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Algebraic Analysis of Solvable Lattice Models

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Dedicated to

MIKIO SATO

and

LUDWIG D. FADDEEV

Commemorating the Fruitful Exchange and Interaction
Between Their Schools in Kyoto and Leningrad

Contents

Preface

The aim of the present volume is to give a survey of the recent development on the interplay between solvable lattice models in statistical mechanics and representation theory of quantum affine algebras. The original papers on this subject were published in the form of a series and the results are all scattered around. We thus felt that a systematic account was necessary, which develops the materials from scratch, focusing attention on the most fundamental case and without assuming prior knowledge about lattice models nor representation theory.

Schematically, the basic problems of integrable models in field theory or statistical mechanics are to diagonalize the given Hamiltonian, and to compute the correlation functions. By correlation functions we mean a system of functions $\langle\phi_\alpha(x)\rangle$, $\langle\phi_\alpha(x)\phi_\beta(y)\rangle$, $\cdots$ obtained as vacuum expectations of the operators in the theory. In the context of lattice statistics they are functions of the lattice sites $x, y, \cdots$; in field theory they are functions of the space-time coordinates or momenta. In principle the totality of the correlation functions has enough information to determine the theory completely.

In a naïve way the Hamiltonian is an infinte dimensional matrix acting on some infinite dimensional space. For instance, in the lattice models the latter is typically given as an infinite tensor product of 'local' spaces, e.g.

$$\cdots \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots .$$

Obviously such a Hamiltonian cannot be defined literally because of the difficulty of divergence. In fact, an arbitrary vector in this huge space is not meaningful; what make sense are only those eigenvectors which have finite energy (= finite eigenvalues). They can be thought of as constituting a non-trivial space, which we will refer to as the *space of states*.

At present one knows very few systems whose correlation functions can be described explicitly. The representative examples are 1) the Ising model, and 2) conformal field theory. The Ising model is a two-dimensional lattice model. Its correlations (on the lattice or in the continuum limit) can be characterized by classical non-linear systems such as the Painlevé equations or soliton equations. The conformal field theory deals with critical, or massless, systems in the continuum. Their correlation functions belong to the linear world, giving a good class of generalized hypergeometric functions. The success in the Ising model or conformal field theory is largely related to the fact that their spaces of states have clear mathematical structures: in the Ising model they are the fermion Fock spaces, and in conformal field theory they are the highest weight representations of infinite-dimensional Lie algebras.

Beyond the Ising model, a large class of solvable lattice models have been known; they are built on the solutions of the Yang-Baxter equation. Our main example— the six-vertex model and its spin-chain equivalent, the XXZ model— is one of the most typical models of this sort. However, until recently the space of states and correlation functions have not been understood very well for these more general class of models.

One of the key insights to this problem came from the corner transfer matrix method introduced by Baxter in 1976. The calculation of the one-point functions is reduced to counting the multiplicities of the eigenvalues of the corner transfer matrix. Among others, in the study of the Hard Hexagon model, it led to a remarkable connection with the Rogers-Ramanujan identities. It was then recognized that, in many interesting cases including the Hard-Hexagon model, the spectra of the corner transfer matrices can be described in terms of the characters of affine Lie algebras. Despite the close similarity to certain structure in conformal field theory, this finding has remained a curiosity for some years. Its combinatorial aspect was subsequently clarified by the theory of crystal bases for quantum affine algebras.

Another key emerged through the recent symmetry approach to massive integrable field theories. Bernard and others realized that these theories possess hidden non-Abelian symmetries by the Yangians. It was hoped to exploit these symmetries to understand the integrability in the massive case, following the spirit of conformal field theory. In the latter case a central role was played by the notion of vertex operators and the Knizhnik-Zamolodchikov (KZ) equations for the correlation functions. It was then found that these structures admit a remarkable deformation: by Smirnov, who showed that the form factors he has constructed over the years satisfy the deformed KZ equations; and by Frenkel and Reshetikhin, who studied the vertex operators for quantum affine algebras and derived the q-deformed KZ equations for their matrix elements.

For lattice models the space on which the corner transfer matrix is acting can be viewed as 'half' of the space of states. The appearance of the Lie algebra characters suggests that this half can be identified with a highest weight representation of the quantum affine algebra, which we expect to govern the symmtries of the models. Our first goal in this volume is to explain that it is indeed so. Let $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$ be the direct sum of level one integrable representations of the quantum affine algebra $U_q(\widehat{sl}_2)$. Then the space of states for the six-vertex model has the structure $\mathcal{H} \otimes \mathcal{H}^*$, the tensor product being understood in a certain completed sense. Thus we are upgrading the dimension counting by the characters to a structural understanding of the space of states. This picture will lead to the description of the correlation functions and the form factors in terms of the q-deformed vertex operators, and, via bosonization, to the integral formulas for them. This will be our second goal.

Our expositions are organized as follows. The first three Chapters are devoted to the standard subjects concerning solvable lattice models in statistical mechanics. Our main examples are the spin 1/2 XXZ chain and the six-vertex model. The setting for these models and their mutual equivalence are explained in Chapter 1 and Chapter 2, respectively. In Chapter 3 we discuss the integrability of the models. The role of the Yang-Baxter equation and the commuting transfer matrices are clarified. The rest of the Chapter is devoted to the introduction of the quantum affine algebra $U_q(\widehat{sl}_2)$, and the representation theoretical interpretation of the Yang-Baxter equation. In Chapter 4 we introduce the main objects, the corner transfer matrices and the vertex operators. By a physical argument we then show how the correlation functions can be written as the trace of products of the vertex operators, and derive difference equations for them. Having these as physical motivations, we restart our mathematical discussions from the next Chapters. Chapter 5 is devoted to the Frenkel-Jing bosonization of the level 1 module of $U_q(\widehat{sl}_2)$. In Chapter 6 we derive the formulas for the vertex operators using bosons. In Chapter 7 we reformulate the physical setting in representation theoretical terms, such as the space of states, vacuum, translation, Hamiltonian and its eigenstates. To derive the formulas for the correlation functions and the form factors we need to calculate the trace of products of vertex operators. This computation is carried out in Chapter 8, and its application is given in Chapter 9. The limit of the XXX model is briefly discussed in Chapter 10. We note that the formulas in Chapters 8–10 are presented here for the first time in such details. The last Chapter 11 is devoted to the discussion of the other types of models, and related works. In the Appendix we collect basic formulas for reader's reference. The bibliography is far from being exhaustive. We have limited the citations to only those which are directly related to the discussions.

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Chapter 1

The spin 1/2 XXZ model for $\Delta < -1$

We introduce a one-dimensional spin 1/2 XXZ Hamiltonian, and formulate the problems we are going to address. The model exhibits distinct features depending on the three regions of the parameter Δ in the Hamiltonian. Focusing attention on the case $\Delta < -1$, we explain the different 'vacuum sectors', which will be the starting point of the subsequent discussions about the space of states. In the end we mention Baxter's exact result for the one point function.

1.1 Quantum Hamiltonian

In this Chapter we consider a simple quantum mechanical model in one dimension, called the XXZ model. In physical terms, it describes a system on a one-dimensional lattice, where each lattice point k carries 'quantum spins' σ_k^α interacting with its neighbors. Mathematically it is formulated as follows.

Let $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$ be a two-dimensional vector space with the distinguished basis v_+, v_-. Let further

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli spin operators acting on V in this basis. When copies V_k of V are involved, we indicate by σ_k^α ($\alpha = x, y, z$) the spin operators on V_k; they are understood to act as identity elsewhere.

The *XXZ Hamiltonian* is an operator

$$\mathrm{H}_{XXZ} = -\frac{1}{2}\sum_k (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z). \tag{1.1}$$

Here $k \in \mathbf{Z}$, and Δ is a real number. The quantum spin chain specified by the Hamiltonian (1.1) is called the XXZ model. In the case $\Delta = 1$ the Hamiltonian acquires the explicit symmetry under the group $SL(2)$.[1] In this case the model is called the XXX model. In the general case the parameter Δ represents the anisotropy of the interaction in the (x, y, z)-direction. The XXX and the XXZ models are among the best studied quantum spin chains in one dimension. It is not our aim here to touch upon their long history. The interested readers are referred to the literature that can be found e.g. in [13, 49].

The physical problems of interest that we wish to address in these lectures are the following: to diagonalize the Hamiltonian, and to compute the matrix elements of a local operator with respect to its eigenvectors. Here by a 'local operator' we mean a linear combination of products of finitely many spin operators σ_k^α. In particular we are interested in determining the vacuum vectors (i.e., the lowest eigenvectors), the excitations over them, and further the correlation functions— the vacuum-to-vacuum matrix elements of local operators.

Here a basic question arises. As it stands the Hamiltonian (1.1) is an operator 'acting' on the infinite tensor product space $\otimes_{k\in\mathbf{Z}} V_k$; this is a system of infinite degrees of freedom. How should one understand the proper meaning of such an infinite system?

The usual approach to handle such 'infinity' is to start from a finite tensor product of size N with a certain (e.g., cyclic) boudary condition. The traditional Bethe Ansatz method [77, 78] provides a way to describe the eigenvalues and eigenvectors. After solving the problems for finite N, one proceeds to analyze the large lattice limit $N \to \infty$. Considerable information has been gained this way as for the diagonalization of the Hamiltonian. In contrast, very little has been known about the correlation functions. Virtually the only exception is Baxter's result on the one point function which we will discuss shortly.

In these lectures we wish to explain an alternative approach developed in the series of papers [23]–[30]. Rather than studying the individual eigenvectors, one tries here to capture the whole 'space of states' as a mathematical

[1] Namely if $g \in SL(2)$ and $\mathrm{H}_{XXX,k} = \sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \sigma_k^z\sigma_{k+1}^z$ then $(g\otimes g)\mathrm{H}_{XXX,k}(g^{-1}\otimes g^{-1}) = \mathrm{H}_{XXX,k}$.

entity, on the basis of the representation theory of quantum groups. At present its applicability is limited to the region of the parameter $\Delta < -1$ (see below for its meaning). Nevertheless it has a particularly appealing aspect: It allows an explicit description of the correlation functions. With this point in mind, we shall review in the rest of this chapter some of the basic features about the Hamiltonian and its eigenvectors.

1.2 Three regions in Δ

Let us explain a few known facts on the XXZ Hamiltonian [13, 49]. Among other things we discuss briefly the distinction between the three regions of the parameter Δ:

$$\Delta < -1, \quad |\Delta| \le 1, \quad 1 < \Delta.$$

Let N be a positive and even integer. Consider the Hamiltonian (1.1) wherein $k \in \mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$. This amounts to considering a finite chain of length N under the cyclic boundary condition $\sigma^\alpha_{N+1} = \sigma^\alpha_1$. Let us denote the Hamiltonian by H_N. This is a $2^N \times 2^N$ Hermitian matrix. Therefore its eigenvalues are all real. Let us number them as $h_1 \le h_2 \le \cdots \le h_{2^N}$.

Define the total spin operator

$$S_N = \frac{1}{2} \sum_{k \in \mathbf{Z}_N} \sigma^z_k.$$

The total space $W_N = \otimes_{k \in \mathbf{Z}_N} V_k$ splits into the eigenspaces of S_N:

$$W_N = \oplus_s W_N^{(s)}, \quad W_N^{(s)} = \{v \in W_N \mid S_N v = sv\},$$

where s ranges over integers from $-N/2$ to $N/2$. Since H_N and S_N commute, one can consider the diagonalization of H_N in each subspace $W_N^{(s)}$ with fixed 'total spin' s.

The spectrum of H_N exhibits distinct features depending on the three regions.

Case 1 $\Delta > 1$: The two lowest eigenvalues h_1 and h_2 are equal. The corresponding eigenspace is the direct sum of $W_N^{(N/2)} = \mathbf{C}\Omega_+$ and $W_N^{(-N/2)} = \mathbf{C}\Omega_-$, where the vacuum vectors $\Omega_\epsilon = \otimes_{k \in \mathbf{Z}_N} v_\epsilon$ are simple pure tensors independent of Δ. Accordingly the correlation functions, which are the matrix elements of operators $\sigma^{\alpha_1}_{k_1} \cdots \sigma^{\alpha_n}_{k_n}$, are also independent of Δ and are readily calculable. For that reason this region is uninteresting to us.

Case 2 $|\Delta| \leq 1$: The lowest eigenvalue belongs to $W_N^{(0)}$. Since the dimension of $W_N^{(0)}$ increases rapidly as $N \to \infty$, the explicit form of the lowest eigenvector is very complicated. Moreover, in the limit, infinitely many eigenvalues collapse to the lowest one, i.e., the gaps above the lowest one tend to zero for those.

Case 3 $\Delta < -1$: The lowest and the next lowest eigenvalues degenerate in the infinite lattice limit. More precisely, $h_1 < h_2$ holds for finite N but in the limit $N \to \infty$ the gap $h_2 - h_1$ tends to zero. On the other hand, the gap $h_3 - h_2$ remains non-zero in the limit. Such is called the *mass gap.* Because the mass gap is non-zero, this region is called *massive.* In this sense Case 2 is a *massless* region. The eigenvectors corresponding to h_1 and h_2 belong to $W_N^{(0)}$. Their explicit forms are complicated.

1.3 The anisotropic limit

From now on, we restrict our consideration to the massive region $\Delta < -1$. To gain insight into the nature of the eigenstates, it is instructive to look at the extreme limit when the anisotropy parameter Δ tends to $-\infty$. In this limit one can see what is happening directly on the infinite lattice, without recourse to the finite lattice Hamiltonian H_N.

Adding a suitable constant and rescaling, let us modify the Hamiltonian as

$$\frac{1}{|\Delta|}(H_{XXZ} + const.) = H_0 - \varepsilon H_1, \qquad \varepsilon = \frac{1}{|\Delta|}, \tag{1.2}$$

$$H_0 = \frac{1}{2}\sum_k (\sigma_k^z \sigma_{k+1}^z + 1), \tag{1.3}$$

$$H_1 = \sum_k (\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+),$$

with $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$. Here the shift by a constant is so chosen as to make the lowest eigenvalue of H_0 be 0.

Let us consider the Hamiltonian H_0 (1.3) obtained as the limit $\Delta \to -\infty$. In contrast to the original Hamiltonian H_0 is much simplified, since it is already diagonal with respect to the pure tensor vectors

$$|p\rangle = \cdots \otimes v_{p(k+1)} \otimes v_{p(k)} \otimes v_{p(k-1)} \otimes \cdots.$$

If we consider the model on the infinite lattice, then p runs through all maps

$\mathbf{Z} \to \{+,-\}$.[1] The important point is that not all of these vectors have finite energy (=eigenvalues). To see this, note that each $(\sigma_k^z\sigma_{k+1}^z + 1)/2$ takes the value 0 or 1 according to whether $p(k) = -p(k+1)$ or $p(k) = p(k+1)$. Therefore, H_0 has a finite eigenvalue on $|p\rangle$ if and only if

$$p(k) = -p(k+1) \quad \text{for almost all } k \in \mathbf{Z}.$$

Let us call a map p satisfying this condition a *path*. Hence for the limiting Hamiltonian H_0 we have the following picture: The eigenvectors which have finite energy can be identified with paths. They constitue a space $\mathcal{F}_0$ which we call the *space of states* for H_0 (the suffix 0 referring to the limit $\Delta = -\infty$). It splits further into the direct sum of four 'sectors'

$$\mathcal{F}_0 = \oplus_{i,j=0,1}\mathcal{F}_0^{(i,j)},$$

where $\mathcal{F}_0^{(i,j)}$ is the span of the paths subject to a fixed 'boundary condition'

$$\begin{aligned} p(k) &= (-1)^{k+i} \quad \text{if } k \gg 1, \\ &= (-1)^{k+j} \quad \text{if } -k \gg 1. \end{aligned}$$

In particular, the sector $\mathcal{F}^{(i,i)}$ $(i = 0,1)$ contains the 'vacuum path', or the vacuum vector

$$\bar{p}^{(i)}(k) = (-1)^{k+i} \quad \forall k \in \mathbf{Z}.$$

It is the unique vector belonging to the minimum eigenvalue 0. The general vectors in $\mathcal{F}^{(i,i)}$ can be regarded as finite excitations over the vacuum vector. We note that a local operator sends one sector into itself, since it cannot change the $\pm$ infinitely many times. In this sense the different sectors are separate from each other.

Intuitively we expect that, for nonzero but small enough $\varepsilon = |\Delta|^{-1}$, the space of states be still a 'span' of the paths, involving possibly infinite linear combinations. To have an idea of what it should mean, let us try to expand the vacuum vector in powers of ε

$$|\text{vac}\rangle = \Omega_0 + \frac{\varepsilon}{2}\Omega_1 + \left(\frac{\varepsilon}{2}\right)^2\Omega_2 + \left(\frac{\varepsilon}{2}\right)^3\Omega_3 + \cdots \tag{1.4}$$

where Ω_0 corresponds to one of the vacuum paths $\bar{p}^{(i)}$ of $\mathcal{F}_0^{(i,i)}$. Demanding it be an eigenvector of the full Hamiltonian $H_0 - \varepsilon H_1$, we can determine the

[1] We have displayed the tensor components in the decreasing order $\cdots 2, 1, 0, -1, -2, \cdots$ from left to right, in accordance with the convention to be used later.

vectors Ω_j in the expansion. Note that the operator $\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+$ exchanges $v_\pm \otimes v_\mp$ with $v_\mp \otimes v_\pm$ and kills $v_\pm \otimes v_\pm$. Hence

$$|n\rangle = (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+)\Omega_0$$

represents the path whose components at the n^{th} and $(n+1)^{\text{th}}$ positions are flipped from those of the vacuum path. Define similarly $|n_1, n_2\rangle$, and so forth. We find that the first two terms besides Ω_0 are given by

$$\Omega_1 = \sum_n |n\rangle, \qquad \Omega_2 = 2\sum_n |n, n+2\rangle + \sum_{n_1 < n_2 - 2} |n_1, n_2\rangle.$$

More generally it should be clear that for any r the Ω_r are linear combinations of vectors obtained by flipping $\pm$ from Ω_0 by at most $2r$ times.

We expect that the eigenvectors other than the vacuum are also analogous infinite linear combinations of the paths. In practice, however, such a direct treatment in the infinite lattice meets a difficulty. For nonzero ε and in the infinite lattice limit, the original Hamiltonian has a continuous spectrum except for the vacuum. After rescaling (1.3) has discrete eigenvalues $\{0, 1, 2, \ldots\}$. This means that there are infinitely many distinct eigenvalues of H_{XXZ} for $\varepsilon \neq 0$ that degenerate to the same eigenvalue of (1.3). Hence even for $\varepsilon = 0$ the general eigenvalue should necessarily be infinite linear combinations of the paths, and we cannot tell a priori how to start the ε-expansions of these eigenvectors. For this purpose one needs to use the eigenvectors obtained on the finite lattice by Bethe Ansatz.

In any event, we will not attempt to justify the picture about the space of states by tracing the limit $N \to \infty$. What we will actually do is to construct a mathematical model directly in the infinite lattice limit that reflects the features mentioned above. The solution to this problem will be given in Chapter 7 by using the representation theory of $U_q(\widehat{sl}(2))$.

1.4 One point function $\langle \mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle$

The perturbative expansion of the vacuum vector (1.4) holds equally well for a finite lattice of length N with periodic boundary condition.[2] Making use of it, let us compute the expectation value of the operator σ_1^z to the order ε^4. Without loss of generality we assume that the first component of Ω_0 is v_+:

[2] If we start from the finite lattice with periodic boundary condition and pass to the infinite lattice limit, we would see only the sectors $\mathcal{F}_0^{(i,j)}$ with $i = j$.

$\sigma_1^z\Omega_0 = \Omega_0$. Multiplying a scalar to $|\mathrm{vac}\rangle$ if necessary, we also assume that the vector Ω_0 never appears in Ω_n with $n > 0$. With this convention, the formula for Ω_4 is not necessary to compute $\langle\mathrm{vac}|\mathrm{vac}\rangle$. For Ω_3 it is sufficient to know that $\Omega_3 = -\Omega_1+$(terms different from Ω_1) for the same reason. Since σ_1^z is diagonal on the paths, a similar argument applies to the computation of $\langle\mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle$. We then find that

$$\langle\mathrm{vac}|\mathrm{vac}\rangle = 1 + \left(\frac{\varepsilon}{2}\right)^2 N + \left(\frac{\varepsilon}{2}\right)^4 \frac{N(N-1)}{2} + \cdots,$$

$$\langle\mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle = 1 + \left(\frac{\varepsilon}{2}\right)^2 (N-4) + \left(\frac{\varepsilon}{2}\right)^4 \left(\frac{N(N-9)}{2} - 4\right) + \cdots,$$

to obtain

$$\frac{\langle\mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle}{\langle\mathrm{vac}|\mathrm{vac}\rangle} = 1 - 4\varepsilon^2 - 4\varepsilon^4 + \cdots.$$

Continuing further one finds that each coefficient of the expansion in ε stabilizes to a finite value as $N \to \infty$. However as one goes higher in the power of ε the combinatorial complication becomes enormous, eventually making the computations impracticable.

In fact an exact formula for this quantity has been known by Baxter. In [8] it is called the spontaneous staggered polarization. Introduce the parametrization

$$\Delta = \frac{q+q^{-1}}{2}$$

so that $\varepsilon = -q/(1+q^2)$. The region of our interest $\Delta < -1$ corresponds to q being real and $-1 < q < 0$.[1] Baxter's formula is

$$\langle\mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle = \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2. \tag{1.5}$$

On the basis of the representation theoretical approach, we will give an integral formula of the correlation function $\langle\mathrm{vac}|L|\mathrm{vac}\rangle$ for an *arbitrary* local operator L (see (9.4)). The formula (1.5) is included as the simplest special case.

Remark. As we mentioned in the introduction, the corner transfer matrix enables us to compute the one point functions. To avoid confusion we remark

[1]The other region $\Delta > 1$ or $|\Delta| \leq 1$ corresponds respectively to $0 < q < 1$ or to q being complex and $|q| = 1$.

that these one point functions are those of the face models (see Chapter 11). In the latter language, Baxter's spontaneous staggered polarization discussed above is the nearest neighbor *two* point function.

Chapter 2

The six-vertex model in the anti-ferroelectric regime

In this Chapter we introduce another well-known lattice model called the six-vertex model. While the XXZ model is a *one*-dimensional system in *quantum* mechanics, the six-vertex model is a *two*-dimensional system in *classical* statistical mechanics. In fact the two models are equivalent, in the sense to be elaborated on in the next Chapter. Here we shall formulate our problems from the point of view of the six-vertex model.

2.1 Vertex model

Consider a two-dimensional square lattice. In the following discussion, such geometrical attributes as angles or lengths will be irrelevant. We shall use only the topological or combinatorial structure of the lattice. Although our ultimate interest lies in the infinite lattice, we start with a finite lattice in order to fix our ideas. Thus let us draw M vertical and N horizontal lines on the plane. For simplicity we assume that both M and N are even. We call an intersection of two lines a *vertex*, and a line segment limited by two neighboring vertices an *edge*. As with the XXZ model, let us impose the cyclic boundary condition on both ends. This means that for each horizontal or vertical line we join the edge at an end with the edge at the opposite end of the same line. Hence the lattice is actually wound on the torus rather than placed on the plane.

A model in classical statistical mechanics is built on our lattice in the following way. First, with each edge j we associate a variable ε_j taking the values + or −. Unlike the quantum spin chain, ε_j is an ordinary commuting variable. We shall refer to the ε_j as (classical) spin variables and the values ± as spins. A *configuration* C is an assignment of spins $\varepsilon_j = \pm$ for all j. Hence there are altogether 2^{2MN} configurations.

Next we introduce a probability measure in the set of all configurations by assigning a statistical weight $W(C)$ to each configuration C. The probability for a configuration C to take place is $Z_{M,N}^{-1}W(C)$, where

$$Z_{M,N} = \sum_C W(C). \tag{2.1}$$

The normalization factor (2.1) is called the *partition function* of the model. To define $W(C)$ we prepare a set of positive real numbers $R_{\varepsilon_1,\varepsilon_2}^{\varepsilon'_1,\varepsilon'_2}$ $(\varepsilon'_1,\cdots,\varepsilon_2 = \pm)$ called *Boltzmann weights.* A configuration C gives rise to a configuration of the spin variables around each vertex v. Denote this configuration by $\varepsilon'_1(C,v)$, $\varepsilon'_2(C,v)$, $\varepsilon_1(C,v)$, $\varepsilon_2(C,v)$ (see the figure below).

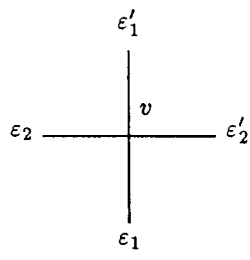

Figure 2.1: A configuration round a vertex

The weight $W(C)$ is given as the product over all the vertices

$$W(C) = \prod_v R_{\varepsilon_1(C,v),\varepsilon_2(C,v)}^{\varepsilon'_1(C,v),\varepsilon'_2(C,v)}. \tag{2.2}$$

The six-vertex model is specified by giving the Boltzmann weights according to the following rule.

$$R_{+,+}^{+,+} = R_{-,-}^{-,-} = a,$$
$$R_{+,-}^{+,-} = R_{-,+}^{-,+} = b,$$

$$R^{+,-}_{-,+} = R^{-,+}_{+,-} = c,$$
$$R^{\varepsilon_1',\varepsilon_2'}_{\varepsilon_1,\varepsilon_2} = 0 \qquad \text{if } \varepsilon_1' + \varepsilon_2' \neq \varepsilon_1 + \varepsilon_2.$$

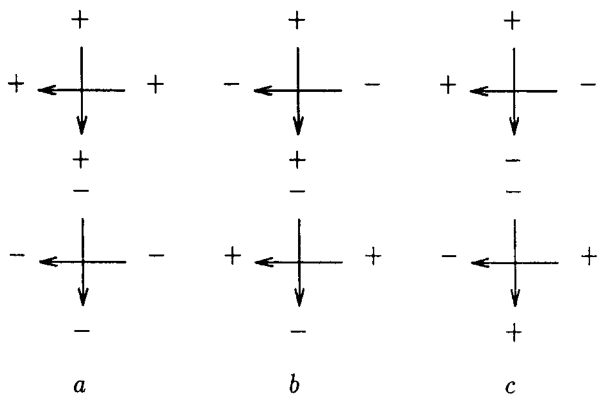

Figure 2.2: Six vertex configurations

Here a, b, c are positive real numbers. The last condition says that the sums of spins are conserved at each vertex in the NE-SW diagonal direction. The Boltzmann weights are chosen to be symmetric under the reversal of spins $\varepsilon_j \to -\varepsilon_j$.

2.2 Ground states and low-temperature expansion

As will be discussed in the next Chapter, the XXZ model with the parameter Δ is equivalent to the six-vertex model with the parameter a, b, c if

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{2.3}$$

The region $\Delta < -1$ corresponds to the region $c > a + b$. We restrict our consideration to this case.

In the region mentioned above, there are two configurations that maximize $W(C)$. We call them ground state configurations. They are those in which all the vertices have the type c configuration around them. In other words the spin variables take the same value along the NE-SW diagonal and alternating values $\cdots + - + - + - \cdots$ along the horizontal direction. We pick a reference edge 1 and call $C^{(i)}$ the one with $\varepsilon_1 = (-1)^i$ $(i = 0, 1)$.

The partition function (2.1) is one of the basic quantities of physical interest. For finite M, N it is simply a polynomial in the three variables a, b, c. We are interested in the behavior of $Z_{M,N}$ when $M, N \to \infty$. It turns out that the limit

$$\kappa = \lim_{M,N\to\infty} (Z_{M,N})^{\frac{1}{MN}} \tag{2.4}$$

is finite. This is called the partition function per site.

Since the weights $W(C)$ are homogeneous in the Boltzmann weights, changing the latter by a common multiple does not lead to any essential change. Let us normalize $c = 1$ for the time being. Then a and b can be regarded as small parameters in terms of which we can expand the physical quantities. Such an expansion is called the low-temperature series expansion. In the physical context the Boltzmann weights a, b, c appear in the form

$$a = e^{-E_a/kT}, \quad b = e^{-E_b/kT}, \quad c = e^{-E_c/kT},$$

where E_a, E_b, E_c are the energies of the local configurations, k the Boltzmann constant, and T the absolute temperature. With E_c, E_a, E_b fixed, the limit $T \to 0$ corresponds to the ratios a/c and b/c tending to zero. Whence the name 'low-temperature'.

Counting the degrees of a and b to be 1, suppose we try to compute (2.4) up to some fixed degree K. Take M and N sufficiently large, and consider only configurations C of degree less than or equal to K. The ground state configurations are the only ones of degree 0: $W(C^{(i)}) = 1$. In general the configurations can be classified into two groups: Given a configuration C let $d_i(C)$ be the number of edges j such that the value of the spin variable ε_j in C is different from that in $C^{(i)}$. We say C belongs to the i-th sector if $d_i(C) < d_{1-i}(C)$. Because of the spin reversal symmetry, the partition function $Z_{M,N}$ (up to degree K) is equal to twice the sum of $W(C)$ over C belonging to one of the sectors. Since this factor 2 is unimportant in the large lattice limit, one can compute the expansion in a, b of κ by using only one sector.

Set $L = MN$. The first few terms in the expansions of $Z_{M,N}$ and κ are

$$\frac{1}{2} Z_{M,N} = 1 + La^2b^2 + La^2b^2(a^2 + b^2)$$

$$+\frac{L(L+1)}{2}a^4b^4 + La^2b^2(a^4+b^4) + \cdots, \tag{2.5}$$

$$\kappa = 1 + a^2b^2 + a^2b^2(a^2+b^2) + a^4b^4 + a^2b^2(a^4+b^4) + \cdots. \tag{2.6}$$

In principle this computation can be carried through to any order, giving κ an unambiguous meaning as a power series in a, b. As for the exact result for κ, see the end of 4.2.

2.3 The correlation function

The *correlation functions* of the six-vertex model is defined as follows. Take n arbitrary edges $j_1, \cdots, j_n$. Consider the following ratio

$$\langle \varepsilon_{j_1} \cdots \varepsilon_{j_n} \rangle = \frac{\sum_C \varepsilon_{j_1}(C) \cdots \varepsilon_{j_n}(C) W(C)}{\sum_C W(C)}. \tag{2.7}$$

Here $\varepsilon_j(C)$ signifies the value of the spin variable on the edge j in the configuration C. Fix a ground-state sector $i = 0$ or 1. Given a degree K, we choose the cyclic lattice of sufficiently large size M, N and compute (2.7) with the summation restricted to the i-th sector. It has a similar low-temperature expansion as with (2.5). We denote the quantity obtained in this way by $\langle \varepsilon_{j_1} \cdots \varepsilon_{j_n} \rangle_i$, and call it the n-point correlation function (in the i-th sector). Notice that it is *not* the same as taking the unrestricted sum in (2.7). For instance if $n = 1$ and the reference edge is 1, then the former is a series of the form $(-1)^i(1 + \cdots)$ while the latter is trivially 0 by the spin reversal symmetry.

The equivalence between the XXZ model and the six-vertex model to be discussed in the next Chapter entails the following relation between the correlation functions. For convenience let us number the horizontal (resp. vertical) lines by integers from bottom to top (resp. right to left). Consider n distinct horizontal edges lying between two neighboring vertical lines. Calling j_k the horizontal edge on the j_k-th line, consider the n-point correlation function for the six-vertex model $\langle \varepsilon_{j_1} \cdots \varepsilon_{j_n} \rangle_i$. Then the statement is

$$\langle \varepsilon_{j_1} \cdots \varepsilon_{j_n} \rangle_i = \langle i | \sigma^z_{j_1} \cdots \sigma^z_{j_n} | i \rangle \tag{2.8}$$

where the right hand side signifies the correlation function of the XXZ model as explained in Chapter 1. In particular Baxter's formula (1.5) applies to the simplest correlation of the six-vertex model.

2.4 Transfer matrix

Basically a quantum spin chain is an eigenvalue problem of a large (infinite!) matrix such as H_{XXZ}. Let us explain that classical statistical systems can be put on an equal footing by using *transfer matrices.*

The partition function (2.2) is a sum over two dimensional configurations, the sums extending over the vertical and the horizontal edges. Pick a particular column. (See Figure 2.2.) Consider the sum over the vertical edges in this column, while fixing the configurations of the horizontal edges to the left and right of this column:

$$T_{\varepsilon_1\cdots\varepsilon_N}^{\varepsilon'_1\cdots\varepsilon'_N} = \sum_{\nu_1,\cdots,\nu_N} R_{\nu_1\varepsilon_1}^{\nu_2\varepsilon'_1} R_{\nu_2\varepsilon_2}^{\nu_3\varepsilon'_2} \cdots R_{\nu_N\varepsilon_N}^{\nu_1\varepsilon'_N}. \tag{2.9}$$

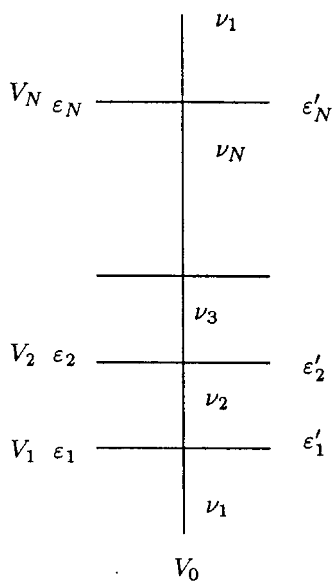

Figure 2.3: Transfer matrix

Regard the indices $\varepsilon_j, \varepsilon'_j$ as the labels of the basis of $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$. Then

(2.9) gives rise to a matrix T acting on the N-fold tensor product $V^{\otimes N}$. We call $T = T_{\mathrm{col}}$ the column transfer matrix, or simply the transfer matrix.

To get the partition function we have to perform the sum over the horizontal edges. Clearly the sum over the horizontal edges between two successive columns amounts to matrix multiplication of T's. In view of the cyclic boundary condition it follows that

$$Z_{M,N} = \mathrm{tr}\,(T^M) = \lambda_1^M \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^M + \cdots\right)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots$ are eigenvalues of T. This makes it manifest that the study of the partition function reduces to the eigenvalue problem of T. The correlation functions is related to the eigenvectors of T but we skip the discussion here.

Operating with T amounts to counting the contribution from vertices on a column. Similarly, counting a contribution from a single vertex amounts to operating with a matrix $R \in \mathrm{End}\,(V \otimes V)$

$$R\,(v_{\varepsilon_1'} \otimes v_{\varepsilon_2'}) = \sum_{\varepsilon_1 \varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2}\, R_{\varepsilon_1 \varepsilon_2}^{\varepsilon_1' \varepsilon_2'}.$$

We shall refer to it as R-matrix. Often it is necessary to consider the tensor product $V_{k_1} \otimes \cdots \otimes V_{k_n}$ of copies V_k of $V = \mathbf{C}^2$, arranged in some order. In this situation we denote by R_{jk} $(j \neq k)$ the operator acting on V_j and V_k as R and as identity elsewhere. Namely let us write $R = \sum a_i \otimes b_i$ with $a_i, b_i \in \mathrm{End}\,(V)$. Then for instance on $V_1 \otimes V_2 \otimes V_3$ we have

$$R_{13} = \sum a_i \otimes \mathrm{id} \otimes b_i,$$

and on $V_3 \otimes V_1 \otimes V_2$ we have

$$R_{13} = \sum b_i \otimes a_i \otimes \mathrm{id}\,.$$

We use $P \in \mathrm{End}\,(V)$ for the transposition; $P(u \otimes v) = v \otimes u$. (Do not confuse this with the weight lattice that will be defined in 3.4.) Note that

$$P_{12} = P_{21}, R_{21} = P_{12} R_{12} P_{12}.$$

Sometimes we use the transposition acting from $V_1 \otimes V_2$ to $V_2 \otimes V_1$. We abuse the notation P for this operator (e.g. (3.25)).

The relation between the R and transfer matrices can be made explicit as follows. Define the monodromy matrix $\mathcal{T}$ acting on the $(N+1)$-fold tensor product $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ $(V_k = V)$

$$\mathcal{T} = R_{01} \cdots R_{0N} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.10}$$

In (2.10) the partition into 2×2 blocks is according to the base v_+, v_- of V_0. The transfer matrix is expressed as

$$T = A + D = \mathrm{tr}_{V_0}(\mathcal{T}).$$

Note that it is thus natural to associate the two-dimensional space V_k to each line k. (See Figure 2.2.)

Chapter 3

Solvability and Symmetry

The solvability of the XXZ model and the six-vertex model is based on the Yang-Baxter equation (YBE) satisfied by the R matrix. We will review the well known fact that the YBE implies commutativity of the transfer matrices— a property that is a manifestation of abelian symmetries of the models. It turns out that our R matrix is dictated by the quantum group $U_q(\widehat{sl}_2)$. We suggest that it leads to non-abelian symmetries.

3.1 Commuting Hamiltonians

A characteristic feature common to classical or quantum integrable systems is the existence of an infinite number of 'commuting integrals', or 'conservation laws'. In the context of the XXZ model, this amounts to the following statement: There exist a hierarchy of independent operators H_1, H_2, H_3, $\cdots$, including $H_{XXZ} = H_1$ as the first member, such that they are mutually commutative,

$$[H_m, H_n] = 0 \qquad \forall m, n. \tag{3.1}$$

Correspondingly, in the six-vertex model there exists a family of transfer matrices $T(\zeta)$ depending on a parameter ζ in such a way that

$$[T(\zeta), T(\zeta')] = 0 \qquad \forall \zeta, \zeta'. \tag{3.2}$$

Moreover the two models are connected by the relation

$$H_n = const. \left(\zeta \frac{d}{d\zeta}\right)^n \log T(\zeta)|_{\zeta=1}. \tag{3.3}$$

The commutativity of the 'higher Hamiltonians' (3.1) is a direct consequence of that of the transfer matrices (3.2). Since H_{XXZ} and $T(\zeta)$ commute, they share the same eigenvectors in common. Thanks to the relation (3.3), knowing the eigenvalues of $T(\zeta)$ is the same as knowing those of the H_n. Thus the two models are equivalent in this sense.

We shall sketch below the mechanism how these properties come about.

3.2 Yang-Baxter equation

Let us begin by re-parameterizing the Boltzmann weights of the six-vertex model:

$$a = \frac{1}{\kappa}, \qquad b = \frac{1}{\kappa}\frac{(1-\zeta^2)q}{1-q^2\zeta^2}, \qquad c = \frac{1}{\kappa}\frac{(1-q^2)\zeta}{1-q^2\zeta^2}. \tag{3.4}$$

The parameter κ simply accommodates the overall scale of the Boltzmann weights. For a reason to be explained later we make a specific choice of κ as a function of q, ζ

$$\kappa(\zeta) = \zeta\frac{(q^4\zeta^2;q^4)_\infty (q^2\zeta^{-2};q^4)_\infty}{(q^4\zeta^{-2};q^4)_\infty (q^2\zeta^2;q^4)_\infty} \tag{3.5}$$

where $(z;p)_\infty = \prod_{n=1}^{\infty}(1-zp^n)$. It follows from (3.4) that

$$\Delta \equiv \frac{a^2+b^2-c^2}{ab} = \frac{q+q^{-1}}{2}$$

is independent of ζ. Hence ζ plays the role of a coordinate on the manifold $\{(a:b:c) \in \mathbf{P}^2 \mid \Delta = const.\}$. We will refer to ζ as the spectral parameter. Normally we fix q and regard $a, b, \cdots$ as functions of ζ. In the matrix form the parameterization reads

$$R(\zeta) = \frac{1}{\kappa(\zeta)}\begin{pmatrix} 1 & & & \\ & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \\ & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \\ & & & 1 \end{pmatrix}, \tag{3.6}$$

where the matrix structure is relative to the basis $v_\varepsilon \otimes v_{\varepsilon'}$ arranged in the order $(\varepsilon, \varepsilon') = (+,+), (+,-), (-,+), (-,-)$.

We have introduced the parameterization (3.4) in a rather *ad hoc* manner. The virtue of it is that for a fixed q the R matrix satisfies the following

Yang-Baxter Equation On $V_1 \otimes V_2 \otimes V_3$ we have

$$R_{12}(\zeta_1/\zeta_2)R_{13}(\zeta_1/\zeta_3)R_{23}(\zeta_2/\zeta_3) = R_{23}(\zeta_2/\zeta_3)R_{13}(\zeta_1/\zeta_3)R_{12}(\zeta_1/\zeta_2). \tag{3.7}$$

Using (3.4) a direct verification of YBE is certainly possible. But we prefer to discuss later the conceptual meaning of YBE and the origin of the formula (3.6). (See 3.5.)

We record here further properties of the R matrix:

Initial Condition

$$R(1) = P. \tag{3.8}$$

Unitarity Relation On $V_1 \otimes V_2$ we have

$$R_{12}(\zeta_1/\zeta_2)R_{21}(\zeta_2/\zeta_1) = 1. \tag{3.9}$$

Crossing Symmetry

$$R(\zeta_2/\zeta_1)_{\varepsilon_2 \varepsilon_1'}^{\varepsilon_2' \varepsilon_1} = R(-q^{-1}\zeta_1/\zeta_2)_{-\varepsilon_1\, \varepsilon_2}^{-\varepsilon_1'\, \varepsilon_2'}. \tag{3.10}$$

Notice that we demand the formulas (3.8–3.10) as written, without introducing extra scalar factors. They are true if and only if $\kappa(\zeta)$ is chosen to satisfy $\kappa(1) = 1$ and

$$\kappa(\zeta)\kappa(\zeta^{-1}) = 1, \qquad \frac{\kappa(-q^{-1}\zeta)}{\kappa(\zeta^{-1})} = \frac{(1-q^{-2}\zeta^2)q}{1-\zeta^2}. \tag{3.11}$$

The solutions of the equations (3.11) are not unique. Among them the formula (3.5) is characterized as the unique solution which is analytic in the region $q^2 \le |\zeta^2| \le q^{-2}$. Later we will find that $R(\zeta)$ with precisely this scalar factor arises in the commutation relations of vertex operators (A.2) and (A.2). We remark also that (3.5) coincides with the known exact result for the partition function per site of the six-vertex model with the normalization $a = 1$; In other words, with the choice (3.4) the partition function per site of the model is simply 1.

3.3 Z-invariant lattice

As it will turn out, when we consider the R matrix $R(\zeta)$ it is more natural to attach independent spectral parameters ζ_1, ζ_2 to the first and the second tensor components of $V \otimes V$, and regard $R(\zeta) = R(\zeta_1/\zeta_2)$. Graphically the Boltzmann weights are represented by crossings as in Fig.3.1.

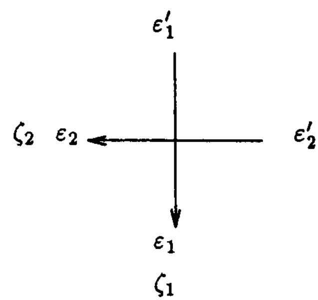

Figure 3.1: Boltzmann weights

Often it becomes necessary to rotate this figure. To avoid possible confusion we assign orientation to the lines represented by 'arrows' on them: In the above 'normal position' the vertical lines are supposed to point downward and the horizontal lines to the left.[1] The YBE (3.7), unitarity (3.9) and crossing symmetry (3.10) are represented by the following figures 3.2, 3.3, 3.4, respectively.

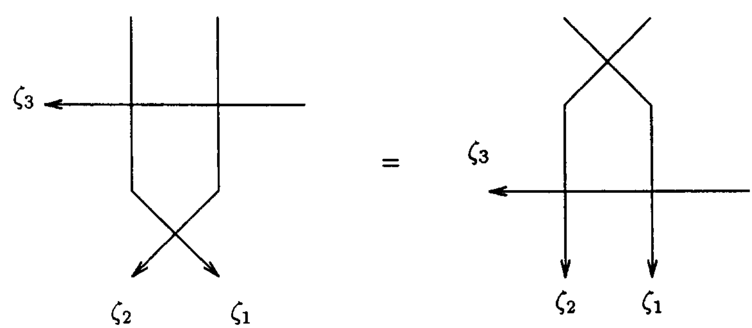

Figure 3.2: Yang-Baxter equation

[1]In the literature the spins $\pm$ on the edges are sometimes represented by 'arrows'. They are not to be confused with our arrows which are used only to represent the orientation.

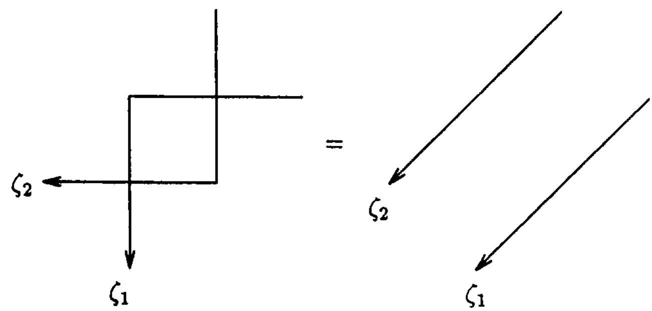

Figure 3.3: Unitarity relation

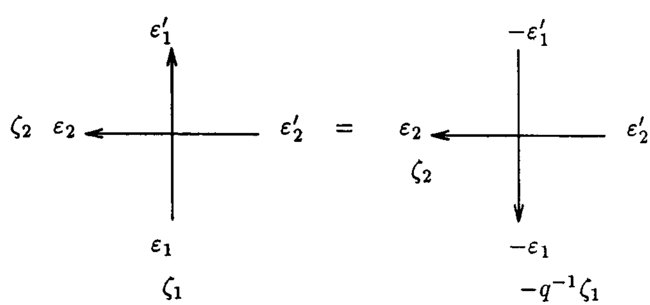

Figure 3.4: Crossing symmetry

In formulating the six-vertex model we orient the lines of the lattice accordingly, so that the horizontal ones point to the left and vertical ones downward. Let us attach spectral parameters independently line by line, $\zeta_V^{(i)}$ to the i-th column and $\zeta_H^{(j)}$ to the j-th row. At the crossing of the i-th column and the j-th row we associate the Boltzmann weights $R_{\epsilon_1\epsilon_2}^{\epsilon'_1\epsilon'_2}(\zeta_V^{(i)}/\zeta_H^{(j)})$. This generalizes the homogeneous model we considered in the last Chapter, where $\zeta_V^{(i)} = \zeta_V$ (resp. $\zeta_H^{(j)} = \zeta_H$) are chosen to be the same for all columns (resp. rows). Though we are interested in the homogeneous model, the idea of independently varying the spectral parameters is crucial to us. The usefulness of such an inhomogeneous model was emphasized by Baxter. In [11] such a model is called Z-invariant since its partition function Z can be shown to be invariant under sliding the rows (or columns) through each other (cf. the argument below). The role of YBE is made particularly clear in the Z-invariant model.

In this setting, let us derive the commutativity (3.2) of the transfer matrices. Let

$$\mathcal{T}_0(\zeta|\zeta_H^{(1)},\cdots,\zeta_H^{(N)}) = R_{0\,1}(\zeta/\zeta_H^{(1)})\cdots R_{0\,N}(\zeta/\zeta_H^{(N)})$$

be the monodromy matrix of the inhomogeneous model, regarded as an operator on $V_0\otimes V_{0'}\otimes V_1\otimes\cdots\otimes V_N$ that acts as identity on $V_{0'}$. We suppress the row variables $\zeta_H^{(j)}$. Define similarly $\mathcal{T}_{0'}(\zeta)$ which acts as identity on V_0. Note that

$$\begin{aligned}&\mathcal{T}_0(\zeta)\mathcal{T}_{0'}(\zeta')\\ &= \left(R_{0\,1}(\zeta/\zeta_H^{(1)})R_{0'\,1}(\zeta'/\zeta_H^{(1)})\right)\cdots\left(R_{0\,N}(\zeta/\zeta_H^{(N)})R_{0'\,N}(\zeta'/\zeta_H^{(N)})\right).\end{aligned}$$

Applying YBE

$$R_{00'}(\zeta/\zeta')\cdot R_{0j}(\zeta/\zeta_j)R_{0'j}(\zeta'/\zeta_j)\cdot R_{00'}(\zeta/\zeta')^{-1} = R_{0'j}(\zeta'/\zeta_j)R_{0j}(\zeta/\zeta_j)$$

we find

$$R_{00'}(\zeta/\zeta')\cdot\mathcal{T}_0(\zeta)\mathcal{T}_{0'}(\zeta')\cdot R_{00'}(\zeta/\zeta')^{-1} = \mathcal{T}_{0'}(\zeta')\mathcal{T}_0(\zeta).$$

Upon taking the trace of both sides over $V_0\otimes V_{0'}$, we obtain the commutativity mentioned above. The following figure illustrates this process.

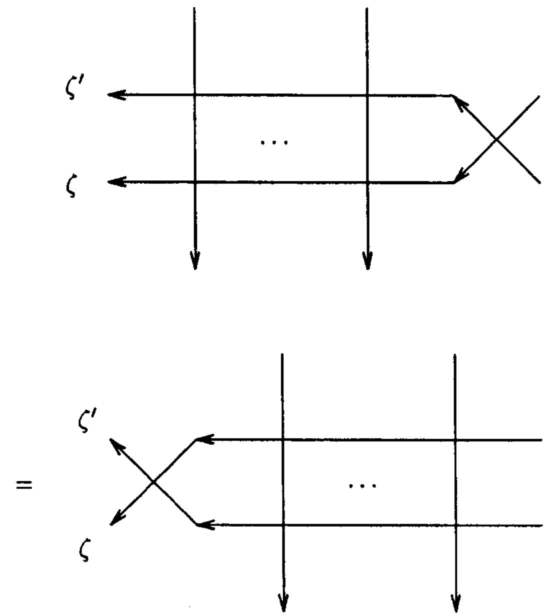

Figure 3.5: Commutativity of the transfer matrices

Next let us establish the relation with the *XXZ* model. We return to the homogeneous model and regard the transfer matrix as a function of $\zeta = \zeta_V/\zeta_H$. First notice the initial condition (3.8) for the R matrix. This implies that the transfer matrix at $\zeta = 1$ reduces to the translation operator

$$T(1)v_{\varepsilon_1} \otimes v_{\varepsilon_2} \otimes \cdots \otimes v_{\varepsilon_N} = v_{\varepsilon_N} \otimes v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_{N-1}}.$$

Since the $T(\zeta)$ commute, it makes sense to define the operators H_n by

$$\log\Big(T(1)^{-1}T(\zeta)\Big) = \sum_{n=1}^{\infty} H_n(\zeta - 1)^n.$$

Clearly they also mutually commute. On the other hand, expanding $R(\zeta)$ at $\zeta = 1$ we find

$$R(\zeta)P = 1 + (1-\zeta)(h + const.) + \cdots \qquad (\zeta \to 1), \tag{3.12}$$

$$h = \frac{q}{1-q^2}\left(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta\sigma^z \otimes \sigma^z\right). \tag{3.13}$$

From this it follows that

$$\zeta \frac{d}{d\zeta} \log T(\zeta)\Big|_{\zeta=1} = \frac{2q}{1-q^2} H_{XXZ} + const. \tag{3.14}$$

In this way the abelian symmetries described in 3.1 emerge from the Yang-Baxter equation.

We remark that in exactly the same way as the column transfer matrix $T_{\text{col}}(\zeta)$ the row transfer matrices $T_{\text{row}}(\zeta)$ can be formulated. As spectral parameters enter only through the ratio, there is no essential distinction between the row and column formulation; in particular the $T_{\text{row}}(\zeta)$ also form a commutative family.

3.4 Quantum affine algebra $U_q(\widehat{sl}_2)$

We wish to elucidate the origin of the R matrix in the framework of the representation theory of quantum groups. To us the relevant quantum group is the quantum affine algebra $U = U_q(\widehat{sl}_2)$. For an introductory guide to the subject the reader is referred e.g. to [36]. Let us recall below some basic notions.

Consider a free abelian group on the letters Λ_0, Λ_1, δ:

$$P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta.$$

We call P the weight lattice, Λ_i $(i = 0,1)$ the fundamental weights and δ the null root. Define the simple roots α_i $(i = 0,1)$ and an element ρ by

$$\alpha_0 + \alpha_1 = \delta, \quad \Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}, \quad \rho = \Lambda_0 + \Lambda_1.$$

Let (h_0, h_1, d) be an ordered basis of $P^* = \text{Hom}(P, \mathbf{Z})$ dual to $(\Lambda_0, \Lambda_1, \delta)$. We define a symmetric bilinear form $(\ ,\ ) : P \times P \to \frac{1}{2}\mathbf{Z}$ by

$$(\Lambda_0, \Lambda_0) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1,$$
$$(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0.$$

Regarding $P^* \subset P$ via this bilinear form we have the identification

$$h_0 = \alpha_0, \quad h_1 = \alpha_1, \quad d = \Lambda_0.$$

In the following definition of $U_q\left(\widehat{sl}_2\right)$ we fix a complex number $q \neq 0, \pm 1$. For definiteness we take q to be real and $-1 < q < 0$, though in most cases it is sufficient to assume that $q^n \neq 1$ for $n = 1, 2, \cdots$. We use the symbol (q-integer)

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

By definition the quantum affine algebra $U_q(\widehat{sl}_2)$ is an algebra with 1 over $\mathbf{C}$, defined on the generators e_i, f_i ($i = 0, 1$) and q^h ($h \in P^*$) and through the defining relations:

$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \\
&q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i, \\
&[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \\
&e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j), \\
&f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j).
\end{aligned}$$

Here $t_i = q^{h_i}$. We will write $U' = U_q'(\widehat{sl}_2)$ for the subalgebra of U generated by e_i, f_i, t_i ($i = 0, 1$).

We choose the following Hopf algebra structure (Δ, a, ϵ):

Coproduct

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i. \tag{3.15}$$

Antipode

$$a(q^h) = q^{-h}, \quad a(e_i) = -t_i^{-1} e_i, \quad a(f_i) = -f_i t_i.$$

Counit

$$\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0.$$

For completeness we list the axioms for these maps, see e.g. [71, 2, 36].

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \epsilon(xy) = \epsilon(x)\epsilon(y), \quad a(xy) = a(y)a(x),$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{3.16}$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{3.17}$$

$$m \circ (a \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes a) \circ \Delta. \tag{3.18}$$

Here $m(x \otimes y) = xy$ denotes the multiplication map. It follows that

$$\Delta a(x) = (a \otimes a)\, \Delta'(x) \tag{3.19}$$

where $\Delta'(x) = \sigma \circ \Delta(x)$, $\sigma(a \otimes b) = b \otimes a$. In the case of our U the following property also holds:

$$a^2(x) = q^{-2\rho} x q^{2\rho} \qquad \forall x \in U. \tag{3.20}$$

Hereafter we will consider only left modules unless otherwise stated explicitly. For a U-module M the weight space is defined by

$$M_\nu = \{\, v \in M \mid q^h\, v = q^{\langle h,\nu\rangle}\, v \,\}$$

where ν is a $\mathbf{C}$-linear form on $\mathbf{C}h_0 \oplus \mathbf{C}h_1 \oplus \mathbf{C}d$. In these lectures we shall consider only *weight modules*, i.e. those which are the direct sums of weight spaces: $M = \oplus_\nu M_\nu$. Though the elements $h \in P$ are not in the algebra U, they make sense as operators on weight modules.

There are two important classes of representations of U:

(1) Highest weight modules,

(2) Evaluation modules.

The highest weight modules will be important in the description of the space of states. We will discuss it in Chapter 5. Here let us consider the evaluation modules which are relevant to the R matrix.

The algebra U has the standard Hopf subalgebra $U_q(sl_2)$ generated by e_1, f_1, t_1. The evaluation modules are constructed from finite-dimensional modules of $U_q(sl_2)$ by introducing spectral parameters (in the language of [44] it is the 'affinization'). Let us consider the example of $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$ which is a $U_q(sl_2)$-module with the action

$$\begin{aligned} e_1 v_+ &= 0, \quad e_1 v_- = v_+, \\ f_1 v_+ &= v_-, \quad f_1 v_- = 0, \\ t_1 v_\pm &= q^{\pm 1} v_\pm. \end{aligned}$$

Let ζ be an indeterminate, and consider

$$\begin{aligned} V_\zeta &= V \otimes \mathbf{C}[\zeta, \zeta^{-1}] = V_\zeta^{(+)} \oplus V_\zeta^{(-)}, \\ V_\zeta^{(\pm)} &= \mathrm{span}\,\{v_\pm \otimes \zeta^{2n}, v_\mp \otimes \zeta^{2n-1}\ (n \in \mathbf{Z})\}. \end{aligned}$$

We equip V_ζ with a U-module structure by setting

$$e_0(v_\varepsilon \otimes \zeta^m) = (f_1 v_\varepsilon) \otimes \zeta^{m+1}, \quad e_1(v_\varepsilon \otimes \zeta^m) = (e_1 v_\varepsilon) \otimes \zeta^{m+1},$$
$$f_0(v_\varepsilon \otimes \zeta^m) = (e_1 v_\varepsilon) \otimes \zeta^{m-1}, \quad f_1(v_\varepsilon \otimes \zeta^m) = (f_1 v_\varepsilon) \otimes \zeta^{m-1},$$
$$t_0 = t_1^{-1}, \quad t_1(v_\varepsilon \otimes \zeta^m) = (t_1 v_\varepsilon) \otimes \zeta^m.$$

The action of q^d is fixed by demanding $q^d(v_\pm \otimes \zeta^0) = v_\pm \otimes \zeta^0$. In general we have

$$q^d(v_\varepsilon \otimes \zeta^m) = q^{m/2+(\pm 1-\varepsilon)/4} v_\varepsilon \otimes \zeta^m \qquad \text{for } v_\varepsilon \otimes \zeta^m \in V_\zeta^{(\pm)}.$$

In view of the relation

$$\rho = 2d + \frac{1}{2}h_1$$

this amounts to setting

$$\rho = \zeta\frac{d}{d\zeta} \pm \frac{1}{2} \qquad \text{on } V_\zeta^{(\pm)}.$$

Each of $V_\zeta^{(\pm)}$ is irreducible under U. If we consider the multiplication by ζ in addition to the action of U, then V_ζ itself is irreducible. Henceforth we will call V_ζ the evaluation module (more precisely, the one associated with the two-dimensional module V). The element $t_0 t_1$ is in the center of U. A U-module M is said to be of level k if $t_0 t_1$ acts as a scalar q^k on M. Hence, by the definition, the evaluation modules have level 0.

We remark that, by taking ζ to be a nonzero complex number, one can also define a U'-module on V by the same formula as above except the action of q^d.

3.5 R matrix as an intertwiner

Recall that for a Hopf algebra one can define the tensor product of two representations using the coproduct. In the sequel, for an element $x \in U$, we shall often write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \tag{3.21}$$

as an abbreviation for the expression of the form $\Delta(x) = \sum_i x_i' \otimes x_i''$ (cf. the sigma notation [71]). Given U-modules M_1, M_2 we define the action of $x \in U$ on $M_1 \otimes M_2$ by

$$x(v_1 \otimes v_2) = \sum x_{(1)} v_1 \otimes x_{(2)} v_2 \tag{3.22}$$

where we used (3.21). The co-associativity axiom (3.16) implies that for any three modules M_1, M_2, M_3 one can canonically identify

$$(M_1 \otimes M_2) \otimes M_3 \simeq M_1 \otimes (M_2 \otimes M_3).$$

On the other hand, it is not obvious whether or not $M_1 \otimes M_2$ is isomorphic to $M_2 \otimes M_1$.[1] This is because the coproduct (3.15) is not symmetric under switching the tensor components.

Now let us consider this question for the evaluation modules V_ζ. Namely we compare

$$V_{\zeta_1} \otimes V_{\zeta_2} \quad \text{and} \quad V_{\zeta_2} \otimes V_{\zeta_1}. \tag{3.23}$$

Suppose there exists an intertwiner for (3.23), that is, a linear operator $\check{R} : V_{\zeta_1} \otimes V_{\zeta_2} \to V_{\zeta_2} \otimes V_{\zeta_1}$ commuting with the action of U and multiplication by ζ_1, ζ_2. This condition is equivalent to a set of linear equations for $\check{R}$

$$\check{R}\ \Delta(x) = \Delta(x)\ \check{R} \qquad \forall x \in U. \tag{3.24}$$

Solving them one finds that up to scalar there is a unique solution

$$\check{R}(\zeta_1/\zeta_2) = P\, R(\zeta_1/\zeta_2) \tag{3.25}$$

where $R(\zeta_1/\zeta_2)$ denotes the R matrix as given by the formula (3.6) and $P : V_{\zeta_1} \otimes V_{\zeta_2} \to V_{\zeta_2} \otimes V_{\zeta_1}$ is the transposition as explained in 2.4. Thus the R matrix arises as the intertwiner giving the equivalence of (3.23).

Let us show that the YBE follows automatically. In terms of the $\check{R}$ matrix, the YBE is equivalent to the relation

$$\begin{aligned} &\left(\check{R}(\zeta_2/\zeta_3) \otimes \mathrm{id}\right) \left(\mathrm{id} \otimes \check{R}(\zeta_1/\zeta_3)\right) \left(\check{R}(\zeta_1/\zeta_2) \otimes \mathrm{id}\right) \\ &= \left(\mathrm{id} \otimes \check{R}(\zeta_1/\zeta_2)\right) \left(\check{R}(\zeta_1/\zeta_3) \otimes \mathrm{id}\right) \left(\mathrm{id} \otimes \check{R}(\zeta_2/\zeta_3)\right), \end{aligned}$$

which states the equality between the following two maps given by composition

$$\begin{aligned} A \ : \ & V_{\zeta_1} \otimes V_{\zeta_2} \otimes V_{\zeta_3} \overset{\check{R}(\zeta_1/\zeta_2)\otimes \mathrm{id}}{\longrightarrow} V_{\zeta_2} \otimes V_{\zeta_1} \otimes V_{\zeta_3} \\ & \overset{\mathrm{id}\otimes \check{R}(\zeta_1/\zeta_3)}{\longrightarrow} V_{\zeta_2} \otimes V_{\zeta_3} \otimes V_{\zeta_1} \overset{\check{R}(\zeta_2/\zeta_3)\otimes \mathrm{id}}{\longrightarrow} V_{\zeta_3} \otimes V_{\zeta_2} \otimes V_{\zeta_1} \end{aligned}$$

and

$$\begin{aligned} B \ : \ & V_{\zeta_1} \otimes V_{\zeta_2} \otimes V_{\zeta_3} \overset{\mathrm{id}\otimes \check{R}(\zeta_2/\zeta_3)}{\longrightarrow} V_{\zeta_1} \otimes V_{\zeta_3} \otimes V_{\zeta_2} \\ & \overset{\check{R}(\zeta_1/\zeta_3)\otimes \mathrm{id}}{\longrightarrow} V_{\zeta_3} \otimes V_{\zeta_1} \otimes V_{\zeta_2} \overset{\mathrm{id}\otimes \check{R}(\zeta_1/\zeta_2)}{\longrightarrow} V_{\zeta_3} \otimes V_{\zeta_2} \otimes V_{\zeta_1}. \end{aligned}$$

---

[1] In fact it is not true in general, e.g. if q is a root of 1.

Clearly both A and B commute with the action of U. On the other hand it is known that the tensor products $V_{\zeta_1} \otimes \cdots \otimes V_{\zeta_m}$ are irreducible for all m [17] (note that the ζ_j are independent indeterminates). Hence any two intertwiners between them must be unique up to scalar, and we have $A = c \times B$. Since both send $v_+ \otimes v_+ \otimes v_+$ to itself times a common scalar, we must have $c = 1$. This proves YBE.

Later we shall mainly quote the intertwining property (3.24) as that of the R matrix rather than $\check{R}$, which means the commutativity of the diagram

$$\begin{array}{ccc} V_{\zeta_1} \otimes V_{\zeta_2} & \stackrel{x_{(1)} \otimes x_{(2)}}{\longrightarrow} & V_{\zeta_1} \otimes V_{\zeta_2} \\ \downarrow R(\zeta_1/\zeta_2) & & \downarrow R(\zeta_1/\zeta_2) \\ V_{\zeta_1} \otimes V_{\zeta_2} & \stackrel{x_{(2)} \otimes x_{(1)}}{\longrightarrow} & V_{\zeta_1} \otimes V_{\zeta_2} \end{array} \tag{3.26}$$

In the first line the action of $x \in U$ is via the coproduct (3.21), and in the last line via the opposite coproduct

$$\Delta'(x) = \sigma\Delta(x) = \sum x_{(2)} \otimes x_{(1)}, \qquad \sigma(x \otimes y) = y \otimes x.$$

Often the summation symbol is omitted in the diagram.

3.6 Dual modules and crossing symmetry

The other properties of the R matrix (3.8), (3.9), (3.10) also have natural meaning. The initial condition (3.8) simply states that up to scalar the identity is the unique intertwiner for $V_\zeta \otimes V_\zeta$ into itself. Likewise the unitarity (3.9) means that the composition $V_{\zeta_1} \otimes V_{\zeta_2} \to V_{\zeta_2} \otimes V_{\zeta_1} \to V_{\zeta_1} \otimes V_{\zeta_2}$ is proportional to the identity. The crossing symmetry (3.10) is related to the dual module as we shall explain below.

Let M be a left U-module. The dual space M^* of M is naturally a *right* U-module by

$$\langle v^* x, v\rangle = \langle v^*, xv\rangle.$$

If ϕ is an anti-automorphism of the algebra U, a left U-action on M^* is given by

$$\langle xv^*, v\rangle = \langle v^*, \phi(x)v\rangle. \tag{3.27}$$

The dual space with this action is denoted $M^{*\phi}$. Suppose further that ϕ is an anti-coalgebra homomorphism in the sense

$$\Delta\phi(x) = (\phi \otimes \phi)\,\Delta'(x). \tag{3.28}$$

Then we have the canonical isomorphism of U-modules

$$(M_1 \otimes M_2)^{*\phi} \simeq M_2^{*\phi} \otimes M_1^{*\phi}.$$

A convenient choice for ϕ is the antipode $\phi = a$. (See (3.19).) In this case we have the canonical identification

$$\begin{aligned} \mathrm{Hom}_U(L, M \otimes N) &= \mathrm{Hom}_U(M^{*a} \otimes L, N), && (3.29) \\ \mathrm{Hom}_U(L \otimes N, M) &= \mathrm{Hom}_U(L, M \otimes N^{*a}). && (3.30) \end{aligned}$$

Of course all that have been discussed apply equally to U'-modules.

Now we consider the dual of the evaluation module V_ζ. Let $V^* = \mathbf{C}v_+^* \oplus \mathbf{C}v_-^*$ be the dual space equipped with the dual basis $\langle v_\varepsilon^*, v_{\varepsilon'} \rangle = \delta_{\varepsilon\varepsilon'}$. We regard

$$\begin{aligned} V_\zeta^{*a} &= V^* \otimes \mathbf{C}[\zeta, \zeta^{-1}] = V_\zeta^{(+)*a} \oplus V_\zeta^{(-)*a}, \\ V_\zeta^{(\pm)*a} &= \mathrm{span}\,\{v_\pm^* \otimes \zeta^{2n}, v_\mp^* \otimes \zeta^{2n-1}\ (n \in \mathbf{Z})\} \end{aligned}$$

as the dual of V_ζ via the pairing

$$\langle v_\varepsilon^* \otimes \zeta^m, v_{\varepsilon'} \otimes \zeta^n \rangle = \delta_{\varepsilon\varepsilon'} \delta_{m+n,0}$$

and the U-module structure defined by the antipode a. Notice that we have

$$\rho = \zeta \frac{d}{d\zeta} \mp \frac{1}{2} \qquad \text{on} \quad V_\zeta^{(\pm)*a}.$$

It turns out that the following is an isomorphism of U-modules:

$$\begin{aligned} V_{-q^{-1}\zeta}^{(\pm)} &\xrightarrow{\sim} V_\zeta^{(\mp)*a}, && (3.31) \\ v_\varepsilon \otimes \zeta^n &\mapsto v_{-\varepsilon}^* \otimes \zeta^n. && (3.32) \end{aligned}$$

For instance, in $V_{-q^{-1}\zeta}^{(\pm)}$, we have

$$f_0(v_- \otimes \zeta^n) = -qv_+ \otimes \zeta^{n-1},$$

while in $V_\zeta^{(\mp)*a}$

$$\begin{aligned} \langle f_0 v_+^* \otimes \zeta^n, v_- \otimes \zeta^m \rangle &= \langle v_+^* \otimes \zeta^n, -f_0 t_0 v_- \otimes \zeta^m \rangle \\ &= \langle v_+^* \otimes \zeta^n, -qv_+ \otimes \zeta^{m-1} \rangle \\ &= -q\delta_{n+m-1,0} \\ &= \langle -qv_-^* \otimes \zeta^{n-1}, v_- \otimes \zeta^m \rangle, \end{aligned}$$

showing $f_0(v_+^* \otimes \zeta^n) = -qv_-^* \otimes \zeta^{n-1}$.

In a similar manner we have

$$\begin{array}{ccc} V_{-q\zeta}^{(\pm)} & \xrightarrow{\sim} & V_{\zeta}^{(\mp)*a^{-1}}, \\ v_\epsilon \otimes \zeta^n & \mapsto & v_{-\epsilon}^* \otimes \zeta^n. \end{array}$$

Now consider the intertwiner

$$R(\zeta_1/\zeta_2)^{-1}P \ : \ V_{\zeta_2} \otimes V_{\zeta_1} \to V_{\zeta_1} \otimes V_{\zeta_2}.$$

Write $R(\zeta_1/\zeta_2)^{-1}$ in the form $\sum \varphi_i \otimes \psi_i$ with $\varphi_i \in \mathrm{End}\,(V_{\zeta_1})$ and $\psi_i \in \mathrm{End}\,(V_{\zeta_2})$. Define

$$\left(R(\zeta_1/\zeta_2)^{-1}\right)^{t_1} = \sum \varphi_i^t \otimes \psi_i$$

where $\varphi_i^t \in \mathrm{End}\,(V_{\zeta_1}^{*a})$ denotes the transpose of φ_i:

$$\langle \varphi_i^t(v^*), v\rangle = \langle v^*, \varphi_i(v)\rangle \qquad \text{for } v^* \in V_{\zeta_1}^{*a},\ v \in V_{\zeta_1}.$$

Define $Q : V \to V^*$ by

$$Qv_\pm = v_\mp^*.$$

Thanks to (3.29) and (3.30), $P\left(R(\zeta_1/\zeta_2)^{-1}\right)^{t_1}$ gives an intertwiner

$$V_{\zeta_1}^{*a} \otimes V_{\zeta_2} \longrightarrow V_{\zeta_2} \otimes V_{\zeta_1}^{*a}.$$

Combining this with (3.32) and comparing the two possible intertwiners

$$V_{-q^{-1}\zeta_1} \otimes V_{\zeta_2} \overset{Q\otimes \mathrm{id}}{\rightarrow} V_{\zeta_1}^{*a} \otimes V_{\zeta_2} \longrightarrow V_{\zeta_2} \otimes V_{\zeta_1}^{*a} \overset{\mathrm{id}\otimes Q^{-1}}{\rightarrow} V_{\zeta_2} \otimes V_{-q^{-1}\zeta_1},$$

we arrive at the crossing symmetry

$$R(-q^{-1}\zeta_1/\zeta_2) = (\text{scalar}) \times (\mathrm{id} \otimes Q^{-1})\left(R(\zeta_1/\zeta_2)^{-1}\right)^{t_1}(Q \otimes \mathrm{id}).$$

We remark that this sort of functorial properties can be best described by the universal R matrix [24].

3.7 Abelian and non-abelian Symmetries

Our discussions so far can be summarized by the following scheme:

$$U_q(\widehat{sl}_2) \overset{3.5}{\Longrightarrow} \text{YBE} \overset{3.3}{\Longrightarrow} \text{Commuting Transfer Matrices.}$$

If you wish the last item can be called *abelian symmetries* of the Hamiltonian H_{XXZ};

$$[H_n, H_{XXZ}] = 0 \quad \forall n. \tag{3.33}$$

There is a well established technology for studying the eigenvalues of the transfer matrix on the basis of the YBE and the abelian symmetries. The Bethe Ansatz method makes it possible to write down all the eigenvalues and eigenvectors in a certain specific form, using solutions to a system of algebraic equations (the Bethe Ansatz equations) (e.g. [13, 49]). Alternatively one can derive functional relations satisfied by the transfer matrices as functions of the spectral parameter [13]. Because of the commutativity of the transfer matrices all the eigenvalues satisfy the same functional relations. In either case the equations are exact on a finite (usually periodic) lattice. The main problem in these approaches is to handle the Bethe Ansatz equations in the infinite lattice limit where the number of roots become also infinite. Such calculations have been carried through for numerous models including the six-vertex/XXZ models as very special cases, and the partition function per site and the 'elementary excitations' have been found exactly. Usually when these quantities are obtained the model is regarded as being 'solved'. However this is not the end of the story. We would like to go further to clarify the structure of the eigenvectors as a mathematical entity, and ultimately that of the correlation functions. We wish to reserve the word 'solved' till we understand these aspects.

We know that building a solvable model amounts to finding a solution to the YBE. From the quantum group point of view, the R matrices are nothing but the intertwiners between tensor products of evaluation modules. Thus there is more than a good reason to expect that quantum groups should play a vital role in 'solving' the models thus constructed. In this connection let us note the following point. Consider the monodromy matrix $\mathcal{T}(\zeta)$ (2.10)

$$\mathcal{T}(\zeta) = R_{01}(\zeta) \cdots R_{0N}(\zeta), \qquad (\zeta = \zeta_V/\zeta_H),$$

acting on $V_{0,\zeta_V} \otimes V_{N,\zeta_H} \otimes \cdots \otimes V_{1,\zeta_H}$. Set $\Delta^{(n)} = (\Delta \otimes \mathrm{id})\Delta^{(n-1)}$, $\Delta^{(1)} = \Delta$, and for $x \in U$ write

$$\Delta^{(N)}(x) = \sum x_{(0)} \otimes x_{(N)} \otimes \cdots \otimes x_{(1)}.$$

From (3.26) we have the commutative diagram

$$\begin{array}{ccc} V_0 \otimes V_N \otimes \cdots \otimes V_1 & \stackrel{x_{(0)} \otimes x_{(N)} \otimes \cdots \otimes x_{(1)}}{\longrightarrow} & V_0 \otimes V_N \otimes \cdots \otimes V_1 \\ \downarrow \mathcal{T}(\zeta) & & \downarrow \mathcal{T}(\zeta) \\ V_0 \otimes V_N \otimes \cdots \otimes V_1 & \stackrel{x_{(N)} \otimes x_{(N-1)} \otimes \cdots \otimes x_{(0)}}{\longrightarrow} & V_0 \otimes V_N \otimes \cdots \otimes V_1 \end{array}$$

In the large lattice limit, neglecting the boundary effects we would conclude that $T(1)^{-1}T(\zeta)$ acting on

$$\cdots \otimes V_{k+1} \otimes V_k \otimes V_{k-1} \otimes \cdots$$

commutes with the action of U' given by

$$\Delta^{(\infty)}(x) = \sum \cdots \otimes x_{(k+1)} \otimes x_{(k)} \otimes x_{(k-1)} \otimes \cdots.$$

In particular the XXZ Hamiltonian would commute with the action of U;

$$[\Delta^{(\infty)}(x), H_{XXZ}] = 0.$$

This would mean that the algebra U plays the role of infinite dimensional *non-abelian symmetries* in contrast to the abelian symmetries (3.33)

Chapter 4

Correlation functions—physical derivation

We are now in a position to introduce the central object in our approach — the *vertex operators*. We wish to explain how they naturally arise when we study the correlation functions via Baxter's corner transfer matrix method, and how they can be used to derive difference equations for the correlation functions. The arguments presented in this Chapter will rely on some physical intuition. They are meant to motivate the mathematical constructions to be developed later.

4.1 Corner Transfer Matrix

First recall that we are working in the region $a, b, c > 0$ and $\Delta < -1$. In the parameterization (3.4) this means

$$-1 < q < 0, \quad 1 < \zeta < (-q)^{-1}. \tag{4.1}$$

We shall discuss the correlation functions in this region.

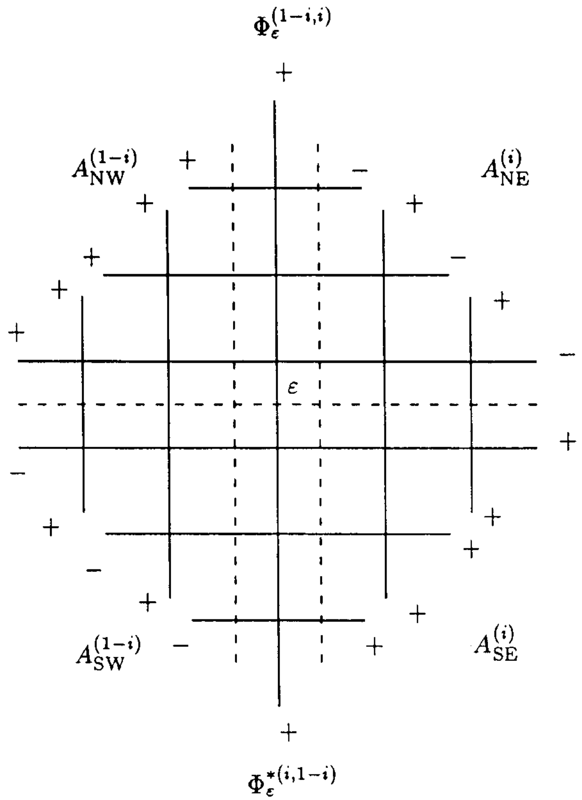

Figure 4.1: Subdivision of the lattice into quadrants

To simplify the presentation, let us first concentrate on the simplest case of the one point correlation function discussed in (1.5). See also (2.8). Consider an edge '0' such that $\varepsilon_0 = (-1)^{i+1}$ in the i-th ground state ($i = 0, 1$). Then the one-point function as defined in (2.7), (2.8) is

$$\langle \varepsilon_0 \rangle_i = P_+^{(i)} - P_-^{(i)},$$

where $P_\varepsilon^{(i)}$ ($\varepsilon = +$ or $-$) denotes the probability that the spin on the edge 0 takes the value ε in the i-th ground state sector. The quantity we are interested in is thus

$$P_\varepsilon^{(i)} = \frac{\sum'_C \prod_v R^{\varepsilon'_1(C,v)\,\varepsilon'_2(C,v)}_{\varepsilon_1(C,v)\,\varepsilon_2(C,v)}}{\sum_C \prod_v R^{\varepsilon'_1(C,v)\,\varepsilon'_2(C,v)}_{\varepsilon_1(C,v)\,\varepsilon_2(C,v)}}. \tag{4.2}$$

The sum in the numerator ranges over all the configurations C such that $\varepsilon_0(C) = \varepsilon$. In addition, the sums are restricted to the i-th sector as explained in 2.2.

An alternative and equivalent way is to define (4.2) as the limit of a similar ratio with the sum over configurations taken on the finite lattice, wherein the spins on the boundary of the lattice are fixed to the i-th ground state configuration. When we pass to the infinite lattice limit, we expect that the detailed shape of the lattice should not matter in so far as the edge 0 is kept 'deep inside'. For our purposes it is convenient to choose a lattice of the shape depicted in fig.4.1.

We will label the horizontal lines from bottom to top as $-N+1, \cdots, N$ where now $2N$ is the total number of lines. Similarly we label the vertical lines from right to left as $-N, \cdots, N$. Let us subdivide this lattice into 6 pieces, consisting of four corners and two half columns, separated by the 'seams' indicated by broken lines in the figure. The idea is to perform the sum (4.2) in two steps, first over the spins not lying on the seams, and then over the ones on the seams.

To this end, following Baxter we introduce the *corner transfer matrix* (CTM) associated with each corner. As an example look at the NW quadrant (see fig.4.2). On the NE boundary the spins are fixed to the $(1-i)$-th ground state. In fig.4.2, they are set to $+$. Let N be the number of rows in the corner. Fixing a configuration $\mathbf{p} = (p_1, \cdots, p_N)$ of the horizontal boundary spins and $\mathbf{p}' = (p'_1, \cdots, p'_N)$ for the vertical one, we consider the sum over the spin configurations in the interior of this corner

$$\left(A_{NW}^{(1-i)}\right)^{p'_N,\cdots,p'_1}_{p_N,\cdots,p_1} = \sum_{\text{interior edges}} \prod R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}. \tag{4.3}$$

By definition $A_{NW}^{(1-i)}$ is a 2^N by 2^N matrix whose $(\mathbf{p},\mathbf{p}')$-entry is given by the sum (4.3).

The other CTM's $A_{SW}^{(1-i)}$, $A_{SE}^{(i)}$, $A_{NE}^{(i)}$ are defined in a similar way.

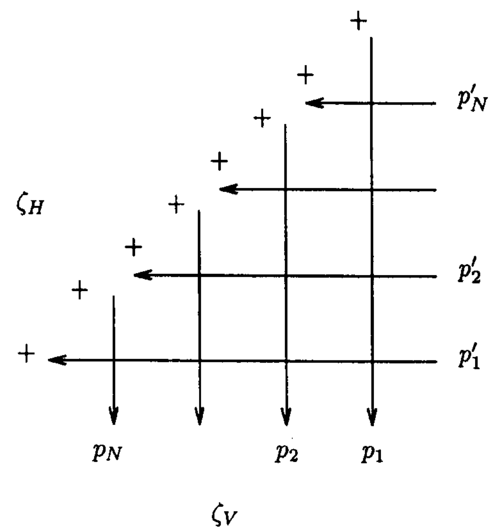

Figure 4.2: Corner transfer matrix $A_{NW}^{(1-i)}$

In the same spirit, define a half-column transfer matrix corresponding to the upper half column by

$$\left(\Phi_{UP\varepsilon}^{(1-i\,i)}\right)_{p_N,\cdots,p_1}^{p'_N,\cdots,p'_1} = \sum_{\nu_1,\cdots,\nu_{N-1}} \prod_{j=1}^{N} R_{\nu_{j-1}p_j}^{\nu_j\ p'_j} \tag{4.4}$$

where $\nu_0 = \varepsilon$ and $\nu_N = (-1)^{N+1-i}$. Similarly define

$$\left(\Phi_{LOW\varepsilon}^{(i\,1-i)}\right)_{p_0,\cdots,p_{-N+1}}^{p'_0,\cdots,p'_{-N+1}} = \sum_{\nu_{-1},\cdots,\nu_{-N+1}} \prod_{j=-N+1}^{0} R_{\nu_{j-1}p'_j}^{\nu_j\ p_j} \tag{4.5}$$

where $\nu_0 = \varepsilon$ and $\nu_{-N} = (-1)^{N+1-i}$. In anticipation of the representation theoretical interpretation, we call them *vertex operators* (VO's). Later we shall introduce a different type of VO's related to the particle creation and annihilation operators. To make distinction we shall also refer to the VO's introduced here as VO's of *type I*.

All these operators in fact depend on the spectral parameters through the ratio $\zeta = \zeta_V/\zeta_H$. In the sequel this dependence will be exhibited as $A_{NW}^{(1-i)}(\zeta)$,

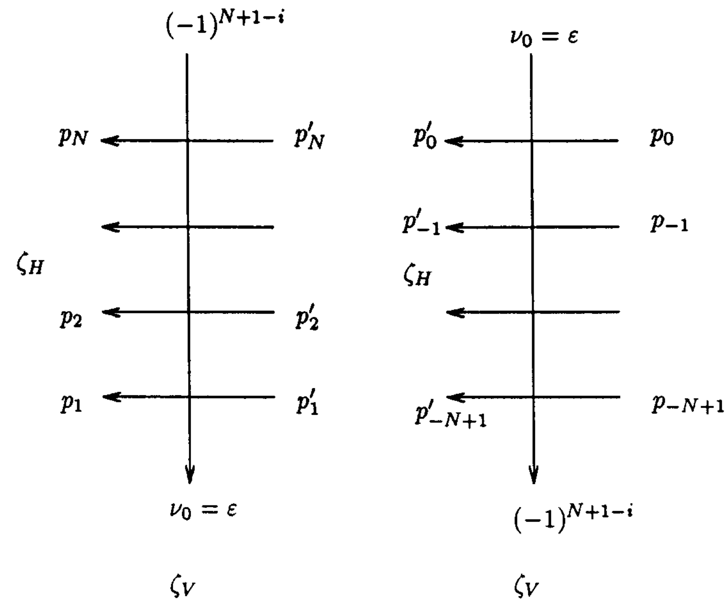

Figure 4.3: Vertex operators $\Phi^{(1-i\,i)}_{UP\varepsilon}$ (left) and $\Phi^{(i\,1-i)}_{LOW\varepsilon}$ (right)

etc. The crossing symmetry for the R matrix entails relations among these operators. Writing $A^{(i)}(\zeta) = A^{(i)}_{NW}(\zeta)$ and $\Phi^{(1-i,i)}_{\varepsilon}(\zeta) = \Phi^{(1-i,i)}_{UP\varepsilon}(\zeta)$ we have

$$A^{(1-i)}_{SW}(\zeta) = \mathcal{R}A^{(1-i)}(-q^{-1}\zeta^{-1}), \tag{4.6}$$

$$A^{(i)}_{SE}(\zeta) = \mathcal{R}A^{(i)}(\zeta)\mathcal{R}, \tag{4.7}$$

$$A^{(i)}_{NE}(\zeta) = A^{(i)}(-q^{-1}\zeta^{-1})\mathcal{R}, \tag{4.8}$$

$$\Phi^{(i\,1-i)}_{LOW\varepsilon}(\zeta) = \mathcal{R}\Phi^{(i,1-i)}_{-\varepsilon}(\zeta)\mathcal{R}. \tag{4.9}$$

Here $\mathcal{R} = \sigma^x \otimes \cdots \otimes \sigma^x$ signifies the spin reversal operator $v_{\varepsilon_N} \otimes \cdots \otimes v_{\varepsilon_1} \mapsto v_{-\varepsilon_N} \otimes \cdots \otimes v_{-\varepsilon_1}$.

Having introduced these fancy operators, one might wonder if they are of any use at all. It turns out, however, that a remarkable simplification takes place in the infinite lattice limit. The heart of the matter is Baxter's discovery on the CTM, which we now explain. Multiplying a scalar factor let us normalize $A^{(i)}(\zeta)$ to make its largest eigenvalue equal to 1. Then the statement is that in the infinite lattice limit we have

$$\lim A^{(i)}(\zeta) = \zeta^{-D^{(i)}}, \tag{4.10}$$

where $D^{(i)}$ is an operator independent of ζ and has a discrete spectrum (recall that we are working in the region $\zeta > 1$)

$$\mathrm{Spec}(D^{(i)}) = \{0, 1, 2, \cdots\}. \tag{4.11}$$

Baxter's argument leading to (4.10–4.11) is based on the Yang-Baxter equation and is described in Chapter 13 of [13]. Apparently he was led to find these properties by observing a drastic simplification in the low temperature series expansion of the CTM [9, 10]. We wish to emphasize here that such a low temperature series makes sense only in the 'massive regime', and that working in this regime is essential. Otherwise, even the very existence of the CTM in the infinite lattice limit (4.10) would become obscure.

The limiting CTM (4.10) is an operator on the space $\mathcal{H}^{(i)}$ spanned by its eigenvectors. In the same way we expect that the VO also tends to some well defined operator (which we denote by the same letter)

$$\Phi_\varepsilon^{(1-i,i)}(\zeta) \; : \; \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(1-i)}.$$

Notice that unlike the CTM's they carry one space to another because of the change in the boundary conditions.

4.2 Properties of Vertex Operators

Let us examine more closely the structure which emerged in the previous sections. The basic ingredients are the spaces $\mathcal{H}^{(i)}$ and the operators $D^{(i)}$, $\Phi^{(1-i,i)}(\zeta)$.

Naïvely one can think of the space $\mathcal{H}^{(i)}$ as the limit of a subspace of the N-fold tensor product $V^{\otimes N}$. In this picture $\mathcal{H}^{(i)}$ is spanned by *half-infinite* pure tensor vectors

$$\cdots \otimes v_{p(3)} \otimes v_{p(2)} \otimes v_{p(1)}, \qquad p(j) = (-)^{j+i} \;\; (j \gg 0).$$

The ground state in this sector corresponds to the sequence $p(j) = (-1)^{j+i}$ $(\forall j = 1, 2, \cdots)$.

The discreteness of the spectrum (4.11) means that $\mathcal{H}^{(i)}$ is a $\mathbf{Z}$-graded vector space:

$$\mathcal{H}^{(i)} = \oplus_{r\in\mathbf{Z}}\mathcal{H}^{(i)}_r, \qquad \mathcal{H}^{(i)}_r = \{v \in \mathcal{H}^{(i)} \mid D^{(i)}v = r\,v\}.$$

As it turns out the eigenvalues r are highly degenerate. Their multiplicities $\dim\mathcal{H}^{(i)}_r$ can be determined by Baxter's method. The relevant calculation can be found in [13], Sect.13.7, where the more general case of the eight-vertex model is treated rather than the six-vertex model. The result is common to both cases and is given by

$$\mathrm{tr}_{\mathcal{H}^{(i)}}(t^{D^{(i)}}) \equiv \sum_{r\in\mathbf{Z}} \dim\mathcal{H}^{(i)}_r\, t^r = \prod_{n=1}^{\infty}\frac{1}{1-t^{2n-1}}. \tag{4.12}$$

The formula (4.12) tells in particular $\dim\mathcal{H}^{(i)}_0 = 1$, which means that up to a scalar there is a unique eigenvector belonging to the maximum eigenvalue of the CTM. In the sequel we fix a nonzero $u^{(i)} \in \mathcal{H}^{(i)}_0$ and refer to it as the highest weight vector.

For a systematic treatment it is often useful to use the language of R matrices. Set

$$\widehat{R}(\zeta) = R(\zeta)P.$$

Neglecting the boundary conditions one can write the CTM as

$$\begin{aligned} A^{(i)}(\zeta) &= (\widehat{R}_{N\,N+1}(\zeta))(\widehat{R}_{N-1\,N}(\zeta)\widehat{R}_{N\,N+1}(\zeta))\cdots \\ &\quad\times(\widehat{R}_{12}(\zeta)\widehat{R}_{23}(\zeta)\cdots\widehat{R}_{N\,N+1}(\zeta)). \end{aligned} \tag{4.13}$$

Expanding (4.13) to the first order we obtain the formal expression (up to an additive constant)

$$D^{(i)} = h_{12} + 2h_{23} + 3h_{34} + \cdots$$

where h is the density of the Hamiltonian (3.13). (See 2.4. for the notations $h_{12}, h_{23}, \cdots$.) This characteristic form for the 'CTM Hamiltonian' $D^{(i)}$ was noted in [9, 10].

Next let us examine the VO. In terms of the R matrix the VO can be written as

$$(\Phi^{(1-i,i)}_\varepsilon(\zeta))_{\mathbf{p}\mathbf{p}'} = \left(\widehat{R}_{12}(\zeta)\widehat{R}_{23}(\zeta)\cdots\widehat{R}_{N\,N+1}(\zeta)\right)^{p'_1\cdots p'_N\varepsilon'}_{\varepsilon p_1\cdots p_N} \tag{4.14}$$

where $\mathbf{p} = (p_1, \cdots, p_N)$, $\mathbf{p}' = (p'_1, \cdots, p'_N)$, and we set $\varepsilon' = (-)^{N+1-i}$. From this expression some symmetries are apparent. The spin-reversal symmetry of the R matrix

$$R(\zeta)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} = R(\zeta)_{-\varepsilon_1\, -\varepsilon_2}^{-\varepsilon'_1\, -\varepsilon'_2}$$

implies

$$\Phi_\varepsilon^{(1,0)}(\zeta) = \mathcal{R}\Phi_{-\varepsilon}^{(0,1)}(\zeta)\mathcal{R}$$

where $\mathcal{R} = \sigma^x \otimes \cdots \otimes \sigma^x$. From the relation $R(-\zeta) = -(\sigma^z \otimes 1)\, R(\zeta)\, (\sigma^z \otimes 1)$ it follows that

$$\widehat{R}_{12}(-\zeta) \cdots \widehat{R}_{N\,N+1}(-\zeta) = (-1)^N \sigma_1^z\, \widehat{R}_{12}(\zeta) \cdots \widehat{R}_{N\,N+1}(\zeta)\, \sigma_{N+1}^z.$$

Hence we have the parity relation

$$\Phi_\varepsilon^{(1-i,i)}(-\zeta) = (-1)^{1-i} \varepsilon\, \Phi_\varepsilon^{(1-i,i)}(\zeta). \tag{4.15}$$

Let us 'derive' the main properties of VO's. First consider fig 4.4. It shows the effect of a successive application of the Yang-Baxter equation. Ignoring the crossing in the left hand side, which represents the R-matrix, as if it disappears to infinity, we get

$$\Phi_{\varepsilon_2}^{(i,1-i)}(\zeta_2)\Phi_{\varepsilon_1}^{(1-i,i)}(\zeta_1) = \sum_{\varepsilon'_1, \varepsilon'_2} R(\zeta_1/\zeta_2)_{\varepsilon_1 \varepsilon_2}^{\varepsilon'_1 \varepsilon'_2} \Phi_{\varepsilon'_1}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon'_2}^{(1-i,i)}(\zeta_2). \tag{4.16}$$

This argument does not tell why the scalar multiple of R has to be chosen as we did. We will come back to this point later.

Next we derive a relation between the CTM and the vertex operator. Differentiation of the YBE

$$\widehat{R}_{j\,j+1}(\zeta)\widehat{R}_{j+1\,j+2}(\xi\zeta)\widehat{R}_{j\,j+1}(\xi) = \widehat{R}_{j+1\,j+2}(\xi)\widehat{R}_{j\,j+1}(\xi\zeta)\widehat{R}_{j+1\,j+2}(\zeta)$$

at $\xi = 1$ yields

$$\begin{aligned} &\widehat{R}_{j\,j+1}(\zeta)\widehat{R}_{j+1\,j+2}(\zeta)h_{j\,j+1} - h_{j+1\,j+2}\widehat{R}_{j\,j+1}(\zeta)\widehat{R}_{j+1\,j+2}(\zeta) \\ &= \zeta\frac{d\widehat{R}_{j\,j+1}(\zeta)}{d\zeta}\widehat{R}_{j+1\,j+2}(\zeta) - \widehat{R}_{j\,j+1}(\zeta)\zeta\frac{d\widehat{R}_{j+1\,j+2}(\zeta)}{d\zeta}. \end{aligned}$$

Multiply $j \times \widehat{R}_{12}(\zeta) \cdots \widehat{R}_{j-1\,j}(\zeta)$ from the left, $\widehat{R}_{j+2\,j+3}(\zeta) \cdots \widehat{R}_{N\,N+1}(\zeta)$ from the right, and sum over $j = 1, 2, \cdots$. This gives

$$\left(\widehat{R}_{12}(\zeta) \cdots \widehat{R}_{N\,N+1}(\zeta)\right)(h_{12} + 2h_{23} + \cdots + h_{N-1\,N})$$

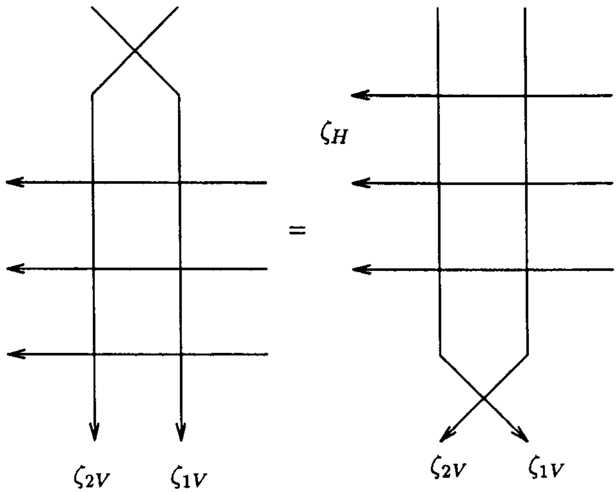

Figure 4.4: Commutation relations of the vertex operators

$$
\begin{aligned}
&- (h_{12} + 2h_{23} + \cdots + h_{N-1\,N}) \left(\widehat{R}_{12}(\zeta) \cdots \widehat{R}_{N\,N+1}(\zeta)\right) \\
&= \zeta \frac{d}{d\zeta} \left(\widehat{R}_{12}(\zeta) \cdots \widehat{R}_{N\,N+1}(\zeta)\right) + \text{boundary terms.}
\end{aligned}
$$

Comparing with (4.14) we find

$$
D^{(1-i)} \Phi_\varepsilon^{(1-i,i)}(\zeta) - \Phi_\varepsilon^{(1-i,i)}(\zeta) D^{(i)} = \zeta \frac{d}{d\zeta} \Phi_\varepsilon^{(1-i,i)}(\zeta). \tag{4.17}
$$

We have followed the argument of [73, 70, 74] where $D^{(i)}$ is referred to as the boost operator. Rewriting (4.17) in the exponentiated form, we have

$$
\xi^{-D^{(1-i)}} \circ \Phi_\varepsilon^{(1-i,i)}(\zeta) \circ \xi^{D^{(i)}} = \Phi_\varepsilon^{(1-i,i)}(\zeta/\xi), \tag{4.18}
$$

The VO's are invertible in the following sense:

$$
\sum_{\varepsilon=\pm} \Phi_{-\varepsilon}^{(i,1-i)}(-q^{-1}\zeta) \Phi_\varepsilon^{(1-i,i)}(\zeta) \;=\; \mathrm{id}\,, \tag{4.19}
$$

$$
\Phi_\varepsilon^{(1-i,i)}(\zeta) \Phi_{-\varepsilon'}^{(i,1-i)}(-q^{-1}\zeta) \;=\; \delta_{\varepsilon\varepsilon'} \times \mathrm{id}\,. \tag{4.20}
$$

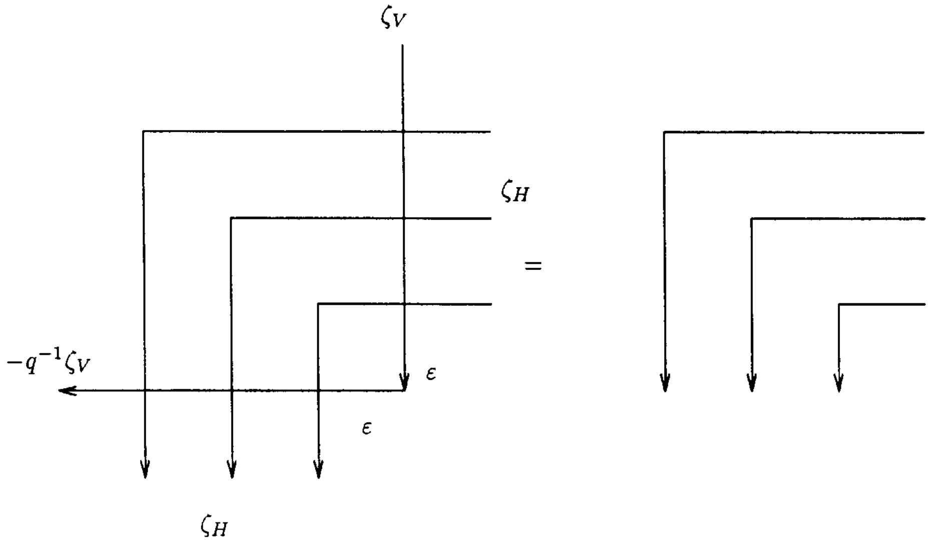

Figure 4.5: The inversion property

Notice that (4.20) follows from (4.19) and the commutation relation (4.16). To see this it suffices to set $\zeta_1 = -q^{-1}\zeta_2$ and use

$$R(-q^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The relation (4.19) is a direct consequence of the unitarity and crossing symmetry for the R matrix. By the crossing symmetry, $\Phi_{-\varepsilon}^{(i,1-i)}(-q^{-1}\zeta)$ is represented by the half-row in fig.4.5. Then the argument should be clear from fig.4.5.

Recall that we have chosen the normalizing scalar $\kappa(\zeta)$ (3.5) to satisfy (3.9) and (3.10) without introducing extra scalar factors. As we show below, the R-matrix appearing in (4.16) must satisfy these conditions as well. In fact, using (4.16) twice we must have

$$\begin{aligned}&\sum_{\varepsilon_1',\varepsilon_1'',\varepsilon_2',\varepsilon_2''} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} R(\zeta_2/\zeta_1)_{\varepsilon_2'\varepsilon_1'}^{\varepsilon_2''\varepsilon_1''} \Phi_{\varepsilon_2''}^{(i,1-i)}(\zeta_2)\Phi_{\varepsilon_1''}^{(1-i,i)}(\zeta_1)\\
&= \sum_{\varepsilon_1',\varepsilon_2'} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} \Phi_{\varepsilon_1'}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon_2'}^{(1-i,i)}(\zeta_2)\\
&= \Phi_{\varepsilon_2}^{(i,1-i)}(\zeta_2)\Phi_{\varepsilon_1}^{(1-i,i)}(\zeta_1).\end{aligned}$$

Therefore, we have (3.9). Notice further that

$$\begin{aligned}&\sum_{\varepsilon_1',\varepsilon_2,\varepsilon_2'} R(\zeta_1/\zeta_2)_{\varepsilon_1''\varepsilon_2''}^{\varepsilon_1'\varepsilon_2} R(-q^{-1}\zeta_2/\zeta_1)_{\varepsilon_1\varepsilon_2}^{-\varepsilon_1'\varepsilon_2'} \Phi_{\varepsilon_2'}^{(1-i,i)}(\zeta_2)\\
&= \sum_{\varepsilon_1',\varepsilon_1'',\varepsilon_2,\varepsilon_2'} R(\zeta_1/\zeta_2)_{\varepsilon_1''\varepsilon_2''}^{\varepsilon_1'\varepsilon_2} R(-q^{-1}\zeta_2/\zeta_1)_{\varepsilon_1\varepsilon_2}^{-\varepsilon_1''\varepsilon_2'}\\
&\times \Phi_{\varepsilon_1'}^{(1-i,i)}(\zeta_1)\Phi_{-\varepsilon_1''}^{(i,1-i)}(-q^{-1}\zeta_1)\Phi_{\varepsilon_2'}^{(1-i,i)}(\zeta_2)\\
&= \sum_{\varepsilon_1',\varepsilon_2} R(\zeta_1/\zeta_2)_{\varepsilon_1''\varepsilon_2''}^{\varepsilon_1'\varepsilon_2} \Phi_{\varepsilon_1'}^{(1-i,i)}(\zeta_1)\Phi_{\varepsilon_2}^{(i,1-i)}(\zeta_2)\Phi_{\varepsilon_1}^{(1-i,i)}(-q^{-1}\zeta_1)\\
&= \Phi_{\varepsilon_2''}^{(1-i,i)}(\zeta_2)\Phi_{\varepsilon_1''}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon_1}^{(1-i,i)}(-q^{-1}\zeta_1)\\
&= \Phi_{\varepsilon_2''}^{(1-i,i)}(\zeta_2)\delta_{\varepsilon_1'',-\varepsilon_1}.\end{aligned}$$

Therefore, we have

$$\sum_{\varepsilon_1',\varepsilon_2} R(\zeta_1/\zeta_2)_{\varepsilon_1''\varepsilon_2''}^{\varepsilon_1'\varepsilon_2} R(-q^{-1}\zeta_2/\zeta_1)_{\varepsilon_1\varepsilon_2}^{-\varepsilon_1'\varepsilon_2'} = \delta_{\varepsilon_1'',-\varepsilon_1}\delta_{\varepsilon_2',\varepsilon_2''}.$$

Because of (3.9) this is equivalent to (3.10). This gives a strong evidence that the R-matrix in (4.16) must be equal to the one given in (3.6).

If we regard $\Phi_\varepsilon^{(1-i,i)}(\zeta)$ as a formal series $\sum_{j\in\mathbf{Z}} \Phi_{\varepsilon,j}^{(1-i,i)}\zeta^{-j}$, then (4.18) tells that the coefficients are linear operators

$$\Phi_{\varepsilon,j}^{(1-i,i)} \ : \ \mathcal{H}_r^{(i)} \longrightarrow \mathcal{H}_{r-j}^{(1-i)} \qquad \forall r. \tag{4.21}$$

In particular, counting degrees and taking (4.15) into account we must have

$$\Phi_\varepsilon^{(1-i,i)}(\zeta)u^{(i)} = const.\, u^{(1-i)} + O(\zeta) \qquad \text{for } \varepsilon = (-1)^{i+1}.$$

Adjusting a scalar multiple we shall henceforth redefine the VO's to make the above constant= 1:

$$\Phi_\varepsilon^{(1-i,i)}(\zeta)u^{(i)} = u^{(1-i)} + O(\zeta) \qquad \text{for } \varepsilon = (-1)^{i+1}. \tag{4.22}$$

The invertibility then holds with id being multiplied by a certain scalar g^{-1} (see (4.28)), to be determined in the next section.

Let us summarize here the properties of the CTM's and the VO's discussed so far.

Character $\mathcal{H}^{(i)}$ is a **Z**-graded vector space with the character

$$\mathrm{tr}_{\mathcal{H}^{(i)}}\left(t^{D^{(i)}}\right) = \prod_{n=1}^{\infty} \frac{1}{1-t^{2n-1}}. \tag{4.23}$$

Homogeneity

$$\xi^{-D^{(1-i)}} \circ \Phi_\varepsilon^{(1-i,i)}(\zeta) \circ \xi^{D^{(i)}} = \Phi_\varepsilon^{(1-i,i)}(\zeta/\xi). \tag{4.24}$$

Commutation Relation

$$\Phi_{\varepsilon_2}^{(i,1-i)}(\zeta_2)\Phi_{\varepsilon_1}^{(1-i,i)}(\zeta_1) = \sum_{\varepsilon_1',\varepsilon_2'} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'} \Phi_{\varepsilon_1'}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon_2'}^{(1-i,i)}(\zeta_2). \tag{4.25}$$

Normalization Fixing $u^{(i)} \in \mathcal{H}_0^{(i)}$ we have

$$\Phi_\varepsilon^{(1-i,i)}(\zeta)u^{(i)} = u^{(1-i)} + O(\zeta) \qquad \text{for } \varepsilon = (-1)^{1-i}. \tag{4.26}$$

Invertibility

$$\begin{aligned} \sum_{\varepsilon=\pm} \Phi_{-\varepsilon}^{(i,1-i)}(-q^{-1}\zeta)\Phi_\varepsilon^{(1-i,i)}(\zeta) &= g^{-1} \times \mathrm{id}, \\ \Phi_\varepsilon^{(1-i,i)}(\zeta)\Phi_{-\varepsilon'}^{(i,1-i)}(-q^{-1}\zeta) &= g^{-1}\delta_{\varepsilon\varepsilon'} \times \mathrm{id} \end{aligned} \tag{4.27}$$

where

$$g = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{4.28}$$

Z_2 Symmetry There exists a linear isomorphism $\nu : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$ such that $\nu \circ D^{(0)} = D^{(1)} \circ \nu$, $\nu(u^{(0)}) = u^{(1)}$ and

$$\Phi_\varepsilon^{(1,0)}(\zeta) = \nu \circ \Phi_{-\varepsilon}^{(0,1)}(\zeta) \circ \nu. \tag{4.29}$$

Parity

$$\Phi_\varepsilon^{(1-i,i)}(-\zeta) = (-1)^{1-i}\varepsilon\, \Phi_\varepsilon^{(1-i,i)}(\zeta). \tag{4.30}$$

4.3 The one point function

Now let us return to the one point function (4.2). Having introduced the CTM's and VO's, what remains to be done is to take the sum over the spins on the seams. This amounts to multiplying these matrices and taking the trace. Note that (4.11) implies in the limit

$$A^{(i)}_{NE}(\zeta)A^{(i)}_{SE}(\zeta) = (-q)^{D^{(i)}}\mathcal{R},$$
$$A^{(i)}_{SW}(\zeta)A^{(i)}_{NW}(\zeta) = \mathcal{R}\,(-q)^{D^{(i)}}.$$

Using (4.18) one can express (4.2) as

$$\begin{aligned} P^{(i)}_{\varepsilon} &= \mathcal{N}^{-1}\mathrm{tr}\left(A^{(i)}_{NE}(\zeta)A^{(i)}_{SE}(\zeta)\Phi^{(i,1-i)}_{LOW\,\varepsilon}(\zeta)A^{(1-i)}_{SW}(\zeta)A^{(1-i)}_{NW}(\zeta)\Phi^{(1-i,i)}_{UP\varepsilon}(\zeta)\right) \\ &= \mathcal{N}^{-1}\mathrm{tr}\left((-q)^{D^{(i)}}\Phi^{(i,1-i)}_{-\varepsilon}(\zeta)(-q)^{D^{(1-i)}}\Phi^{(1-i,i)}_{\varepsilon}(\zeta)\right) \\ &= \mathcal{N}^{-1}\mathrm{tr}\left(q^{2D^{(i)}}\Phi^{(i,1-i)}_{-\varepsilon}(-q^{-1}\zeta)\Phi^{(1-i,i)}_{\varepsilon}(\zeta)\right). \end{aligned} \tag{4.31}$$

Here $\mathcal{N}$ is a normalization factor, to be determined by the condition $P^{(i)}_{+} + P^{(i)}_{-} = 1$.

We wish to show that the properties of the VO's contain enough information to determine the one point function.[1] Set

$$F^{(i)}_{\varepsilon_1\varepsilon_2}(\zeta) = \mathrm{tr}_{\mathcal{H}^{(i)}}\left(q^{2D^{(i)}}\Phi^{(i,1-i)}_{\varepsilon_1}(\zeta_1)\Phi^{(1-i,i)}_{\varepsilon_2}(\zeta_2)\right)$$

where $\zeta = \zeta_2/\zeta_1$. Here we used (4.24) to conclude that the dependence of the trace function on the variables ζ_1 and ζ_2 is only through the ratio ζ. We have

$$P^{(i)}_{\varepsilon} = \frac{F^{(i)}_{-\varepsilon\,\varepsilon}(-q)}{F^{(i)}_{+-}(-q) + F^{(i)}_{-+}(-q)}.$$

Using the cyclicity of trace and (4.24) we have

$$\begin{aligned} F^{(i)}_{\varepsilon_1\varepsilon_2}(\zeta) &= \mathrm{tr}_{\mathcal{H}^{(1-i)}}\left(\Phi^{(1-i,i)}_{\varepsilon_2}(\zeta_2)q^{2D^{(i)}}\Phi^{(i,1-i)}_{\varepsilon_1}(\zeta_1)\right) \\ &= \mathrm{tr}_{\mathcal{H}^{(1-i)}}\left(q^{2D^{(1-i)}}\Phi^{(1-i,i)}_{\varepsilon_2}(q^{-2}\zeta_2)\Phi^{(i,1-i)}_{\varepsilon_1}(\zeta_1)\right) \\ &= F^{(1-i)}_{\varepsilon_2\varepsilon_1}(q^2\zeta^{-1}) \end{aligned} \tag{4.32}$$

[1] See however the remark at the end of this section.

The relation (4.16) implies

$$F^{(i)}_{\epsilon_2\epsilon_1}(\zeta^{-1}) = \sum_{\epsilon'_1,\epsilon'_2} R(\zeta^{-1})^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2} F^{(i)}_{\epsilon'_1\epsilon'_2}(\zeta) \tag{4.33}$$

From (4.29) we have

$$F^{(i)}_{\epsilon_1\epsilon_2}(\zeta) = F^{(1-i)}_{-\epsilon_1\,-\epsilon_2}(\zeta)$$

Set

$$f_\pm(\zeta) = F^{(0)}_{+-}(\zeta) \pm F^{(0)}_{-+}(\zeta). \tag{4.34}$$

Then (4.32) implies

$$f_+(\zeta) = f_+(q^2\zeta^{-1}), \tag{4.35}$$

while (4.30) implies

$$f_-(\zeta) = f_+(-\zeta).$$

Therefore, (4.33) reduces to the relation

$$f_+(\zeta) = \frac{1}{\kappa(\zeta)}\frac{q+\zeta}{1+q\zeta} f_+(\zeta^{-1}), \tag{4.36}$$

where $\kappa(\zeta)$ is given in (3.5). Combining (4.35) and (4.36) we obtain a difference equation

$$f_+(q^2\zeta) = \kappa(\zeta)\frac{1+q\zeta}{\zeta+q} f_+(\zeta). \tag{4.37}$$

To solve this equation let

$$\varphi(z) = \frac{(q^6z;q^4,q^4)_\infty}{(q^4z^{-1};q^4,q^4)_\infty}\frac{(q^2z^{-1};q^4,q^4)_\infty}{(q^8z;q^4,q^4)_\infty}, \qquad \frac{\varphi(\zeta^{-2})}{\varphi(q^{-4}\zeta^{-2})} = \frac{\zeta}{\kappa(\zeta)}$$

where $(z;q_1,q_2)_\infty = \prod_{m,n\geq 0}(1-zq_1^m q_2^n)$. If we assume that $f_+(\zeta)$ is analytic in the annulus $|q| < |\zeta| < |q^{-1}|$, then the solution of (4.37) is unique up to constant and is given by

$$f_+(\zeta) = const. \times \frac{\varphi(\zeta^{-2})}{(-q^3\zeta^{-1};q^2)_\infty(-q\zeta;q^2)_\infty}. \tag{4.38}$$

Specializing to $\zeta = -q$ we find

$$(-1)^{1-i}(P^{(i)}_+ - P^{(i)}_-) = \frac{f_-(-q)}{f_+(-q)} = \frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}.$$

This reproduces Baxter's formula for the spontaneous staggered polarization (1.5).

It should be clear to the reader that the derivation of the difference equation can be extended straightforwardly to the case of general correlation functions. We shall come to a more systematic treatment in the following section.

At the same time we notice a difficulty common to solving difference equations. If $f_+(\zeta)$ is a nontrivial solution of (4.37), then obviously the general solution is given by $f_+(\zeta) \times \gamma(\zeta)$ where $\gamma(\zeta)$ is an arbitrary pseudo-constant (i.e. a function satisfying $\gamma(q^2\zeta) = \gamma(\zeta)$). This situation is in sharp contrast with holonomic *differential* equations, where the solution space is finite dimensional. In order to fix a unique solution to the difference equation we have imposed in the above an analyticity condition, which is to be verified separately. Here we will not discuss this point any further, but it must be taken care of when we construct a mathematical model for the space of states.

Let us show that the scalar g (4.28) can be determined from the other properties. The argument is similar to the calculation of the one point function. Instead of the trace we look at the 'highest-to-highest' matrix elements

$$f^{(i)}(\zeta)_{\varepsilon_1\varepsilon_2} = \langle u^{(i)}|\Phi_{\varepsilon_1}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon_2}^{(1-i,i)}(\zeta_2)|u^{(i)}\rangle,$$

where $\zeta = \zeta_2/\zeta_1$. Set

$$\phi_\pm^{(i)}(\zeta) = f^{(i)}(\zeta)_{+-} \pm f^{(i)}(\zeta)_{-+}.$$

Then we have

$$\begin{aligned}
\phi_-^{(0)}(\zeta) &= \phi_+^{(0)}(-\zeta), \\
\phi_-^{(1)}(\zeta) &= -\phi_+^{(1)}(-\zeta), \\
\phi_-^{(0)}(\zeta) &= \phi_+^{(1)}(\zeta).
\end{aligned}$$

Therefore, using (4.25) we obtain

$$\frac{\phi_+^{(0)}(\zeta)}{\phi_+^{(0)}(\zeta^{-1})} = \kappa(\zeta)\frac{1+q\zeta}{\zeta+q}.$$

In addition, from (4.22) we have $\phi_+^{(0)}(\zeta) = 1 + O(\zeta)$. This is a Riemann-Hilbert factorization problem. Assuming again suitable analyticity properties we can solve this equation. The result is

$$f_{+-}^{(0)}(\zeta) = f_{-+}^{(1)}(\zeta) = \frac{(q^6\zeta^2;q^4)_\infty}{(q^4\zeta^2;q^4)_\infty},$$
$$f_{-+}^{(0)}(\zeta) = f_{+-}^{(1)}(\zeta) = -q\zeta\frac{(q^6\zeta^2;q^4)_\infty}{(q^4\zeta^2;q^4)_\infty}.$$

Setting $\zeta = -q^{-1}$ and comparing with (4.27) we find (4.28).

We remark that a similar graphical argument leads to the difference equations for the partition function per site, see [13], p.384. If we normalize the Boltzmann weights by $a = 1$, then they are exactly (3.11). The simplest solution (3.5) of the difference equations coincides with the correct result obtained by Bethe Ansatz.

4.4 Trace functions and difference equations

Now let us consider the *trace function*

$$F_n^{(i)}(x;\zeta_1,\cdots,\zeta_n)_{\varepsilon_1\cdots\varepsilon_n} = \mathrm{tr}_{\mathcal{H}^{(i)}}\left(x^D\Phi_{\varepsilon_1}^{(i,i+1)}(\zeta_1)\cdots\Phi_{\varepsilon_n}^{(i+n-1,i+n)}(\zeta_n)\right). \tag{4.39}$$

Here n is even and $x \in \mathbf{C}$ is a parameter satisfying $|x| < 1$. The index i in VO is to be read modulo 2. In exactly the same way as for the one point function, the general n-point functions can be written in terms of (4.39). Let $E_{\varepsilon\varepsilon'}$ denote the matrix unit with 1 as the $(\varepsilon,\varepsilon')$-entry and 0 elsewhere. In the spin-chain language consider the local operator $E_{\varepsilon_n\varepsilon_n'}\otimes\cdots\otimes E_{\varepsilon_1\varepsilon_1'}$. Then its expectation value in the i-th ground state sector is given by

$$\begin{aligned}&\langle E_{\varepsilon_n\varepsilon_n'}\otimes\cdots\otimes E_{\varepsilon_1\varepsilon_1'}\rangle_{(i)}\\ &= F^{(i)}_{-\varepsilon_1\cdots-\varepsilon_n,\varepsilon_n'\cdots\varepsilon_1'}(q^2;-q\zeta_1,\cdots,-q\zeta_n,\zeta_n,\cdots,\zeta_1)/\mathrm{tr}\left(q^{2D^{(i)}}\right).\end{aligned}$$

We shall also regard (4.39) as a function which takes values in $V\otimes\cdots\otimes V$. From the properties of VO one finds immediately the following:

$$\begin{aligned}&F_n^{(i)}(x;\cdots,\zeta_{j+1},\zeta_j,\cdots)\\ &\quad= P_{j\,j+1}R_{j\,j+1}(\zeta_j/\zeta_{j+1})F_n^{(i)}(x;\cdots,\zeta_j,\zeta_{j+1},\cdots),\end{aligned} \tag{4.40}$$
$$F_n^{(i)}(x;\zeta_1,\cdots,x\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_n} = F_n^{(i+1)}(x;\zeta_n,\zeta_1,\cdots,\zeta_{n-1})_{\varepsilon_n,\varepsilon_1,\cdots,\varepsilon_{n-1}},$$

$$F_n^{(1)}(x;\zeta_1,\cdots,\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_n} = F_n^{(0)}(x;\zeta_1,\cdots,\zeta_n)_{-\varepsilon_1,\cdots,-\varepsilon_n},$$
$$F_n^{(i)}(x;-\zeta_1,\cdots,-\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_n} = (-1)^{n/2}\varepsilon_1\cdots\varepsilon_n F_n^{(i)}(x;\zeta_1,\cdots,\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_n} \tag{4.41}$$

$$F_n^{(i)}(x;\zeta_1,\cdots,\zeta_j,\zeta_{j+1},\cdots,\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_j,\varepsilon_{j+1},\varepsilon_n}|_{\zeta_{j+1}=-q^{-1}\zeta_j}$$
$$= \delta_{\varepsilon_j,-\varepsilon_{j+1}} g^{-1} F_{n-2}^{(i)}(x;\zeta_1,\cdots,\zeta_{j-1},\zeta_{j+2},\cdots,\zeta_n)_{\varepsilon_1,\cdots,\varepsilon_{j-1},\varepsilon_{j+2},\cdots,\varepsilon_n} \tag{4.42}$$

Except for the last one, these formulas are similar to Smirnov's axioms for the form factors of massive integrable field theories [65].[1] Combining (4.40) and (4.41) we have

$$\begin{aligned} F^{(i)}(x;\zeta_1,\cdots,x\zeta_j,\cdots,\zeta_n) &= A_j(\zeta_1,\cdots,\zeta_n)F^{(i+1)}(x;\zeta_1,\cdots,\zeta_j,\cdots,\zeta_n), \\ A_j(\zeta_1,\cdots,\zeta_n) &= R_{j\,j+1}(x\zeta_j/\zeta_{j+1})^{-1}\cdots R_{jn}(x\zeta_j/\zeta_n)^{-1} \\ &\quad \times R_{1j}(\zeta_1/\zeta_j)\cdots R_{j-1\,j}(\zeta_{j-1}/\zeta_j). \end{aligned}$$

Hence the combination

$$G^{(\pm)}(x;\zeta_1,\cdots,\zeta_n) = F^{(0)}(x;\zeta_1,\cdots,\zeta_j,\cdots,\zeta_n) \pm F^{(1)}(x;\zeta_1,\cdots,\zeta_j,\cdots,\zeta_n)$$

satisfies the qKZ equation

$$G^{(\pm)}(x;\zeta_1,\cdots,x\zeta_j,\cdots,\zeta_n) = \pm A_j(\zeta_1,\cdots,\zeta_n)G^{(\pm)}(x;\zeta_1,\cdots,\zeta_j,\cdots,\zeta_n). \tag{4.43}$$

We remark that by specializing $x = 0$ only the highest weight contributes to the trace, and hence (4.39) reduces to the 'highest-to-highest' matrix element

$$f_n^{(i)}(\zeta_1,\cdots,\zeta_n) = \langle u_i|\Phi^{(i,i+1)}(\zeta_1)\cdots\Phi^{(i+n-1,i+n)}(\zeta_n)|u_i\rangle. \tag{4.44}$$

Unlike the trace functions, these are *power series* in the variables $\zeta_2/\zeta_1, \cdots, \zeta_n/\zeta_{n-1}$.

Along with the CTM the VO's already appeared in Baxter's paper [12], where they are called half-column and half-row transfer matrices. Apparently there has been no attempt to combine them with the idea of varying the spectral parameters which plays such a crucial role in the Z-invariant

[1] The precise analogues of the form factors are the trace functions involving type II vertex operators. See the discussion in 11.3.

inhomogeneous lattice [11]. All the graphical arguments we have been using so far apply equally well to the more general cases, e.g. the eight-vertex model [30]. We will develop in Chapters 5,6 a systematic method of obtaining correlation functions for the six-vertex model. For the eight-vertex model an analogous technique is not known, which makes it difficult to construct solutions of the difference equations for the correlation functions except for the one-point function [39].

Chapter 5

Level one modules and bosonization

From now on we turn to the mathematical construction of the structures that emerged in the previous chapter. For that purpose we need to prepare more about the representation theory of quantum affine algebra $U = U_q(\widehat{sl}_2)$. Our goal in this chapter is a concrete realization of the level one highest weight modules using free bosons.

5.1 Highest weight modules

As mentioned earlier, an important class of representations of U are the highest weight modules. We shall consider here only the irreducible modules with dominant integral highest weights. Thus let $\Lambda \in P$ be such that (h_0, Λ), (h_1, Λ) are both non-negative integers. For each such Λ there exists a unique nonzero U-module $V(\Lambda)$ characterized by the following properties.

$$\begin{aligned}
&\text{there exists a vector } v_\Lambda \in V(\Lambda) \text{ such that} \\
&e_i v_\Lambda = 0, \quad t_i v_\Lambda = q^{(h_i,\Lambda)} v_\Lambda, \quad f_i^{(h_i,\Lambda)+1} v_\Lambda = 0 \qquad (i = 0, 1), \\
&V(\Lambda) = U\, v_\Lambda.
\end{aligned}$$

Moreover they are irreducible and (except in the trivial case $\Lambda = 0$) are infinite dimensional .

By the definition $V(\Lambda)$ is spanned by elements of the form

$$f_{i_1} \cdots f_{i_N} v_\Lambda, \qquad (i_1, \cdots, i_N = 0, 1).$$

The element ρ acts on it as

$$\rho f_{i_1} \cdots f_{i_N} v_\Lambda = (-N + (\rho, \Lambda)) f_{i_1} \cdots f_{i_N} v_\Lambda.$$

The grading defined by $D = -\rho + (\rho, \Lambda)$

$$V(\Lambda) = \oplus_{r=0}^{\infty} V(\Lambda)_r, \qquad V(\Lambda)_r = \{v \in V(\Lambda) \mid Dv = rv\}$$

is called the principal grading. The principally specialized character

$$\chi_\Lambda(t) = \mathrm{tr}_{V(\Lambda)} \left(t^D\right) \tag{5.1}$$

is given by the same formula as in the affine Lie algebra case [43].

The module $V(\Lambda)$ has level $(h_0 + h_1, \Lambda)$, and there are only two that has level 1, namely $V(\Lambda_i)$ $(i = 0, 1)$. These are the cases of our interest. Their characters are given by

$$\chi_{\Lambda_i}(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^{2n-1}}. \tag{5.2}$$

Later we shall also consider the (restricted) dual module

$$V^*(\Lambda) = \oplus_{r=0}^{\infty} (V(\Lambda)_r)^*.$$

As a right U-module $V^*(\Lambda)$ is a highest weight module of highest weight Λ with level $(\Lambda, h_0 + h_1)$. Viewed as a left U-module via the antipode, $V^{*a}(\Lambda)$ is a lowest weight module of lowest weight $-\Lambda$ with level $-(\Lambda, h_0 + h_1)$. Sometimes we use the bra-ket notation for the pairing $\langle u|v\rangle = \langle u, v\rangle$ for $u \in V^*(\Lambda)$, $v \in V(\Lambda)$. We will write $v_\Lambda = |\Lambda\rangle$. We fix also $\langle\Lambda| \in (V(\Lambda)_0)^*$ such that $\langle\Lambda|\Lambda\rangle = 1$.

5.2 Drinfeld's generators

The level one modules have much more concrete realization in terms of *bosons*. Let us explain the construction following the work of [31]. For that purpose the original generators of U are not convenient, and we need to use another set of generators given by Drinfeld.

For convenience we shall enlarge U' by adjoining a central element $\gamma^{1/2}$ such that $(\gamma^{1/2})^2 = t_0 t_1$. Thus $\gamma = q^k$ on level k-modules. Drinfeld introduced the following generators

$$a_k \ (k \in \mathbf{Z}\backslash\{0\}), \quad x_k^{\pm} \ (k \in \mathbf{Z}), \ \gamma^{\pm 1/2}, \ K^{\pm 1}.$$

They satisfy the following defining relations:

$$[a_k, a_l] = \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{5.3}$$

$$Ka_kK^{-1} = a_k, \quad Kx_k^{\pm}K^{-1} = q^{\pm 2}x_k^{\pm}, \tag{5.4}$$

$$[a_k, x_l^{\pm}] = \pm\frac{[2k]}{k}\gamma^{\mp|k|/2}x_{k+l}^{\pm}, \tag{5.5}$$

$$x_{k+1}^{\pm}x_l^{\pm} - q^{\pm 2}x_l^{\pm}x_{k+1}^{\pm} = q^{\pm 2}x_k^{\pm}x_{l+1}^{\pm} - x_{l+1}^{\pm}x_k^{\pm}, \tag{5.6}$$

$$[x_k^+, x_l^-] = \frac{\gamma^{\frac{k-l}{2}}\psi_{k+l} - \gamma^{\frac{l-k}{2}}\varphi_{k+l}}{q - q^{-1}}, \tag{5.7}$$

where ψ_k, φ_k are defined by

$$\sum_{k=0}^{\infty}\psi_k z^{-k} = K\exp\{(q - q^{-1})\sum_{k=1}^{\infty}a_k z^{-k}\}, \tag{5.8}$$

$$\sum_{k=0}^{\infty}\varphi_{-k} z^{k} = K^{-1}\exp\{-(q - q^{-1})\sum_{k=1}^{\infty}a_{-k} z^{k}\}. \tag{5.9}$$

The old generators are related to them via

$$t_1 = K, \quad x_0^+ = e_1, \quad x_0^- = f_1,$$
$$t_0 = \gamma K^{-1}, \quad x_1^- = e_0 t_1, \quad x_{-1}^+ = t_1^{-1}f_0.$$

In terms of the generating functions (called *currents*)

$$X^{\pm}(z) = \sum_{n\in\mathbf{Z}} x_n^{\pm} z^{-n-1},$$

the defining relations are written more compactly as follows:

$$[a_k, X^{\pm}(z)] = \pm\frac{[2k]}{k}\gamma^{\mp|k|/2}z^k X^{\pm}(z), \tag{5.10}$$

$$(z - q^{\pm 2}w)X^{\pm}(z)X^{\pm}(w) + (w - q^{\pm 2}z)X^{\pm}(w)X^{\pm}(z) = 0, \tag{5.11}$$

$$[X^+(z), X^-(w)] = K\exp\{(q - q^{-1})\sum_{k=1}^{\infty}a_k\gamma^{k/2}z^{-k}\}\frac{\delta(z/\gamma w)}{(q - q^{-1})zw}$$
$$-K^{-1}\exp\{-(q - q^{-1})\sum_{k=1}^{\infty}a_{-k}\gamma^{k/2}z^{k}\}\frac{\delta(\gamma z/w)}{(q - q^{-1})zw}. \tag{5.12}$$

Here $\delta(z) = \sum_{n\in\mathbf{Z}} z^n$ is a formal series, and we have used the formula

$$\delta(z/w)f(z) = \delta(z/w)f(w).$$

To obtain the full algebra U we impose the following.

$$q^d \gamma^{1/2} q^{-d} = \gamma^{1/2}, \quad q^d K q^{-d} = K,$$
$$q^d x_k^{\pm} q^{-d} = q^k x_k^{\pm}, \tag{5.13}$$
$$q^d a_k q^{-d} = q^k a_k. \tag{5.14}$$

5.3 Realization of level one modules

In the sequel we concentrate on the level one modules. Since $\gamma = q$, (5.3) becomes

$$[a_k, a_l] = \delta_{k+l,0} \frac{[2k][k]}{k}.$$

Thus apart from a normalization the a_k constitute free bosons. Abbreviating α_1 to α we set

$$V'(\Lambda_i) = \mathbf{C}[a_{-1}, a_{-2}, \cdots] \otimes (\oplus_{n \in \mathbf{Z}} \mathbf{C} e^{\Lambda_i + n\alpha}).$$

Here the e^λ are formal symbols satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$. On this space we define the action of a_k ($k \neq 0$) by

$$\begin{aligned} a_k(f \otimes e^\beta) &= a_k f \otimes e^\beta \quad \text{if } k < 0, \\ &= [a_k, f] \otimes e^\beta \quad \text{if } k > 0 \end{aligned}$$

where $f \in \mathbf{C}[a_{-1}, a_{-2}, \cdots]$ and $\beta = \Lambda_i + n\alpha$. Let further e^α and ∂ be the operators acting on $V'(\Lambda_i)$ as

$$\begin{aligned} e^\alpha (f \otimes e^\beta) &= f \otimes e^{\beta+\alpha}, \\ \partial (f \otimes e^\beta) &= (\alpha, \beta) f \otimes e^\beta. \end{aligned}$$

The actions of the other generators are as follows:

$$K = q^\partial, \qquad \gamma = q,$$
$$X^\pm(z) = \exp\Big(\pm \sum_{n=1}^{\infty} \frac{a_{-n}}{[n]} q^{-n/2} z^n\Big) \exp\Big(\mp \sum_{n=1}^{\infty} \frac{a_n}{[n]} q^{-n/2} z^{-n}\Big) e^{\pm\alpha} z^{\pm\partial}, \tag{5.15}$$
$$q^d(1 \otimes e^\beta) = q^{-(\beta,\beta)/2 + i/4}(1 \otimes e^\beta) \quad \text{for } \beta = \Lambda_i + n\alpha,\ n \in \mathbf{Z}. \tag{5.16}$$

The action of q^d on general vectors are defined via (5.14).

The statement is that with these definitions $V'(\Lambda_i)$ coincides with the irreducible level one highest weight module $V(\Lambda_i)$ where the highest weight vector is given by

$$|\Lambda_i\rangle = 1 \otimes e^{\Lambda_i}.$$

We shall verify below that these formulas indeed give a representation of U.

Let us check the defining relations (5.10), (5.11), (5.12). For the calculation we need the following normal ordering symbol : :

$$\begin{aligned} :a_k a_l: &= a_k a_l \quad \text{if } k < 0, \\ &= a_l a_k \quad \text{if } k > 0, \\ :\alpha\partial: &= :\partial\alpha: = \alpha\partial, \end{aligned}$$

as well as the standard formulas

$$e^A e^B = e^{[A,B]} e^B e^A \quad \text{if } [A,B] \text{ is a scalar}, \tag{5.17}$$

$$\exp\Big(\pm \sum_{n=1}^{\infty} \frac{z^n}{n}\Big) = (1-z)^{\mp 1}. \tag{5.18}$$

The relation (5.10) immediately follows from (5.3) and (5.17). To show (5.11) and (5.12), we compute the normal-ordered products of the currents

$$X^+(z)X^+(w) = z^2\Big(1-\frac{w}{z}\Big)\Big(1-\frac{w}{q^2 z}\Big) :X^+(z)X^+(w):, \tag{5.19}$$

$$X^-(z)X^-(w) = z^2\Big(1-\frac{w}{z}\Big)\Big(1-\frac{q^2 w}{z}\Big) :X^-(z)X^-(w):, \tag{5.20}$$

$$X^+(z)X^-(w) = \frac{1}{z^2\Big(1-\frac{qw}{z}\Big)\Big(1-\frac{w}{qz}\Big)} :X^+(z)X^-(w):, \tag{5.21}$$

$$X^-(w)X^+(z) = \frac{1}{w^2\Big(1-\frac{qz}{w}\Big)\Big(1-\frac{z}{qw}\Big)} :X^+(z)X^-(w):. \tag{5.22}$$

To illustrate the use of the formulas (5.17) and (5.18), let us compute (5.19). To rewrite the product $X^+(z)X^+(w)$ into the normal-ordered form, we must reverse the order of the products

$$\exp\Big(-\sum_{n=1}^{\infty} \frac{a_n}{[n]} q^{-n/2} z^{-n}\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{a_{-n}}{[n]} q^{-n/2} w^n\Big)$$

and

$$z^{\partial} e^{\alpha}.$$

Putting

$$A = -\sum_{n=1}^{\infty} \frac{a_n}{[n]} q^{-n/2} z^{-n}, \qquad B = \sum_{n=1}^{\infty} \frac{a_{-n}}{[n]} q^{-n/2} w^n,$$

in (5.17) and using (5.18), we find

$$\begin{aligned} [A, B] &= -\sum_{n=1}^{\infty} \frac{[2n][n]}{n} \frac{q^{-n}}{[n]^2} \left(\frac{w}{z}\right)^n \\ &= -\sum_{n=1}^{\infty} \frac{(w/z)^n + (w/q^2 z)^n}{n} \\ &= \left(1 - \frac{w}{z}\right)\left(1 - \frac{w}{q^2 z}\right). \end{aligned}$$

Therefore, we have

$$\exp\left(-\sum_{n=1}^{\infty} \frac{a_n}{[n]} q^{-n/2} z^{-n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{a_{-n}}{[n]} q^{-n/2} w^n\right) =$$
$$\left(1 - \frac{w}{z}\right)\left(1 - \frac{w}{q^2 z}\right) : \exp\left(-\sum_{n=1}^{\infty} \frac{a_n}{[n]} q^{-n/2} z^{-n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{a_{-n}}{[n]} q^{-n/2} w^n\right) : .$$

Similarly, we have $z^{\partial} e^{\alpha} = z^2 : z^{\partial} e^{\alpha} :$. Thus we obtain (5.19).

The relation (5.11) immediately follows from (5.19) and (5.20). Noting that

$$\frac{1}{z^2\left(1 - \frac{qw}{z}\right)\left(1 - \frac{w}{qz}\right)} - \frac{1}{w^2\left(1 - \frac{qz}{w}\right)\left(1 - \frac{z}{qw}\right)} = \frac{\delta\left(\frac{z}{qw}\right) - \delta\left(\frac{w}{qz}\right)}{(q - q^{-1}) z w},$$

we can derive the relation (5.12) from (5.21) and (5.22).

Let us verify the relations (5.13). They are equivalent to

$$q^d X^{\pm}(z) q^{-d} = q^{-1} X^{\pm}(q^{-1} z). \tag{5.23}$$

Using (5.14) we can reduce (5.23) to

$$q^d e^{\alpha} q^{-d} = q^{-1} e^{\alpha} q^{-\partial}. \tag{5.24}$$

With the normalization $q^d(1 \otimes e^{\Lambda_i}) = 1 \otimes e^{\Lambda_i}$ we are thus led to (5.16).

5.4 Principal vs. homogeneous pictures

Before closing this chapter let us give a remark about the different gradings of U. Recall that on weight modules the action of $h \in P$ makes sense. In this sense the elements d and ρ act as derivations on U:

$$[d, e_i] = \delta_{i0} e_i, \quad [d, f_i] = -\delta_{i0} f_i, \quad [d, q^h] = 0,$$
$$[\rho, e_i] = e_i, \quad [\rho, f_i] = -f_i, \quad [\rho, q^h] = 0.$$

We refer to the grading of U defined by d (resp. ρ) as the homogeneous (resp. principal) grading. The $0 \leftrightarrow 1$ symmetry is manifest in the principal grading.

The evaluation module V_ζ considered in Chapter 3 is adapted to the principal grading operator in the sense that ρ is realized as differentiation in ζ. We call it the evaluation module in the *principal picture*.

In contrast, (5.23) shows that the realization of the level one modules is adapted to the homogeneous grading. For that reason, in the construction of the vertex operators we will choose to work with the evaluation module in the *homogeneous picture*. By definition it is $V_z^{(h)} = V \otimes \mathbf{C}[z, z^{-1}]$ equipped with the following U-module structure:

$$\begin{aligned}
&e_0(v_\varepsilon \otimes z^m) = (f_1 v_\varepsilon) \otimes z^{m+1}, \quad e_1(v_\varepsilon \otimes z^m) = (e_1 v_\varepsilon) \otimes z^m,\\
&f_0(v_\varepsilon \otimes z^m) = (e_1 v_\varepsilon) \otimes z^{m-1}, \quad f_1(v_\varepsilon \otimes z^m) = (f_1 v_\varepsilon) \otimes z^m,\\
&t_0 = t_1^{-1}, \quad t_1(v_\varepsilon \otimes z^m) = (t_1 v_\varepsilon) \otimes z^m,\\
&d = z\frac{d}{dz}.
\end{aligned} \tag{5.25}$$

The dual module $V_z^{(h)*a} = V^* \otimes \mathbf{C}[z, z^{-1}]$ is defined similarly as in the principal picture, where the pairing is $\langle v_\varepsilon^* \otimes z^m, v_{\varepsilon'} \otimes z^n \rangle = \delta_{\varepsilon\varepsilon'} \delta_{m+n,0}$. The analog of the isomorphism (3.32) takes the form

$$\begin{array}{ccc}
V_{q^{\mp 2}z}^{(h)} & \xrightarrow{\sim} & V_z^{(h)*a^{\pm 1}}, \\
v_+ \otimes z^n & \mapsto & v_-^* \otimes z^n \\
v_- \otimes z^n & \mapsto & -q^{\pm 1} v_+^* \otimes z^n.
\end{array} \tag{5.26}$$

In fact each irreducible piece $V_\zeta^{(\pm)}$ of V_ζ is equivalent to $V_z^{(h)}$ by the map

$$\begin{array}{ccc}
V_\zeta^{(\pm)} & \xrightarrow{\bar{C}(\zeta)} & V_z^{(h)}, \qquad \zeta^2 = z, \\
v_\varepsilon \otimes \zeta^m & \mapsto & v_\varepsilon \otimes \zeta^{m+(\pm 1-\varepsilon)/2}.
\end{array} \tag{5.27}$$

Let

$$P\overline{R}(z_1/z_2)\ :\ V_{z_1}^{(h)} \otimes V_{z_2}^{(h)} \longrightarrow V_{z_2}^{(h)} \otimes V_{z_1}^{(h)}$$

be an intertwiner in the homogeneous picture. The above equivalence (5.27) implies that $\overline{R}(z)$ is related to the R matrix $R(\zeta)$ (3.6) in the principal picture by the formula

$$\overline{R}(z_1/z_2) = \text{scalar } \times \left(\bar{C}(\zeta_1) \otimes \bar{C}(\zeta_2)\right) R(\zeta_1/\zeta_2) \left(\bar{C}(\zeta_1) \otimes \bar{C}(\zeta_2)\right)^{-1}.$$

Normalizing as $\overline{R}(z_1/z_2)v_+ \otimes v_+ = v_+ \otimes v_+$ we have explicitly

$$\overline{R}(z) = \begin{pmatrix} 1 & & & \\ & \dfrac{(1-z)q}{1-q^2 z} & \dfrac{(1-q^2)}{1-q^2 z} & \\ & \dfrac{(1-q^2)z}{1-q^2 z} & \dfrac{(1-z)q}{1-q^2 z} & \\ & & & 1 \end{pmatrix}.$$

Here again the matrix structure is relative to the basis $v_\varepsilon \otimes v_{\varepsilon'}$ in the order $(\varepsilon, \varepsilon') = (+,+), (+,-), (-,+), (-,-)$.

Chapter 6

Vertex operators

Along with the highest weight modules, another important ingredients are the (q-deformed) vertex operators in the sense of Frenkel-Reshetikhin [32]. We proceed to constructing the vertex operators on the basis of the results of the previous chapter.

6.1 The notion of vertex operators

In this chapter, except in the last section, we will deal with the homogeneous picture. Thus let $V_z^{(h)}$ be the evaluation module (5.25) associated with the two-dimensional representation. Consider the intertwiners of U-modules

$$\widetilde{\Phi}^{(1-i,i)}(z) \quad : \quad V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_z^{(h)}, \tag{6.1}$$

$$\widetilde{\Psi}^{(1-i,i)}(z) \quad : \quad V(\Lambda_i) \longrightarrow V_z^{(h)} \otimes V(\Lambda_{1-i}). \tag{6.2}$$

We call (6.1), (6.2) vertex operators of type I and type II, respectively. Their precise meaning is as follows. For instance, the type I operator is a formal series

$$\begin{aligned} \widetilde{\Phi}^{(1-i,i)}(z) &= \sum_{\varepsilon} \widetilde{\Phi}_{\varepsilon}^{(1-i,i)}(z) \otimes v_{\varepsilon}, \\ \widetilde{\Phi}_{\varepsilon}^{(1-i,i)}(z) &= \sum_{n \in Z} \widetilde{\Phi}_{\varepsilon n}^{(1-i,i)} z^{-n}, \end{aligned} \tag{6.3}$$

whose coefficients are linear maps

$$\widetilde{\Phi}_{\varepsilon n}^{(1-i,i)} \ : \ V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}),$$

such that it commutes with the action of U in the sense that

$$\sum \tilde{\Phi}^{(1-i,i)}_{\varepsilon n} xv \otimes (v_\varepsilon \otimes z^{-n}) = \Delta(x) \left\{ \sum \tilde{\Phi}^{(1-i,i)}_{\varepsilon n} v \otimes (v_\varepsilon \otimes z^{-n}) \right\}$$

for all $x \in U$ and $v \in V(\Lambda_i)$.

Here the action on $v_\varepsilon \otimes z^{-n}$ is understood in $V^{(h)}_z$. The type II operator is defined analogously. When there is no fear of confusion we will suppress the superscripts $(1-i,i)$ that indicate the relevant spaces.

The VO's have been introduced and studied by Frenkel-Reshetikhin [32] in a more general setting where highest weight modules have arbitrary level. It was shown that the product of VO's

$$\tilde{\Phi}_{\varepsilon_1}(z_1)\cdots\tilde{\Phi}_{\varepsilon_n}(z_n)$$

is analytic[1] in the region $|z_1| \gg \cdots \gg |z_n|$, extends to a meromorphic function on $(\mathbf{C}\backslash\{0\})^n$, and that its highest-to-highest matrix elements satisfy the qKZ equation.

In our situation such operators exist, and are unique up to multiplication by a scalar. (See [32, 21] for a general statement.) In the sequel we adopt the normalization

$$\langle\Lambda_1|\tilde{\Phi}_-(z)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\tilde{\Phi}_+(z)|\Lambda_1\rangle = 1, \tag{6.4}$$

$$\langle\Lambda_1|\tilde{\Psi}_-(z)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\tilde{\Psi}_+(z)|\Lambda_1\rangle = 1. \tag{6.5}$$

More generally an intertwiner of the form $V(\Lambda_i) \to V(\Lambda_{i+n}) \otimes V^{(h)}_{z_1} \otimes \cdots \otimes V^{(h)}_{z_n}$ is unique up to scalar.[2] This is a special feature of the present situation, reflecting the fact that the two-dimensional module is 'perfect' with respect to the level one modules [44].

6.2 Type I vertex operator

We now construct the type I VO (6.1). We define the components of $\tilde{\Phi}(z)$ as in (6.3). From the intertwining relation

$$\Delta(x) \circ \tilde{\Phi}(z) = \tilde{\Phi}(z) \circ x \quad \text{for } x \in U \tag{6.6}$$

[1] In the sense of matrix elements.

[2] The suffix $i+n$ is to be read modulo 2.

with the choice $x = t_1, q^d$ we obtain

$$\begin{aligned} K\widetilde{\Phi}_\pm(z)K^{-1} &= q^{\mp 1}\widetilde{\Phi}_\pm(z), & (6.7)\\ q^{-d}\widetilde{\Phi}_\pm(z)q^d &= \widetilde{\Phi}_\pm(qz). & (6.8)\end{aligned}$$

With the choice $x = f_1, e_1, f_0, e_0$ we obtain respectively

$$\widetilde{\Phi}_+(z)x_0^- - q^{-1}x_0^-\widetilde{\Phi}_+(z) = 0, \quad (6.9)$$

$$\widetilde{\Phi}_+(z) = \widetilde{\Phi}_-(z)x_0^- - qx_0^-\widetilde{\Phi}_-(z), \quad (6.10)$$

$$K\widetilde{\Phi}_-(z) = \widetilde{\Phi}_+(z)x_0^+ - x_0^+\widetilde{\Phi}_+(z), \quad (6.11)$$

$$\widetilde{\Phi}_-(z)x_0^+ - x_0^+\widetilde{\Phi}_-(z) = 0, \quad (6.12)$$

$$(qzK)^{-1}\widetilde{\Phi}_-(z) = \widetilde{\Phi}_+(z)x_{-1}^+ - x_{-1}^+\widetilde{\Phi}_+(z), \quad (6.13)$$

$$\widetilde{\Phi}_-(z)x_{-1}^+ - x_{-1}^+\widetilde{\Phi}_-(z) = 0, \quad (6.14)$$

$$\widetilde{\Phi}_+(z)x_1^- - qx_1^-\widetilde{\Phi}_+(z) = 0, \quad (6.15)$$

$$q^2z\widetilde{\Phi}_+(z) = \widetilde{\Phi}_-(z)x_1^- - q^{-1}x_1^-\widetilde{\Phi}_-(z). \quad (6.16)$$

We will solve these equations for $\widetilde{\Phi}_\pm(z)$.

First, assuming (6.9)–(6.16), let us derive

$$[X^+(w), \widetilde{\Phi}_-(z)] = 0, \quad (6.17)$$

$$[a_n, \widetilde{\Phi}_-(z)] = \frac{q^{7n/2}[n]}{n}z^n\widetilde{\Phi}_-(z), \quad (6.18)$$

$$[a_{-n}, \widetilde{\Phi}_-(z)] = \frac{q^{-5n/2}[n]}{n}z^{-n}\widetilde{\Phi}_-(z), \quad (6.19)$$

where $n \in \mathbf{Z}_{>0}$. From (6.10) and (6.16) we obtain

$$q^2z(\widetilde{\Phi}_-(z)x_0^- - qx_0^-\widetilde{\Phi}_-(z)) - (\widetilde{\Phi}_-(z)x_1^- - q^{-1}x_1^-\widetilde{\Phi}_-(z)) = 0. \quad (6.20)$$

Set

$$\begin{aligned} A(w) &= \exp\Big((q-q^{-1})\sum_{n=1}^{\infty} a_n q^{n/2}w^{-n}\Big),\\ B(w) &= \exp\Big(-(q-q^{-1})\sum_{n=1}^{\infty} a_{-n}q^{n/2}w^{n}\Big).\end{aligned}$$

Using (5.12) we have

$$\begin{aligned} (q-q^{-1})[X^+(w), x_0^-] &= w^{-1}KA(w) - w^{-1}K^{-1}B(w), & (6.21)\\ (q-q^{-1})[X^+(w), x_1^-] &= q^{-1}KA(w) - qK^{-1}B(w). & (6.22)\end{aligned}$$

Thanks to (6.12) and (6.14), the equation (6.17) is already true for the coefficients of w^0 and w^{-1}. Commuting (6.20) with $X^+(w)$ and using (6.21), (6.22), we then obtain

$$\begin{aligned}
&\frac{q^2 z}{w}\tilde{\Phi}_-(z)(KA(w) - K^{-1}B(w)) - \frac{q^3 z}{w}(KA(w) - K^{-1}B(w))\tilde{\Phi}_-(z) \\
&-\tilde{\Phi}_-(z)(q^{-1}KA(w) - qK^{-1}B(w)) + (q^{-2}KA(w) - K^{-1}B(w))\tilde{\Phi}_-(z) \\
&= 0, \qquad (6.23)
\end{aligned}$$

which holds to the orders w^0 and w^{-1}. Thus we obtain (6.18) and (6.19) for $n = 1$. Starting from this, taking commutators with $a_{\pm 1}$ repeatedly and using (5.5), one can show (6.17) to all orders. This in turn implies that (6.23) is true to all orders. Finally, expanding (6.23) in w, we get (6.18) and (6.19).

The commutation relations (6.17), (6.18), (6.19), the normalization (6.4) and the relation (6.7) altogether determine the operator $\tilde{\Phi}_-(z)$ uniquely:

$$\begin{aligned}
\tilde{\Phi}_-^{(1-i,i)}(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{a_{-n}}{[2n]} q^{7n/2} z^n\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{-5n/2} z^{-n}\Big) \\
&\quad \times e^{\alpha/2}(-q^3 z)^{(\partial+i)/2}. \qquad (6.24)
\end{aligned}$$

The other component $\tilde{\Phi}_+^{(1-i,i)}(z)$ is determined via (6.10).

Yet we must check that (6.24), (6.10) satisfy the rest of the intertwining relations. We use the following formulas for the operator products.

$$\begin{aligned}
X^+(w)\tilde{\Phi}_-(z) &= \tilde{\Phi}_-(z)X^+(w) = (w - q^3 z) : X^+(w)\tilde{\Phi}_-(z) :, \\
X^-(w)\tilde{\Phi}_-(z) &= \frac{1}{w(1 - q^4 z/w)} : X^-(w)\tilde{\Phi}_-(z) :, \\
\tilde{\Phi}_-(z)X^-(w) &= \frac{-1}{q^3 z(1 - w/q^2 z)} : \tilde{\Phi}_-(z)X^-(w) : .
\end{aligned}$$

We need to show (6.9), (6.11), (6.13), (6.15), (6.16). To prove (6.9) and (6.15), we need to compute

$$\begin{aligned}
\tilde{\Phi}_-(z)x_m^- x_n^- &= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} w_1^m w_2^n \tilde{\Phi}_-(z)X^-(w_1)X^-(w_2) \\
&= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} \frac{w_1^{m+2} w_2^n (1 - w_2/w_1)(1 - q^2 w_2/w_1)}{q^6 z^2 (1 - w_1/q^2 z)(1 - w_2/q^2 z)} \\
&\quad \times : \tilde{\Phi}_-(z)X^-(w_1)X^-(w_2) : .
\end{aligned}$$

Using the equalities

$$\frac{w_1 - w_2}{q^2 z(1 - w_1/q^2 z)(1 - w_2/q^2 z)} = \frac{1}{1 - w_1/q^2 z} - \frac{1}{1 - w_2/q^2 z}$$

and renaming the integration variables for the second term noting

$$: \tilde{\Phi}_-(z) X^-(w_1) X^-(w_2) :=: \tilde{\Phi}_-(z) X^-(w_2) X^-(w_1) :,$$

we have

$$\begin{aligned}\tilde{\Phi}_-(z) x_m^- x_n^- &= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} : \tilde{\Phi}_-(z) X^-(w_1) X^-(w_2) : \\ &\times \frac{w_1^{m+1} w_2^n - q^2 w_1^m w_2^{n+1} + q^2 w_1^{n+1} w_2^m - w_1^n w_2^{m+1}}{q^4 z(1 - w_1/q^2 z)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}x_m^- \tilde{\Phi}_-(z) x_n^- &= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} : \tilde{\Phi}_-(z) X^-(w_1) X^-(w_2) : \\ &\times \left(\frac{-w_1^n w_2^{m+1} + w_1^{n+1} w_2^m}{q^3 z(1 - w_1/q^2 z)} + \frac{q(-w_1^m w_2^n + w_1^{m-1} w_2^{n+1})}{1 - q^4 z/w_1} \right), \\ x_n^- x_m^- \tilde{\Phi}_-(z) &= \oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} : \tilde{\Phi}_-(z) X^-(w_1) X^-(w_2) : \\ &\times \frac{1}{1 - q^4 z/w_1} \left(-w_1^n w_2^m + q^2 w_1^{n-1} w_2^{m+1} + w_1^{m-1} w_2^{n+1} - q^2 w_1^m w_2^n \right).\end{aligned}$$

The equalities (6.9) and (6.15) immediately follow from these formulas. The equality (6.16) is equivalent to (6.20). The latter is proved by a similar calculation.

Finally, let us prove (6.11) and (6.13). We have

$$\begin{aligned}[\tilde{\Phi}_+(z), x_{-1}^+] &= \tilde{\Phi}_-(z)[x_0^-, x_{-1}^+] - q[x_0^-, x_{-1}^+]\tilde{\Phi}_-(z) \\ &= -q^{1/2} \tilde{\Phi}_-(z) K^{-1} a_{-1} + q^{3/2} K^{-1} a_{-1} \tilde{\Phi}_-(z) \\ &= -q^{3/2} K^{-1} [\tilde{\Phi}_-(z), a_{-1}] \\ &= (qzK)^{-1} \tilde{\Phi}_-(z),\end{aligned}$$

showing (6.13). The proof of (6.11) is similar.

6.3 Type II vertex operator

The construction of the type II operator (6.2) is quite parallel. On $V(\Lambda_i)$ they are given by the formulas

$$\begin{aligned}\widetilde{\Psi}_+^{(1-i,i)}(z) &= \exp\Big(-\sum_{n=1}^{\infty}\frac{a_{-n}}{[2n]}q^{n/2}z^n\Big)\exp\Big(\sum_{n=1}^{\infty}\frac{a_n}{[2n]}q^{-3n/2}z^{-n}\Big)\\ &\quad\times e^{-\alpha/2}(-qz)^{(-\partial+i)/2}(-q)^{i-1},\\ \widetilde{\Psi}_-^{(1-i,i)}(z) &= \widetilde{\Psi}_+^{(1-i,i)}(z)x_0^+ - qx_0^+\widetilde{\Psi}_+^{(1-i,i)}(z).\end{aligned}$$

The intertwining relations are

$$\begin{aligned}&\widetilde{\Psi}_-(z) = \widetilde{\Psi}_+(z)x_0^+ - qx_0^+\widetilde{\Psi}_+(z),\\ &\widetilde{\Psi}_-(z)x_0^+ - q^{-1}x_0^+\widetilde{\Psi}_-(z) = 0,\\ &\widetilde{\Psi}_+(z)x_1^- - x_1^-\widetilde{\Psi}_+(z) = 0,\\ &z\widetilde{\Psi}_+(z)K = \widetilde{\Psi}_-(z)x_1^- - x_1^-\widetilde{\Psi}_-(z),\\ &\widetilde{\Psi}_+(z)x_0^- - x_0^-\widetilde{\Psi}_+(z) = 0,\\ &K^{-1}\widetilde{\Psi}_+(z) = \widetilde{\Psi}_-(z)x_0^- - x_0^-\widetilde{\Psi}_-(z),\\ &(qz)^{-1}\widetilde{\Psi}_-(z) = q\widetilde{\Psi}_+(z)x_{-1}^+ - x_{-1}^+\widetilde{\Psi}_+(z),\\ &q^{-1}\widetilde{\Psi}_-(z)x_{-1}^+ - x_{-1}^+\widetilde{\Psi}_-(z) = 0.\end{aligned}$$

For the proof, we use the following product formulas.

$$X^-(w)\widetilde{\Psi}_+(z) = \widetilde{\Psi}_+(z)X^-(w) = (w-qz) : X^-(w)\widetilde{\Psi}_+(z) :, \tag{6.25}$$

$$X^+(w)\widetilde{\Psi}_+(z) = \frac{1}{w(1-\frac{z}{w})} : X^+(w)\widetilde{\Psi}_+(z) :, \tag{6.26}$$

$$\widetilde{\Psi}_+(z)X^+(w) = \frac{-1}{qz(1-\frac{w}{q^2z})} : \widetilde{\Psi}_+(z)X^+(w) : . \tag{6.27}$$

We omit the details of the proof.

6.4 Commutation relations

Consider the product of the vertex operators

$$(\widetilde{\Phi}^{(i,1-i)}(z_1)\otimes \mathrm{id}_{V_{z_2}})\widetilde{\Phi}^{(1-i,i)}(z_2) : V(\Lambda_i) \to V(\Lambda_i)\otimes V_{z_1}^{(h)}\otimes V_{z_2}^{(h)}.$$

This is an intertwiner. Let us write it as

$$\overset{1}{\tilde{\Phi}}{}^{(i,1-i)}(z_1)\,\overset{2}{\tilde{\Phi}}{}^{(1-i,i)}(z_2) = \sum_{\epsilon_1,\epsilon_2} \tilde{\Phi}^{(i,1-i)}_{\epsilon_1}(z_1)\tilde{\Phi}^{(1-i,i)}_{\epsilon_2}(z_2) \otimes v_{\epsilon_1} \otimes v_{\epsilon_2}.$$

In this notation, we have

$$\overset{2}{\tilde{\Phi}}{}^{(i,1-i)}(z_2)\,\overset{1}{\tilde{\Phi}}{}^{(1-i,i)}(z_1) = \sum_{\epsilon_1,\epsilon_2} \tilde{\Phi}^{(i,1-i)}_{\epsilon_2}(z_2)\tilde{\Phi}^{(1-i,i)}_{\epsilon_1}(z_1) \otimes v_{\epsilon_1} \otimes v_{\epsilon_2}.$$

This is an intertwiner

$$V(\Lambda_i) \longrightarrow V(\Lambda_i) \otimes V^{(h)}_{z_1} \otimes V^{(h)}_{z_2} \tag{6.28}$$

where in the right hand side the action of $x \in U$ is given by $\sum x_{(1)} \otimes x_{(3)} \otimes x_{(2)}$, with $(\Delta \otimes \mathrm{id}\,)\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

Consider now $\overline{R}(z_1/z_2)\,\overset{1}{\tilde{\Phi}}{}^{(i,1-i)}(z_1)\,\overset{2}{\tilde{\Phi}}{}^{(1-i,i)}(z_2)$ where $\overline{R}$ operates on $V^{(h)}_{z_1} \otimes V^{(h)}_{z_2}$. This gives another intertwiner of the form (6.28). Thanks to the uniqueness of VO's mentioned above, it must be the same up to scalar as $\overset{2}{\tilde{\Phi}}{}^{(i,1-i)}(z_2)\,\overset{1}{\tilde{\Phi}}{}^{(1-i,i)}(z_1)$. The scalar can be determined by computing

$$\begin{aligned} \tilde{\Phi}^{(i,1-i)}_{-}(z_1)\tilde{\Phi}^{(1-i,i)}_{-}(z_2) &= (-q^3 z_1)^{1/2}\frac{(q^2 z_2/z_1;q^4)_\infty}{(q^4 z_2/z_1;q^4)_\infty} \\ &\quad \times :\tilde{\Phi}^{(i,1-i)}_{-}(z_1)\tilde{\Phi}^{(1-i,i)}_{-}(z_2):, \end{aligned} \tag{6.29}$$

$$:\tilde{\Phi}^{(i,1-i)}_{-}(z_1)\tilde{\Phi}^{(1-i,i)}_{-}(z_2): \;=\; \left(\frac{z_1}{z_2}\right)^{(1-2i)/2} :\tilde{\Phi}^{(i,1-i)}_{-}(z_2)\tilde{\Phi}^{(1-i,i)}_{-}(z_1):, \tag{6.30}$$

where we used

$$\sum_{n=1}^{\infty} \frac{1}{n}\frac{[n]}{[2n]} z^n = \log\frac{(q^3 z;q^4)_\infty}{(qz;q^4)_\infty}.$$

We thus find

$$\begin{aligned} &\overline{R}(z_1,z_2)\,\overset{1}{\tilde{\Phi}}{}^{(i,1-i)}(z_1)\,\overset{2}{\tilde{\Phi}}{}^{(1-i,i)}(z_2) \\ &= \left(\frac{z_1}{z_2}\right)^{1-i}\frac{(q^4 z_1/z_2;q^4)_\infty}{(q^2 z_1/z_2;q^4)_\infty}\frac{(q^2 z_2/z_1;q^4)_\infty}{(q^4 z_2/z_1;q^4)_\infty}\,\overset{2}{\tilde{\Phi}}{}^{(i,1-i)}(z_2)\,\overset{1}{\tilde{\Phi}}{}^{(1-i,i)}(z_1). \end{aligned} \tag{6.31}$$

In the same way we have

$$\widetilde{\Psi}_{+}^{(i,1-i)}(z_1)\widetilde{\Psi}_{+}^{(1-i,i)}(z_2) = (-qz_1)^{1/2}\frac{(z_2/z_1;q^4)_\infty}{(q^2z_2/z_1;q^4)_\infty} :\widetilde{\Psi}_{+}^{(i,1-i)}(z_1)\widetilde{\Psi}_{+}^{(1-i,i)}(z_2):, \quad (6.32)$$

$$\widetilde{\Psi}_{+}^{(i,1-i)}(z_1)\widetilde{\Phi}_{-}^{(1-i,i)}(z_2) = (-qz_1)^{-1/2}\frac{(q^5z_2/z_1;q^4)_\infty}{(q^3z_2/z_1;q^4)_\infty} :\widetilde{\Psi}_{+}^{(i,1-i)}(z_1)\widetilde{\Phi}_{-}^{(1-i,i)}(z_2):, \quad (6.33)$$

$$\widetilde{\Phi}_{-}^{(i,1-i)}(z_2)\widetilde{\Psi}_{+}^{(1-i,i)}(z_1) = (-q^3z_2)^{-1/2}\frac{(qz_1/z_2;q^4)_\infty}{(z_1/qz_2;q^4)_\infty} :\widetilde{\Phi}_{-}^{(i,1-i)}(z_2)\widetilde{\Psi}_{+}^{(1-i,i)}(z_1):. \quad (6.34)$$

From these identities follow the commutation relations

$$\overline{R}(z_1,z_2)\overset{2}{\widetilde{\Psi}}{}^{(i,1-i)}(z_2)\overset{1}{\widetilde{\Psi}}{}^{(1-i,i)}(z_1) = -\left(\frac{z_1}{z_2}\right)^{i}\frac{(q^4z_1/z_2;q^4)_\infty}{(q^2z_1/z_2;q^4)_\infty}\frac{(q^2z_2/z_1;q^4)_\infty}{(q^4z_2/z_1;q^4)_\infty}\overset{1}{\widetilde{\Psi}}{}^{(i,1-i)}(z_1)\overset{2}{\widetilde{\Psi}}{}^{(1-i,i)}(z_2), \quad (6.35)$$

$$\overset{1}{\widetilde{\Psi}}{}^{(i,1-i)}(z_1)\overset{2}{\widetilde{\Phi}}{}^{(1-i,i)}(z_2) = \left(\frac{z_1}{z_2}\right)^{1-i}\frac{(qz_2/z_1;q^4)_\infty}{(q^3z_2/z_1;q^4)_\infty}\frac{(q^3z_1/z_2;q^4)_\infty}{(qz_1/z_2;q^4)_\infty}\overset{2}{\widetilde{\Phi}}{}^{(i,1-i)}(z_2)\overset{1}{\widetilde{\Psi}}{}^{(1-i,i)}(z_1). \quad (6.36)$$

Notice that the type I and type II operators mutually commute up to a scalar factor.

6.5 Dual vertex operators

Let us consider the intertwiners of the form

$$\begin{aligned}\widetilde{\Phi}^{*(1-i,i)}(z) &: V(\Lambda_i)\otimes V_z^{(h)} \to V(\Lambda_{1-i}),\\ \widetilde{\Psi}^{*(1-i,i)}(z) &: V_z^{(h)}\otimes V(\Lambda_i) \to V(\Lambda_{1-i}).\end{aligned}$$

We call them dual vertex operators. Define their components by

$$\begin{aligned}\widetilde{\Phi}_{\varepsilon}^{*(1-i,i)}(z)|v\rangle &= \widetilde{\Phi}^{*(1-i,i)}(z)(|v\rangle\otimes v_\varepsilon),\\ \widetilde{\Psi}_{\varepsilon}^{*(1-i,i)}(z)|v\rangle &= \widetilde{\Psi}^{*(1-i,i)}(z)(v_\varepsilon\otimes|v\rangle).\end{aligned}$$

We choose the normalization

$$\langle\Lambda_1|\widetilde{\Phi}^*_+(z)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\widetilde{\Phi}^*_-(z)|\Lambda_1\rangle = 1,$$
$$\langle\Lambda_1|\widetilde{\Psi}^*_+(z)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\widetilde{\Psi}^*_-(z)|\Lambda_1\rangle = 1.$$

Using the relations (3.29-3.30) and the isomorphism (5.26) we obtain

$$\widetilde{\Phi}^{*(1-i,i)}_\varepsilon(z) = (-q)^{i+\frac{\varepsilon-1}{2}}\widetilde{\Phi}^{(1-i,i)}_{-\varepsilon}(q^{-2}z), \tag{6.37}$$
$$\widetilde{\Psi}^{*(1-i,i)}_\varepsilon(z) = (-q)^{-i+\frac{1-\varepsilon}{2}}\widetilde{\Psi}^{(1-i,i)}_{-\varepsilon}(q^{2}z). \tag{6.38}$$

We have also

$$\widetilde{\Phi}^{(i,1-i)}_\varepsilon(z)\widetilde{\Phi}^{*(1-i,i)}_{\varepsilon'}(z) = g^{-1}\delta_{\varepsilon\varepsilon'}, \tag{6.39}$$
$$\widetilde{\Psi}^{(i,1-i)}_\varepsilon(z_1)\widetilde{\Psi}^{*(1-i,i)}_{\varepsilon'}(z_2) = g\delta_{\varepsilon\varepsilon'}\frac{1}{1-z_2/z_1} + \text{ regular at } z_1 = z_2 \tag{6.40}$$

where g is as in (4.28).

Let us prove the second. First we consider the case $\varepsilon' = -\varepsilon$. Using the commutation relation, we have

$$\widetilde{\Psi}^{(i,1-i)}_\varepsilon(z_1)\widetilde{\Psi}^{(1-i,i)}_\varepsilon(q^2z_2)$$
$$= -\left(\frac{q^2\zeta_2}{\zeta_1}\right)^i \frac{(z_1/z_2;q^4)_\infty(q^6z_2/z_1;q^4)_\infty}{(q^2z_1/z_2;q^4)_\infty(q^4z_2/z_1;q^4)_\infty}\widetilde{\Psi}^{(i,1-i)}_\varepsilon(q^2z_2)\widetilde{\Psi}^{(1-i,i)}_\varepsilon(z_1).$$

Suppose first that $\varepsilon = +$ and consider $\widetilde{\Psi}^{(i,1-i)}_+(q^2z_2)\widetilde{\Psi}^{(1-i,i)}_+(z_1)$ given by (6.32). Its simple pole at $z_1 = z_2$ is cancelled by the factor $(z_1/z_2;q^4)_\infty$. Therefore, this is regular. For $\varepsilon = -$, we need to compute the product with x_0^+ by using (6.26) and (6.27). An explicit calculation shows that there is no further pole at $z_1 = z_2$. We have shown $\widetilde{\Psi}^{(i,1-i)}_{-\varepsilon}(z_1)\widetilde{\Psi}^{*(1-i,i)}_\varepsilon(z_2)$ is regular at $z_1 = z_2$. Next consider the case $\varepsilon' = \varepsilon$. We have

$$\widetilde{\Psi}^{(i,1-i)}_-(z_1)\widetilde{\Psi}^{*(1-i,i)}_-(z_2)$$
$$= (\widetilde{\Psi}^{(i,1-i)}_+(z_1)x_0^+ - qx_0^+\widetilde{\Psi}^{(i,1-i)}_+(z_1))(-q)^{1-i}\widetilde{\Psi}^{(1-i,i)}_+(q^2z_2)$$
$$= (-qz_1)^{1/2}\frac{(q^2z_2/z_1;q^4)_\infty}{(q^4z_2/z_1;q^4)_\infty}\oint\frac{dw}{2\pi i} : \widetilde{\Psi}^{(i,1-i)}_+(z_1)\widetilde{\Psi}^{(1-i,i)}_+(q^2z_2)X^+(w) :$$
$$\times(-q)^{1-i}\left\{\frac{-1}{qz_1(1-\frac{w}{q^2z_1})} - q\frac{1}{w(1-\frac{z_1}{w})}\right\}\frac{1}{w(1-\frac{q^2z_2}{w})}.$$

The pole at $z_1 = z_2$ appears only in the first term of the integration. Using

$$: \widetilde{\Psi}_{+}^{(i,1-i)}(z)\widetilde{\Psi}_{+}^{(1-i,i)}(q^2z)X^{+}(q^2z) := (-q)^{i-1/2}z^{1/2},$$

we have (6.40). The other case is shown similarly.

Using (6.31) we obtain also

$$\sum_{\varepsilon=\pm} \widetilde{\Phi}_{\varepsilon}^{*(i,1-i)}(z)\widetilde{\Phi}_{\varepsilon}^{(1-i,i)}(z) = g^{-1}.$$

6.6 Principal picture

By the isomorphism (5.27) between the principal and the homogeneous pictures, the corresponding VO's are connected by

$$\begin{aligned}
\Phi_{\varepsilon}^{(1-i,i)}(\zeta) &= \zeta^{\frac{\varepsilon+1}{2}-i}\widetilde{\Phi}_{\varepsilon}^{(1-i,i)}(\zeta^2), \\
\Psi_{\varepsilon}^{(1-i,i)}(\zeta) &= \zeta^{\frac{\varepsilon+1}{2}-i}\widetilde{\Psi}_{\varepsilon}^{(1-i,i)}(\zeta^2).
\end{aligned}$$

This amounts to choosing the normalization

$$\begin{aligned}
\langle\Lambda_1|\Phi_-(\zeta)|\Lambda_0\rangle = 1, \qquad & \langle\Lambda_0|\Phi_+(\zeta)|\Lambda_1\rangle = 1, \\
\langle\Lambda_1|\Psi_-(\zeta)|\Lambda_0\rangle = 1, \qquad & \langle\Lambda_0|\Psi_+(\zeta)|\Lambda_1\rangle = 1.
\end{aligned}$$

The parity property (4.30) is manifest.

By the definition of highest weight modules, there is a $\mathbf{Z}_2$-symmetry exchanging 0 and 1. This means that we have the isomorphism of vector spaces

$$\begin{aligned}
&\nu : V(\Lambda_0) \longrightarrow V(\Lambda_1), \\
&\nu(|\Lambda_0\rangle) = |\Lambda_1\rangle, \qquad \nu(xu) = x\nu(u) \quad x \in U, u \in V(\Lambda_0).
\end{aligned}$$

Unfortunately this symmetry is not at all clear in the bosonization as it is based on the homogeneous picture. It is apparent rather in the principal picture. In fact we have the relations

$$\nu\Phi_{\varepsilon}^{(0,1)}(\zeta)\nu = \Phi_{-\varepsilon}^{(1,0)}(\zeta), \qquad \nu\Psi_{\varepsilon}^{(0,1)}(\zeta)\nu = \Psi_{-\varepsilon}^{(1,0)}(\zeta).$$

This implies, for instance,

$$\langle\Lambda_i|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_n}(\zeta_n)|\Lambda_j\rangle = \langle\Lambda_{1-i}|\Phi_{-\varepsilon_1}(\zeta_1)\cdots\Phi_{-\varepsilon_n}(\zeta_n)|\Lambda_{1-j}\rangle.$$

In the appendix, we summarize the properties of VO using the principal picture.

One of the important differences between type I and type II operators lies in the structure of poles. This is manifest from the formulas (A.16): As an operator the product $\Phi_{\varepsilon_1}(\zeta_1)\Phi_{\varepsilon_2}(\zeta_2)$ of type I operators is analytic up to $|\zeta_2/\zeta_1| < q^{-2}$ including $\zeta_1 = \zeta_2$, whereas for the type II operators $\Psi^*_{\varepsilon_1}(\zeta_1)\Psi^*_{\varepsilon_2}(\zeta_2)$ is analytic only in $|\zeta_2/\zeta_1| < q^2$ and has poles at $\zeta_2/\zeta_1 = \pm q$.

We end this chapter by giving a remark on the symmetry of matrix elements. Let ω be the involutive anti-automorphism of U given as follows:

$$\omega(q^h) = q^h, \qquad \omega(e_i) = -f_i t_i, \qquad \omega(f_i) = -t_i^{-1} e_i.$$

It can be shown by a standard argument that the irreducible highest weight module $V(\Lambda)$ admits a unique non-degenerate symmetric bilinear form (,) with the properties

$$(v_\Lambda, v_\Lambda) = 1, \qquad (u, xv) = (\omega(x)u, v) \quad \forall x \in U,\ u, v \in V(\Lambda).$$

Then the following relations hold for any $u, v \in V(\Lambda_i)$:

$$(u, \Phi_{\varepsilon,n} v) = (\Phi_{-\varepsilon,-n} u, v), \qquad (u, \Psi^*_{\varepsilon,n} v) = (\Psi^*_{-\varepsilon,-n} u, v). \tag{6.41}$$

This can be verified by induction on the degree of u, v, but we omit the details. Let

$$\overline{\omega} : V(\Lambda) \longrightarrow V^*(\Lambda)$$

be the identification map via (,), i.e. one such that $\overline{\omega}(|\Lambda\rangle) = \langle\Lambda|$ and $\overline{\omega}(xu) = \overline{\omega}(u)\omega(x)$ for $x \in U$ and $u \in V(\Lambda)$. Then the properties (6.41) can be rephrased as

$$\overline{\omega}(\Phi_{\varepsilon_1,n_1} \cdots \Phi_{\varepsilon_k,n_k}|\Lambda_i\rangle) = \langle\Lambda_i|\Phi_{-\varepsilon_k,-n_k} \cdots \Phi_{-\varepsilon_1,-n_1}$$

and likewise for Ψ^*. In particular we have

$$\langle\Lambda_i|\Phi_{\varepsilon_1}(\zeta_1) \cdots \Phi_{\varepsilon_k}(\zeta_k)|\Lambda_j\rangle = \langle\Lambda_j|\Phi_{-\varepsilon_k}(\zeta_k^{-1}) \cdots \Phi_{-\varepsilon_1}(\zeta_1^{-1})|\Lambda_i\rangle.$$

The same relation holds if we replace $\Phi_\varepsilon(\zeta)$ by $\Psi^*_\varepsilon(\zeta)$.

Chapter 7

Space of states— mathematical picture

In this chapter we return to the structures introduced in Chapter 4. We begin by identifying $\mathcal{H}^{(i)}$, $D^{(i)}$ and $\Phi^{(1-i,i)}$ with objects in representation theory. Using these as building blocks we shall define such notions as space of states, transfer matrix, Hamiltonian, vacuum and excited states.

7.1 Space of states

Recall the formula (4.23) for the character of the space $\mathcal{H}^{(i)}$. The basic observation is that it coincides with the principally specialized character of the level one modules (5.2):

$$\mathrm{tr}_{\mathcal{H}^{(i)}}\left(t^{D^{(i)}}\right) = \mathrm{tr}_{V(\Lambda_i)}\left(t^{-\rho+i/2}\right).$$

This may seem rather accidental at first sight. Nevertheless this type of coincidence between the characters of affine Lie algebras and the spectra of CTM has been observed for a wide class of models [18, 19, 20]. Subsequently it was established systematically by the theory of crystal bases [44]. We are thus led to make the following identification:

$$\mathcal{H}^{(i)} = V(\Lambda_i),$$
$$D^{(i)} = -\rho + \frac{i}{2}.$$

Moreover, one expects that the VO defined as a half-column transfer matrix should commute with the action of U. The argument goes in much the same way as with the full-column transfer matrix at the end of Chapter 3. The only difference is that for the upper half column the boundary on the top goes away in the limit but the central site does not. This is the reason why VO is a map of the form $\mathcal{H}^{(i)} \to \mathcal{H}^{(1-i)} \otimes V$. Thus we identify the VO's defined as the half-column transfer matrices with $\Phi^{(1-i,i)}(\zeta)$ defined mathematically as intertwiners of type I. With this identification we have already verified all the properties of VO listed in Chapter 4.

Roughly speaking, what we are doing is to regard the half-infinite tensor product as representing the level one module

$$\cdots \otimes V_{\zeta_H} \otimes V_{\zeta_H} \sim V(\Lambda_i),$$

where the choice of the i-th boundary condition is implied in the left hand side. Then how should one understand the tensor product extending in both directions?

To answer this question let us consider the following anti-automorphism

$$b(x) = (-q)^{\rho} a(x)(-q)^{-\rho} \qquad x \in U.$$

Explicitly we have

$$b(e_i) = q t_i^{-1} e_i, \quad b(f_i) = q^{-1} f_i t_i, \quad b(q^h) = q^{-h}.$$

A convenient feature of b is that $b^2 = \mathrm{id}.$, and that the following isomorphism holds:

$$C : V_\zeta \longrightarrow V_\zeta^{*b} \qquad v_\pm \otimes \zeta^n \mapsto v_\mp^* \otimes \zeta^n.$$

Moreover it is an anti-coalgebra homomorphism, so that

$$V_{\zeta_H} \otimes V_{\zeta_H} \otimes \cdots \overset{\mathcal{R}}{\simeq} V_{\zeta_H}^{*b} \otimes V_{\zeta_H}^{*b} \otimes \cdots \tag{7.1}$$

$$= (\cdots \otimes V_{\zeta_H} \otimes V_{\zeta_H})^{*b} \sim V(\Lambda_i)^{*b} \tag{7.2}$$

where we put $\mathcal{R} = C \otimes C \otimes \cdots$. Hence, assuming the i-th (resp. j-th) boundary condition to the left (resp. right) end, we are led to the identification

$$\cdots \otimes V_{\zeta_H} \otimes V_{\zeta_H} \otimes V_{\zeta_H} \otimes V_{\zeta_H} \otimes \cdots \sim V(\Lambda_i) \otimes V(\Lambda_j)^{*b}. \tag{7.3}$$

Notice that $V(\Lambda_j)^{*b}$ is a lowest weight module of level -1, and hence the right hand side has level 0. We now define:[1]

[1]In [23] $\mathcal{F}$ was defined using antipode a in place of b. Here we followed the convention of [59] which seems convenient to make contact with the naïve picture.

Space of states

$$\begin{aligned} \mathcal{F} &= \mathcal{H} \otimes \mathcal{H}^{*b} = \oplus_{i,j=0,1} \mathcal{F}^{(i,j)}, \\ \mathcal{H} &= V(\Lambda_0) \oplus V(\Lambda_1), \\ \mathcal{F}^{(i,j)} &= V(\Lambda_i) \otimes V(\Lambda_j)^{*b}. \end{aligned} \tag{7.4}$$

Note that this is a level 0 module. To be precise the space of states should be a proper completion of the algebraic tensor product (7.4), for operating with the VO's necessarily produces infinite sums. As we have explicit formulas for the space and VO's anyway, we will not discuss these points here. Henceforth we regard the VO's as acting on the direct sum $\mathcal{H}$, and drop the superscripts $(1-i, i)$.

7.2 Translation and local operators

At first glance the argument leading to the identification (7.4) seems artificial, as it breaks the translational invariance of the problem. One may also take

$$V(\Lambda_{i'}) \otimes V^{\otimes n} \otimes V(\Lambda_{j'})^{*b} \tag{7.5}$$

as an alternative of the $\mathcal{F}^{(i,j)}$ (with an appropriate choice of i', j'), since after all what matters is only the boundary condition. Fortunately these questions can be resolved by using the type I operators.

To simplify the notations we will suppress the index i and write $D = -\rho + i/2$, etc.. Using the isomorphisms

$$\begin{aligned} V(\Lambda_i) &\longrightarrow V(\Lambda_{1-i}) \otimes V, & u &\mapsto \sqrt{g} \sum_\varepsilon \Phi_\varepsilon(1) u \otimes v_\varepsilon, \\ V \otimes V(\Lambda_j)^{*b} &\longrightarrow V(\Lambda_{1-j})^{*b}, & v_\varepsilon \otimes u^* &\mapsto \sqrt{g} \Phi_{-\varepsilon}(1)^t u^*, \end{aligned}$$

we may identify

$$V(\Lambda_i) \otimes V(\Lambda_j)^{*b} \xrightarrow{\sim} V(\Lambda_{1-i}) \otimes V \otimes V(\Lambda_j)^{*b} \xrightarrow{\sim} V(\Lambda_{1-i}) \otimes V(\Lambda_{1-j})^{*b}. \tag{7.6}$$

It is also clear that (7.5) can be identified with $\mathcal{F}^{(i,j)}$. Composing (7.6) we define

Translation operator

$$T = g \sum_\varepsilon \Phi_\varepsilon(1) \otimes \Phi_{-\varepsilon}(1)^t,$$

whose inverse is

$$T^{-1} = g \sum_{\varepsilon} \Phi^*_{\varepsilon}(1) \otimes \Phi^*_{-\varepsilon}(1)^t.$$

Using the middle realization of (7.6), one can define the local operators on V as operators on $\mathcal{F}^{(ij)}$. Let us label the tensor components from the middle as $1, 2, \cdots$ for the left half, $0, -1, \cdots$ for the right half. Then the operators acting on the site 1 are defined by

$$\begin{aligned} E_{\varepsilon\varepsilon'} &= g\, \Phi^*_{\varepsilon}(1)\Phi_{\varepsilon'}(1) \otimes \mathrm{id}, \\ \sigma_1^{\pm} &= E_{\pm\mp}, \qquad \sigma_1^z = E_{++} - E_{--}. \end{aligned} \tag{7.7}$$

Here $\sigma^{\pm} = (\sigma^x \pm i\sigma^y)/2$. More generally we set

$$\sigma_n^{\alpha} = T^{-(n-1)}\, \sigma_1^{\alpha}\, T^{n-1} \quad (n \in \mathbf{Z}).$$

For instance

$$\begin{aligned} \sigma_2^{\pm} &= g^2 \sum_{\varepsilon} \Phi^*_{\varepsilon}(1)\Phi^*_{\pm}(1)\Phi_{\mp}(1)\Phi_{\varepsilon}(1) \otimes \mathrm{id}, \\ \sigma_0^{\pm} &= g\, \mathrm{id} \otimes \left(\Phi^*_{\pm}(1)\Phi_{\mp}(1)\right)^t. \end{aligned}$$

7.3 Transfer matrix

Let us consider how the column transfer matrix $T(\zeta) = T_{\mathrm{col}}(\zeta)$ looks like in this picture. From the graphical definition of VO, its elements are given by

$$\begin{aligned} T(\zeta)^{p'_N,\cdots,p'_{-N+1}}_{p_N,\cdots,p_{-N+1}} &= \sum_{\varepsilon} (\Phi_{UP\varepsilon}(\zeta))^{p'_N,\cdots,p'_1}_{p_N,\cdots,p_1} \times (\Phi_{LOW\varepsilon}(\zeta))^{p'_0,\cdots,p'_{-N+1}}_{p_0,\cdots,p_{-N+1}} \\ &= \sum_{\varepsilon} (\Phi_{\varepsilon}(\zeta))^{p'_N,\cdots,p'_1}_{p_N,\cdots,p_1} \times (\Phi_{-\varepsilon}(\zeta))^{-p_0,\cdots,-p_{-N+1}}_{-p'_0,\cdots,-p'_{-N+1}}. \end{aligned}$$

On the other hand, if we regard $\Phi(\zeta) = \sum_{\varepsilon} \Phi_{\varepsilon}(\zeta) \otimes v_{\varepsilon}$ as a map

$$(\cdots \otimes V_{\zeta_H} \otimes V_{\zeta_H}) \longrightarrow (\cdots \otimes V_{\zeta_H} \otimes V_{\zeta_H}) \otimes V_{\zeta_V},$$

then its transpose

$$V^{*b}_{\zeta_V} \otimes \left(V^{*b}_{\zeta_H} \otimes V^{*b}_{\zeta_H} \otimes \cdots\right) \longrightarrow \left(V^{*b}_{\zeta_H} \otimes V^{*b}_{\zeta_H} \otimes \cdots\right)$$

is a map

$$v_{\varepsilon} \otimes (v_{p'_1} \otimes v_{p'_2} \otimes \cdots) \mapsto \sum (v_{p_1} \otimes v_{p_2} \otimes \cdots)\,(\Phi_{-\varepsilon}(\zeta))^{-p_0,\cdots,-p_{-N+1}}_{-p'_0,\cdots,-p'_{-N+1}}.$$

Here we have used the identification (7.2).

This motivates us to define

Transfer matrix

$$T(\zeta) = g\sum_{\varepsilon} \Phi_\varepsilon(\zeta) \otimes (\Phi_{-\varepsilon}(\zeta))^t . \qquad (7.8)$$

The translation operator is $T = T(1)$ as expected. Conversely, using the homogeneity of VO we can write

$$T(\zeta) = \zeta^D T \zeta^{-D}$$

where D denotes its action on $\mathcal{H} \otimes \mathcal{H}^{*b}$, i.e. $D \otimes \mathrm{id} - \mathrm{id} \otimes D^t$. Hence the transfer matrices are manifestly commutative with each other. Notice also that

$$T(\zeta)^{-1} = g\sum_{\varepsilon} \Phi^*_\varepsilon(\zeta) \otimes (\Phi^*_{-\varepsilon}(\zeta))^t. \qquad (7.9)$$

Comparing with the formula (3.14) we define

XXZ Hamiltonian

$$H = \frac{1-q^2}{2q}\zeta\frac{d}{d\zeta}\log T(\zeta)\Big|_{\zeta=1}. \qquad (7.10)$$

7.4 Vacuum

Often it is convenient to regard a state $f \in \mathcal{F}$ as a linear map on $\mathcal{H}$ according to the relation $\mathcal{H} \otimes \mathcal{H}^{*b} \simeq \mathrm{End}(\mathcal{H})$. [1] In this language the left action of U takes the form

$$x.f = \sum x_{(1)} \circ f \circ b(x_{(2)}), \qquad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}. \qquad (7.11)$$

A linear operator of the form $\mathcal{O} = \phi \otimes \psi$ ($\phi \in \mathrm{End}(\mathcal{H})$ and $\psi \in \mathrm{End}(\mathcal{H}^{*b})$) operates on a state f as

$$f \mapsto \phi \circ f \circ \psi^t. \qquad (7.12)$$

The dual space is $\mathcal{F}^* \simeq \mathcal{H} \otimes \mathcal{H}^{*b}$ via the pairing

$$\langle f, g\rangle = \mathrm{tr}_{\mathcal{H}}(f \circ g).$$

[1] Strictly speaking '$\simeq$' should be understood properly as $\mathcal{H}$ is infinite dimensional.

The right action of U is given by

$$f.x = \sum b(x_{(2)}) \circ f \circ x_{(1)}. \tag{7.13}$$

Let $P^{(i)}$ denote the projection $\mathcal{H} \to V(\Lambda_i)$. We call the element

$$|\mathrm{vac}\rangle_{(i)} = \chi^{-1/2}(-q)^{D^{(i)}} P^{(i)} \in \mathcal{F}$$

the *vacuum* in the i-th sector, where χ denotes the principally specialized character (5.1)

$$\chi = \chi_{\Lambda_i}(q^2) = \prod_{n=1}^{\infty}(1 - q^{4n-2})^{-1}. \tag{7.14}$$

When regarded as an element of $\mathcal{F}^*$ it is denoted by ${}_{(i)}\langle\mathrm{vac}|$. In what follows we will suppress $P^{(i)}$ which should be clear from the context. For an operator $\mathcal{O}$ on $\mathcal{F}$ its vacuum expectation value is the coupling of ${}_{(i)}\langle\mathrm{vac}|$ and $\mathcal{O}|\mathrm{vac}\rangle_{(i)}$. In particular if $\mathcal{O}$ is of the type $\mathcal{O} = \phi \otimes \mathrm{id}$ or $= \mathrm{id} \otimes \phi^t$, then its vacuum expectation value reads

$${}_{(i)}\langle\mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle_{(i)} = \frac{\mathrm{tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\phi\right)}{\mathrm{tr}_{V(\Lambda_i)}\left(q^{2D^{(i)}}\right)}. \tag{7.15}$$

Physically one expects that the vacuum vectors are translationally invariant and a singlet (i.e. belongs to the trivial representation of U). These are immediate consequences of the definition. In fact,

$$\begin{aligned} T|\mathrm{vac}\rangle_{(i)} &= \chi^{-1/2} g \sum_{\varepsilon} \Phi_\varepsilon(\zeta)(-q)^{D^{(i)}}\Phi_{-\varepsilon}(\zeta) \\ &= \chi^{-1/2} g \sum_{\varepsilon} \Phi_\varepsilon(\zeta)\Phi_{-\varepsilon}(-q\zeta)(-q)^{D^{(1-i)}} \\ &= |\mathrm{vac}\rangle_{(1-i)}, \end{aligned}$$

where we used (A.4.4). Similarly,

$$\begin{aligned} x.|\mathrm{vac}\rangle_{(i)} &= \chi^{-1/2}\sum x_{(1)}(-q)^{D^{(i)}}\, b(x_{(2)}) \\ &= \chi^{-1/2}\sum x_{(1)}a(x_{(2)})(-q)^{D^{(i)}} \\ &= \varepsilon(x)|\mathrm{vac}\rangle_{(i)}. \end{aligned}$$

In the third line we used (3.18).

7.5 Eigenstates

In order to construct general eigenstates we employ the type II operators. For $\xi_1, \cdots, \xi_n$ with $|\xi_j| = 1$ we define the n-particle states

$$|\xi_n, \cdots, \xi_1\rangle_{\varepsilon_n, \cdots, \varepsilon_1; (i)} = g^{-n/2}\chi^{-1/2}\Psi^*_{\varepsilon_n}(\xi_n)\cdots\Psi^*_{\varepsilon_1}(\xi_1)(-q)^{D^{(i)}}, \quad (7.16)$$

$$_{(i);\varepsilon_1, \cdots, \varepsilon_n}\langle\xi_1, \cdots, \xi_n| = g^{-n/2}\chi^{-1/2}(-q)^{D^{(i)}}\Psi_{\varepsilon_1}(\xi_1)\cdots\Psi_{\varepsilon_n}(\xi_n). \quad (7.17)$$

These are vectors in $\mathcal{F}^{(i+n,i)}$, $\mathcal{F}^{*(i+n,i)}$ respectively. The product of operators such as (7.16) is understood as analytic continuation from the domain of convergence $|\xi_n| \gg \cdots \gg |\xi_1|$ to the unit circle. The series obtained by expanding the operators in power of the spectral parameters are not absolutely convergent on the unit circles $|\xi_i| = 1$ because in the analytic continuation we encounter poles.

Using the commutation relation with the type I operators (A.4.3) one verifies that

$$\begin{aligned} T(\zeta)|\xi_n, \cdots, \xi_1\rangle_{\varepsilon_n, \cdots, \varepsilon_1; (i)} &= c\sum_{\varepsilon} \Phi_\varepsilon(\zeta)\Psi^*_{\varepsilon_n}(\xi_n)\cdots\Psi^*_{\varepsilon_1}(\xi_1)(-q)^{D^{(i)}}\Phi_\varepsilon(\zeta) \\ &= c\prod_{j=1}^{n} \tau(\zeta/\xi_j)\,\Psi^*_{\varepsilon_n}(\xi_n)\cdots\Psi^*_{\varepsilon_1}(\xi_1)(-q)^{D^{(1-i)}} \\ &\qquad\qquad \times \sum_{\varepsilon}\Phi_\varepsilon(-q^{-1}\zeta)\Phi_\varepsilon(\zeta) \\ &= \prod_{j=1}^{n}\tau(\zeta/\xi_j)\,|\xi_n, \cdots, \xi_1\rangle_{\varepsilon_n, \cdots, \varepsilon_1; (1-i)}. \end{aligned}$$

Here we put $c = g^{1-n/2}\chi^{-1/2}$ and used $\tau(-q\zeta) = \tau(\zeta^{-1}) = \tau(\zeta)^{-1}$. Therefore (7.16) are eigenstates of $T(\zeta)^2$. Likewise for the bra vectors. In particular the eigenvalues of the translation operator T^2 and the Hamiltonian H on the one-particle states $|\xi\rangle_{\varepsilon;(i)}$ are given by

$$\begin{aligned} T^2|\xi\rangle_{\varepsilon;(i)} &= \tau(\xi)^{-2}|\xi\rangle_{\varepsilon;(i)}, \\ H|\xi\rangle_{\varepsilon;(i)} &= \epsilon(\xi)|\xi\rangle_{\varepsilon;(i)}, \end{aligned}$$

where

$$\begin{aligned} \tau(\xi) &= \xi^{-1}\frac{\Theta_{q^4}(q\xi^2)}{\Theta_{q^4}(q\xi^{-2})}, \\ \epsilon(\xi) &= \frac{1-q^2}{2q}\xi\frac{d}{d\xi}\log\tau(\xi). \end{aligned}$$

Let us rewrite $\tau(\xi)$ and $\epsilon(\xi)$ in terms of Jacobi's elliptic functions [35, 13]. We set

$$\xi = -ie^{i\theta}, \qquad q = -e^{-\pi K'/K},$$

where K, K' are the complete elliptic integrals. Because we are working in the region $-1 < q < 0$, the nome of the elliptic functions is chosen to be $-q$. In this parametrization we have

$$\tau(\xi) = \mathrm{sn}\left(\frac{2K}{\pi}\theta\right) + i\,\mathrm{cn}\left(\frac{2K}{\pi}\theta\right),$$

or equivalently

$$\tau(\xi) = e^{-ip(\theta)}, \qquad p(\theta) = \mathrm{am}\left(\frac{2K}{\pi}\theta\right) - \frac{\pi}{2}. \tag{7.18}$$

By differentiation we also obtain

$$\epsilon(\xi) = \frac{2K}{\pi}\sinh\frac{\pi K'}{K}\,\mathrm{dn}\left(\frac{2K}{\pi}\theta\right). \tag{7.19}$$

These formulas coincide with the momentum/energy of the elementary excitations derived from the Bethe Ansatz method [7].

Let us consider the action of U' on the eigenstates. For $f \in \mathrm{End}\,(\mathcal{H})$ we define the left and right adjoint actions of $x \in U'$ by

$$\mathrm{ad}\,x.f = \sum x_{(1)}\,f\,a(x_{(2)}), \qquad f.\mathrm{ad}^r x = \sum a^{-1}(x_{(2)})\,f\,x_{(1)},$$

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. They are related to the left (resp. right) actions (see (7.11) (resp. (7.13))) on $\mathcal{F}$ (resp. $\mathcal{F}^*$) via

$$x.\left(f(-q)^D\right) = (\mathrm{ad}\,x.f)(-q)^D, \qquad \left((-q)^D f\right).x = (-q)^D\,(f.\mathrm{ad}^r x).$$

Moreover they are compatible with the composition of maps in the sense that

$$\begin{aligned}\mathrm{ad}\,x.(fg) &= \sum \left(\mathrm{ad}\,(x_{(1)}).f\right)\left(\mathrm{ad}\,(x_{(2)}).g\right),\\ (fg).\mathrm{ad}^r x &= \sum \left(f.\mathrm{ad}\,(x_{(2)})\right)\left(g.\mathrm{ad}\,(x_{(1)})\right).\end{aligned}$$

Using (3.17), (3.18) one can verify

$$\begin{aligned}\mathrm{ad}\,x.\Psi^*_\varepsilon(\xi) &= \Psi^*(\xi)\,(xv_\varepsilon \otimes \cdot),\\ \Psi_\varepsilon(\xi).\mathrm{ad}^r x &= (\langle v^*_\varepsilon x, \cdot\rangle \otimes \mathrm{id}\,)\,\Psi(\xi).\end{aligned}$$

Therefore, the left and right hand sides of the following share the same transformation properties under the action of U':

$$\begin{aligned} |\xi_n,\cdots,\xi_1\rangle_{\varepsilon_n,\cdots,\varepsilon_1;(i)} &\longleftrightarrow v_{\varepsilon_n}\otimes\cdots\otimes v_{\varepsilon_1} \in V_{\xi_n}\otimes\cdots\otimes V_{\xi_1}, \\ {}_{(i);\varepsilon_1,\cdots,\varepsilon_n}\langle\xi_1,\cdots,\xi_n| &\longleftrightarrow v^*_{\varepsilon_1}\otimes\cdots\otimes v^*_{\varepsilon_n} \in V^*_{\xi_1}\otimes\cdots\otimes V^*_{\xi_n}. \end{aligned}$$

In other words the eigenstates give rise to the embedding

$$V_{\xi_n}\otimes\cdots\otimes V_{\xi_1} \subset \mathcal{H}^{(i)}\otimes\mathcal{H}^{(i)*b} \text{ if } n \text{ is even},$$

and

$$V_{\xi_n}\otimes\cdots\otimes V_{\xi_1} \subset \mathcal{H}^{(1-i)}\otimes\mathcal{H}^{(i)*b} \text{ if } n \text{ is odd}.$$

In [28] Faddeev and Takhtajan pointed out that the particles of the XXX model in the anti-ferromagnetic regime transform as a two-dimensional representation of $sl(2)$. This means that, in the same way as above, the n-particle states constitute the n-fold tensor of $\mathbf{C}^2$ as a representation of $sl(2)$. However the action of the full *affine* algebra $\widehat{sl}_2$ was not discussed there.

We expect that these eigenstates satisfy further the orthogonality

$${}_{(i);\varepsilon_1,\cdots,\varepsilon_m}\langle\xi_1,\cdots,\xi_m|\xi'_n,\cdots,\xi'_1\rangle_{\varepsilon'_n,\cdots,\varepsilon'_1;(i)} = 0 \qquad (m\neq n), \tag{7.20}$$

and the completeness relations

$$\begin{aligned} \mathrm{id}_{\mathcal{F}} = \sum_{i=0,1}\sum_{n\geq 0}\sum_{\varepsilon_n,\cdots,\varepsilon_1} \frac{1}{n!}\oint\frac{d\xi_n}{2\pi i\xi_n}\cdots\oint\frac{d\xi_1}{2\pi i\xi_1} \\ \times|\xi_n,\cdots,\xi_1\rangle_{\varepsilon_n,\cdots,\varepsilon_1;(i)}\,{}_{(i);\varepsilon_1,\cdots,\varepsilon_n}\langle\xi_1,\cdots,\xi_n|. \end{aligned} \tag{7.21}$$

Notice that thanks to the commutation relation (A.2) the combination

$$\sum_{\varepsilon_n,\cdots,\varepsilon_1} |\xi_n,\cdots,\xi_1\rangle_{\varepsilon_n,\cdots,\varepsilon_1;(i)}\,{}_{(i);\varepsilon_1,\cdots,\varepsilon_n}\langle\xi_1,\cdots,\xi_n|$$

is symmetric in the variables $\xi_1,\cdots,\xi_n$.

We will come back to these relations (including the case $m = n$ of (7.20)) after deriving the integral formulas for the trace of VO's.

Chapter 8

Traces of vertex operators

We calculate the trace functions by using the bosonization of the vertex operators. The resulting expressions are integrals of certain meromorphic functions written in an infinite product form.

8.1 Calculating the trace

We will calculate the trace functions;

$$\operatorname{tr}_{V(\Lambda_i)}\left(x^D y^h \Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_m}(\zeta_m)\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)\right). \tag{8.1}$$

In this section, the parameter x is assumed to be sufficiently small. After establishing that (8.1) is meromorphic in all the variables, we specialize x to q^2 by analytic continuation in order to obtain the correlation functions discussed in Chapter 4.

We use the boson realization of the spaces $V(\Lambda_i)$ $(i = 0, 1)$ discussed in Chapter 5;

$$V(\Lambda_i) = \mathbf{C}[a_{-1}, a_{-2}, \cdots] \otimes \left(\oplus_{l\in\mathbf{Z}} e^{\Lambda_i + l\alpha}\right).$$

To avoid confusion, let us use the (bra)cket notation such as $a_{-1}|0\rangle$ to designate the vectors in $\mathbf{C}[a_{-1}, a_{-2}, \cdots]$.

For convenience, we introduce operators $\varphi(\zeta)$, $\psi^*(\xi)$, $\chi^-(w)$ and $\chi^+(v)$ which are all of the form

$$e^{\sum_{n=1}^\infty A_n a_{-n}} e^{\sum_{n=1}^\infty B_n a_n} e^{c\alpha} f^{\partial}.$$

The choices of A_n, B_n, c and f are given in the following table. In the subsequent calculation we use different letters ζ, ξ, w and v for the variables of these four operators, in order to distinguish contributions from each of them.

	A_n	B_n	c	f
$\varphi(\zeta)$	$\frac{q^{7n/2}}{[2n]}\zeta^{2n}$	$-\frac{q^{-5n/2}}{[2n]}\zeta^{-2n}$	$\frac{1}{2}$	$(-q^3\zeta^2)^{1/2}$
$\psi^*(\zeta)$	$-\frac{q^{5n/2}}{[2n]}\xi^{2n}$	$\frac{q^{-7n/2}}{[2n]}\xi^{-2n}$	$-\frac{1}{2}$	$(-q^3\xi^2)^{-1/2}$
$\chi^-(w)$	$-\frac{q^{5n/2}}{[n]}w^n$	$\frac{q^{-3n/2}}{[n]}w^{-n}$	-1	$(q^2w)^{-1}$
$\chi^+(v)$	$\frac{q^{3n/2}}{[n]}(-v)^n$	$-\frac{q^{-5n/2}}{[n]}(-v)^{-n}$	1	$-q^2v$

Notice that $\chi^-(w) = X^-(q^2w)$ and $\chi^+(v) = X^+(-q^2v)$. The operators $\Phi_\varepsilon^{(1-i,i)}(\zeta)$ and $\Psi_\varepsilon^{*(1-i,i)}(\xi)$ are given as follows.

$$\Phi_-^{(1-i,i)}(\zeta) = (-q^3)^{i/2}\varphi(\zeta), \tag{8.2}$$

$$\Phi_+^{(1-i,i)}(\zeta) = (-q^3)^{i/2}q^2\zeta\oint\frac{dw}{2\pi i}(\varphi(\zeta)\chi^-(w)-q\chi^-(w)\varphi(\zeta)), \tag{8.3}$$

$$\Psi_-^{*(1-i,i)}(\xi) = (-q^3)^{i/2}\xi\psi^*(\xi), \tag{8.4}$$

$$\Psi_+^{*(1-i,i)}(\xi) = (-q^3)^{i/2}q\oint\frac{dv}{2\pi i}(\psi^*(\xi)\chi^+(v)-q\chi^+(v)\psi^*(\xi)). \tag{8.5}$$

Here the integration symbol $\oint\frac{dw}{2\pi i}$ means taking the (-1)-th coefficient in the Laurent expansion in w.

We use the formula

$$\begin{aligned}&\mathrm{tr}_{V(\Lambda_i)}\left(x^D e^{\sum_{n=1}^\infty A_n a_{-n}} e^{\sum_{n=1}^\infty B_n a_n} e^{c\alpha} f^{\partial}\right) \qquad (8.6)\\ &= \delta_{c,0}(x^2)_\infty^{-1} f^i\Theta_{x^4}(-x^{1+2i}f^2)\exp\left(\sum_{m=1}^\infty\sum_{n=1}^\infty x^{2mn}A_nB_n\frac{[2n][n]}{n}\right),\end{aligned}$$

where $(z)_\infty$ and $\Theta_p(z)$ will be defined in (8.17).

Eq.(8.6) can be verified as follows. Clearly we must have $\delta_{c,0}$ since the operator $e^{c\alpha}$ shifts e^β to $e^{\beta+c\alpha}$. We can calculate the contributions in the trace for the tensor components (i) $\sum_{k=0}^\infty \mathbf{C}a_{-n}^k|0\rangle$ $(n\geq 1)$ and (ii) $\sum_{l=1}^\infty \mathbf{C}e^{\Lambda_i+l\alpha}$, separately. Notice that the operators D and ∂ are diagonal on the bases $a_{-n_1}\cdots a_{-n_r}|0\rangle\otimes e^{\Lambda_i+l\alpha}$ with the eigenvalues

$$D : 2\sum_{j=1}^r n_j + 2l^2 - l + 2il$$

and

$$\partial : i + 2l,$$

respectively. Therefore, for (ii) we have

$$\sum_{l \in \mathbf{Z}} x^{2l^2 - l + 2il} f^{i+2l} = f^i \Theta_{x^4}(-x^{1+2i} f^2).$$

For (i) we use

$$e^{A_n a_{-n}} e^{B_n a_n} a_{-n}^k |0\rangle = e^{A_n a_{-n}} \left(a_{-n} + B_n \frac{[2n][n]}{n} \right)^k |0\rangle.$$

Taking the coefficient of $a_{-n}^k|0\rangle$, multiplying x^{2kn} (coming from the term $2\sum_{j=1}^r n_j$ in the eigenvalue of D) and summing up with respect to k we get

$$\sum_{k=0}^{\infty} r^k \sum_{l=0}^{k} \frac{k!}{(l!)^2 (k-l)!} s^l = \frac{e^{\frac{rs}{1-r}}}{1-r}$$

where

$$r = x^{2n} \quad \text{and } s = A_n B_n \frac{[2n][n]}{n}.$$

Let us introduce $h_i(z)$ $(i = 1, 2, 3)$ by

$$h_1(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \right) = 1 - z, \tag{8.7}$$

$$h_2(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{[2n] z^n}{n[n]} \right) = (1 - qz)(1 - q^{-1} z), \tag{8.8}$$

$$h_3(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{[n] z^n}{n[2n]} \right) = \frac{(qz; q^4)_\infty}{(q^3 z; q^4)_\infty}. \tag{8.9}$$

We also use $h_i^{(\infty)}(z) = \prod_{n=0}^{\infty} h_i(x^{2n} z)$.

We wish to calculate $\mathrm{tr}_{V(\Lambda_i)} \left(x^D y^\partial \mathcal{O} \right)$ where $\mathcal{O}$ is a product (in a certain order) of the operators

$$\begin{aligned} &\varphi(\zeta_j) \quad (1 \le j \le m), \\ &\psi^*(\xi_k) \quad (1 \le k \le n), \\ &\chi^-(w_a) \quad (a \in A), \\ &\chi^+(v_b) \quad (b \in B), \end{aligned} \tag{8.10}$$

where we define the index sets

$$\begin{aligned} A &= \{j; 1 \le j \le m, \varepsilon_j = +\}, \\ B &= \{k; 1 \le k \le n, \mu_k = +\}. \end{aligned} \tag{8.11}$$

In fact, we must take care not only of products but also of sums and integrals of the operators as we see in (8.3) and (8.5). We will come back to this point later.

The procedure of the calculation goes as follows. First, we normal-order the product, and then apply the formula (8.6). Thus we get a Laurent series in the variables ζ_j, ξ_k, w_a and v_b. We will see that the series has a certain domain of convergence, and is equal to a meromorphic function.

Let us explain this procedure in detail by taking the product $\varphi(\zeta_1)\varphi(\zeta_2)$ as an example.

The normal-ordering gives rise to

$$\varphi(\zeta_1)\varphi(\zeta_2) = c_{\varphi\varphi}(\zeta_1)h_{\varphi\varphi}(\zeta_1,\zeta_2) : \varphi(\zeta_1)\varphi(\zeta_2) :, \tag{8.12}$$

$$c_{\varphi\varphi}(\zeta_1) = (-q^3\zeta_1)^{1/2}, \tag{8.13}$$

$$h_{\varphi\varphi}(\zeta_1,\zeta_2) = h_3(q\zeta_2^2/\zeta_1^2). \tag{8.14}$$

The factor $c_{\varphi\varphi}(\zeta_1)$ comes from the product $(-q^3\zeta_1)^{\partial/2}e^{\alpha/2}$, and $h_{\varphi\varphi}(\zeta_1,\zeta_2)$ from $\exp\left(\sum -q^{-5n/2}\zeta_1^{-2n}a_n/[2n]\right)\exp\left(\sum q^{7n/2}\zeta_2^{2n}a_{-n}/[2n]\right)$. In summing up the series $h_{\varphi\varphi}(\zeta_1,\zeta_2)$, we used the product formula (8.9). The convergence domain is known to be $|\zeta_2/\zeta_1| < q^{-2}$.

In applying (8.6), we must take into account both of the contributions from

$$(A_n, B_n) = \left(\frac{q^{7n/2}}{[2n]}\zeta_2^{2n}, -\frac{q^{-5n/2}}{[2n]}\zeta_1^{-2n}\right)$$

and

$$(A_n, B_n) = \left(\frac{q^{7n/2}}{[2n]}\zeta_1^{2n}, -\frac{q^{-5n/2}}{[2n]}\zeta_2^{-2n}\right).$$

They are $h_3^{(\infty)}(x^2q\zeta_2^2/\zeta_1^2)$ and $h_3^{(\infty)}(x^2q\zeta_1^2/\zeta_2^2)$, respectively. The convergence domain for the product of these two series is known to be $xq^2 < |\zeta_2/\zeta_1| < (xq^2)^{-1}$.

As a whole, the contribution from the product $\varphi(\zeta_1)\varphi(\zeta_2)$ is

$$(-q^3\zeta_1)^{1/2}h_3^{(\infty)}(q\zeta_2^2/\zeta_1^2)h_3^{(\infty)}(x^2q\zeta_1^2/\zeta_2^2),$$

and the convergence domain is

$$xq^2 < |\zeta_2/\zeta_1| < q^{-2}.$$

For each pair of the operators in (8.10), we want to calculate the contribution of the normal-ordering (8.12) and that of the trace operation (8.6). As above, the latter is easily obtained from the former. We give the list of the c-terms (8.13) and the h-terms (8.14) in the tables below.

Table of c-terms

	$\varphi(\zeta')$	$\psi^*(\xi')$	$\chi^-(w')$	$\chi^+(v')$
$\varphi(\zeta)$	$(-q^3\zeta^2)^{1/2}$	$(-q^3\zeta^2)^{-1/2}$	$(-q^3\zeta^2)^{-1}$	$-q^3\zeta^2$
$\psi^*(\xi)$	$(-q^3\xi^2)^{-1/2}$	$(-q^3\xi^2)^{1/2}$	$-q^3\xi^2$	$(-q^3\xi^2)^{-1}$
$\chi^-(w)$	w^{-1}	w	w^2	w^{-2}
$\chi^+(v)$	v	v^{-1}	v^{-2}	v^2

Table of h-terms ($z = \zeta^2$, $z' = \zeta'^2$, $u = -\xi^2$, $u' = -\xi'^2$)

	$\varphi(\zeta')$	$\psi^*(\xi')$	$\chi^-(w')$	$\chi^+(v')$
$\varphi(\zeta)$	$h_3(qz'/z)$	$h_3(-u'/z)^{-1}$	$h_1(w'/z)^{-1}$	$h_1(-q^{-1}v'/z)$
$\psi^*(\xi)$	$h_3(-z'/u)^{-1}$	$h_3(q^{-1}u'/u)$	$h_1(-q^{-1}w'/u)$	$h_1(q^{-2}v'/u)^{-1}$
$\chi^-(w)$	$h_1(q^2z'/w)^{-1}$	$h_1(-qu'/w)$	$h_2(qw'/w)$	$h_2(-v'/w)^{-1}$
$\chi^+(v)$	$h_1(-qz'/v)$	$h_1(u'/v)^{-1}$	$h_2(-w'/v)^{-1}$	$h_2(q^{-1}v'/v)$

Let us discuss the convergence domain. As we discussed, for each pair of $\varphi(\zeta_j)$ and $\varphi(\zeta_{j'})$ such that the former sits to the left of the latter in the product, the condition for the convergence is $x^2q^4 < |z_{j'}/z_j| < q^{-4}$. From the table of h-terms, we can list such a condition for each pair. The condition differs for each pair. Nevertheless, we can observe the following common feature; For any pair z_L and z_R among the variables z_j $(1 \le j \le m)$, u_k $(1 \le k \le n)$, w_a $(a \in A)$, v_b $(b \in B)$, where the operator corresponding to z_L sits to the left of the operator corresponding to z_R, the condition for the convergence is of the form

$$x^2q^{j_1} < |z_R/z_L| < q^{j_2}$$

for some j_1 and j_2. Therefore, if x is sufficiently small, we can take a non-empty domain of convergence common to all the variables. In this domain, the Laurent series sums up to a meromorphic function as we can read from the tables of the c-terms and the h-terms.

In the formulas (8.3) and (8.5) by the integration symbol we meant to take the (-1)-th coefficient in the Laurent series expansion of the operator. Now, after the calculation of the trace, we can take the (-1)-th coefficient in the Laurent series expansion of the meromorphic function by an actual integration with respect to an appropriate cycle in the convergence domain. Thus, we have obtained an integral formula for the trace function. In the next section, we write down the integrand and the cycles explicitly.

8.2 Result

Let us summarize the result for the trace. Notice the $\mathbf{Z}_2$ symmetry

$$\begin{aligned} &\mathrm{tr}_{V(\Lambda_0)}\left(x^D y^h \Phi_{\epsilon_1}(\zeta_1)\cdots\Phi_{\epsilon_m}(\zeta_m)\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)\right) \\ &= \mathrm{tr}_{V(\Lambda_1)}\left(x^D y^{1-h}\Phi_{-\epsilon_1}(\zeta_1)\cdots\Phi_{-\epsilon_m}(\zeta_m)\Psi^*_{-\mu_n}(\xi_n)\cdots\Psi^*_{-\mu_1}(\xi_1)\right). \end{aligned} \tag{8.15}$$

Recall the definition of the index set A and B in (8.10). Set

$$s = \sharp A, \quad t = \sharp B.$$

Because of the Kronecker symbol in (8.6), the trace (8.1) vanishes unless

$$m - n = 2(s-t). \tag{8.16}$$

In the sequel the indices j, k, a, b are understood to run over

$$1 \le j \le m, \quad 1 \le k \le n, \quad a \in A, \quad b \in B,$$

respectively. We set

$$z_j = \zeta_j^2, \quad u_k = -\xi_k^2.$$

We use the following notations:

$$\begin{aligned} \{z\} &= (z;q^4,x^2)_\infty, \qquad (z)_\infty = (z;x^2)_\infty, \\ \Theta_p(z) &= (z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty = \sum_{n\in\mathbf{Z}}(-z)^n p^{n(n-1)/2}, \end{aligned} \tag{8.17}$$

$$
\begin{aligned}
\gamma_\sigma(z) &= \frac{\{(-q)^{1+\sigma}x^2z\}}{\{(-q)^{3+\sigma}x^2z\}}\frac{\{(-q)^{1+\sigma}z^{-1}\}}{\{(-q)^{3+\sigma}z^{-1}\}} \qquad (\sigma=+,0,-),\\
C^{mn}_{st} &= \delta_{m-n,2(s-t)}(-q)^N\left(\frac{\{q^2x^2\}}{\{q^4x^2\}}\right)^m\left(\frac{\{x^2\}}{\{q^2x^2\}}\right)^n\\
&\quad\times(x^2)^{s+t-1}_\infty(q^2)^s_\infty(q^{-2})^t_\infty,\\
N &= -\frac{m}{2}\left(\frac{m}{2}+1\right)+\frac{n}{2}\left(\frac{n}{2}+1\right)+ms-nt.
\end{aligned}
$$

Then for $i=0,1$ we have

$$
\begin{aligned}
&\mathrm{tr}_{V(\Lambda_i)}\left(x^Dy^h\Phi_{\epsilon_1}(\zeta_1)\cdots\Phi_{\epsilon_m}(\zeta_m)\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)\right)\\
&=C^{mn}_{st}\frac{\prod_{j<j'}\gamma_+(z_j/z_{j'})\prod_{k<k'}\gamma_-(u_{k'}/u_k)}{\prod_{j,k}\gamma_0(z_j/u_k)}\\
&\times\prod_j\zeta_j^{-j+(1+\epsilon_j)/2+i}\prod_k\xi_k^{k-(1+\mu_k)/2-i}\\
&\times\prod_a\oint_C\frac{dw_a}{2\pi iw_a}\prod_b\oint_{\widetilde{C}}\frac{dv_b}{2\pi iv_b}F_i(z,u,w,v)
\end{aligned}
\tag{8.18}
$$

where the integrand is given by

$$
\begin{aligned}
F_i(z,u,w,v)&=\prod_{j<a}(qz_j-q^{-1}w_a)\prod_{j>a}(z_j-w_a)\prod_{j,a}\frac{1}{(w_a/z_j)_\infty(q^2z_j/w_a)_\infty}\\
&\times\prod_{k<b}(qv_b^{-1}-q^{-1}u_k^{-1})\prod_{k>b}(v_b^{-1}-u_k^{-1})\prod_{k,b}\frac{1}{(q^{-2}v_b/u_k)_\infty(u_k/v_b)_\infty}\\
&\times\prod_{j,b}\frac{\Theta_{x^2}(-q^{-1}v_b/z_j)}{(x^2)_\infty}\prod_{k,a}\frac{\Theta_{x^2}(-qu_k/w_a)}{(x^2)_\infty}\\
&\times\prod_{a,b}\frac{(x^2)^2_\infty}{\Theta_{x^2}(-qv_b/w_a)\Theta_{x^2}(-q^{-1}v_b/w_a)}\\
&\times\prod_{a<a'}\frac{(q^2w_a/w_{a'})_\infty(q^2w_{a'}/w_a)_\infty}{w_{a'}-q^2w_a}\frac{w_{a'}^{-1}\Theta_{x^2}(w_{a'}/w_a)}{(x^2)_\infty}\\
&\times\prod_{b<b'}(v_{b'}-q^{-2}v_b)(x^2q^{-2}v_b/v_{b'})_\infty(x^2q^{-2}v_{b'}/v_b)_\infty\frac{v_{b'}\Theta_{x^2}(v_b/v_{b'})}{(x^2)_\infty}\\
&\times\Theta_{x^4}\left(-(-1)^nx^{1+2i}q^{m-n}y^2\frac{\prod z_j\prod v_b^2}{\prod u_k\prod w_a^2}\right)\\
&\times\left((-1)^t(-q)^{(m-n)/2}y\frac{\prod v_b}{\prod w_a}\right)^i.
\end{aligned}
$$

The contours C, $\tilde{C}$ are anti-clockwise circuit surrounding the poles of the integrand by the following rule.

$$C \ : \ q^2x^{2l}z_j \ (l \geq 0) \ \text{are inside and} \ x^{-2l}z_j \ (l \geq 0) \ \text{are outside} \ , \tag{8.19}$$

$$\begin{aligned} \tilde{C} \ &: \ x^{2l}u_k \ (l \geq 0) \ \text{are inside and} \ q^2x^{-2l}u_k \ (l \geq 0) \ \text{are outside} \ , \\ &\quad -q^{\pm1}x^{2l}w_a \ (l \geq 1) \ \text{are inside and} \ -q^{\pm1}x^{-2l}w_a \ (l \geq 0) \ \text{are outside} \ . \end{aligned} \tag{8.20}$$

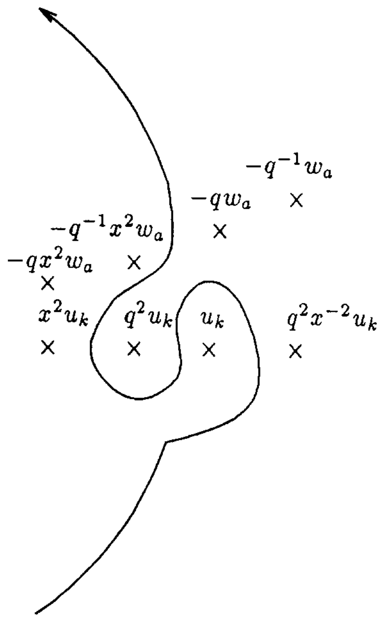

Figure 8.1: The contour $\tilde{C}$

The contours C and $\tilde{C}$ are determined by the following argument. They are chosen by examining the domain of convergence for the integration variables w_a and v_b as discussed in 8.1. In general, the product of operators in (8.1) gives rise to a sum of products of the operators listed in (8.10). This is because (8.3) and (8.5) contain two terms. Different products have different domains of convergence. Therefore, the integral formula contains integrands with different contours. This is cumbersome though not serious. If we can choose a contour commonly belonging to the convergence domains, we can sum the integrands to get a single integrand. In other words, we can use the following formula instead of (8.3) or (8.5);

$$\Phi_+^{(1-i,i)}(\zeta) = (-q^3)^{i/2}\oint_C \frac{dw}{2\pi i w}\frac{(1-q^2)w^2\zeta}{q(w-\zeta^2)(w-q^2\zeta^2)} : \varphi(\zeta)\chi^-(w) :,$$
$$\Psi_+^{*(1-i,i)}(\xi) = (-q^3)^{i/2}\oint_{\tilde{C}} \frac{dv}{2\pi i v}\frac{(1-q^2)v\xi^2}{(v-\xi^2)(v-q^2\xi^2)} : \psi^*(\xi)\chi^+(v) : .$$

Let us examine if this is true. First, consider w_a. For the product $\varphi(\zeta_a)\chi^-(w_a)$, the convergence domain is

$$x^2q^2 < \left|\frac{w_a}{z_a}\right| < 1,$$

and for $\chi^-(w_a)\varphi(\zeta_a)$, it is

$$q^2 < \left|\frac{w_a}{z_a}\right| < x^{-2}.$$

Therefore, we can choose the contour C for the variable w_a in the common domain

$$q^2 < \left|\frac{w_a}{z_a}\right| < 1.$$

Next, consider v_b. For the product $\psi^*(\xi_b)\chi^+(v_b)$, the convergence domain is

$$x^2 < \left|\frac{v_b}{u_b}\right| < q^2,$$

and for $\chi^+(v_b)\psi^*(\xi_b)$, it is

$$1 < \left|\frac{v_b}{u_b}\right| < x^{-2}q^2.$$

In this case, there is no common domain. Nevertheless, we can take the contour $\tilde{C}$ for the variable v_b in such a way that the pole at u_b lies inside of $\tilde{C}$ and the pole $q^{-2}u_b$ outside. In other words, we choose $\tilde{C}$ in such a way that it surrounds u_k inside but q^2u_k should be outside it for all k .

For the product of type I operators alone the formula above simplifies as follows.

$$\mathrm{tr}_{V(\Lambda_i)}\left(x^D y^h \Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_m}(\zeta_m)\right)$$

$$
\begin{aligned}
&= C_s^m \prod_{j<j'} \gamma_+(z_j/z_{j'}) \prod_j \zeta_j^{-j+(1+\varepsilon_j)/2+i} \prod_a \oint_C \frac{dw_a}{2\pi i w_a} \\
&\times \prod_{j<a}(qz_j - q^{-1}w_a) \prod_{j>a}(z_j - w_a) \prod_{j,a} \frac{1}{(w_a/z_j)_\infty (q^2 z_j/w_a)_\infty} \\
&\times \prod_{a<a'} \frac{(q^2 w_a/w_{a'})_\infty (q^2 w_{a'}/w_a)_\infty}{w_{a'} - q^2 w_a} \frac{w_{a'}^{-1} \Theta_{x^2}(w_{a'}/w_a)}{(x^2)_\infty} \\
&\times \Theta_{x^4}\left(-x^{1+2i} q^m y^2 \frac{\prod z_j}{\prod w_a^2}\right) \times \left((-q)^{m/2} y \prod w_a^{-1}\right)^i ,
\end{aligned}
\tag{8.21}
$$

where

$$
C_s^m = C_{s0}^{m\,0} = \delta_{m,2s}(-q)^{m^2/4-m/2} \left(\frac{\{q^2x^2\}}{\{q^4x^2\}}\right)^m (x^2)_\infty^{m/2-1} (q^2)_\infty^{m/2}.
$$

8.3 Examples

In the case $\varepsilon_j = \mu_k = -$ for all j, k (and hence $m = n$), the formula (8.18) involves no integrals. Accordingly we have

$$
\begin{aligned}
&\operatorname{tr}_{V(\Lambda_i)}\left(x^D y^h \Phi_-(\zeta_1)\cdots\Phi_-(\zeta_n)\Psi_-^*(\xi_n)\cdots\Psi_-^*(\xi_1)\right) \\
&= (x^2)_\infty^{n-1} \frac{\prod_{j<j'}\gamma_+(\zeta_j^2/\zeta_{j'}^2)\prod_{k<k'}\gamma_-(\xi_{k'}^2/\xi_k^2)}{\prod_{j,k}\gamma_0(-\zeta_j^2/\xi_k^2)} \\
&\times \left(\prod_j \zeta_j^{-j+i} \prod_k \xi_k^{k-i}\right) y^i \Theta_{x^4}\left(-x^{1+2i} y^2 \frac{\prod \zeta_j^2}{\prod \xi_k^2}\right).
\end{aligned}
\tag{8.22}
$$

In Appendix (A.4.7), we give the explicit formulas for the traces in the simplest cases, which can be determined by solving the difference equations and examining the analyticity by the integral formulas. The formulas (A.18) and (A.19) are special cases of (8.22) but are listed for comparison. Notice that the formula (A.18) vanishes at $x = q^2$. In general, we have (see 8.4)

$$
\operatorname{tr}_{V(\Lambda_i)}\left(q^{2D}\Psi_{\mu_1}(\zeta_1)\cdots\Psi_{\mu_n}(\zeta_n)\right) = 0.
$$

We will give the derivation of the formulas (A.17) and (A.18). Essentially it is a repetition of the argument in 4.2. In 4.2 the parameter x was set to

q^2. We also assumed a certain analyticity property, which we derive here from the integral formula in this section.

Set

$$f_\pm(\zeta) = \mathrm{tr}_{V(\Lambda_0)}\left(x^D(\Phi_+^{(0,1)}(\zeta_1)\Phi_-^{(1,0)}(\zeta_2) \pm \Phi_-^{(0,1)}(\zeta_1)\Phi_+^{(1,0)}(\zeta_2))\right),$$

where $\zeta = \zeta_2/\zeta_1$. Then, as in 4.2, we have

$$f_-(\zeta) = f_+(-\zeta), \tag{8.23}$$

$$\frac{f_+(x\zeta)}{f_+(\zeta)} = \frac{1+q\zeta}{q+\zeta}\kappa(\zeta). \tag{8.24}$$

From (A.6) we have

$$f_+(-q) = \frac{2g^{-1}}{(x)_\infty}.$$

It is easy to check that (A.17) gives a solution to these equations. On the other hand, the solution is unique if we further demand that it is holomorphic for $q < |\zeta| < x^{-1}q^{-1}$. Let us show this property by using the integral formula. Specializing to the present situation and setting $z = \zeta^2$, we have

$$\mathrm{tr}_{V(\Lambda_0)}\left(x^D\Phi_\pm^{(0,1)}(\zeta_1)\Phi_\mp^{(1,0)}(\zeta_2)\right) = \frac{\{q^2x^2\}^2(q^2)_\infty\gamma_+(z^{-1})}{\{q^4x^2\}^2}\times$$
$$\times\oint_C \frac{dw}{2\pi i w}\frac{\Theta_{x^4}(-xq^2z^{-1}/w^2)}{(wz)_\infty(w)_\infty(q^2z^{-1}/w)_\infty(q^2/w)_\infty}\times\begin{cases}(1-w)\\ \zeta(q/z-w/q)\end{cases}.$$

The contour for w is pinched when $z = q^2, x^2q^2, \ldots$, or $z = q^{-2}, x^{-2}q^{-2}, \ldots$, where the integral may have poles. However, the pole at $z = q^{-2}$ is cancelled by the zero of $\gamma_+(z^{-1})$. Therefore, $f_+(\zeta)$ is holomorphic if $q < |\zeta| < x^{-1}q^{-1}$.

The derivation of (A.18) is similar. Instead of (A.6), we use (A.9).

8.4 Orthogonality of the eigenvectors

We will state the orthogonality relation (see (7.20) for the case $m \neq n$) for the eigenvectors created by the type II vertex operator. We also give a sketch of proof, although we have not worked out the complete details of the proof.

Before going into the general case, let us consider the special case,

$$G^{(i)}_{\mu,\mu'}(\xi,\xi') = \mathrm{tr}_{V(\Lambda_i)}\left(x^D\Psi_\mu(\xi)\Psi^*_{\mu'}(\xi')\right).$$

This is a meromorphic function of ξ and ξ' as given in (A.18). We are interested in the poles of $G^{(i)}_{\mu,\mu'}(\xi,\xi')$ that are located near the submanifold $|\xi| = |\xi'|$ when x is close to q^2. The formula (A.18) shows that there are poles at $\xi^2 = \xi'^2$ and $\xi^2 = q^4x^{-2}\xi'^2$. From (A.9) we see that

$$G^{(i)}_{\mu,\mu'}(\xi,\xi') \sim \delta_{\mu\mu'}\frac{1}{1-\frac{\xi'^2}{\xi^2}} \times g\,\mathrm{tr}_{V(\Lambda_i)}\left(x^D\right) \qquad \xi^2 \to \xi'^2.$$

On the other hand,

$$\begin{aligned}\mathrm{tr}_{V(\Lambda_i)}\left(x^D\Psi_\mu(\xi)\Psi^*_{\mu'}(\xi')\right) &= \mathrm{tr}_{V(\Lambda_{1-i})}\left(x^D\Psi^*_{\mu'}(\xi')\Psi_\mu(x\xi)\right)\\ &= G^{(1-i)}_{-\mu',-\mu}(-q\xi',-q^{-1}x\xi).\end{aligned}$$

Therefore we have

$$G^{(i)}_{\mu,\mu'}(\xi,\xi') \sim -\delta_{\mu\mu'}\frac{1}{1-\frac{x^2\xi^2}{q^4\xi'^2}} \times g\,\mathrm{tr}_{V(\Lambda_{1-i})}\left(x^D\right) \qquad \xi^2 \to q^4x^{-2}\xi'^2.$$

Since $\mathrm{tr}_{V(\Lambda_i)}\left(x^D\right) = \mathrm{tr}_{V(\Lambda_{1-i})}\left(x^D\right) = (x;x^2)^{-1}_\infty$, these two poles cancel each other at $x = q^2$. This is easily seen from (A.18) since it is identically zero at $x = q^2$. A question arises: Is the norm of the eigenvector ${}_{(i);\mu}\langle\xi|\xi'\rangle_{(i);\mu'}$ identically zero? The answer is NO, but

$${}_{(i);\mu}\langle\xi|\xi'\rangle_{(i);\mu'} = \delta_{\mu\mu'}\delta(\xi^2/\xi'^2),$$

where $\delta(\xi^2/\xi'^2)$ is Dirac's delta function in the sense given below.

We assume the reader knows the definition of Dirac's delta functions as a sum of boundary values of holomorphic functions. Take the real coordinate θ on $S^1 = \{\xi; |\xi| = 1\}$ where $\xi = e^{i\theta}$. By $\delta(\xi^2)$ we mean the delta function on S^1 supported on $\xi^2 = 1$;

$$\delta(\xi^2) = \frac{1}{1-e^{2i(\theta+i0)}} - \frac{1}{1-e^{2i(\theta-i0)}}.$$

We have

$$\text{b.v.}\int_C \frac{dw}{2\pi i w}\frac{1}{(1-\frac{\xi_1^2}{w})(1-\frac{w}{\xi_2^2})} = \frac{1}{1-e^{2i(\theta_1-\theta_2+i0)}}.$$

Here the integration contour C is such that $\xi_1^2 = e^{2i\theta_1}$ is inside and $\xi_2^2 = e^{2i\theta_2}$ is outside. The symbol b.v. means taking the boundary value near the real manifold $|\xi_1| = |\xi_2| = 1$, the integral defines a function holomorphic in the imaginary direction $|\xi_1| < |\xi_2|$, i.e., $\mathrm{Im}\,\theta_1 > \mathrm{Im}\,\theta_2$. Therefore, we take the boundary value of the meromorphic function $\frac{1}{1-\xi_1^2/\xi_2^2}$ from that direction. In other words, the left hand side can be replaced by the integral on the real manifold $S^1 = \{w = e^{i\lambda}\}$,

$$\int \frac{d\lambda}{2\pi} \frac{1}{(1 - e^{i(2\theta_1 - \lambda + i0)})(1 - e^{i(\lambda - 2\theta_2 + i0)})}.$$

With this understood let us proceed to the general case.

Consider the trace function

$$\mathrm{tr}_{V(\Lambda_i)}\left(x^D \Psi_{\mu_n}(\xi_n) \cdots \Psi_{\mu_1}(\xi_1)\right).$$

The integral formula tells us that this is a meromorphic function. We claim that it is identically zero if $x = q^2$. The reason is as follows. Set

$$\begin{aligned} F^{(i)}(x; \xi_n, \cdots, \xi_1) &= \mathrm{tr}_{V(\Lambda_i)}\left(x^D \Psi(\xi_n) \cdots \Psi(\xi_1)\right) \\ &= \sum_{\mu_n, \cdots, \mu_1} \mathrm{tr}_{V(\Lambda_i)}\left(x^D \Psi_{\mu_n}(\xi_n) \cdots \Psi_{\mu_1}(\xi_1)\right) v_{\mu_1} \otimes \cdots \otimes v_{\mu_n}. \end{aligned}$$

This is a meromorphic function taking its values in $V \otimes \cdots \otimes V$. Suppose that $(\xi_n, \cdots, \xi_1) \in (\mathbf{C}\backslash\{0\})^n$ is a regular point of $F^{(i)}(x; \xi_n, \cdots, \xi_1)$. Consider the action of U' on $V \otimes \cdots \otimes V$,

$$\rho = \rho_{\xi_1} \otimes \cdots \otimes \rho_{\xi_n},$$

where, for a comlex number $\xi \in \mathbf{C}\backslash\{0\}$, ρ_ξ denotes the representation of U' such that

$$\begin{aligned} \rho(t_0)^{-1} = \rho(t_1) &= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \\ \rho(e_0) = \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}, &\quad \rho(f_1) = \begin{pmatrix} 0 & 0 \\ \xi^{-1} & 0 \end{pmatrix}, \\ \rho(e_1) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, &\quad \rho(f_0) = \begin{pmatrix} 0 & \xi^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The vector $F^{(i)}(q^2; \xi_n, \cdots, \xi_1)$ behaves as a singlet, i.e., a vector in the one-dimensional representation, under the action of U'. For example, take $e_j \in$

U'. Because of the intertwining property of $\Psi(\xi)$ we have

$$\begin{aligned}&\rho(e_j)F^{(i)}(x;\xi_n,\cdots,\xi_1)\\&=\mathrm{tr}_{V(\Lambda_i)}\left(x^D\Psi(\xi_n)\cdots\Psi(\xi_1)e_j\right)-\rho(t_j)\,\mathrm{tr}_{V(\Lambda_i)}\left(x^De_j\Psi(\xi_n)\cdots\Psi(\xi_1)\right).\end{aligned}$$

Using the properties

$$\begin{aligned}&x^De_j=x^{-1}e_jx^D,\\&\text{weight of }\left(\mathrm{tr}_{V(\Lambda_i)}(x^De_j\Psi(\xi_n)\cdots\Psi(\xi_1))\right)=\text{ weight of }e_j,\end{aligned}$$

we have

$$\begin{aligned}&\rho(t_j)\,\mathrm{tr}_{V(\Lambda_i)}\left(x^De_j\Psi(\xi_n)\cdots\Psi(\xi_1)\right)\\&=q^2x^{-1}\mathrm{tr}_{V(\Lambda_i)}\left(x^D\Psi(\xi_n)\cdots\Psi(\xi_1)e_j\right)\end{aligned}$$

Therefore, letting $x\to q^2$ we have

$$\rho(t_j)F^{(i)}(q^2;\xi_n,\cdots,\xi_1)=0.$$

On the other hand, if $(\xi_n,\cdots,\xi_1)$ is generic, the representation ρ has no singlet [17]. Therefore, the vector $F^{(i)}(q^2;\xi_n,\cdots,\xi_1)$ itself is zero.

Now, consider the meromorphic function

$$\begin{aligned}&G_{\mu_1,\cdots,\mu_m\mu'_n,\cdots,\mu'_1}(x;\xi_1,\cdots,\xi_m,\xi'_n,\cdots,\xi'_1)\\&=\mathrm{tr}_{V(\Lambda_i)}\left(x^D\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_m}(\xi_m)\Psi^*_{\mu'_n}(\xi'_n)\cdots\Psi^*_{\mu'_1}(\xi'_1)\right),\end{aligned}$$

and set

$$\begin{aligned}&G^{(i)}(x;\xi_1,\cdots,\xi_m,\xi'_n,\cdots,\xi'_1)\\&=\sum_{\mu_1,\cdots,\mu_m,\mu'_n,\cdots,\mu'_1}G_{\mu_1,\cdots,\mu_m\mu'_n,\cdots,\mu'_1}(x;\xi_1,\cdots,\xi_m,\xi'_n,\cdots,\xi'_1)\times\\&\times v_{\mu_1}\otimes\cdots\otimes v_{\mu_m}\otimes v_{-\mu'_n}\otimes\cdots\otimes v_{-\mu'_1}.\end{aligned}$$

We will examine when x is close to q^2 the singularity structure of G near the real submanifold

$$|\xi_j|=|\xi'_k|=1\quad\forall j,k,\tag{8.25}$$

which is equal to $(S^1)^{m+n}$. Using the integral formula, we can show that in the vicinity of (8.25) the only poles are $\xi_j^2=\xi_k'^2$ or $\xi_k'^2=x^2q^{-4}\xi_j^2$. These poles are simple. They arise because the contour for an integration variable, say v, is pinched by the simple poles of the integrand of the following types: $v = \xi_j^2$ and $v = \xi_k'^2$, $v = q^{-2}\xi_j^2$ and $v = x^{-2}q^2\xi_k'^2$, or $v = x^2q^{-2}\xi_j^2$ and $v = q^2\xi_k'^2$. Looking at the contours more closely, we see that the pinching by the poles $v = \xi_j^2$ and $v = \xi_k'^2$ is such that the integral has a well-defined boundary value on (8.25) from the imaginary direction $|\xi_j^2| > |\xi_k'^2|$. Similarly, the pinching by the poles $v = q^{-2}\xi_j^2$ and $v = x^{-2}q^2\xi_k'^2$, (or $v = x^2q^{-2}\xi_j^2$ and $v = q^2\xi_k'^2$) is such that the integral has a well-defined boundary value from $|\xi_k'^2| > |\xi_j^2|$. Summing up, the integral has a well-defined boundary value,

$$\text{b.v.}\, G^{(i)}(q^2;\xi_1,\cdots,\xi_m,\xi_n',\cdots,\xi_1') \tag{8.26}$$

on $(S^1)^{m+n}$.

For an element w of the n-th symmetric group let

$$w = w_{i_s j_s}\cdots w_{i_1 j_1}$$

be a reduced expression in terms of elementary transpositions w_{ij} $(i < j)$. We set

$$R_w = R_{i_s j_s}(\xi_{i_s}/\xi_{j_s})\cdots R_{i_1 j_1}(\xi_{i_1}/\xi_{j_1}).$$

With this notation we claim that (8.26) is equal to 0 if $m \neq n$, or equal to

$$\begin{aligned}&\chi g^n \sum_{w\in S_n} \operatorname{sgn} w\, R_w \prod_{j=1}^{n}\left(\frac{\xi_j'}{\xi_{w(j)}}\right)^{i+j+\frac{1-\mu_j}{2}} \delta(\xi_j'^2/\xi_{w(j)}^2)\\ &\times v_{\mu_1}\otimes\cdots\otimes v_{\mu_n}\otimes v_{-\mu_{w(n)}}\otimes\cdots\otimes v_{-\mu_{w(1)}},\end{aligned} \tag{8.27}$$

if $m = n$.

To show (8.27) the key observation is the residue formula

$$\begin{aligned}&\operatorname{tr}_{V(\Lambda_i)}\left(x^D\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_m}(\xi_m)\Psi^*_{\mu_n'}(\xi_n')\cdots\Psi_{\mu_1'}(\xi_1')\right)\\ &= \frac{g\delta_{\mu_m,\mu_n'}(\frac{\xi_n'}{\xi_m})^{i+n+\frac{1-\mu_n'}{2}}}{1-\xi_n'^2/\xi_m^2}\\ &\times \operatorname{tr}_{V(\Lambda_i)}\left(x^D\Psi(\xi_1)\cdots\Psi(\xi_{m-1})\Psi^*(\xi_{n-1}')\cdots\Psi(\xi_1')\right) + (\text{regular}).\end{aligned} \tag{8.28}$$

This is formally a consequence of (A.9), and can also be derived from the integral formula (8.18). Because of the R-matrix symmetry the calculation of the residues at the poles $\xi_j^2 = \xi_k'^2$ can be reduced to the special case (8.28). In any event, the residues of these poles are obtained inductively. Noting this and using the fact that $G^{(i)}(q^2;\xi_1,\cdots,\xi_m,\xi_n'\cdots,\xi_1')$ is zero as a meromorphic function, we have (8.27).

Chapter 9

Correlation functions and form factors

In this section we study various specializations of the formula in the previous chapter. They are the correlation functions, i.e., the vacuum-to-vacuum matrix elements of local operators for the XXZ model, the form factors, i.e., the matrix elements of Pauli spins with respect to the eigenvectors of the XXZ Hamiltonian, and the matrix elements of the products of type I vertex operators with respect to the highest weight vectors. We also discuss the completeness of the eigenvectors created by the type II vertex operators.

9.1 Correlation functions

First let us consider the correlation functions for local operators. We will specialize the trace function (8.1) to

$$x = q^2, \quad y = 1,$$

in order to apply the formula (7.15).

Recall that the matrix unit operator $E^{(r)}_{\varepsilon'\varepsilon}$ acting on the site r is realized as (7.7):

$$E^{(r)}_{\varepsilon'\varepsilon} = T^{-(r-1)}\,(g\Phi^*_{\varepsilon'}(1)\Phi_\varepsilon(1) \otimes \mathrm{id}\,)\,T^{r-1}. \tag{9.1}$$

We generalize the situation slightly by introducing the spectral parameters ζ_j for each column j. Let us set

$$E^{(r)}_{\varepsilon'\varepsilon}(\zeta_1,\cdots,\zeta_r) = \mathrm{Ad}\,T(\zeta_1)^{-1}\cdots \mathrm{Ad}\,T(\zeta_{r-1})^{-1}\,(g\Phi^*_{\varepsilon'}(\zeta_r)\Phi_\varepsilon(\zeta_r)\otimes \mathrm{id}\,) \tag{9.2}$$

Then, by using (7.8), (7.9), (7.12), (A.6), we have the following formula for the correlation function

$$\begin{aligned}&{}_{(i)}\langle \mathrm{vac}|E^{(n)}_{\varepsilon'_n\varepsilon_n}(\zeta_1,\cdots,\zeta_n)\cdots E^{(1)}_{\varepsilon'_1\varepsilon_1}(\zeta_1)|\mathrm{vac}\rangle_{(i)} = \chi^{-1}g^n\times\\ &\times \mathrm{tr}_{V(\Lambda_i)}\left(q^{2D}\Phi_{-\varepsilon'_1}(-q^{-1}\zeta_1)\cdots\Phi_{-\varepsilon'_n}(-q^{-1}\zeta_n)\Phi_{\varepsilon_n}(\zeta_n)\cdots\Phi_{\varepsilon_1}(\zeta_1)\right).\end{aligned}$$

The integral formula for this is obtained from (8.21) by letting $m\to 2n$ and specializing variables as

$$\zeta_1,\cdots,\zeta_m \longrightarrow -q^{-1}\zeta_1,\cdots,-q^{-1}\zeta_n,\zeta_n,\cdots,\zeta_1.$$

Notice that the specialization $\zeta_j\to -q^{-1}\zeta_k$ $(j<k)$ in (8.21) causes no pinching of the integration contour (8.19) because of the factor $\prod_{j<a}(qz_j - q^{-1}w_a)\prod_{a<k}(z_k-w_a)$ in the integrand.

The resulting formula reads as follows. Set $z_j=\zeta_j^2$ again, and set further

$$\begin{aligned}&h(z) = (q^2z;q^2)_\infty(q^2z^{-1};q^2)_\infty,\\ &A' = \{j \mid 1\le j\le n,\ \varepsilon'_j=-1\},\quad A=\{j\mid 1\le j\le n,\ \varepsilon_j=+1\},\\ &s'=\sharp A',\quad s=\sharp A.\end{aligned}$$

The selection rule (8.16) this case reads as

$$s'+s=n.$$

For each $a'\in A'$ (resp. $a\in A$) prepare an integration variable $w'_{a'}$ (resp. w_a). Writing $A'=\{a'_1,\cdots,a'_{s'}\}$, $A=\{a_1,\cdots,a_s\}$ with $a'_1<\cdots<a'_{s'}$, $a_1<\cdots<a_s$, we arrange them as

$$(\eta_1,\cdots,\eta_n) = (w'_{a'_1},\cdots,w'_{a'_{s'}},w_{a_s},\cdots,w_{a_1}). \tag{9.3}$$

Then we have

$$\begin{aligned}&{}_{(i)}\langle \mathrm{vac}|E^{(n)}_{\varepsilon'_n\varepsilon_n}(\zeta_1,\cdots,\zeta_n)\cdots E^{(1)}_{\varepsilon'_1\varepsilon_1}(\zeta_1)|\mathrm{vac}\rangle_{(i)}\\ &=\delta_{s'+s,n}\times(-1)^{N+s}(-q)^N g\prod_{a'\in A'}\oint_{C'}\frac{dw'_{a'}}{2\pi i(w'_{a'}-z_{a'})}\prod_{a\in A}\oint_C\frac{dw_a}{2\pi i(w_a-z_a)}\\ &\times\prod_j\zeta_j^{(\varepsilon_j-\varepsilon'_j)/2}\prod_{\substack{a'<j\le n\\ a'\in A'}}\frac{z_j-q^2w'_{a'}}{z_j-w'_{a'}}\prod_{\substack{a<j\le n\\ a\in A}}\frac{w_a-q^2z_j}{w_a-z_j}\prod_{j<k}\frac{\eta_k-\eta_j}{\eta_k-q^2\eta_j}\\ &\times\frac{h(1)^n\prod_{j<k}h(z_j/z_k)h(\eta_j/\eta_k)}{\prod_{j,k}h(\eta_j/z_k)}\Theta_{q^8}\left(-q^{2+4i}\prod_j z_j^2/\eta_j^2\right)\left(\frac{\prod z_j}{\prod\eta_j}\right)^i.\end{aligned} \tag{9.4}$$

Here

$$N=\sum_{a'\in A'}a'+\sum_{a\in A}a-\frac{n(n+1)}{2}.$$

The contours C', C are such that

$$\begin{array}{lll} C' & : & q^{2l}z_j(j\geq 0) \text{ are inside and } q^{-2l}z_j(l\geq 1) \text{ are outside,} \\ C & : & q^{2l}z_j(l\geq 1) \text{ are inside and } q^{-2l}z_j(j\geq 0) \text{ are outside.} \end{array}$$

Notice that the formula reduces to 1 if $n=0$ because of the equality

$$\Theta_{q^8}\left(-q^{2+4i}\right)=g^{-1}.$$

The correlators of the homogeneous lattice are given by setting $\zeta_j=1$ for all j.

9.2 Form factors

In much the same way as with the correlation functions, the form factors of a local operator $\mathcal{O}$

$$_{(i)}\langle\mathrm{vac}|\mathcal{O}|\xi_n,\cdots,\xi_1\rangle_{\mu_n,\cdots,\mu_1;(i)}\quad (n:\ \text{even})$$

are specializations of the formula (8.18). Note that the cyclic property of the trace and (A.5) imply the relation

$$\begin{array}{l} _{(i);\mu_{m+1},\cdots,\mu_n}\langle\xi_{m+1},\cdots,\xi_n|\mathcal{O}|\xi_1,\cdots,\xi_m\rangle_{\mu_1,\cdots,\mu_m;(i)}= \\ _{(i+n-m)}\langle\mathrm{vac}|\mathcal{O}|\xi_1,\cdots,\xi_m,-q\xi_{m+1},\cdots,-q\xi_n\rangle_{\mu_1,\cdots,\mu_m,-\mu_{m+1},\cdots,-\mu_n;(i+n-m)} \end{array}$$

as a meromorphic function in $\xi_1,\cdots,\xi_n$. We consider here the form factors of the Pauli spin operators $\mathcal{O}=\sigma_1^\alpha$. Thanks to the $\mathbf{Z}_2$ symmetry

$$_{(i)}\langle\mathrm{vac}|\sigma_1^-|\xi_n,\cdots,\xi_1\rangle_{\mu_n,\cdots,\mu_1;(i)}={}_{(1-i)}\langle\mathrm{vac}|\sigma_1^+|\xi_n,\cdots,\xi_1\rangle_{-\mu_n,\cdots,-\mu_1;(1-i)} \tag{9.5}$$

it suffices to consider σ_1^+ and σ_1^z. We give below the results for the n-particle form factors of σ_1^α where n is even and $\alpha=+,z$. Here we use

$$\begin{aligned} \gamma(u) &= \frac{(q^4u;q^4;q^4)_\infty(u^{-1};q^4;q^4)_\infty}{(q^6u;q^4;q^4)_\infty(q^2u^{-1};q^4;q^4)_\infty}, \\ \rho &= \frac{(q^4;q^4;q^4)_\infty}{(q^6;q^4;q^4)_\infty}. \end{aligned}$$

As before we set

$$B = \{k; 1 \le k \le n, \mu_k = +\}, \qquad t = \sharp B,$$

and the indices b, b' run over the set B.

$$\begin{aligned}
&{}_{(i)}\langle \mathrm{vac}|\sigma_1^\alpha|\xi_n, \cdots, \xi_1\rangle_{\mu_n,\cdots,\mu_1;(i)} \\
&= \frac{\prod_{k<k'}\gamma(u_{k'}/u_k)}{\prod_k(-qu_k;q^4)_\infty(-q^3u_k^{-1};q^4)_\infty}\prod_k \xi_k^{k-(1+\mu_k)/2-i} \\
&\times \prod_b \oint_{\widetilde{C}} \frac{dv_b}{2\pi i v_b} \prod_{k<b}(qv_b^{-1} - q^{-1}u_k^{-1}) \prod_{k>b}(v_b^{-1} - u_k^{-1}) \\
&\times \prod_{k,b} \frac{1}{(u_k/v_b;q^4)_\infty(q^{-2}v_b/u_k;q^4)_\infty} \prod_b (-q^{-1}v_b;q^2)_\infty(-q^3/v_b;q^2)_\infty \\
&\times \prod_{b<b'} v_{b'}(v_{b'} - q^{-2}v_b)(v_b/v_{b'};q^2)_\infty(q^2v_{b'}/v_b;q^2)_\infty \times G_i^\alpha(v,u).
\end{aligned}$$

Here $u_k = -\xi_k^2$ and

$$\begin{aligned}
G_i^+(v,u) &= C_{n,i}^+ \prod_b (-v_b)^i \times \Theta_{q^8}\left(-q^{2-n+4i}\frac{\prod_b v_b^2}{\prod_k u_k}\right), \\
C_{n,i}^+ &= \delta_{t,n/2-1}(-q)^{-\frac{n^2}{4}+\frac{(3-i)n}{2}-1}(1-q^{-2})^{\frac{n}{2}-1}(q^2;q^4)_\infty(q^4;q^4)_\infty^{n-1}\rho^n, \\
G_i^z(v,u) &= C_{n,i}^z \oint_{C^+ + C^-} \frac{dw}{2\pi i w} \frac{w^{1-i}\prod_b(-v_b)^i}{(w;q^2)_\infty(q^2/w;q^2)_\infty} \\
&\quad\times \frac{\prod_k(-qu_k/w;q^4)_\infty(-q^3w/u_k;q^4)_\infty}{\prod_b(-q^{-1}v_b/w;q^2)_\infty(-q^3w/v_b;q^2)_\infty}\Theta_{q^8}\left(-q^{2-n+4i}\frac{\prod_b v_b^2}{w^2\prod_k u_k}\right), \\
C_{n,i}^z &= \delta_{t,n/2}(-q)^{-\frac{n^2}{4}+\frac{(1-i)n}{2}}(1-q^{-2})^{\frac{n}{2}}(q^2;q^4)_\infty^3(q^4;q^4)_\infty^{n+1}\rho^n.
\end{aligned}$$

The contours are

$$\begin{aligned}
\widetilde{C} &: \quad q^{4l}u_k\ (l \ge 0)\ \text{ are inside and } q^{2-4l}u_k\ (l \ge 0)\ \text{ are outside },\\
C^\pm &: \quad q^{2l+1\pm1}\ (l \ge 0)\ \text{ are inside and } q^{-2l-1\pm1}\ (l \ge 0)\ \text{ are outside },\\
&\quad -q^{-1+2l}v_b\ (l \ge 0)\ \text{ are inside and } -q^{-3-2l}v_b\ (l \ge 0)\ \text{ are outside }.
\end{aligned}$$

We remark that because of (8.22) the 2-particle form factors for σ_1^+ has the simple form

$${}_{(i)}\langle \mathrm{vac}|\sigma_1^+|\xi_2, \xi_1\rangle_{--;(i)}/(-q)^{1-i}\xi_1^{1-i}\xi_2^{2-i}$$

$$= (q^2;q^4)_\infty (q^4;q^4)^3_\infty \rho^2 \frac{\gamma(\xi_2^2/\xi_1^2)}{\prod_{k=1,2}\Theta_{q^4}(q^3/\xi_k^2)} \times \Theta_{q^8}\left(-\xi_1^{-2}\xi_2^{-2}q^{4i}\right).$$

9.3 Matrix elements

We will give the integral formulas for the matrix elements [1],

$$\langle \Lambda_i|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_m}(\zeta_m)|\Lambda_{i+m}\rangle$$

and

$$\langle \Lambda_i|\Psi^*_{\mu_1}(\xi_1)\cdots\Psi^*_{\mu_m}(\xi_m)|\Lambda_{i+m}\rangle.$$

If m is even, we set $x=0$ in (8.21). Then only the highest weight vector contributes to the trace, and we obtain the highest-to-highest matrix element. The case of odd m can be handled by specializing this further to $\zeta_m = 0$. To see this, consider for example

$$\langle \Lambda_0|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_{m-1}}(\zeta_{m-1})|\Lambda_1\rangle \quad (m:\text{even}). \tag{9.6}$$

This is non-zero only if $\sharp\{j;\varepsilon_j=+\} = \sharp\{j;\varepsilon_j=-\}+1$. If it is the case, we get (9.6) by specializing

$$\langle \Lambda_0|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_{m-1}}(\zeta_{m-1})\Phi_-(\zeta_m)|\Lambda_0\rangle$$

to $\zeta_m = 0$.

The integral formula reads as follows. Following the notation in (8.21) we have

$$\begin{aligned}
&\langle \Lambda_0|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_m}(\zeta_m)|\Lambda_m\rangle \\
&= \langle \Lambda_1|\Phi_{-\varepsilon_1}(\zeta_1)\cdots\Phi_{-\varepsilon_m}(\zeta_m)|\Lambda_{1+m}\rangle \\
&= (-q)^{s^2-\sum_{a\in A} a}(1-q^2)^s \prod_{j=1}^m \zeta_j^{(1+\varepsilon_j)/2-j+2s} \prod_{j<j'} \frac{\left(q^2\zeta_{j'}^2/\zeta_j^2;q^4\right)_\infty}{\left(q^4\zeta_{j'}^2/\zeta_j^2;q^4\right)_\infty} \\
&\times \prod_{a\in A}\oint_C \frac{dw_a}{2\pi i} \frac{\prod_{a<a'}(w_a-w_{a'})(w_a-q^2w_{a'})}{\prod_{j\le a}(\zeta_j^2-w_a)\prod_{a\le j}(w_a-q^2\zeta_j^2)} \left(\delta_{m,2s}\prod_{a\in A} w_a + \delta_{m,2s-1}\right)
\end{aligned} \tag{9.7}$$

The contour C is given by $|q^2\zeta_j^2| < |w_a| < |\zeta_j^2|$ for all j, a.

[1] For convenience we extend the suffix for the highest weights by $\Lambda_i = \Lambda_{i+2}$

For instance we have

$$\begin{aligned}&\langle\Lambda_m|\Phi_{(-1)^m}(1)\cdots\Phi_+(1)\Phi_-(1)|\Lambda_0\rangle\\&=(1-q^2)^s\left(\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\right)^{m(m-1)/2}\prod_{k=1}^s\oint_{q^2<|w_k|<1}\frac{dw_k}{2\pi i}\\&\times\frac{\prod_{k<k'}(w_k-w_{k'})(w_k-q^2w_{k'})}{\prod_{k=1}^s(1-w_k)^{2k-1}(w_k-q^2)^{m-2k+2}}\times\left(\delta_{m,2s}\prod_{k=1}^s w_a+\delta_{m,2s-1}\right)\end{aligned}$$

We conjecture that the m-th root of this quantity has a limit when $m\to\infty$. (See Section 2 and Appendix 3 in [23].)

In principle, we can obtain a formula for the matrix element

$$\langle\Lambda_i|\Phi_{\varepsilon_1}(\zeta_1)\cdots\Phi_{\varepsilon_m}(\zeta_m)|\Lambda_{i+m}\rangle,$$

that contains no integration: Using the R-matrix symmetry (A.1), we can reduce the general case to the extreme case $\varepsilon_1=\cdots=\varepsilon_s=+$ and $\varepsilon_{s+1}=\cdots=\varepsilon_m=-$. For the latter, the integrals in (9.7) can be performed successively. Consider the integration for w_1. The only pole in the variable w_1 that lies outside of the contour C is $w_1=\zeta_1^2$. By taking the residue at $w_1=\zeta_1^2$, the integrand reduces in such a way that the pole at $w_2=\zeta_1^2$ is cancelled by the zero arising from the factor w_1-w_2. Thus we continue the process of integration. The explicit formula is as follows. Let $i,j=0,1$, $i-j\equiv m \bmod 2$, and

$$s=\left[\frac{m+1-i}{2}\right].$$

Then we find

$$\begin{aligned}&\langle\Lambda_i|\Phi_+(\zeta_1)\cdots\Phi_+(\zeta_s)\Phi_-(\zeta_{s+1})\cdots\Phi_-(\zeta_m)|\Lambda_j\rangle\\&=(-q)^{s(s-1+2i)/2}\prod_{1\le k\le s}\zeta_k^{k-m+1-j}\prod_{s<k\le m}\zeta_k^{k-m+j}\\&\times\prod_{1\le k<k'\le m}\frac{(q^6\zeta_{k'}^2/\zeta_k^2;q^4)_\infty}{(q^4\zeta_{k'}^2/\zeta_k^2;q^4)_\infty}\prod_{1\le k<k'\le s}(\zeta_k^2-q^2\zeta_{k'}^2)\prod_{s<k<k'\le m}(\zeta_k^2-q^2\zeta_{k'}^2).\end{aligned}$$

The matrix elements for the type II operators are quite similar. We have

$$\begin{aligned}&\langle\Lambda_j|\Psi_-^*(\xi_n)\cdots\Psi_-^*(\xi_{t+1})\Psi_+^*(\xi_t)\cdots\Psi_+^*(\xi_1)|\Lambda_i\rangle\\&=(-q)^{-t(t-1+2i)/2}\prod_{1\le k\le t}\xi_k^{-k+1+i}\prod_{t<k\le n}\xi_k^{-k+2-i}\qquad(9.8)\\&\times\prod_{1\le k<k'\le n}\frac{(\xi_k^2/\xi_{k'}^2;q^4)_\infty}{(q^{-2}\xi_k^2/\xi_{k'}^2;q^4)_\infty}\prod_{1\le k<k'\le t}(\xi_{k'}^2-q^{-2}\xi_k^2)\prod_{t<k<k'\le n}(\xi_{k'}^2-q^{-2}\xi_k^2)\end{aligned}$$

where $i,j = 0,1$, $i - j \equiv n \bmod 2$, and

$$t = \left[\frac{n+j}{2}\right].$$

9.4 Completeness relation

The completeness relation (7.21) amounts to the statement that for any $f, f' \in \mathcal{F}$ we have

$$\begin{aligned}\langle f, f'\rangle = \sum_{i=0,1}\sum_{n\geq 0}\sum_{\mu_n,\cdots,\mu_1} \chi^{-1}\frac{g^{-n}}{n!}\oint\frac{d\xi_n}{2\pi i\xi_n}\cdots\oint\frac{d\xi_1}{2\pi i\xi_1} \\ \times \mathrm{tr}_{V(\Lambda_i)}\left(f\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)(-q)^D\right) \\ \times \mathrm{tr}_{V(\Lambda_i)}\left((-q)^D\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_n}(\xi_n)f'\right).\end{aligned}$$

Here the integration is taken on the unit circle $|\xi_k| = 1$. Let

$$|u\rangle \in V(\Lambda_i), \quad |v\rangle \in V(\Lambda_j), \quad \langle u'| \in V^*(\Lambda_i), \quad \langle v'| \in V^*(\Lambda_j)$$

be arbitrary weight vectors. Taking $f = |u\rangle\langle v'|$ and $f' = |v\rangle\langle u'|$ we obtain

$$\begin{aligned}\langle u'|u\rangle\langle v'|v\rangle = \sum_{\substack{n\geq 0\\ n\equiv i-j \bmod 2}} \chi^{-1}\frac{g^{-n}}{n!}(-q)^{\deg u+\deg u'}\oint\frac{d\xi_n}{2\pi i\xi_n}\cdots\oint\frac{d\xi_1}{2\pi i\xi_1} \\ \times \sum_{\mu_n,\cdots,\mu_1}\langle v'|\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)|u\rangle\langle u'|\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_n}(\xi_n)|v\rangle.\end{aligned} \tag{9.9}$$

At present we do not have a mathematical proof of these statements. Here we give a plausibility check of the simplest case choosing $|u\rangle$, etc. to be the highest weight vectors. Eq.(9.9) then reduces to the equalities

$$\begin{aligned}\chi &= \sum_{\substack{n\geq 0\\ n \text{ even}}}\frac{g^{-n}}{n!}\oint\frac{d\xi_n}{2\pi i\xi_n}\cdots\oint\frac{d\xi_1}{2\pi i\xi_1} \qquad (9.10)\\ &\quad\times\sum_{\mu_n,\cdots,\mu_1}\langle\Lambda_0|\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)|\Lambda_0\rangle\langle\Lambda_0|\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_n}(\xi_n)|\Lambda_0\rangle \\ &= \sum_{\substack{n\geq 0\\ n \text{ odd}}}\frac{g^{-n}}{n!}\oint\frac{d\xi_n}{2\pi i\xi_n}\cdots\oint\frac{d\xi_1}{2\pi i\xi_1} \qquad (9.11)\\ &\quad\times\sum_{\mu_n,\cdots,\mu_1}\langle\Lambda_1|\Psi^*_{\mu_n}(\xi_n)\cdots\Psi^*_{\mu_1}(\xi_1)|\Lambda_0\rangle\langle\Lambda_0|\Psi_{\mu_1}(\xi_1)\cdots\Psi_{\mu_n}(\xi_n)|\Lambda_1\rangle.\end{aligned}$$

Let us estimate the order of the summand as $q \to 0$. From the explicit formulas in the extreme case (9.8) one verifies that

$$\langle \Lambda_0 | \Psi^*_\mp(\xi_{2t}) \cdots \Psi^*_\mp(\xi_{t+1}) \Psi^*_\pm(\xi_t) \cdots \Psi^*_\pm(\xi_1) | \Lambda_0 \rangle = O(q^{(3t^2 \pm t)/2}), \tag{9.12}$$

$$\langle \Lambda_1 | \Psi^*_\mp(\xi_{2t-1}) \cdots \Psi^*_\mp(\xi_{t'+1}) \Psi^*_\pm(\xi_{t'}) \cdots \Psi^*_\pm(\xi_1) | \Lambda_0 \rangle = O(q^{(3t^2 - 3t)/2}). \tag{9.13}$$

In the last line $t' = t$ for the upper sign and $t' = t-1$ for the lower sign. Define $a_{\mu_1,\cdots,\mu_n}(\xi_1,\cdots,\xi_n)$ by

$$\langle \Lambda_n | \Psi^*_{\mu_n}(\xi_n) \cdots \Psi^*_{\mu_1}(\xi_1) | \Lambda_0 \rangle = \prod_{k<k'} \frac{(\xi_k^2/\xi_{k'}^2; q^4)_\infty}{(q^{-2}\xi_k^2/\xi_{k'}^2; q^4)_\infty} \times a_{\mu_1,\cdots,\mu_n}(\xi_1,\cdots,\xi_n).$$

They satisfy

$$\begin{aligned} a_{\cdots\mp\pm\cdots}(\cdots,\xi_{k+1},\xi_k,\cdots) &= -\frac{1-q^2\xi^2}{q\xi(1-\xi^2)} a_{\cdots\pm\mp\cdots}(\cdots,\xi_k,\xi_{k+1},\cdots) \\ &\quad + \frac{\xi(1-q^2)}{q(1-\xi^2)} a_{\cdots\pm\mp\cdots}(\cdots,\xi_{k+1},\xi_k,\cdots), \end{aligned}$$

where $\xi = \xi_k/\xi_{k+1}$. Thus each time the neighboring $+-$ or $-+$ is interchanged one picks q^{-1} at worst. Let S_n stand for the sum $\sum_{\mu_n,\cdots,\mu_1} \cdots$ in (9.10), (9.11), respectively. Using (9.12) or (9.13), and taking into account the remark at the end of Chapter 6, we find that

$$\begin{aligned} S_n &= O(q^{t(t+1)}) \qquad (n = 2t), \\ &= O(q^{t(t+1)-2}) \qquad (n = 2t-1). \end{aligned}$$

For small n, the above recursion formula along with (9.8) allows us to calculate the $a_{\mu_1,\cdots,\mu_n}(\xi_1,\cdots,\xi_n)$ explicitly:

$$a_+(\xi_1) = 1$$

$$a_{-+}(\xi_1,\xi_2) = 1, \quad a_{+-}(\xi_1,\xi_2) = -q^{-1}\frac{\xi_1}{\xi_2},$$

$$a_{++-}(\xi_1,\xi_2,\xi_3) = -q^{-1}\frac{\xi_2}{\xi_3}\left(1 - q^{-2}\frac{\xi_1^2}{\xi_2^2}\right),$$

$$a_{+-+}(\xi_1,\xi_2,\xi_3) = 1 - q^{-4}\frac{\xi_1^2}{\xi_3^2},$$

$$a_{-++}(\xi_1,\xi_2,\xi_3) = -q^{-1}\frac{\xi_1}{\xi_2}\left(1 - q^{-2}\frac{\xi_2^2}{\xi_3^2}\right).$$

Hence if the first two terms are taken, (9.10), (9.11) becomes respectively

$$\begin{aligned}\chi &= 1+\frac{g^{-2}}{2}(1+q^{-2})\oint\frac{d\xi}{2\pi i\xi}\frac{(\xi^2;q^4)_\infty}{(q^{-2}\xi^2;q^4)_\infty}\frac{(\xi^{-2};q^4)_\infty}{(q^{-2}\xi^{-2};q^4)_\infty}+\cdots\\ &= 1+q^2+q^4+\cdots\end{aligned}$$

and

$$\begin{aligned}\chi &= g^{-1}+\frac{g^{-3}}{6}\prod_{k=1}^{3}\oint\frac{d\xi_k}{2\pi i\xi_k}\prod_{1\le k<k'\le 3}\frac{(\xi_k^2/\xi_{k'}^2;q^4)_\infty}{(q^{-2}\xi_k^2/\xi_{k'}^2;q^4)_\infty}\frac{(\xi_{k'}^2/\xi_k^2;q^4)_\infty}{(q^{-2}\xi_{k'}^2/\xi_k^2;q^4)_\infty}\\ &\quad\times q^{-8}\left(1+2q^2+2q^6+q^8-q^4\sum_{1\le k\neq k'\le 3}\xi_k^2/\xi_{k'}^2\right)+\cdots\\ &= 1+q^2+q^4+2q^6+2q^8+\cdots,\end{aligned}$$

which agrees with the formula for χ (7.14).

Chapter 10

The XXX limit $q \to -1$

In this chapter we give integral formulas for the correlation functions and the form factors for the XXX model by taking the limit $q \to -1$ in the formulas for the XXZ model.

10.1 The XXX limit and the continuum limit

We begin by a remark on the meaning of the limit $q \to -1$. Let us transform the XXZ Hamiltonian (1.1) in the naïve picture into an equivalent form, by conjugation under the operator $K = \prod_{k:\mathrm{odd}} \sigma_k^z$. Since σ^x, σ^y anti-commute with σ^z, we have

$$K \mathrm{H}_{XXZ} K^{-1} = \frac{1}{2} \sum_k \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \Delta \sigma_k^z \sigma_{k+1}^z \right).$$

Namely the result is again the XXZ Hamiltonian but with Δ negated and the overall sign in front is changed. In particular, in the limit $q \to -1$, the transformed one gives rise to the negative of the XXX Hamiltonian $-H_{XXX}$. Of course, such a difference by an overall sign is irrelevant for a finite lattice. It matters however in the large lattice limit, where one focuses attention on the *lowest* eigenvalues of the Hamiltonian and the 'finite' deviations from them. For instance the ground state energy for $-H_{XXX}$ is the largest eigenvalue in the context of H_{XXX}. The Hamiltonian $-H_{XXX}$ is usually referred to as the XXX Hamiltonian in the anti-ferromagnetic regime. We will obtain below its correlation functions and the form factors as the limit of the corresponding quantities for the XXZ model. We will find that the parameters $\zeta_j^2, -\xi_j^2$ (entering the type I and type II operators, respectively) must be scaled near the point 1.

Let us mention here about a similar but different limit, namely the limit to the continuum field theory. Consider for example the two-point correlation function $\langle \sigma_{N+1}^z \sigma_1^z \rangle$. Inserting the identity operator (7.21) we have

$$
{}_{(i)}\langle \mathrm{vac}|\sigma_{N+1}^z \sigma_1^z|\mathrm{vac}\rangle_{(i)} = \sum_{\substack{n\geq 0,\\ n:\mathrm{even}}} \sum_{\varepsilon_n,\cdots,\varepsilon_1} \frac{1}{n!} \oint \frac{d\xi_n}{2\pi i \xi_n} \cdots \oint \frac{d\xi_1}{2\pi i \xi_1} \prod_{j=1}^n \tau(\xi_j)^{-N}
$$
$$
\times {}_{(i+N)}\langle \mathrm{vac}|\sigma_1^z|\xi_n, \cdots, \xi_1\rangle_{\varepsilon_n,\cdots,\varepsilon_1;(i+N)\,(i);\varepsilon_1,\cdots,\varepsilon_n}\langle \xi_1, \cdots, \xi_n|\sigma_1^z|\mathrm{vac}\rangle_{(i)}.
$$

To take the continuum limit we let $q \to -1$ and $N \to \infty$ at the same time. Because of the oscillatory nature of $\tau(\xi) = e^{-ip(\theta)}$ (7.18), the main contribution comes from the point $p(\theta) = 0$, or equivalently $\xi = 1$, as opposed to the choice in the XXX limit $-\xi^2 = 1$. Setting $\xi = (-q)^{-\beta/\pi i}$ with β fixed, we find that the momentum and energy functions (7.18), (7.19) scale respectively to

$$
p(\theta) = -k' \sinh\beta + \cdots, \qquad \epsilon(\xi) = \pi k' \cosh\beta + \cdots. \tag{10.1}
$$

Here $k' = \sqrt{1-k^2}$ denotes the conjugate modulus for the elliptic functions with nome $-q$. Setting $\pi K'/K = \varepsilon$ we have in the limit $\varepsilon \to 0$ (i.e. $K' \to \frac{\pi}{2}$ and $K \to \infty$),

$$
k' \sim 4e^{-\pi^2/2\varepsilon}.
$$

In view of (10.1), we must let the lattice spacing N tend to ∞ by keeping $k'N$ finite. This means

$$
N \sim const.\, e^{\pi^2/2\varepsilon}.
$$

Naturally we expect that the limiting relativistic field theory is the $SU(2)$-invariant Thirring model (or the chiral Gross-Neveu model) treated in [65]. In this volume we will not discuss this limit any further.

10.2 Scaling

For later use let us first list the behaviors of basic functions. In the limit $p \to 1$ we have

$$
(p^z; p)_\infty \sim (p;p)_\infty \frac{(1-p)^{1-z}}{\Gamma(z)}, \tag{10.2}
$$

$$
\Theta_p(-p^z) \sim (2\pi)^{1/2}(1-p)^{-1/2}. \tag{10.3}
$$

We use also the following asymptotics:

$$\gamma_\sigma(z) \sim (q^4;q^4)_\infty \frac{(1-q^4)^{1/2-\sigma/4}}{\Gamma\left(\frac{1}{2}+\frac{\sigma}{4}\right)} \frac{A_\sigma(\beta)}{A_\sigma(\frac{\pi i}{2})}, \qquad z=(q^2)^{-\beta/\pi i}, \tag{10.4}$$

where for $\sigma = +,0,-$ we set

$$A_\sigma(\beta) = \exp\left(-\int_0^\infty \frac{dx}{x}\frac{\sinh^2 x(1-\beta/\pi i)}{\sinh 2x \cosh x} e^{-\sigma x}\right).$$

To verify (10.4) we note first that

$$\log \prod_{n_1,\cdots,n_m\geq 0} (1-zp_1^{n_1}\cdots p_m^{n_m}) = -\sum_{k=1}^\infty \frac{1}{(1-p_1^k)\cdots(1-p_m^k)}\frac{z^k}{k}.$$

Substituting $-q = e^{-\varepsilon}$, $z = e^{2\beta/\pi i}$ and using the definition of $\gamma_\sigma(z)$ (8.17) we have

$$\log\frac{\gamma_\sigma(z)}{\gamma_\sigma(q^{-2})} = -\sum_{k=1}^\infty \frac{\sinh^2(k\varepsilon(1-\beta/\pi i))}{\sinh 2k\varepsilon \cosh k\varepsilon}\frac{e^{-k\varepsilon\sigma}}{k},$$
$$\log\frac{\gamma_\sigma(q^{-2})}{((-q)^{2+\sigma};q^4)_\infty} = \sum_{k=1}^\infty \frac{\sinh^2(k\varepsilon/2)}{\sinh 2k\varepsilon \cosh k\varepsilon}\frac{e^{-k\varepsilon\sigma}}{k}.$$

Setting $x = k\varepsilon$ and letting $\varepsilon \to 0$ we find that the left hand sides of these equations tend to $\log A_\sigma(\beta)$ and $-\log A_\sigma(\pi i/2)$, respectively. The formula (10.4) follows from these and (10.2).

10.3 Critical values of the correlators

The XXZ Hamiltonian reduces to the XXX Hamiltonian when $q=-1$, i.e., $\Delta = -1$. To get a non-trivial limit of the corresponding R-matrix, we must scale the spectral parameter ζ at $\zeta = 1$: We set

$$\zeta = (q^2)^{-\beta/2\pi i}.$$

The R-matrix then scales to

$$P + \frac{\beta}{2\pi i} I,$$

where P is the transposition and I is the identity. For simplicity we ignored a scalar factor in the scaling. We employ the same scaling for the variables $z_j = \zeta_j^2$ in the integral formula (9.4).

Let us examine the behavior of the integrand of (9.4) in the limit $q \to -1$. The crucial factors are the function h in the last line. We have

$$\frac{h(z)}{(q^2;q^2)_\infty^2} \sim \frac{1}{\Gamma(1-\frac{\beta}{\pi i})\Gamma(1+\frac{\beta}{\pi i})} \text{ if } z = (q^2)^{-\beta/\pi i}.$$

If we keep z away from $z = 1$ and let $q \to -1$, then the left hand side diverges. Since we scale the variables z_j it is also necessary to scale the integration variables η_j in (9.3). In other words, in the limit $q \to -1$, the main contribution to the integral comes from the integration near $\eta_j = 1$. Therefore, we put

$$w'_{a'} = (q^2)^{-\alpha'_{a'}/\pi i}, \quad w_a = (q^2)^{-\alpha_a/\pi i}.$$

As in (9.3) we arrange them as

$$(\gamma_1, \cdots, \gamma_n) = (\alpha'_{a'_1}, \cdots, \alpha'_{a'_{s'}}, \alpha_{a_s}, \cdots, \alpha_{a_1})$$

In the limit $q \to -1$ we find the following.

$$\begin{aligned}
&\lim {}_{(i)}\langle \mathrm{vac}|E^{(n)}_{\varepsilon'_n\varepsilon_n}(\zeta_n)\cdots E^{(1)}_{\varepsilon'_1\varepsilon_1}(\zeta_1)|\mathrm{vac}\rangle_{(i)} \\
&= (-)^M \prod_{a\in A'}\int_{C^+}\frac{d\alpha'_{a'}}{2\pi i(\alpha'_{a'}-\beta_{a'})}\prod_{a\in A}\int_{C^-}\frac{d\alpha_a}{2\pi i(\alpha_a-\beta_a)} \\
&\times \prod_{\substack{a'\in A'\\ a'<j\le n}}\frac{\beta_j-\alpha'_{a'}+\pi i}{\beta_j-\alpha'_{a'}}\prod_{\substack{a\in A\\ a<j\le n}}\frac{\alpha_a-\beta_j+\pi i}{\alpha_a-\beta_j} \\
&\times \prod_{j<k}\left(\frac{\sinh(\gamma_k-\gamma_j)}{\gamma_k-\gamma_j+\pi i}\frac{\sinh(\beta_j-\beta_k)}{\beta_j-\beta_k}\right)\prod_{j,k}\frac{\gamma_j-\beta_k}{\sinh(\gamma_j-\beta_k)}.
\end{aligned}$$

Here

$$M = \sum_{a'\in A'} a' + \sum_{a\in A} a + s - \frac{n(n+1)}{2}.$$

The contours $C^\pm$ are such that for all k and $\alpha \in C^\pm$ we have

$$\begin{aligned}
C^+ &: \operatorname{Im}\beta_k < \operatorname{Im}\alpha < \operatorname{Im}\beta_k + \pi, \\
C^- &: \operatorname{Im}\beta_k - \pi < \operatorname{Im}\alpha < \operatorname{Im}\beta_k,
\end{aligned}$$

and their orientation is from $\mathrm{Re}\,\alpha = \infty$ to $\mathrm{Re}\,\alpha = -\infty$. As we expect, the result is independent of the sector $i = 0, 1$. Setting all the spectral parameters β_k to zero, we get the integral formula for the correlation functions for the XXX model. These formulas were obtained in [60, 50].

10.4 Form factors in the limit

We expect that

$$_{(i)}\langle \mathrm{vac}|\sigma_1^\alpha|\xi_n, \cdots, \xi_1\rangle_{\mu_n,\cdots,\mu_1;(i)}(d\xi_n\cdots d\xi_1)^{\frac{1}{2}}$$

scales to a finite limit. The correct scaling is given by

$$\begin{aligned} u_k\left(= -\xi_k^2\right) &= (q^2)^{-\beta_k/\pi i}, \\ v_b &= (q^2)^{-\alpha_b/\pi i + 1/2}. \end{aligned}$$

The shift 1/2 in the exponent of v_b is chosen to make the formulas neat. The crucial point is that the scaling is taken at $u_k \sim 1$ and $v_b \sim 1$.

To see that $u_k \sim 1$ is the correct scaling let us examine the simplest case

$$\begin{aligned} &_{(0)}\langle \mathrm{vac}|\sigma_1^+|\xi_2, \xi_1\rangle_{--;(0)} \\ &= -q\xi_1\xi_2^2(q^2;q^4)_\infty(q^4;q^4)_\infty\gamma_+(q^{-2})\frac{\gamma_-(u_2/u_1)\Theta_{q^8}(-1/u_1u_2)}{\prod_{k=1,2}(-q^3/u_k;q^4)_\infty(-qu_k;q^4)_\infty}. \end{aligned}$$

Notice the following behavior in the limit $q \to -1$;

$$\begin{aligned} (q^2;q^4)_\infty &\sim C_1\,(q^4;q^4)_\infty\varepsilon^{\frac{1}{2}}, \\ \gamma_+(q^{-2}) &\sim C_2\,(q^4;q^4)_\infty\varepsilon^{\frac{1}{4}}, \\ \gamma_-(u_2/u_1) &\sim C_3\,(q^4;q^4)_\infty\varepsilon^{\frac{3}{4}}, \\ (-q^3/u_k;q^4)_\infty(-qu_k;q^4)_\infty &\sim C_4\,(q^4;q^4)_\infty^2\varepsilon, \\ \Theta_{q^8}(-1/u_1u_2) &\sim C_5\varepsilon^{-\frac{1}{2}}, \end{aligned}$$

where C_j are finite. Therefore, we have

$$_{(0)}\langle \mathrm{vac}|\sigma_1^+|\xi_2, \xi_1\rangle_{--;(0)} \sim C_6\,\varepsilon^{-1}.$$

Since

$$(d\xi_2 d\xi_1)^{\frac{1}{2}} \sim \frac{2\varepsilon}{\pi i}(d\beta_2 d\beta_1)^{\frac{1}{2}},$$

the scaling for u_k is correct. In fact, if we keep u_k away from 1 in the scaling process, the factor ${}_{(0)}\langle\mathrm{vac}|\sigma_1^+|\xi_2,\xi_1\rangle_{--;(0)}$ is rapidly decreasing.

We scale the form factor at $u_k \sim 1$ in general. Since $u_k = -\xi_k^2$, there are two choices of the limiting value of ξ_k; $\xi_k = \pm i$. Since

$$|\xi_n,\cdots,\xi_1\rangle_{\varepsilon_n,\cdots,\varepsilon_1;(i)}\,{}_{(i);\varepsilon_1,\cdots,\varepsilon_n}\langle\xi_1,\cdots,\xi_n|$$

is even with respect to the sign change $\xi_k \to -\xi_k$ separately, without loss of generality we set $\xi_k \sim i$ in the scaling.

Now that the scaling for u_k is fixed it is easy to see that the scaling for the integration variables v_b (and w for σ_1^z) must be taken at 1. We have the following.

$$\begin{aligned}
&{}_{(i)}\langle\mathrm{vac}|\sigma_1^\alpha|\xi_n,\cdots,\xi_1\rangle_{\mu_n,\cdots,\mu_1;(i)}\\
&\sim \varepsilon^{-n/2}\prod_{k<k'}\frac{A_-(\beta_{k'}-\beta_k)}{A_-(\pi i/2)\Gamma(1/4)}\prod_k\frac{\pi i}{\sinh(\pi i/4-\beta_k/2)}\\
&\times\prod_b\int_{\overline{C}}\frac{d\alpha_b}{2\pi i}\prod_{k<b}(\alpha_b-\beta_k+\frac{\pi i}{2})\prod_{b<k}(\beta_k-\alpha_b+\frac{\pi i}{2})\\
&\times\prod_{k,b}\Gamma\left(-\frac14+\frac{\alpha_b-\beta_k}{2\pi i}\right)\Gamma\left(-\frac14-\frac{\alpha_b-\beta_k}{2\pi i}\right)\\
&\times\prod_b\sinh\alpha_b\prod_{b<b'}(\alpha_b-\alpha_{b'}+\pi i)\sinh(\alpha_b-\alpha_{b'})\\
&\times\overline{G}^\alpha(\alpha,\beta).
\end{aligned}$$

Here

$$\begin{aligned}
\overline{G}^+(\alpha,\beta) &= i^{n^2/4+n-2i}\left(\frac{A_+(\pi i/2)}{\Gamma(3/4)}\right)^{-n/2}2^{-3n^2/4+n/2}\pi^{-7n^2/8+9n/4-1},\\
\overline{G}^z(\alpha,\beta) &= (-1)^{n(n+2)/8+1}\left(\frac{A_+(\pi i/2)}{\Gamma(3/4)}\right)^{-n/2}2^{-3n^2/4-n}\pi^{-7n^2/8-n/4}\\
&\times\int_{\overline{C}_1+\overline{C}_2}\frac{d\alpha}{2\pi i}\frac{1}{\sinh\alpha}\prod_b\frac{1}{\sinh(\alpha-\alpha_b)}\prod_k\sinh\left(\frac{\alpha-\beta_k}{2}+\frac{\pi i}{4}\right).
\end{aligned}$$

The contour $\overline{C}$ is shown in the following figure.

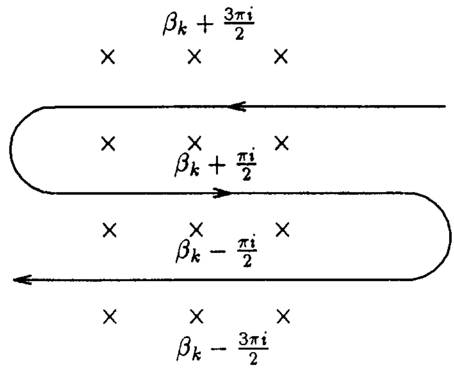

Figure 10.1: The contour $\overline{C}$

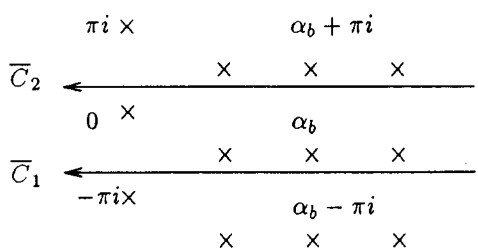

Figure 10.2: The contours $\overline{C}_1$ and $\overline{C}_2$

Chapter 11

Discussions

We end these lectures by mentioning some related works. There are various classes of models to which at least part of the present method is applicable. We will briefly survey these extensions and the difficulties that arise. We also touch upon mathematical works concerning vertex operators and the q-difference equations.

11.1 Other models

So far we have been focusing our attention on the six-vertex model in the anti-ferroelectric regime and its spin-chain equivalent, the XXZ model with $\Delta < -1$. Here we wish to discuss the possibility of extending the present scheme to other known models, and difficulties as well. Specifically we will touch upon the following different categories of models.

(1) Vertex models associated with various quantum affine algebras and their representations,

(2) RSOS models of Andrews-Baxter-Forrester and their Lie-theoretical generalizations,

(3) Eight-vertex model,

(4) Ising-type edge interaction models.

The category (1) is a straightforward generalization of the six-vertex model. In principle the present method is applicable to such models when the underlying finite-dimensional module V is 'perfect' [44]— loosely speaking this means that the highest weight module of a fixed level can be realized as a half-infinite tensor product of V. The representative examples of this sort are the higher spin analog of the six-vertex/XXZ model based on the $(k+1)$-dimensional representation of $U_q(sl_2)$. It has been known from the Bethe Ansatz method [69] that, in the anti-ferroelectric regime, the excitations of these models always carry spin 1/2 irrespective of k, and that the single-particle eigenvalues of the transfer matrix are given by the same function $\tau(\zeta)$ as in the case $k = 1$. Moreover the multi-particle states do not have a simple tensor product form but obey an RSOS type restriction [62]. In [34] these models are studied in the framework of the present method. The relevant highest weight modules are level k integrable representations of $U_q(\widehat{sl}_2)$. The particle picture provided by the VO's is in agreement with the known results mentioned above. The case of $k = 2$ has a particularly nice feature, for the level two modules of $U_q(\widehat{sl}_2)$ have explicit realizations in terms of bosons and fermions [15]. The corresponding VO's and integral formula for the correlation functions are worked out in [33, 16].

As an example of higher rank algebras, the XXZ type model corresponding to the n-dimensional representation of $U_q(sl_n)$ is studied in [51] (see also [22]). On the basis of the Frenkel-Jing bosonization [31], the formulas for the VO's and the spontaneous staggered polarization are derived.

All these works concern the anti-ferroelectric regime of the models. In exactly the same way as for the six-vertex model the correlators are expressed as traces of type I VO's and the eigenstates are described by the type II VO's. However, it is a rather non-trivial problem to solve the difference equations for the correlators and form factors. We will come back to this point in the next subsection. In practice the bosonization is the most efficient method for that. In the case of general level, there exists also a bosonization based on the Wakimoto modules, but the expressions for the currents become far more complicated than for level 1 [54, 57, 64, 5, 6, 46, 1]. To our knowledge the treatment of integrable modules and the calculation of the traces of VO's have not been accomplished so far (see however [48]).

As we already mentioned, our method is not applicable to the massless regime $|\Delta| \leq 1$. The model being critical there, the corner transfer matrix is no longer well defined. We have no direct hint concerning the representation theoretical picture of the space of states.

Next let us come to the second category (2) consisting of *Interaction-Round-A-Face(IRF) models*, or face models for short. In contrast to the vertex models we have been discussing, here the Boltzmann weights are associated with configurations around each *face* (an elementary square) of the lattice. While the local 'spin' variables of the vertex models live on the edges, those of the face models live on the lattice sites. In the models based on Lie theory the states that the local variables assume are identified with the weights of the Lie algebra. The Boltzmann weights of these face models are written in terms of elliptic theta functions [3, 18, 19, 42, 38]. Their true nature was subsequently clarified to be the connection matrices for the qKZ equation [32]. In [40, 30] the present framework is extended to these models, again in the anti-ferroelectric regime usually referred to as regime III. The mathematical formulation is a relative, or coset, version of that of the six-vertex model. In place of the highest weight modules themselves, we consider a tensor product of two such:

$$V(\xi) \otimes V(\eta) = \bigoplus_{\lambda} \Omega_{\xi\eta;\lambda} \otimes V(\lambda),$$

where

$$\Omega_{\xi\eta;\lambda} = \{ v \in V(\xi) \otimes V(\eta) \mid e_i v = 0, \quad t_i v = q^{(\lambda,h_i)} v \ \forall i \}$$

denotes the space of highest weight vectors of a definite weight λ. The corner transfer matrix of a face model splits into blocks in which the local variable at the central site is fixed to a state λ. The $\Omega_{\xi\eta;\lambda}$ is the space on which the corner transfer matrix is acting; in other words it is the piece of the 'left half' of the space of states where the central site is in a fixed state λ. The choice of (ξ, η) accommodates that of the boundary conditions. The whole space of states is $\oplus_\lambda \Omega_{\xi\eta;\lambda} \otimes \Omega^*_{\xi'\eta';\lambda}$. This formulation has been guided by the foregoing results concerning the spectrum of the corner transfer matrix [18, 19, 42, 44, 21]. The VO's for the RSOS models can also be formulated mathematically. Their relation to the VO's for the vertex models is discussed in [30].

Beyond the models discussed so far, the representation theoretical interpretation of the space of states is not known. Nevertheless there are examples for which the corner transfer matrix along with the 'physical' arguments in Chapter 4 enables us to formulate the VO's [39, 30].

The construction of Chapter 4 carries over almost word-to-word to the third category (3), the eight-vertex model. By solving the simplest difference equation, the conjectured formula for the spontaneous staggered polarization for the eight-vertex model can be recovered [39] (see also [61] as for the sl_n case). Very recently an elliptic extention of the quantum affine algebra was proposed [29], which is conjectured to admit a natural deformation of the whole structure including highest weight modules and VO's.

The last category (4) comprises models where the Boltzmann weights are associated with each *edge* of the lattice. The simplest example is the Ising model. In fact it can be formulated as the simplest case of the ABF models as well, so the mathematical formulation of its space of states is known. An equivalent, yet more direct, approach is also possible: to use the Jordan-Wigner fermions. In this case the 'left-half' of the space of states can be identified with the fermion Fock space. This exercise is worked out in [30].

Within the category of edge-interaction models, the Ising model has extensions in two different directions: the Kashiwara-Miwa models [45] and the chiral Potts models [4, 58, 14]. The Boltzmann weights of the Kashiwara-Miwa models are written in terms of elliptic theta functions. Though the mathematical framework for the space of states is not known, the structure of the corner transfer matrix is very similar to the cases we have been discussing, and its spectrum can be interpreted in terms of Lie algebra characters [41]. The difference equations for the correlators are worked out on the basis of the physical arguments in [30]. The chiral Potts models are more difficult to handle. In all the other models the spectral parameters live on $\mathbf{P}^1$ or a genus 1 curve, and enter the Boltzmann weights as the difference $u_1 - u_2$ of the additive spectral parameters u_i.[1] This 'difference property' was crucial for the corner transfer matrix to have the simple structure in the infinite lattice limit. In the case of the chiral Potts models, the spectral parameters live on algebraic curves of high genus, and the difference property dose not make sense. It is a challenge to unravel the structure of the chiral Potts models, in particular that of the corner transfer matrix to begin with.

11.2 The q-KZ equation

In conformal field theory, the vertex operators as intertwiners between highest weight modules first appeared in the paper by Tsuchiya-Kanie [75]. It opened a way to many interesting topics in mathematics centering around the representation theory, differential equations and special functions. Subsequently the q-deformation of vertex operators was discussed by Frenkel and Reshetikhin [32] and the difference-equation version of the Knizhnik-Zamolodchikov (KZ) equations was derived. A rather remarkable fact is that the same difference equations arise in different contexts.

To make comparison, let us recall the results of [32]. For simplicity we restrict the discussion to the case of $U_q(\widehat{sl}_2)$ and the evaluation module V_ζ

[1] We have been using multiplicative spectral parameters $\zeta_i = e^{u_i}$.

with V two-dimensional. Consider the compositions of type I VO's between irreducible highest weight modules $V(\mu_j)$ of level k,

$$V(\mu_n) \longrightarrow V(\mu_{n-1}) \otimes V_{\zeta_n} \longrightarrow \cdots \longrightarrow V(\mu_0) \otimes V_{\zeta_1} \otimes \cdots \otimes V_{\zeta_n}.$$

Define the highest-to-highest matrix elements

$$F(\zeta_1, \cdots, \zeta_n) = \langle \mu_0 | \Phi^{(\mu_0,\mu_1)}(\zeta_1) \cdots \Phi^{(\mu_{n-1},\mu_n)}(\zeta_n) | \mu_n \rangle$$

which are $V^{\otimes n}$-valued functions. Then they satisfy the following system of difference equations[1]

$$\begin{aligned} F(\zeta_1, \cdots, p\zeta_j, \cdots, \zeta_n) &= R^{(-)}_{j-1\,j}(p^{-1}\zeta_{j-1}/\zeta_j)^{-1} \cdots R^{(-)}_{1\,j}(p^{-1}\zeta_1/\zeta_j)^{-1} \\ &\times D_j R^{(-)}_{j\,n}(\zeta_j/\zeta_n) \cdots R^{(-)}_{j\,j+1}(\zeta_j/\zeta_{j+1}) F(\zeta_1, \cdots, \zeta_n). \qquad (11.1) \end{aligned}$$

Here $p = q^{k+2}$, and the R matrix is a scalar multiple of (3.6)

$$R^{(-)}(\zeta) = \rho^{(-)}(\zeta^2) R(\zeta), \qquad \rho^{(-)}(z) = q^{1/2} \frac{(q^4 z^{-1}; q^4)_\infty (z^{-1}; q^4)_\infty}{(q^2 z^{-1}; q^4)^2_\infty}.$$

The D_j are diagonal operators acting on the j-th space

$$\begin{aligned} D_j &= q^{-\psi_j + k - 2(\mu_j, \Lambda_1) + 1/2}, \\ \psi &= \bar{\mu}_0 + \bar{\mu}_n - k\bar{\rho}, \end{aligned}$$

where in the last line $\bar{\mu} = l(\Lambda_1 - \Lambda_0)$ denotes the classical part of $\mu = (k-l)\Lambda_0 + l\Lambda_1$. Let us call (11.1) the q-KZ equation of level k.

The case of level 0 appeared earlier in Smirnov's works on the form factors of massive integrable field theory models. In particular the form factors of the sine-Gordon model satisfy (11.1) with $D_j = 1$. In this context the parameter q is related to the coupling constant of the sine-Gordon model but with $|q| = 1$.

As we have already seen, the trace functions for type I vertex operators also satisfy the q-KZ equations (4.43). The case relevant to the correlation functions is $x = q^2$, so the q-KZ equations are of level $k = -4$ (here $2h = 4$ is twice the dual Coxeter number for $\widehat{sl}_2$). Unlike in Smirnov's case the relevant region is $|q| < 1$.

[1] The original equations are presented in the homogeneous picture in [32]. We have rewritten them in the principal picture.

We noted earlier the difference in the analytic structures between the highest-to-highest matrix elements and the trace functions: the former are power series in the variables ζ_{j+1}/ζ_j but the latter contain both positive and negative powers. The derivation of the q-KZ equations for the former makes serious use of the structure theory of the quantized affine algebras, whereas for the latter it is based simply on the commutation relations of VO's and the cyclic property of the trace. Combining the two methods, Etingof [25] derived two independent difference equations for the trace functions (see also [26, 27] for related works).

The construction of solutions to the q-KZ equations is also an interesting and important problem. At the moment three different types of integral representations are known:

(1) Jackson integral [55, 56, 63, 76, 72, 48]

(2) Smirnov's integrals [65, 37, 47]

(3) Contour integrals from bosonization.

The Jackson integrals (1) are actually infinite sums of special type, rather than genuine integrals. They have been investigated for general highest weight modules and evaluation modules of $U_q(\widehat{sl}_2)$ and $U_q(\widehat{sl}_n)$. In order to get actual solutions one has to determine the possible choices of integration cycles. To our knowledge this issue has not been settled. In [48] the trace functions were studied on the basis of the Wakimoto module construction.

So far Smirnov's integrals (2) have been known only for level 0 case. As opposed to Jackson integrals these are ordinary contour integrals. Here one makes a specific choice of the contours and the integrand admits certain freedom, which accommodates the freedom of choosing different solutions. Finally the integral formulas (3) presented in Chapter 8 apply only to this particular solution.

In this connection we wish to mention Smirnov's works on the classical limit of the form factors. He gave an intriguing interpretation to his integral formulas as the quantization of period integrals on Riemann surfaces [66, 68, 67]. See also [60] for related discussions.

11.3 Related works

In closing let us mention some other works directly related to the present volume.

We defined the eigenstates of the Hamiltonian by applying the type II operators on the vacuum. Though these eigenstates themselves have canonical meaning, the operators that create them are by no means unique. In fact the type II operators are not the most natural ones for the following reason. As operators on the space of states they act only on the left half, i.e. they have the form $\mathcal{O} \otimes \mathrm{id}$. If we apply the type II operators to arbitrary vectors of the highest weight module then there appear negative powers of q which is hard to control. This is because the type II operators do *not* preserve the crystal structure as opposed to the type I operators [21]. From physical considerations the eigenstates should have a well defined limit as q tends to 0, but it is hard to see from the construction using type II operators. Miki introduced two kinds of creation operators which act on both the left and right halves of the space [59]. When acting on the vacuum, each one of these new operators create the same set of eigenstates as do the type II operators. They satisfy some simple commutation relations with each other, and in addition they seem to be well defined as $q \to 0$. Using these he showed that the form factors of the XXZ model satisfy properties analogous to Smirnov's axioms for the form factors of massive field theory models [65].

Lukyanov's works [53] are concerned with massive field theory. Using a method similar to the bosonization for our lattice models, he discussed a construction of the form factors directly in the continuum. He also gave an interpretation of the type I vertex operators as the analogue of the Jost functions in classical inverse scattering method [52].

We remark that Korepin and others have developed an approach to the lattice correlation functions on the basis of the quantum inverse scattering method. For these works the reader is referred to the monograph [49].

Appendix A

List of formulas

A.1 R matrix

(A.1.1) Parameterization

$$R(\zeta) = \frac{1}{\kappa(\zeta)}\begin{pmatrix} 1 & & & \\ & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \\ & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \\ & & & 1 \end{pmatrix},$$

$$\kappa(\zeta) = \zeta\frac{(q^4\zeta^2;q^4)_\infty(q^2\zeta^{-2};q^4)_\infty}{(q^4\zeta^{-2};q^4)_\infty(q^2\zeta^2;q^4)_\infty},$$

where we set

$$(z;p)_\infty = \prod_{n=1}^{\infty}(1-zp^n).$$

(A.1.2) Initial Condition

$$R(1) = P$$

(A.1.3) Unitarity Relation

$$R_{12}(\zeta_1/\zeta_2)R_{21}(\zeta_2/\zeta_1) = 1$$

(A.1.4) Crossing Symmetry

$$R_{21}(\zeta_2/\zeta_1)^{t_1} = \sigma_1^x R_{12}(-q^{-1}\zeta_1/\zeta_2)\sigma_1^x$$

(A.1.5) Convention of indices

$$R(\zeta)\, v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} = \sum_{\varepsilon_1,\varepsilon_2} v_{\varepsilon_1} \otimes v_{\varepsilon_2} R(\zeta)_{\varepsilon_1\,\varepsilon_2}^{\varepsilon_1'\,\varepsilon_2'}.$$

A.2 $U_q(\widehat{sl}_2)$

(A.2.1) Weights, roots

$$\begin{aligned}
&P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta,\\
&\alpha_0 + \alpha_1 = \delta, \quad \Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}, \quad \rho = \Lambda_0 + \Lambda_1,\\
&(\Lambda_0, \Lambda_0) = 0, \quad (\Lambda_0, \alpha_1) = 0, \quad (\Lambda_0, \delta) = 1,\\
&(\alpha_1, \alpha_1) = 2, \quad (\alpha_1, \delta) = 0, \quad (\delta, \delta) = 0.
\end{aligned}$$

(A.2.2) Identification

$$h_0 = \alpha_0, \quad h_1 = \alpha_1, \quad d = \Lambda_0, \quad \rho = 2d + \frac{1}{2}h_1.$$

(A.2.3) Defining relations

$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'},\\
&q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i,\\
&[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}},\\
&e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (i \neq j),\\
&f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j),
\end{aligned}$$

where we set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad t_i = q^{h_i}.$$

(A.2.4) Hopf algebra structure

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h,\\
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,\\
&\epsilon(q^h) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0,\\
&a(q^h) = q^{-h}, \quad a(e_i) = -t_i^{-1} e_i, \quad a(f_i) = -f_i t_i.
\end{aligned}$$

(A.2.5) Axiomatic properties

$$\begin{aligned}
&\Delta(xy) = \Delta(x)\Delta(y), \quad \epsilon(xy) = \epsilon(x)\epsilon(y), \quad a(xy) = a(y)a(x), \\
&(\Delta \otimes \mathrm{id}\,) \circ \Delta = (\mathrm{id}\, \otimes \Delta) \circ \Delta, \\
&(\epsilon \otimes \mathrm{id}\,) \circ \Delta = \mathrm{id}\, = (\mathrm{id}\, \otimes \epsilon) \circ \Delta, \\
&m \circ (a \otimes \mathrm{id}\,) \circ \Delta = \epsilon = m \circ (\mathrm{id}\, \otimes a) \circ \Delta.
\end{aligned}$$

where $m(x \otimes y) = xy$.

(A.2.6) Square of antipode

$$a^2(x) = q^{-2\rho} x q^{2\rho} \qquad \forall x \in U.$$

(A.2.7) Dual modules The dual space M^* of a left U-module M is naturally a *right* U-module by

$$\langle v^* x, v \rangle = \langle v^*, xv \rangle.$$

For an anti-automorphism ϕ the left module structure $M^{*\phi}$ is given by

$$\langle x v^*, v \rangle = \langle v^*, \phi(x) v \rangle.$$

If ϕ is also a coalgebra anti-homomorphism, then

$$\begin{aligned}
&(M_1 \otimes M_2)^{*\phi} \simeq M_2^{*\phi} \otimes M_1^{*\phi}, \\
&\mathrm{Hom}\,_U(L, M \otimes N) = \mathrm{Hom}\,_U(M^{*a} \otimes L, N), \\
&\mathrm{Hom}\,_U(L \otimes N, M) = \mathrm{Hom}\,_U(L, M \otimes N^{*a}).
\end{aligned}$$

(A.2.8) Two-dimensional module $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$:

$$\begin{aligned}
&e_1 v_+ = 0, \quad e_1 v_- = v_+, \\
&f_1 v_+ = v_-, \quad f_1 v_- = 0, \\
&t_1 v_\pm = q^{\pm 1} v_\pm.
\end{aligned}$$

(A.2.9) Evaluation module

$$\begin{aligned}
V_\zeta &= V \otimes \mathbf{C}[\zeta, \zeta^{-1}] = V_\zeta^{(+)} \oplus V_\zeta^{(-)}, \\
V_\zeta^{(\pm)} &= \mathrm{span}\,\{v_\pm \otimes \zeta^{2n}, v_\mp \otimes \zeta^{2n-1} \ (n \in \mathbf{Z})\},
\end{aligned}$$

with the action:

$$\begin{aligned}
&e_0(v_\varepsilon \otimes \zeta^m) = (f_1 v_\varepsilon) \otimes \zeta^{m+1}, \quad e_1(v_\varepsilon \otimes \zeta^m) = (e_1 v_\varepsilon) \otimes \zeta^{m+1},\\
&f_0(v_\varepsilon \otimes \zeta^m) = (e_1 v_\varepsilon) \otimes \zeta^{m-1}, \quad f_1(v_\varepsilon \otimes \zeta^m) = (f_1 v_\varepsilon) \otimes \zeta^{m-1},\\
&t_0 = t_1^{-1}, \quad t_1(v_\varepsilon \otimes \zeta^m) = (t_1 v_\varepsilon) \otimes \zeta^m,\\
&\rho = \zeta \frac{d}{d\zeta} \pm \frac{1}{2}.
\end{aligned}$$

(A.2.10) Dual evaluation module

$$\begin{aligned}
&V_\zeta^{*a} = V^* \otimes \mathbf{C}[\zeta, \zeta^{-1}] = V_\zeta^{(+)*a} \oplus V_\zeta^{(-)*a},\\
&V_\zeta^{(\pm)*a} = \mathrm{span}\,\{v_\pm^* \otimes \zeta^{2n}, v_\mp^* \otimes \zeta^{2n-1}\ (n \in \mathbf{Z})\},\\
&\langle v_\varepsilon^* \otimes \zeta^m, v_{\varepsilon'} \otimes \zeta^n \rangle = \delta_{\varepsilon\varepsilon'}\delta_{m+n,0},\\
&\rho = \zeta \frac{d}{d\zeta} \mp \frac{1}{2} \qquad \text{on} \quad V_\zeta^{(\pm)*a}.
\end{aligned}$$

We have an isomorphism

$$V^{(s)}_{-q^{\mp 1}\zeta} \xrightarrow{\sim} V_\zeta^{(-s)*a^{\pm 1}}, \qquad v_\varepsilon \otimes \zeta^n \mapsto v_{-\varepsilon}^* \otimes \zeta^n.$$

(A.2.11) Drinfeld generators

$$\begin{aligned}
&[a_k, a_l] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},\\
&K a_k K^{-1} = a_k, \quad K x_k^\pm K^{-1} = q^{\pm 2} x_k^\pm,\\
&[a_k, x_l^\pm] = \pm \frac{[2k]}{k} \gamma^{\mp |k|/2} x_{k+l}^\pm,\\
&x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm = q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm,\\
&[x_k^+, x_l^-] = \frac{\gamma^{\frac{k-l}{2}} \psi_{k+l} - \gamma^{\frac{l-k}{2}} \varphi_{k+l}}{q - q^{-1}},\\
&q^d \gamma^{1/2} q^{-d} = \gamma^{1/2}, \quad q^d K q^{-d} = K,\\
&q^d x_k^\pm q^{-d} = q^k x_k^\pm, \quad q^d a_k q^{-d} = q^k a_k.
\end{aligned}$$

where ψ_k, φ_k are defined by

$$\begin{aligned}
&\sum_{k=0}^\infty \psi_k z^{-k} = K \exp\{(q - q^{-1}) \sum_{k=1}^\infty a_k z^{-k}\},\\
&\sum_{k=0}^\infty \varphi_{-k} z^k = K^{-1} \exp\{-(q - q^{-1}) \sum_{k=1}^\infty a_{-k} z^k\}.
\end{aligned}$$

(A.2.12) Relation with Chevalley generators

$$t_1 = K, \quad x_0^+ = e_1, \quad x_0^- = f_1,$$
$$t_0 = \gamma K^{-1}, \quad x_1^- = e_0 t_1, \quad x_{-1}^+ = t_1^{-1} f_0.$$

A.3 Currents and vertex operators

(A.3.1) Currents

$$X^\pm(z) = \sum_{n\in\mathbf{Z}} x_n^\pm z^{-n-1},$$
$$[a_k, X^\pm(z)] = \pm\frac{[2k]}{k}\gamma^{\mp|k|/2} z^k X^\pm(z),$$
$$(z - q^{\pm2}w)X^\pm(z)X^\pm(w) + (w - q^{\pm2}z)X^\pm(w)X^\pm(z) = 0,$$
$$[X^+(z), X^-(w)] = K\exp\{(q-q^{-1})\sum_{k=1}^\infty a_k\gamma^{k/2}z^{-k}\}\frac{\delta(z/\gamma w)}{(q-q^{-1})zw}$$
$$-K^{-1}\exp\{-(q-q^{-1})\sum_{k=1}^\infty a_{-k}\gamma^{k/2}z^k\}\frac{\delta(\gamma z/w)}{(q-q^{-1})zw}.$$

Here $\delta(z) = \sum_{n\in\mathbf{Z}} z^n$.

(A.3.2) Vertex operators

$$\Phi(\zeta) : V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_\zeta, \qquad \Phi(\zeta) = \sum_\varepsilon \Phi_\varepsilon(\zeta) \otimes v_\varepsilon,$$
$$\Psi^*(\zeta) : V_\zeta \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}), \qquad \Psi_\varepsilon^*(\zeta) = \Psi^*(\zeta)(v_\varepsilon \otimes \cdot).$$

They are normalized as

$$\langle\Lambda_1|\Phi_-(\zeta)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\Phi_+(\zeta)|\Lambda_1\rangle = 1,$$
$$\langle\Lambda_1|\Psi_-(\zeta)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\Psi_+(\zeta)|\Lambda_1\rangle = 1.$$

(A.3.3) Fock space

$$V(\Lambda_i) = \mathbf{C}[a_{-1}, a_{-2}, \cdots] \otimes \left(\oplus_{n\in\mathbf{Z}}\mathbf{C}e^{\Lambda_i+n\alpha}\right).$$

(A.3.4) Bosonization

$$X^\pm(z) \;=\; e^{R^\pm(z)}e^{S^\pm(z)}e^{\pm\alpha}z^{\pm\partial},$$

$$\begin{aligned}
\Phi_-^{(1-i,i)}(\zeta) &= e^{P(\zeta^2)}e^{Q(\zeta^2)}e^{\alpha/2}(-q^3\zeta^2)^{(\partial+i)/2}\zeta^{-i},\\
\Phi_+^{(1-i,i)}(\zeta) &= \oint_{C_1}\frac{dw}{2\pi i}\frac{(1-q^2)w\zeta}{q(w-q^2\zeta^2)(w-q^4\zeta^2)} :\Phi_-^{(1-i,i)}(\zeta)X^-(w):,\\
\Psi_-^{*(1-i,i)}(\zeta) &= e^{-P(q^{-1}\zeta^2)}e^{-Q(q\zeta^2)}e^{-\alpha/2}(-q^3\zeta^2)^{(-\partial+i)/2}\zeta^{1-i},\\
\Psi_+^{*(1-i,i)}(\zeta) &= \oint_{C_2}\frac{dw}{2\pi i}\frac{q^2(1-q^2)\zeta}{(w-q^2\zeta^2)(w-q^4\zeta^2)} :\Psi_-^{*(1-i,i)}(\zeta)X^+(w):,
\end{aligned}$$

where

$$P(z)=\sum_{n=1}^\infty \frac{a_{-n}}{[2n]}q^{7n/2}z^n, \qquad Q(z)=-\sum_{n=1}^\infty \frac{a_n}{[2n]}q^{-5n/2}z^{-n},$$
$$R^\pm(z)=\pm\sum_{n=1}^\infty \frac{a_{-n}}{[n]}q^{\mp n/2}z^n, \quad S^\pm(z)=\mp\sum_{n=1}^\infty \frac{a_n}{[n]}q^{\mp n/2}z^{-n}.$$

The contours encircle $w=0$ in such a way that

$$\begin{aligned}
C_1 &: \quad q^4\zeta^2 \text{ is inside and } q^2\zeta^2 \text{ is outside},\\
C_2 &: \quad q^4\zeta^2 \text{ is outside and } q^2\zeta^2 \text{ is inside}.
\end{aligned}$$

A.4 Properties of Vertex operators

(A.4.1) Homogeneity

$$\xi^{-D}\circ\Phi_\varepsilon(\zeta)\circ\xi^D=\Phi_\varepsilon(\zeta/\xi), \qquad \xi^{-D}\circ\Psi^*_\varepsilon(\zeta)\circ\xi^D=\Psi^*_\varepsilon(\zeta/\xi).$$

(A.4.2) Z_2-symmetry

$$\nu\Phi_\varepsilon^{(0,1)}(\zeta)\nu=\Phi_{-\varepsilon}^{(1,0)}(\zeta), \qquad \nu\Psi_\varepsilon^{(0,1)}(\zeta)\nu=\Psi_{-\varepsilon}^{(1,0)}(\zeta).$$

(A.4.3) Commutation relations

$$R(\zeta_1/\zeta_2)\overset{1}{\Phi}(\zeta_1)\overset{2}{\Phi}(\zeta_2) = \overset{2}{\Phi}(\zeta_2)\overset{1}{\Phi}(\zeta_1), \tag{A.1}$$
$$\overset{1}{\Psi}{}^*(\zeta_1)\overset{2}{\Psi}{}^*(\zeta_2) = -\overset{2}{\Psi}{}^*(\zeta_2)\overset{1}{\Psi}{}^*(\zeta_1)R(\zeta_1/\zeta_2), \tag{A.2}$$
$$\overset{1}{\Phi}(\zeta_1)\overset{2}{\Psi}{}^*(\zeta_2) = \tau(\zeta_1/\zeta_2)\overset{2}{\Psi}{}^*(\zeta_2)\overset{1}{\Phi}(\zeta_1). \tag{A.3}$$

where

$$\begin{aligned}
\tau(\zeta) &= \zeta^{-1}\frac{\Theta_{q^4}(q\zeta^2)}{\Theta_{q^4}(q\zeta^{-2})},\\
\Theta_p(z) &= (z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty.
\end{aligned}$$

(A.4.4) Dual VO

$$\begin{aligned}
\Phi^*_\varepsilon(\zeta) &= \Phi_{-\varepsilon}(-q^{-1}\zeta), && \text{(A.4)}\\
\Psi_\varepsilon(\zeta) &= \Psi^*_{-\varepsilon}(-q^{-1}\zeta), && \text{(A.5)}\\
g\sum_\varepsilon \Phi^*_\varepsilon(\zeta)\Phi_\varepsilon(\zeta) &= \mathrm{id}\,, && \text{(A.6)}\\
g\,\Phi_{\varepsilon_1}(\zeta)\Phi^*_{\varepsilon_2}(\zeta) &= \delta_{\varepsilon_1\varepsilon_2}\mathrm{id}\,, && \text{(A.7)}\\
& && \text{(A.8)}
\end{aligned}$$

$$\Psi^{(i,1-i)}_{\varepsilon_1}(\zeta_1)\Psi^{*(1-i,i)}_{\varepsilon_2}(\zeta_2) = \frac{g\delta_{\varepsilon_1\varepsilon_2}}{1-\zeta_2^2/\zeta_1^2}\left(\frac{\zeta_2}{\zeta_1}\right)^{i+\frac{1-\varepsilon_1}{2}} + \cdots \qquad \zeta_1 \to \pm\zeta_2, \tag{A.9}$$

where $\cdots$ means regular terms and

$$g = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{A.10}$$

(A.4.5) Parity

$$\begin{aligned}
\Phi^{(1-i,i)}(-\zeta) &= (-1)^{i+\frac{1+\varepsilon}{2}}\Phi^{(1-i,i)}_\varepsilon(\zeta), && \text{(A.11)}\\
\Psi^{*(1-i,i)}(-\zeta) &= (-1)^{i+\frac{1-\varepsilon}{2}}\Psi^{*(1-i,i)}_\varepsilon(\zeta). && \text{(A.12)}
\end{aligned}$$

(A.4.6) Two-point functions

$$\begin{aligned}
\langle\Lambda_0|\Phi_\pm(\zeta_1)\Phi_\mp(\zeta_2)|\Lambda_0\rangle &= \frac{(q^6\zeta_2^2/\zeta_1^2;q^4)_\infty}{(q^4\zeta_2^2/\zeta_1^2;q^4)_\infty}\times\begin{cases}1\\ \frac{-q\zeta_2}{\zeta_1}\end{cases}, && \text{(A.13)}\\
\langle\Lambda_0|\Psi^*_\mp(\zeta_1)\Psi^*_\pm(\zeta_2)|\Lambda_0\rangle &= \frac{(\zeta_2^2/\zeta_1^2;q^4)_\infty}{(q^{-2}\zeta_2^2/\zeta_1^2;q^4)_\infty}\times\begin{cases}1\\ \frac{\zeta_2}{-q\zeta_1}\end{cases}, && \text{(A.14)}\\
\langle\Lambda_0|\Phi_\pm(\zeta_1)\Psi^*_\pm(\zeta_2)|\Lambda_0\rangle &= \frac{(q^3\zeta_2^2/\zeta_1^2;q^4)_\infty}{(q\zeta_2^2/\zeta_1^2;q^4)_\infty}\times\begin{cases}1\\ \frac{\zeta_2}{\zeta_1}\end{cases}, && \text{(A.15)}\\
\langle\Lambda_0|\Psi^*_\mp(\zeta_1)\Phi_\mp(\zeta_2)|\Lambda_0\rangle &= \frac{(q^3\zeta_2^2/\zeta_1^2;q^4)_\infty}{(q\zeta_2^2/\zeta_1^2;q^4)_\infty}\times\begin{cases}1\\ \frac{\zeta_2}{\zeta_1}\end{cases}. && \text{(A.16)}
\end{aligned}$$

(A.4.7) Trace functions Let $\zeta = \zeta_2/\zeta_1$, $\{z\} = (z;q^4,x^2)_\infty$.

$$\mathrm{tr}_{V(\Lambda_0)}\Big(x^D\Phi_\pm(\zeta_1)\Phi_\mp(\zeta_2)\Big) = \frac{\{q^2x^2\}^2}{\{q^4x^2\}^2}\frac{\{q^2\zeta^2\}}{\{q^4\zeta^2\}}\frac{\{q^2x^2\zeta^{-2}\}}{\{q^4x^2\zeta^{-2}\}}$$

$$\times\frac{1}{2}\left(\frac{(q^2x;x^2)_\infty}{(-q\zeta;x)_\infty(-qx\zeta^{-1};x)_\infty}\pm\frac{(q^2x;x^2)_\infty}{(q\zeta;x)_\infty(qx\zeta^{-1};x)_\infty}\right), \quad \text{(A.17)}$$

$$\mathrm{tr}_{V(\Lambda_0)}\Big(x^D\Psi_\pm(\zeta_1)\Psi_\mp(\zeta_2)\Big)=\frac{\{x^2\}^2}{\{q^2x^2\}^2}\frac{\{\zeta^2\}}{\{q^2\zeta^2\}}\frac{\{x^2\zeta^{-2}\}}{\{q^2x^2\zeta^{-2}\}}$$

$$\times\frac{1}{2}\left(\frac{(q^{-2}x;x^2)_\infty}{(-q^{-1}\zeta;x)_\infty(-q^{-1}x\zeta^{-1};x)_\infty}\pm\frac{(q^{-2}x;x^2)_\infty}{(q^{-1}\zeta;x)_\infty(q^{-1}x\zeta^{-1};x)_\infty}\right),$$

$$\mathrm{tr}_{V(\Lambda_0)}\Big(x^D\Phi_+(\zeta_1)\Psi_-(\zeta_2)\Big)=\frac{\{q\zeta^2\}}{\{q^{-1}\zeta^2\}}\frac{\{q^5x^2\zeta^{-2}\}}{\{q^3x^2\zeta^{-2}\}}\Theta_{x^4}(-xq^{-2}\zeta^2), \quad \text{(A.18)}$$

$$\mathrm{tr}_{V(\Lambda_0)}\Big(x^D\Phi_-(\zeta_1)\Psi_+(\zeta_2)\Big)=\frac{\{q\zeta^2\}}{\{q^{-1}\zeta^2\}}\frac{\{q^5x^2\zeta^{-2}\}}{\{q^3x^2\zeta^{-2}\}}\Theta_{x^4}(-xq^2\zeta^{-2})\times(-q^{-1}\zeta). \quad \text{(A.19)}$$

A.5 Principal vs homogeneous pictures

(A.5.1) Evaluation module

$$V_\zeta^{(\pm)}\xrightarrow{\bar{C}(\zeta)}V_z^{(h)},\qquad \zeta^2=z,$$
$$v_\varepsilon\otimes\zeta^m\ \mapsto\ v_\varepsilon\otimes\zeta^{m+(\pm1-\varepsilon)/2},$$

(A.5.2) R matrix

$$\overline{R}(\zeta_1^2/\zeta_2^2)=\kappa(\zeta_1/\zeta_2)\times$$
$$\times\left(\bar{C}(\zeta_1)\otimes\bar{C}(\zeta_2)\right)R(\zeta_1/\zeta_2)\left(\bar{C}(\zeta_1)\otimes\bar{C}(\zeta_2)\right)^{-1},$$

$$\overline{R}(z)=\begin{pmatrix}1&&&\\&\dfrac{(1-z)q}{1-q^2z}&\dfrac{(1-q^2)}{1-q^2z}&\\&\dfrac{(1-q^2)z}{1-q^2z}&\dfrac{(1-z)q}{1-q^2z}&\\&&&1\end{pmatrix}.$$

(A.5.3) Vertex operators

$$\Phi_\varepsilon^{(1-i,i)}(\zeta)=\zeta^{\frac{\varepsilon+1}{2}-i}\widetilde{\Phi}_\varepsilon^{(1-i,i)}(\zeta^2),$$
$$\Psi_\varepsilon^{(1-i,i)}(\zeta)=\zeta^{\frac{\varepsilon+1}{2}-i}\widetilde{\Psi}_\varepsilon^{(1-i,i)}(\zeta^2).$$

A.6 Space of states

(A.6.1) Space of states

$$\mathcal{F} = \mathcal{H} \otimes \mathcal{H}^{*b} = \oplus_{i,j=0,1} \mathcal{F}^{(i,j)},$$
$$\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1),$$
$$\mathcal{F}^{(i,j)} = V(\Lambda_i) \otimes V(\Lambda_j)^{*b}.$$

where $b(x) = (-q)^\rho a(x)(-q)^{-\rho}$ for $x \in U$.

(A.6.2) Translation operator

$$\begin{aligned} T &= g \sum_\varepsilon \Phi_\varepsilon(1) \otimes \Phi_{-\varepsilon}(1)^t, \\ T^{-1} &= g \sum_\varepsilon \Phi^*_\varepsilon(1) \otimes \Phi^*_{-\varepsilon}(1)^t. \end{aligned}$$

(A.6.3) Local operators

$$\begin{aligned} E_{\varepsilon\varepsilon'} &= g\, \Phi^*_\varepsilon(1)\Phi_{\varepsilon'}(1) \otimes \mathrm{id}, \\ \sigma_1^\pm &= E_{\pm\mp}, \qquad \sigma_1^z = E_{++} - E_{--}, \\ \sigma_n^\alpha &= T^{-(n-1)} \sigma_1^\alpha T^{n-1}. \end{aligned}$$

(A.6.4) Transfer matrix

$$\begin{aligned} T(\zeta) &= g \sum_\varepsilon \Phi_\varepsilon(\zeta) \otimes (\Phi_{-\varepsilon}(\zeta))^t, \\ H &= \frac{1-q^2}{2q} \zeta \frac{d}{d\zeta} \log T(\zeta)\Big|_{\zeta=1}. \end{aligned}$$

(A.6.5) Vacuum

$$|\mathrm{vac}\rangle_{(i)} = \chi^{-1/2}(-q)^{D^{(i)}} P^{(i)},$$
$$\chi = \chi_{\Lambda_i}(q^2) = \prod_{n=1}^\infty (1 - q^{4n-2})^{-1},$$

(A.6.6) Expectation value For $\mathcal{O} = \phi \otimes \mathrm{id}$ or $\mathrm{id} \otimes \phi^t$:

$${}_{(i)}\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle_{(i)} = \frac{\mathrm{tr}_{V(\Lambda_i)} \left(q^{2D^{(i)}} \phi\right)}{\mathrm{tr}_{V(\Lambda_i)} \left(q^{2D^{(i)}}\right)}.$$

(A.6.7) Eigenstates

$$
\begin{aligned}
|\xi_n, \cdots, \xi_1\rangle_{\varepsilon_n, \cdots, \varepsilon_1; (i)} &= g^{-n/2} \chi^{-1/2} \Psi^*_{\varepsilon_n}(\xi_n) \cdots \Psi^*_{\varepsilon_1}(\xi_1)(-q)^{D^{(i)}}, \\
{}_{(i); \varepsilon_1, \cdots, \varepsilon_n}\langle \xi_1, \cdots, \xi_n| &= g^{-n/2} \chi^{-1/2} (-q)^{D^{(i)}} \Psi_{\varepsilon_1}(\xi_1) \cdots \Psi_{\varepsilon_n}(\xi_n).
\end{aligned}
$$

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