/(\mathcal{H}-a)) = H_g^{so(N+1)}(W)\Big/\left(\mathcal{I}_{N+1} - \frac{(N-1)^2}{4}\right)$$

instead. It follows from Lemma 4.5 that for any g the linear basis in this quotient is given by the images of monomials

$$L_{12}^{i_{12}} \dots L_{N,N+1}^{i_{N,N+1}} w,$$

where $w \in W$, $i_{rs} \in \mathbb{N} \cup \{0\}$, $i_{N,N+1} \leq 1$, and monomials have no crossings. It is also clear that all the defining relations of the algebra $R_{g,0}(W)/(\mathcal{H}-a)$ are quadratic.

5 Maximal superintegrability

In this section, we discuss an application of the Dunkl LRL vector to questions on maximal superintegrability. Recall that a quantum Hamiltonian H in N-dimensional space is maximally superintegrable if there exist $2N-1$ differential operators $J_1 = H, J_2, \dots, J_{2N-1}$ such that J_i are algebraically independent and $[H, J_k] = 0$ for any k, $1 \leq k \leq 2N-1$ (cf. [15]).

Let $H_\gamma = H_{\gamma,g}^{\text{loc}}$ be given by

$$H_\gamma = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha(g_\alpha - 1)(\alpha, \alpha)}{x_\alpha^2} + \frac{2\gamma}{r}.$$

Note that when acting on W-invariants the differential operator $H_\gamma = \text{Res}\mathcal{H}_\gamma$, where Res denotes the restriction of a W-invariant operator to W-invariants (cf. [13]).

Let us consider the following algebra:

$$R_N^{cl} = \langle A_i^{cl}, M_{jk}, p^2 : 1 \leq i \leq N, 1 \leq j < k \leq N \rangle \subset \mathbb{C}[x_1, \dots, x_N, p_1, \dots, p_N],$$

where $A_i^{cl} = \sum_{j=1}^N p_j(x_j p_i - x_i p_j)$, $M_{jk} = x_j p_k - x_k p_j$, $p^2 = \sum_{i=1}^N p_i^2$.

Proposition 5.1 *The transcendence degree of the algebra R_N^{cl} is equal to $2N-1$.*

Proof Consider elements p^2, M_{1i}, $2 \leq i \leq N$, $M_{i,i+1}$, $2 \leq i \leq N-1$ and A_1^{cl}. Note that any monomial in these elements has no crossings. It follows from Theorem 4.7 and

its proof that these elements are algebraically independent. Using crossing relations, it is easy to see that any other element M_{ij}, A_i^{cl} can be expressed as a rational function of the above elements. The statement follows.   □

Theorem 5.2 *The Hamiltonian H_γ is maximally superintegrable.*

Proof The group W acts on the algebra R_N^{cl}. Let $(R_N^{cl})^W$ be the subalgebra of W-invariant elements. By Noether's results, $(R_N^{cl})^W$ is a finitely generated domain and the extension $(R_N^{cl})^W \subset R_N^{cl}$ is integral (see, for example, [38, Theorem 13.17]). It follows by Proposition 5.1 that the transcendence degree of the algebra $(R_N^{cl})^W$ is equal to $2N - 1$. Let Q_i, $1 \le i \le 2N - 1$ be a homogeneous basis of $(R_N^{cl})^W$. Let us consider the corresponding elements $Q_i^q \in R_{g,\gamma}(W)$ obtained by taking the Weyl quantisation of Q_i, that is given a monomial in Q_i one gets the corresponding sum of monomials in Q_i^q by replacing $p^2 \to \mathcal{H}$, $M_{kl} \to \mathcal{L}_{kl}$, $A_i^{cl} \to \mathcal{A}_i$ and averaging over all possible orderings of variables entering the original monomial. It follows that Q_i^q is W-invariant. Note that the highest order term of $\rho(Q_i^q) \in D$ is equal to Q_i, where the homomorphism ρ is defined in Proposition 4.2. Therefore, elements $\rho(Q_i^q)$ are algebraically independent. Define differential operators $J_i = \mathrm{Res}\, \rho(Q_i^q)$. It is clear that $[H_\gamma, J_i] = 0$, and differential operators J_i are algebraically independent. Theorem follows.   □

Remark 5.3 In the particular case, $\gamma = 0$ a large set of quantum integrals for H_0 was given in [18, Proposition II.1] and the superintegrability was claimed although a proof of the algebraic independence of the quantum integrals was omitted.

The Dunkl LRL vector can also be used to establish the maximal superintegrability of certain generalisations of the Calogero–Moser systems with the Coulomb potential which do not have full Coxeter symmetry. At $\gamma = 0$, such systems appeared in the works [32, 33]. At special values of parameters, the corresponding Hamiltonians can be obtained via special restrictions of Dunkl operators [31]. Let us recall these settings in more detail.

Pairs (Π, g) where $\Pi = \Pi_V$ is an intersection of some Coxeter reflection hyperplanes in V and g is a special coupling parameter were determined in [31, Theorem 3] so that the ideal of functions vanishing on the Coxeter group orbit $W(\Pi)$ is invariant under the corresponding rational Cherednik algebra. This led to quantum integrability of the Hamiltonian

$$H_\Pi = \Delta_\Pi - \sum_{\alpha \in \mathcal{R}_\Pi} \frac{\widehat{g}_\alpha(\alpha, \alpha)}{(\alpha, x)^2},$$

where $\mathcal{R}_\Pi$ consists of the orthogonal projections of vectors from $\mathcal{R}$ to Π which are non-zero, Δ_Π is the Laplace operator on Π, $x \in \Pi$, and $\widehat{g}_\alpha$ are coupling parameters determined in terms of g [31, Theorem 5, Proposition 2]. The quantum integrals were obtained by a restriction on Π procedure applied to any W-invariant combination of Dunkl operators in a suitable gauge.

In the proof of the next theorem, we apply such a restriction to W-invariant combinations Q of Dunkl operators which may depend on x-variables as well. The restriction

works in the same way as in [31]. In particular, the highest term of the resulting quantum integral is obtained by the restriction of the highest term of Q on $\Pi_V \times \Pi_{V^*}$, where Π_{V^*} is the image of the plane Π_V in the momentum space under the natural isomorphism which identifies $(x_1, \ldots, x_N)$ with $(p_1, \ldots, p_N)$.

Assuming that the pairs (Π, g) are as described we have the following statement on the maximal superintegrability of the corresponding Hamiltonians H_Π as well as these Hamiltonians with an extra Coulomb potential.

Theorem 5.4 *The operator* $H_{\Pi,\gamma} = H_\Pi + \frac{2\gamma}{r}$ *is maximally superintegrable for any* $\gamma \in \mathbb{C}$, *where* $r = (x, x)^{1/2}$, $x \in \Pi$.

Proof Consider a homomorphism π from the ring of polynomials $\mathbb{C}[x_1, \ldots, x_N, p_1, \ldots, p_N]$ to the ring of polynomials on $\Pi_V \times \Pi_{V^*}$ given by the natural restriction. Then, $\pi(R_N^{cl})$ is isomorphic to the algebra R_d^{cl}, where $d = \dim \Pi$; this can be seen by taking coordinates in such a way that plane $\Pi = \Pi_V$ has equations $x_N = \ldots = x_{d+1} = 0$, and similarly in p-space. Therefore, the transcendence degree of $\pi(R_N^{cl})$ is equal to $2d - 1$ by Proposition 5.1. Note that the integral extension $(R_N^{cl})^W \subset R_N^{cl}$ leads to the integral extension $\pi((R_N^{cl})^W) \subset \pi(R_N^{cl})$, hence the transcendence degree of $\pi((R_N^{cl})^W)$ is equal to $2d - 1$. For any element $\bar{Q} \in \pi((R_N^{cl})^W)$, let us choose $Q \in (R_N^{cl})^W$ such that $\pi(Q) = \bar{Q}$. Consider the quantisation Q^q of Q as in the proof of Theorem 5.2. The differential operator $\mathrm{Res}\,\rho(Q^q)$ commutes with the Hamiltonian H_γ, and it leads to the quantum integral for $H_{\Pi,\gamma}$ with the highest term $\bar{Q}$ by the results from [31]. The statement follows. $\square$

As an example, Theorem 5.4 establishes the maximal superintegrability of the following Hamiltonian describing generalisation of the Calogero–Moser system to the case of two sets of particles of any sizes $n, m \in \mathbb{N}$ with coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$:

$$H_{n,m} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^m \frac{\partial^2}{\partial y_j^2} - \sum_{i_1 < i_2}^n \frac{2k(k+1)}{(x_{i_1} - x_{i_2})^2} - \sum_{j_1 < j_2}^m \frac{2k^{-1}(k^{-1}+1)}{(y_{j_1} - y_{j_2})^2}$$
$$- \sum_{i=1}^n \sum_{j=1}^m \frac{2(k+1)}{(x_i - \sqrt{k} y_j)^2} + \frac{2\gamma}{r},$$

where $r = (\sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2)^{1/2}$, $\gamma \in \mathbb{C}$ and $k \in \mathbb{N}$. Integrability of the Hamiltonian $H_{n,m}$ at $\gamma = 0$ and arbitrary $k \in \mathbb{C}^\times$ was established in [32] for $m = 1$ or $n = 1$, and in [33] in general.

As another example, Theorem 5.4 establishes the maximal superintegrability of the following three-dimensional Hamiltonian:

$$H = \Delta - \sum_{i=1}^{3} \frac{(4c+1)(4c+3)}{4x_i^2} - \sum_{\substack{i<j \\ \varepsilon_1 \in \{1,-1\}}}^{3} \frac{2c(c+1)}{(x_i + \varepsilon_1 x_j)^2}$$
$$- \sum_{\varepsilon_2, \varepsilon_3 \in \{1,-1\}} \frac{6}{(x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3)^2} + \frac{2\gamma}{\left(\sum_{i=1}^{3} x_i^2\right)^{1/2}},$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ and $c, \gamma \in \mathbb{C}$. Integrability of this Hamiltonian was established in [31] for $\gamma = 0$.

We refer to [31] for further examples.

6 Concluding remarks

In the case of the Coulomb problem, the LRL vector allows us to derive the spectrum [2] and study scattering [39]. It would be interesting to try to extend this analysis to the Hamiltonians considered in this paper. A development of the representation theory of the algebras $R_{g,\gamma}(W)$ and $H_g^{so(N+1)}(W)$ may be needed which is an interesting direction on its own.

Another open question is on the maximal superintegrability of the generalised Calogero–Moser systems without full Coxeter symmetry such as the Hamiltonian $H_{n,m}$ in the case of general coupling parameter(s), that is a parameter k in the case of $H_{n,m}$.

References

1. Goldstein, H.: More on the prehistory of the Laplace or Runge–Lenz vector. Am. J. Phys. **44**, 1123 (1976). https://doi.org/10.1119/1.10202

2. Pauli, W.: On the hydrogen spectrum from the standpoint of the new quantum mechanics. Z. Physik **36**, 336–363 (1926)
3. Hulthén, L.: Über die quantenmechanische Herleitung der Balmerterme. Zeitschrift für Physik **86**(1–2), 21–23 (1933). https://doi.org/10.1007/BF01340179
4. Györgyi, G., Revai, J.: Hidden symmetry of the Kepler problem. Sov. Phys. JETP **48**, 1445 (1965). http://www.jetp.ac.ru/cgi-bin/e/index/e/21/5/p967?a=list
5. Fock, V.: Hydrogen atom and non-Euclidean geometry. Bull. Acad. Sci. URSS **2**, 169–188 (1935)
6. Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Am. Math. Soc. **311**, 167–183 (1989). http://www.ams.org/journals/tran/1989-311-01/S0002-9947-1989-0951883-8
7. Dunkl, C.F.: Reflection groups and orthogonal polynomials on the sphere. Math. Z. **197**(1), 33–60 (1988). https://doi.org/10.1007/BF01161629
8. Dunkl, C.F., Xu, Y.: Orhogonal Polynomials on Several Variables. Encyclopedia of Mathematics and its Applications, vol. 81. Cambridge University Press, Cambridge (2001)
9. Etingof, P., Ginzburg, V.: Symplectic reflection algebras, Calogero–Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. **147**(2), 243–348 (2002). arxiv:math/0011114
10. Calogero, F.: Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. **12**, 419–436 (1971). https://doi.org/10.1063/1.1665604
11. Moser, J.: Three integrable Hamiltonian systems connected with isospectral deformations. Adv. Math. **16**, 197 (1975). https://doi.org/10.1016/0001-8708(75)90151-6
12. Olshanetsky, M.A., Perelomov, A.M.: Classical integrable finite dimensional systems related to Lie algebras. Phys. Rept. **71**, 313 (1981). https://doi.org/10.1016/0370-1573(81)90023-5; Quantum integrable systems related to Lie algebras. ibid **94**, 313 (1983). https://doi.org/10.1016/0370-1573(83)90018-2
13. Heckman, G.J.: A remark on the Dunkl differential-difference operators. Prog. Math. **101**, 181–191 (1991). https://doi.org/10.1007/978-1-4612-0455-8_8
14. Polychronakos, A.: Exchange operator formalism for integrable systems of particles. Phys. Rev. Lett. **69**, 703–705 (1992). arxiv:hep-th/9202057
15. Wojciechowski, S.: Superintegrability of the Calogero–Moser system. Phys. Lett. A **95**, 279 (1983). https://doi.org/10.1016/0375-9601(83)90018-X
16. Kuznetsov, V.B.: Hidden symmetry of the quantum Calogero–Moser system. Phys. Lett. A **218**, 212 (1996). arXiv:solv-int/9509001
17. Gonera, C.: A note on superintegrability of the quantum Calogero. Phys. Lett. A **237**, 365 (1998). https://doi.org/10.1016/S0375-9601(98)00903-7
18. Caseiro, R., Francoise, J.-P., Sasaki, R.: Quadratic algebra associated with rational Calogero–Moser models. J. Math. Phys. **42**, 5329–5340 (2001). arxiv:hep-th/0102153
19. Ujino, H., Wadati, M., Hikami, K.: The quantum Calogero–Moser model: algebraic structures. J. Phys. Soc. Jpn. **62**, 3035–3043 (1993). https://doi.org/10.1143/JPSJ.62.3035
20. Reshetikhin, N.: Degenerate integrability of the spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems. Lett. Math. Phys. **63**(1), 55–71 (2003). arxiv:math/0202245
21. Reshetikhin, N.: Degenerate integrability of quantum spin Calogero–Moser systems. Lett. Math. Phys. **107**(1), 187–200 (2017). arXiv:1510.00492
22. Fairon, M., Görbe, T.: Superintegrability of Calogero–Moser systems associated with the cyclic quiver. Nonlinearity **34**(11), 7662–7682 (2021). arXiv:2101.05520
23. Feigin, M., Hakobyan, T.: On Dunkl angular momenta algebra. JHEP **11**, 107 (2015). arXiv:1409.2480
24. De Bie, H., Oste, R., Van der Jeugt, J.: On the algebra of symmetries of Laplace and Dirac operators. Lett. Math. Phys. **108**(8), 1905–1953 (2018). arXiv:1701.05760
25. Genest, V.X., Lapointe, A., Vinet, L.: The Dunkl–Coulomb problem in the plane. Phys. Lett. A **379**, 923–927 (2015). arXiv:1405.5742
26. Hakobyan, T., Nersessian, A.: Runge–Lenz vector in Calogero–Coulomb problem. Phys. Rev. A **92**, 022111 (2015). arXiv:1504.00760
27. Hakobyan, T., Lechtenfeld, O., Nersessian, A.: Superintegrability of generalized Calogero models with oscillator or Coulomb potential. Phys. Rev. D **90**, 101701(R) (2014). arXiv:1409.8288
28. Correa, F., Hakobyan, T., Lechtenfeld, O., Nersessian, A.: Spherical Calogero model with oscillator/Coulomb potential: quantum case. Phys. Rev. D **93**, 125009 (2016). arXiv:1604.00027
29. Correa, F., Hakobyan, T., Lechtenfeld, O., Nersessian, A.: Spherical Calogero model with oscillator/Coulomb potential: classical case. Phys. Rev. D **93**, 125008 (2016). arXiv:1604.00026

30. Hakobyan, T., Nersessian, A.: Integrability and separation of variables in Calogero–Coulomb–Stark and two-center Calogero–Coulomb systems. Phys. Rev. D **93**, 045025 (2016). arXiv:1509.01077
31. Feigin, M.: Generalized Calogero–Moser systems from rational Cherednik algebras. Selecta Math. (N.S.) **18**(1), 253–281 (2012). arXiv:0809.3487
32. Chalykh, O., Feigin, M., Veselov, A.: New integrable generalizations of Calogero–Moser quantum problem. J. Math. Phys. **39**(2), 695–703 (1998). https://doi.org/10.1063/1.532347
33. Sergeev, A.N., Veselov, A.P.: Deformed quantum Calogero–Moser problems and Lie superalgebras. Commun. Math. Phys. **245**(2), 249–278 (2004). arxiv:math-ph/0303025
34. Braverman, A., Gaitsgory, D.: Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type. J. Algebra **181**(2), 315–328 (1996). arxiv:hep-th/9411113
35. Humphreys, J.E.: Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics, vol. 29. Cambridge University Press, Cambridge (1990)
36. Feigin, M.: Intertwining relations for the spherical parts of generalized Calogero operators. Theor. Math. Phys. **135**, 497–509 (2003). https://doi.org/10.1023/A:1023231402145
37. Feigin, M., Thompson, D.: The global sections of the sheaf of Cherednik algebras of a smooth quadric (In preparation)
38. Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics, vol. 150. Springer, Berlin (1995)
39. Zwanziger, D.: Algebraic calculation of nonrelativisitic Coulomb phase shifts. J. Math. Phys **8**(9), 1858–1860 (1967)