Algebrability of the set of everywhere surjective functions on $\mathbb{C}$

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Dedicated to the memory of our great friend and colleague, Vladimir Gurariy.

Abstract

We show that the set $\mathcal{L}$ of complex-valued everywhere surjective functions on $\mathbb{C}$ is algebrable. Specifically, $\mathcal{L}$ contains an infinitely generated algebra every non-zero element of which is everywhere surjective. We also give a technique to construct, for every $n \in \mathbb{N}$, n algebraically independent everywhere surjective functions, $f_1, f_2, \dots, f_n$, so that for every non-constant polynomial $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$, $P(f_1, f_2, \dots, f_n)$ is also everywhere surjective.

1 Introduction and background

H. Lebesgue was perhaps the first to produce the somewhat surprising example of a function $f : \mathbb{R} \to \mathbb{R}$ with the property that on every non-trivial interval (a, b), $f((a, b)) = \mathbb{R}$. ([3]; see also the more modern reference [2], where examples of other functions are indicated.) In fact, using a term that we believe was coined by V. Gurariy, the set of such everywhere surjective functions is *lineable;* that is, it contains a vector subspace of the largest possible dimension, 2^c (see [1]). One can follow a similar argument as the one shown in [2] to show that there are everywhere surjective functions $f : \mathbb{C} \to \mathbb{C}$, i.e., functions so that for every non-empty open subset $U \subset \mathbb{C}$, $f(U) = \mathbb{C}$. From now on, we denote by $\mathcal{L}$ the set of everywhere surjective functions from $\mathbb{C}$ to $\mathbb{C}$. In this note we investigate another concept introduced by Gurariy, namely whether it is possible to find an algebra $\mathcal{B}$

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with $\mathcal{B} \subset \mathcal{L} \cup \{0\}$, and we study how "big" $\mathcal{B}$ is. Note that "bigness" for us has one of two possible meanings: The dimension of $\mathcal{B}$ as a complex vector space, or else the cardinality of a minimal system of generators of $\mathcal{B}$ as an algebra. We will be talking about $\mathcal{L}$ being *algebrable* or about the *algebrability* of $\mathcal{L}$.

We construct here finitely and infinitely generated subalgebras of functions, every non-zero element being in $\mathcal{B}$. In order to do this, we work with rings of polynomials in several complex variables, in particular with $\mathbb{C}[z_1, z_2]$, which contains an infinitely generated subalgebra. We show that algebrability is related to the behavior of the set of polynomials in several complex variables and, in fact, we prove some results about polynomials and we generalize certain constructions of algebrability.

We denote by $\dim(\mathcal{B})$ the dimension of $\mathcal{B}$ as a complex vector subspace of the set of functions $\mathcal{F}(\mathbb{C}, \mathbb{C})$. Next, we give a definition that we will need in order to state our main result:

Definition 1.1 (algebrability).
Given a set A , we say that:

1. *A is algebrable if there is an algebra $\mathcal{B} \subset A \cup \{0\}$ so that $\mathcal{B}$ has an infinite minimal system of generators. (Here, by $S = \{z_\alpha\}$* is a minimal set of generators of $\mathcal{B}$, *we mean that $\mathcal{B} = \mathcal{A}(S)$ is the algebra generated by* S, *and for every α_0, $z_{\alpha_0} \notin \mathcal{A}(S \setminus \{z_{\alpha_0}\})$.)*

2. *A is (α, β)-algebrable if there is an algebra $\mathcal{B}$ so that $\mathcal{B} \subset A \cup \{0\}$, dim($\mathcal{B}$)$= \alpha$ and card $(S) = \beta$, where α and β are two cardinal numbers, and S is a minimal system of generators of $\mathcal{B}$.*

Our main result in this note (theorem 2.3) is the following:

Theorem *The set of complex-valued everywhere surjective functions is algebrable. In fact, it contains an infinite dimensional and infinitely generated algebra every non-zero element of which is everywhere surjective.*

Besides constructing infinitely generated algebras, we can start by giving some simple constructions of finitely generated ones. These constructions are useful to understand the nature of the "infinitely generated problem." We begin by constructing a singly generated algebra $\mathcal{B}$, generated by f, any fixed everywhere surjective function, so that every non-zero element of $\mathcal{B}$ is also everywhere surjective. To do this, consider the algebra

$$\mathcal{B} = \mathcal{A}(\{f\}).$$

Let us first show that $S = \{f^n : n \in \mathbb{N}\}$ is a linear independent family, so that $\mathcal{B}$ has dimension $\aleph_0$ as a vector space. Suppose that $\sum_{j=1}^n a_j f^{k_j} \equiv 0$. Since f is everywhere surjective we have that the polynomial $\sum_{j=1}^n a_j z^{k_j} \equiv 0$. Therefore, $a_j = 0$ for every j, and thus the family S is linearly independent. Next, let us see that every $g \in \mathcal{B}$, $g \neq 0$, is everywhere surjective. Any such g can be written as $\sum_{j=1}^n a_j f^{k_j}$ for some $k_1, k_2, \ldots, k_n \in \mathbb{N}$. Take now $\varnothing \neq U \subset \mathbb{C}$ any open subset of $\mathbb{C}$ and any $w \in \mathbb{C}$. We need to show that there is $c \in U$ so that $g(c) = w$. Let $d \in \mathbb{C}$ be a zero of the polynomial $\sum_{j=1}^n a_j z^{k_j} - w$. Since $f \in \mathcal{L}$, there is $c \in U$ so that $f(c) = d$. Thus,

$g(c) = w$, and $g \in \mathcal{L}$. Therefore, we have constructed an algebra of everywhere surjective functions generated by one element, and $\mathcal{L}$ is $(\aleph_0, 1)$-algebrable.

We can go further, and show that $\mathcal{L}$ is $(\aleph_0, n)$-algebrable for every $n \in \mathbb{N}$. In order to see this, for $1 < n \in \mathbb{N}$ consider the set

$$S_n = \{f^n, f^{n+1}, \dots, f^{2n-1}\}.$$

We have that:

1. S_n has cardinality n.
2. $\mathcal{B}_n = \mathcal{A}(S_n)$ is such that for every $h \in S_n$, $h \notin \mathcal{A}(S_n \setminus \{h\})$; thus S_n is a minimal system of generators.
3. By the previous argument it is easy to see that for every $n \in \mathbb{N}$, every non-zero element of $\mathcal{B}_n$ is everywhere surjective, so $\mathcal{B}_n \setminus \{0\} \subset \mathcal{L}$.
4. The set $\{f^n, f^{2n}, f^{3n}, f^{4n}, \dots\} \subset \mathcal{B}_n$, so $\dim(\mathcal{B}_n) = \aleph_0$ and $\mathcal{L}$ is $(\aleph_0, n)$-algebrable.

Moreover, we have that the following holds:

$$\mathbb{C}[z] \equiv \mathcal{A}(\{1, f\}) \supsetneq \mathcal{A}(\{f\}) = \mathcal{B}_1 \supsetneq \mathcal{B}_2 \supsetneq \mathcal{B}_3 \supsetneq \mathcal{B}_4 \supsetneq \mathcal{B}_5 \supsetneq \cdots,$$

since $\mathcal{B}_n$ does not contain the non-zero constants, $f^n \in \mathcal{B}_n \setminus \mathcal{B}_{n+1}$, and $f \notin \mathcal{B}_n$, for $n > 1$, and where the symbol "$\equiv$" means "isomorphism of algebras."

2 Algebrability of $\mathcal{L}$

So far we have essentially been working with polynomials in $\mathbb{C}[z]$. However, $\mathbb{C}[z]$ does not contain an infinitely generated algebra ; i.e. all of its subalgebras are finitely generated. A proof of this known result can be sketched as follows: In $\mathbb{K}[x]$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), suppose that S is a subalgebra that contains $\mathbb{K}$. Let P be a non-constant polynomial in S and let T be the subalgebra of S generated by 1 and P. Then x is algebraic over T, so $\mathbb{K}[x]$ is a finitely generated T-module. Note that T is Noetherian and S is a T-submodule of $\mathbb{K}[x]$, so S is a finitely generated T-module. Hence S is a finitely generated algebra.

Therefore the question of whether $\mathcal{L}$ is algebrable cannot be solved using the one variable methods of the previous section. On the other hand, the problem we want to solve can be reduced to answering the following question about two variables in a positive way:

> *Can one find two algebraically independent surjective functions* $f, g : \mathbb{C} \to \mathbb{C}$*, so that the algebra generated by them,* $\mathcal{A}(f, g)$*, verifies that every element of* $\mathcal{A}(f, g) \setminus \{0\}$ *is onto?*

In other words, is it possible to find two onto functions $f, g : \mathbb{C} \to \mathbb{C}$, so that

$$\mathcal{A}(f, g) \equiv \mathbb{C}[z_1, z_2],$$

and such that every non-zero function $h \in \mathcal{A}(f,g)$ is onto?

As we saw earlier, $\mathbb{C}[z]$ only contains finitely generated algebras. On the other hand, the ring of polynomials in two complex variables $\mathbb{C}[z_1, z_2]$ contains a subalgebra whose minimal system of generators is

$$S = \{z_1, z_1 z_2, z_1 z_2^2, z_1 z_2^3, z_1 z_2^4, z_1 z_2^5, \dots\},$$

i.e., an infinitely generated subalgebra. Therefore, if f and g are two algebraically independent functions, then the algebra generated by them is isomorphic to the ring of polynomials in two complex variables $\mathbb{C}[z_1, z_2]$, i.e. $\mathcal{A}(f,g) \equiv \mathbb{C}[z_1, z_2]$, and therefore $\mathcal{A}(f,g)$ also contains an infinitely generated subalgebra. Let us see that a positive answer to the above question suffices for our purposes. Suppose that one can find such functions f and g verifying that $\mathcal{A}(f,g) \equiv \mathbb{C}[z_1, z_2]$, and with every $0 \neq h \in \mathcal{A}(f,g)$ being onto. Then, consider the algebra $\mathcal{A}(S)$ generated by the minimal system of generators

$$S = \{f, fg, fg^2, fg^3, fg^4, \dots\}.$$

Fix an arbitrary everywhere surjective function $F : \mathbb{C} \to \mathbb{C}$ and define a new algebra, $\mathcal{F}$, by

$$\mathcal{F} = \{H \circ F \ : \ H \in \mathcal{A}(S)\}.$$

This new algebra $\mathcal{F}$ is also infinitely generated, and its minimal system of generators is the infinite set

$$\{f \circ F, (fg) \circ F, (fg^2) \circ F, (fg^3) \circ F, (fg^4) \circ F, \dots\}.$$

The following simple lemma will be needed in what follows. The proof can be found in [1].

Lemma 2.1. *Let $H : \mathbb{C} \to \mathbb{C}$ be an onto function, and let $F : \mathbb{C} \to \mathbb{C}$ be an everywhere surjective function. Then $H \circ F$ is everywhere surjective.*

After this, it is clear that we will be done once we show that there are two algebraically independent onto functions f and g so that every non-zero function $h \in \mathcal{A}(f,g)$ is onto. We construct two surjective functions f and g and we show that given any non-constant polynomial $P \in \mathbb{C}[z_1, z_2]$, $P(f,g)$ is an onto function. In order to show this, we need another simple lemma.

Lemma 2.2. *Let $n \in \mathbb{N}$. If $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$ is a non-constant polynomial, then there exist $m_1, m_2, \dots, m_n \in \mathbb{N}$ so that*

$$Q(z) = P(z^{m_1}, z^{m_2}, \dots, z^{m_n})$$

is onto.

Proof.
We argue by induction on n, noting that the result is the fundamental theorem of algebra for $n = 1$.

Suppose the result is true for some $n \in \mathbb{N}$. Any non-constant polynomial $P \in \mathbb{C}[z_1, z_2, \dots, z_n, z_{n+1}]$ can be written as follows:

$$P(z_1, z_2, \dots, z_n, z_{n+1}) = \sum_{j=0}^{k} z_1^j \cdot S_j(z_2, \dots, z_{n+1})$$

for some $k \in \mathbb{N}$, where $S_j \in \mathbb{C}[z_2, ..., z_{n+1}]$. We now show that we can choose $m_1, m_2, \dots, m_{n+1} \in \mathbb{N}$ so that $Q(z) = P(z^{m_1}, z^{m_2}, \dots, z^{m_{n+1}})$ has degree one or bigger. Denote $d_i = \deg(S_i)$.

- If $k = 0$, then our assertion is true by the induction hypothesis.
- If $k \neq 0$ then we can choose $m_2 = \cdots = m_{n+1} = 1$ and $m_1 \in \mathbb{N}$ such that

  $$m_1 \cdot k + d_k > \max\{m_1(k-1) + d_{k-1}, m_1(k-2) + d_{k-2}, \dots, m_1 + d_1, d_0\}.$$

  By choosing these values, we obtain that

  $$\deg(P(z^{m_1}, z^{m_2}, \dots, z^{m_{n+1}})) = \deg(P(z^{m_1}, z, \dots, z)) \geq 1,$$

  and $Q(z)$ is onto.

■

With this lemma, we can now construct two functions f and g as follows. For every $(p, q) \in \mathbb{N} \times \mathbb{N}$ let $\phi_{p,q}$ be a homeomorphism between the sets

$$U_{p,q} = \{a + ib \in \mathbb{C} : p - 1 < a < p, q - 1 < b < q\} \text{ and } \mathbb{C}.$$

Let us define the following two functions:

$$f(z) = \begin{cases} (\phi_{p,q}(z))^p & \text{if} \quad z \in U_{p,q} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(z) = \begin{cases} (\phi_{p,q}(z))^q & \text{if} \quad z \in U_{p,q} \\ 0 & \text{otherwise.} \end{cases}$$

To show that f and g are algebraically independent, let $P \in \mathbb{C}[z_1, z_2]$ be a non-constant polynomial, and consider Q, m_1, and m_2 as in the previous lemma (for the case $n = 2$.) The function $h = P(f, g)_{|U_{m_1,m_2}}$ verifies that

$$h(\phi_{m_1,m_2}^{-1}(z)) = P(z^{m_1}, z^{m_2}) = Q(z), \text{ for every } z \in \mathbb{C}.$$

Then,

$$h(U_{m_1,m_2}) = h(\phi_{m_1,m_2}^{-1}(\mathbb{C})) = Q(\mathbb{C}) = \mathbb{C}.$$

So h is onto and, in particular, $P(f, g)$ cannot be 0. And we have finally shown the main result:

Theorem 2.3. *The set of complex-valued everywhere surjective functions is algebrable, i.e., it contains an infinite dimensional and infinitely generated algebra every non-zero element of which is everywhere surjective.*

3 Remarks

It is interesting to notice that we can use lemma 2.2 to construct, for every $n \in \mathbb{N}$, n algebraically independent onto functions, $f_1, f_2, \ldots, f_n : \mathbb{C} \to \mathbb{C}$, so that every non-zero function $h \in \mathcal{A}(f_1, f_2, \ldots, f_n)$ is onto. In order to construct such functions, consider (for every $j \in \mathbb{N}$) the "strips" given by the following open sets:

$$U_j = \{a + ib \in \mathbb{C} : j - 1 < a < j\}.$$

Now, consider any bijection between the countable collection $\{U_j : j \in \mathbb{N}\}$ and $\mathbb{N}^n$. Let us now relabel the U_j's, calling them $U_{m_1,m_2,\ldots,m_n}$, where the $n-$ tuple $(m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$. Next, for every $(m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, let $\phi_{m_1,m_2,\ldots,m_n}$ be any homeomorphism between the sets $U_{m_1,m_2,\ldots,m_n}$ and $\mathbb{C}$. Next, define the following functions:

$$f_j(z) = \begin{cases} (\phi_{k_1,k_2,\ldots,k_n}(z))^{k_j} & \text{if} \quad z \in U_{k_1,k_2,\ldots,k_n}, \\ 0 & \text{otherwise}, \end{cases}$$

for every $1 \leq j \leq n$.

We need to show that $f_1, f_2, \ldots, f_n$ are algebraically independent and that any $0 \neq h \in \mathcal{A}(f_1, f_2, \ldots, f_n)$ is onto. For that, take any non-constant polynomial $P \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ and consider $Q, m_1, m_2, \ldots, m_n$ as in lemma 2.2. The function $h = P(f_1, f_2, \ldots, f_n)_{|U_{m_1,m_2,\ldots,m_n}}$ verifies that

$$h(\phi^{-1}_{m_1,m_2,\ldots,m_n}(z)) = P(z^{m_1}, z^{m_2}, \ldots, z^{m_n}) = Q(z), \text{ for every } z \in \mathbb{C}.$$

Then

$$h(U_{m_1,m_2,\ldots,m_n}) = h(\phi^{-1}_{m_1,m_2,\ldots,m_n}(\mathbb{C})) = Q(\mathbb{C}) = \mathbb{C}.$$

Thus, h is onto and, in particular, $P(f_1, f_2, \ldots, f_n)$ cannot be 0. This completes the construction.

From this construction it is clear that for every $n \in \mathbb{N}$, we can also construct n algebraically independent everywhere surjective functions, $g_1, g_2, \ldots, g_n$, so that for every non-constant polynomial $P \in \mathbb{C}[z_1, z_2, \ldots, z_n]$, $P(g_1, g_2, \ldots, g_n)$ is also everywhere surjective. For this it is enough to consider the functions $g_j = f_j \circ F$, where $F : \mathbb{C} \to \mathbb{C}$ is any previously fixed everywhere surjective function and then use lemma 2.1.

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