Algebra versus analysis in the theory of flexible polyhedra

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Introduction

- Two basic theorems of the theory of flexible polyhedra were proven by completely different methods: R. Alexander[1] used analysis, namely, the Stokes theorem, to prove that the total mean curvature remains constant during the flex, while I.Kh. Sabitov[2] used algebra, namely, the theory of resultants, to prove that the oriented volume remains constant during the flex.
- We show that none of these methods can be used to prove the both theorems.
- As a by-product, we prove that the total mean curvature of any polyhedron in the Euclidean 3-space is not an algebraic function of its edge lengths.

[1] Trans. Amer. Math. Soc. **288** (1985), 661–678.
[2] Fundam. Prikl. Mat. **2** (1996), 1235–1246.

Definitions

- A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously due to changes of its dihedral angles only, i. e., if every face remains self-congruent during the flex.
- In other words, a polyhedron P_0 is flexible if it is included in a continuous family $\{P_t\}$, $0 \leqslant t \leqslant 1$, of polyhedra P_t such that, for every t, the corresponding faces of P_0 and P_t are congruent while polyhedra P_0 and P_t are not congruent.
- In general, self-intersections are possible both for P_0 and P_t.
- Without loss of generality we assume that the faces of the polyhedra are triangular.

Existence of flexible polyhedra

- Fleixible octahedra with self-intersections in the Euclidean 3-space $\mathbb{R}^3$ were found and studied by R. Bricard[3].
- Flexible self-intersection free sphere-homeomorphic polyhedra in $\mathbb{R}^3$ were constructed by R. Connelly[4].
- K. Steffen[5] constructed a flexible self-intersection free sphere-homeomorphic polyhedron in $\mathbb{R}^3$ with 9 vertices.

[3] Journ. de Math. (5) **3** (1897), 113–148.
[4] Inst. Hautes Études Sci. Publ. Math. **47** (1977), 333–338.
[5] *M. Berger*, Geometry, I. Springer, Berlin, 1987.

Folklore

- A citation from Marcel Berger[6]: "*Our opinion is that the known examples of bendable polyhedra do not flex very much.*"
- Given two real numbers $0 < r < R$, there exists a flexible sphere-homeomorphic polyhedron P in $\mathbb{R}^3$ such that, in one position, P lies in a ball of radius r while, in another position, P is not contained in any ball of radius R.
- There is a lattice tiling in $\mathbb{R}^3$ such that:
  (i) each tile P is a sphere-homeomorphic flexible polyhedron and
  (ii) any flex of P generates a movement of the whole tiling such that the result is a lattice tiling again.
- In $\mathbb{R}^3$, there are flexible polyhedra of arbitrary genus as well as non-orientable ones.

[6] A Panoramic View of Riemannian Geometry. Springer, Berlin, 2007. Page 148.

Definiton of the total mean curvature

During last 30 years, many non-trivial properties of flexible polyhedra were discovered. We formulate two of them in a form that is convenient for our purposes.

Definition: Let P be a closed oriented polyhedron in $\mathbb{R}^3$, let E be the set of its edges, let $|\ell|$ be the length of edge ℓ, and let $\alpha(\ell)$ be the dihedral angle of P at edge ℓ measured from inside of P. The sum

$$M(P) = \frac{1}{2}\sum_{\ell\in E}|\ell|\bigl(\pi - \alpha(\ell)\bigr)$$

is called the total mean curvature of P.

Alexander's theorems

Theorem 1. *Let P_0 be a flexible oriented polyhedron in $\mathbb{R}^3$ and let $\{P_t\}$, $0 \leqslant t \leqslant 1$, be its flex. The total mean curvature $M(P_t)$ is independent of t.*

Remark: Theorem 1 was obtained by R. Alexander[7] as an obvious corollary of Theorem 2, while the latter was proved with the help of the Stokes theorem, i. e., by means of Analysis.

Theorem 2. *Let P be a closed oriented polyhedron in $\mathbb{R}^3$, let $\mathbf{w}$ be its infinitesimal flex, and let $P(t) = \{\mathbf{r} + t\mathbf{w} | \mathbf{r} \in P\}$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} M\big(P(t)\big) = 0.$$

[7] Trans. Amer. Math. Soc. **288** (1985), 661–678.

Sabitov's theorem

Theorem 3. *If $\{P_t\}$ is a flex of an oriented polyhedron in $\mathbb{R}^3$, then the oriented volume of P_t is constant in t.*

Remark: Theorem 3 was obtained by I.Kh. Sabitov[8] as an obvious corollary of Theorem 4, while the latter was proved with the help of the theory of resultants, i. e., by means of Algebra.

Theorem 4. *For the set $\mathcal{P}_K$ of all (not necessarily flexible) closed oriented polyhedra in $\mathbb{R}^3$ with triangular faces and with a prescribed combinatorial structure K there exists a universal polynomial $\mathfrak{p}_K$ of a single variable whose coefficients are universal polynomials in the edge lengths of a polyhedron $P \in \mathcal{P}_K$ and such that the oriented volume of any $P \in \mathcal{P}_K$ is a root of $\mathfrak{p}_K$.*

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[8] Fundam. Prikl. Mat. **2** (1996), 1235–1246 and Discrete Comput. Geom. **20** (1998), 405–425.

The main results

We found[9] that Theorem 1 cannot be proved by means of Algebra and Theorem 3 cannot be proved by means of Analisis. More precisely, we proved the following two theorems.

Theorem 5. *There is a closed oriented polyhedron P in $\mathbb{R}^3$ with the following properties*:
(i) *the flux across P of some infinitesimal flex of P is non-zero*;
(ii) *P contains no vertex V whose star lies in a plane*;
(iii) *P contains no vertex V such that some three edges of P incident to V lie in a plane.*

Theorem 6. *The total mean curvature of any closed oriented polyhedron in $\mathbb{R}^3$ is not an algebraic function of its edge lengths.*

[9] Aequat. Math. **79**, no. 3 (2010), 229–235.

The end

This talk is based on the author's paper
"Algebra versus analysis in the theory of flexible polyhedra"
published in Aequationes Mathematicae, **79**, no. 3 (2010), 229–235
and freely available at arXiv:0902.0186v2 [math.MG]

Thank you for attention!