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ALGEBRAIC AND GEOMETRIC ISOMONODROMIC DEFORMATIONS

CHARLES F. DORAN

Abstract

Using the Gauss-Manin connection (Picard-Fuchs differential equation) and a result of Malgrange, a special class of algebraic solutions to isomonodromic deformation equations, the *geometric isomonodromic deformations*, is defined from "families of families" of algebraic varieties. Geometric isomonodromic deformations arise naturally from combinatorial strata in the moduli spaces of elliptic surfaces over $\mathbb{P}^1$. The complete list of geometric solutions to the Painlevé VI equation arising in this way is determined. Motivated by this construction, we define another class of algebraic isomonodromic deformations whose monodromy preserving families arise by "pullback" from (rigid) local systems. Using explicit methods from the theory of Hurwitz spaces, all such algebraic Painlevé VI solutions coming from arithmetic triangle groups are classified.

1. Introduction

The classical theory of isomonodromic (i.e., monodromy-preserving) deformations of Fuchsian ordinary differential equations associates to each such family a solution of an auxiliary integrable Pfaffian system — the *isomonodromic deformation equation*.

Algebraic geometry turns out to be a good source of interesting isomonodromic deformations. Consider a family of algebraic varieties, where the base of the family is itself a $\mathbb{P}^1$-fibered variety. Then the Gauss-Manin connection for the periods restricts on each $\mathbb{P}^1$ fiber to a Picard-Fuchs differential equation. Quite generally, the resulting family of Fuchsian equations has constant monodromy, and so provides a solution to the associated isomonodromic deformation equation. In particular, since the equations vary algebraically, it is an *algebraic* solution.

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For sufficiently simple families, every step is computable. The first interesting case to consider is that of elliptic curve families derived from the moduli space of elliptic surfaces over $\mathbb{P}^1$ (with section). Each point in the moduli space corresponds to a pencil of elliptic curves, so the base of the elliptic curve family is tautologically $\mathbb{P}^1$-fibered. Thus the above theory applies. Strata in the moduli space of elliptic surfaces over $\mathbb{P}^1$, which correspond to families with the same number of each Kodaira type of singular fibers, provide such algebraic isomonodromic deformations. In the simplest case, there are four singular fibers, and the Picard-Fuchs equation is of "$4+1$" type (i.e., 5 regular singular points, one of which is apparent). The isomonodromic deformation equation is equivalent to a Painlevé VI nonlinear second order ordinary differential equation. In Section 3.2.2, all such families are determined, along with the resulting algebraic solutions to their Painlevé VI equations.

Our construction of geometric isomonodromic deformations in the elliptic curve case suggests a way to describe many algebraic isomonodromic deformations not necessarily coming from Picard-Fuchs differential equations. This generalization is based on the realization that, in the elliptic case, all we need to compute our geometric solutions are Kodaira's functional invariants for our family of surfaces (i.e., the rational functions $\mathcal{J}(z)$), and the hypergeometric differential equation ${}_2F_1(1/12, 1/12, 2/3)$ over the J-line $\mathbb{P}^1_J$. If t parametrizes our family of surfaces, then the corresponding isomonodromic family of differential equations is just (up to projective equivalence) the family $\mathcal{J}_t^*({}_2F_1(1/12, 1/12, 2/3))$. Since singular fiber types are specified by the ramification behavior of each $\mathcal{J}$ over $\{0, 1, \infty\} \subset \mathbb{P}^1_J$, the (Riemann-Hurwitz) *topological type* of these rational functions remains constant in each stratum. Thus our method generalizes to a "pullback construction" of isomonodromic families of differential equations via special families of rational funtions. In this context, these algebraic isomonodromic deformations correspond to specially parametrized components of certain Hurwitz spaces.

In Section 4.2.2, we characterize the topological types of all pullback algebraic solutions to Painlevé VI equations. We also give a finite classification by topological type of the Painlevé VI algebraic isomonodromic deformations obtained by pulling back hypergeometric equations for arithmetic Fuchsian triangles, and establish the existence of infinitely many such topological types for spherical and planar triangles.

1.1 Calabi-Yau manifolds and Picard-Fuchs differential equations

A *Calabi-Yau manifold* M is a compact complex manifold with trivial canonical bundle. A one dimensional Calabi-Yau manifold is an elliptic curve. A simply connected Calabi-Yau manifold of dimension two is a K3 surface.

Up to complex scaling, a Calabi-Yau n-fold has a unique holomorphic n-form

$$\omega_{n,0} \in H^n(M, \mathbb{C}).$$

The *periods* of this holomorphic n-form are the numbers

$$p_\gamma(M) := \int_\gamma \omega_{n,0}$$

as γ runs through the n-cycles of M. Fix an ordered basis of n-cycles $\{\gamma_1, \ldots, \gamma_k\}$, for k the nth Betti number of M. Then the periods of M define the vector

$$[p_{\gamma_i}(M)]_{i=1}^k \in \mathbb{P}^{k-1}$$

the (projective) *period point* of M.

Example 1.1. Consider the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$ in Weierstrass form. Here $n = 1$ and $k = 2$. The holomorphic one form is the usual differential of the first kind dx/y.

Given a family $\pi : X \to S$ of Calabi-Yau manifolds X_s, by continuous extension of the basis of n-cycles the periods of the Calabi-Yau fibers define a vector of k multi-valued functions on the base manifold S

$$[p_{\gamma_i(s)}(X_s)]_{i=1}^k : S \to \mathbb{P}^{k-1}$$

the *period mapping* associated with the family.

Example 1.2. Consider the family $\mathcal{E}$ of elliptic curves over $\mathbb{P}^1$ defined by the equation

$$\mathcal{E} : y^2 = 4x^3 - \frac{27s}{s-1}x - \frac{27s}{s-1}.$$

The periods of the form dx/y may be given in terms of the hypergeometric function ${}_2F_1$ (see [50], pp. 232–233, for explicit expressions).

Suppose now that $S = \mathbb{P}^1$. The functions $p_{\gamma(s)}(X_s)$ satisfy an ordinary differential equation with regular singular points (i.e., a *Fuchsian* ODE) [3]

$$\frac{d^k f}{ds^k} + P_1(s)\frac{d^{k-1} f}{ds^{k-1}} + \cdots + P_k(s) f = 0, \; P_i(s) \in \mathbb{C}(s). \tag{1.1}$$

The differential equation (1.1) is called the *Picard-Fuchs differential equation* of the family $\pi : X \to \mathbb{P}^1$.

Example 1.3. For the family $\mathcal{E}$ in Example 1.2, the Picard-Fuchs equation is

$$\frac{d^2 f}{ds^2} + \frac{1}{s}\frac{df}{ds} + \frac{(31/144)s - 1/36}{s^2(s-1)^2} f = 0.$$

There is a basis of solutions with local monodromies G_0, G_1, G_∞ about the regular singular points $\{0, 1, \infty\}$ respectively, where

$$G_0 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \; G_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; G_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The period mapping is defined as a map to projective space. If one is interested in isomonodromic deformations it suffices to consider the Picard-Fuchs differential equation only up to “projective equivalence” (see the discussion of *SL-equations* in [28, Section 3.4.]). With this in mind, we define the *projective normal form* of a Fuchsian ordinary differential equation (e.g., that in Equation (1.1) above) to be the unique Fuchsian ordinary differential equation without a $(k-1)$st order derivative

$$\frac{d^k g}{ds^k} + R_2(s)\frac{d^{k-2} g}{ds^{k-2}} + \cdots + R_k(s) g = 0, \; R_i(s) \in \mathbb{C}(s)$$

whose fundamental solutions define the same projective period map as that of Equation (1.1). It is always possible to pass to the projective normal form differential equation by rescaling each fundamental solution of the original equation by the kth root of the Wronskian.

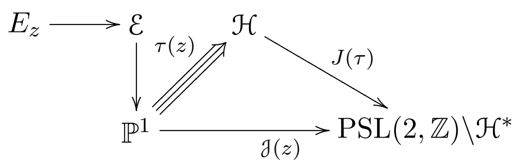
Example 1.4. Suppose now that $k = 2$, i.e., the initial differential equation is

$$\frac{d^2 f}{ds^2} + P_1(s)\frac{df}{ds} + P_2(s) f = 0,$$

then the projective normal form of this differential equation takes the particularly simple form

$$\frac{d^2 g}{ds^2} + \left(P_2(s) - \frac{1}{2}{P_1}'(s) - \frac{1}{4}P_1(s)^2\right) g = 0.$$

In the case of elliptic curves, the natural geometric invariant to consider is Kodaira's *functional invariant* $\mathcal{J}(z)$, defined as the composition of the multi-valued period morphism $\tau(z) = \omega_1(z)/\omega_0(z)$ to the upper half plane and the elliptic modular function $J(\tau)$ to the J-line [33]. In other words, the functional invariant associates to each point of the base of the family the J-invariant of its elliptic curve fiber.



Kodaira proves that $\mathcal{J}(z)$ is a rational function, and in fact, in terms of the Weierstrass presentation (Section 3.2.1), it takes the familiar form

$$\mathcal{J}(z) = \frac{g_2(z)^3}{g_2(z)^3 - 27g_3(z)^2}.$$

We consider the "pullback by $\mathcal{J}$" of the hypergeometric differential equation Λ associated with the arithmetic triangle group $\mathrm{PSL}(2,\mathbb{Z})$. By the rationality of $\mathcal{J}(z)$, the resulting differential equation $\mathcal{J}^*(\Lambda)$ agrees with the Picard-Fuchs equation up to projective equivalence (Proposition 3.11).

1.2 Geometric isomonodromic deformations

Isomonodromic deformation theory is the classical theory, dating back to Poincaré, Garnier [22], and Schlessinger [44], which describes monodromy-preserving deformations of Fuchsian systems and equations. Monodromy-preserving, or *isomonodromic*, deformations of a Fuchsian system on $\mathbb{P}^1$ are controlled by a time-dependent completely integrable Hamiltonian system, the so-called *isomonodromic deformation equation*. Initial conditions are given by specifying a particular Fuchsian system on $\mathbb{P}^1$ to be deformed. Solutions of the isomonodromic deformation equation with these initial conditions describe deformations of the given Fuchsian system which preserve the number of regular singular points and their local monodromies (up to global conjugacy).

The simplest possible case arises while studying isomonodromic deformations of second order Fuchsian equations with one apparent regular singular point (at position $\lambda \in \mathbb{P}^1$) and four non-apparent regular singular points (at positions $0, 1, \infty, t \in \mathbb{P}^1$). In this setting the corresponding isomonodromic deformation equation is the second order nonlinear *Painlevé VI* ordinary differential equation (Equation (2.2)), with dependent variable λ as a function of t.

We call an isomonodromic deformation of Picard-Fuchs differential equations through Picard-Fuchs equations a *geometric isomonodromic deformation*. To find the simplest examples of geometric isomonodromic deformations, i.e., geometric solutions to Painlevé VI, we first associate to an elliptic surface over $\mathbb{P}^1$ its projective normalized Picard-Fuchs equation. We must find examples of one-parameter families of elliptic surfaces over $\mathbb{P}^1$ with four singular fibers (same number and types for each member of the family, normalized to be placed at positions $0, 1, \infty, t$) and a functional invariant with exactly one extra ramification point (at position λ). In fact, there exists a complete classification of elliptic surfaces over $\mathbb{P}^1$ with exactly four singular fibers; only five of these live in one-parameter families satisfying the extra ramification condition. Each then defines an algebraic solution to a Painlevé VI equation, which can be determined explicitly from the Weierstrass equations (Theorem 3.13, Tables 2 and 3).

The property that the periods of families of varieties over a $\mathbb{P}^1$-fibered base define algebraic solutions to isomonodromic deformation equations in no way depends on the smooth fibers being elliptic curves, or even Calabi-Yau manifolds. This can be most easily seen by formulating the whole discussion in the equivalent context of flat connections on vector bundles rather than that of Fuchsian systems *per se*. In fact, Malgrange has done just this, constructing a *universal isomonodromic deformation space* from which all isomonodromic deformations are derived. Interpreting the initial Fuchsian system ∇_0 as a flat connection with regular singular points on $\mathbb{P}^1$, Malgrange found a unique flat connection $\widetilde{\nabla}$ on the universal isomonodromic deformation space which extends it [35] (see also Theorem 2.6 and Corollary 2.7). This result is quite useful: given any flat connection ∇ on a vector bundle over a simply connected $\mathbb{P}^1$-fibered manifold M which restricts to the initial Fuchsian equation ∇_0 over a particular $\mathbb{P}^1$ fiber, and a map φ from M to the universal deformation space taking ∇_0 to $\widetilde{\nabla}_0$, then $\nabla = \varphi^*(\widetilde{\nabla})$. The isomonodromic deformation equation then corresponds to the integrability condition for ∇ as a matrix valued 1-form defining the monodromy preserving family of connections.

In order to apply this theory in our geometric context, we replace the Picard-Fuchs differential equation by the Gauss-Manin connection on the Hodge bundle. It is well known that the Gauss-Manin connection is both flat (integrable) and has only regular singular poles. These are exactly the two conditions needed to apply Malgrange's theorem and hence to define geometric isomonodromic deformations in this general setting. Moreover, the algebraicity of the Gauss-Manin connection tells us at once that all such geometric solutions are algebraic (Section 3.1.3).

1.3 Algebraic isomonodromic deformations by "pullback"

Our capacity to describe the algebraic solutions to isomonodromic deformation equations coming from geometry, even in the simplest case of Painlevé VI solutions, is limited by our ability to compute (projective normalized) Picard-Fuchs differential equations in families. In the case of elliptic surfaces over $\mathbb{P}^1$, we know that the projective normalized Picard-Fuchs equation is determined by Kodaira's functional $\mathcal{J}$-invariant and the hypergeometric differential equation Λ attached to the J-line $\mathbb{P}^1_J$. Moreover, the isomonodromic family is essentially obtained by "pulling back" Λ by the $\mathcal{J}$ of each elliptic surface represented by a point in the chosen combinatorial stratum in moduli space.

This raises the following natural questions: Can other algebraic isomonodromic deformations be obtained by "pulling back" a Fuchsian ordinary differential equation Λ on $\mathbb{P}^1$ by a family of rational functions? In particular, is it possible to obtain more algebraic Painlevé VI solutions in this way? In Section 4 we answer these questions in the affirmative.

First we examine the behavior of regular singular points of a Fuchsian ordinary differential equation $\pi^*(\Lambda)$ under a rational function on the projective line $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ (Section 4.1.1). This relates the local characteristics of each singular point to the ramification behavior of the rational function. In particular, we see that regular singular points with infinite order local monodromy always pull back to regular singular points with infinite order local monodromy. But for points p with finite (≥ 2) order local monodromy, the order of local monodromy about a point $q \in \pi^{-1}(p)$ is multiplicatively related to that at p by the ramification order $\nu_q(\pi)$. We also see how apparent singularities arise under pullback; they correspond to those ramified points which lie over ordinary points for Λ.

As a source of examples of differential equations Λ, we introduce classical hypergeometric differential equations and Fuchsian triangles. In particular, we recall the classification of arithmetic triangles due to Takeuchi (Theorem 4.4). This provides us with a large and interesting class of examples of second order Fuchsian differential equations with three regular singular points, generalizing the natural equation on the J-line in the geometric case.

We begin Section 4.2 with a brief introduction to moduli spaces of n-sheeted covers of the projective line with specified ramification, i.e., the theory of *Hurwitz spaces*. We are particularly interested in one parameter families of covers with ramification over several "fixed" branch points $B \subset \mathbb{P}^1$ and over a single "variable" branch point. The resulting *Hurwitz curve* itself covers $\mathbb{P}^1$, branching just along B, with sheets described in terms of the monodromy of our original covers and a natural action of the group $\mathcal{B}_{0,r}$ of *colored braids with* $r = |B|$ *plaits*. Using this description, at least under the assumption that our original covers had genus zero, it is possible to obtain explicit expressions for the covers in our family as rational functions and an algebraic equation for the Hurwitz curve. We sketch the method due to Couveignes.

After providing motivation from both the geometric case and the Belyi pair interpretation of algebraic Painlevé VI solutions, we provide in Theorem 4.5 a topological characterization of "pullback" algebraic Painlevé VI solutions. This result describes the *topological types* of families of covers (indexed by t), compatible with a given rank two regular local system $\mathcal{L}$ on $\mathbb{P}^1$, for the isomonodromic family of projective normalized pullback equations $\overline{\pi_t^*(\mathcal{L})}$ to determine an algebraic solution to a Painlevé VI equation. Finally, we apply this criterion to the hypergeometric differential equations associated with the various Fuchsian triangles in Section 4.1.2, obtaining a finite list of topological types from the arithmetic Fuchsian triangles (extending that from $(2, 3, \infty)$ in the geometric case) and infinitely many (or none) from each of the planar and spherical cases.

1.4 Main results

For the convenience of the reader we include here the statements of our main results.

First, we have our result classifying the geometric solutions to Painlevé VI equations coming from elliptic surface moduli:

Theorem 3.13. *All of the geometric solutions to the Painlevé* VI *equation coming from moduli of Weierstrass elliptic surfaces are listed in Tables* 2 *and* 3. *Weierstrass equations for the five relevant families appear in Table* 1.

Second, there is our classification of pullback algebraic solutions to Painlevé VI equations:

Theorem 4.5. *The pulled back family of projective normalized local systems* $\overline{\pi_t^*(\mathcal{L})}$ *forms an isomonodromic family with five regular singular points, exactly one apparent* (*at* $\lambda(t)$), *and hence determines an algebraic solution* $\lambda(t)$ *to the Painlevé* VI *equation* $\mathrm{PVI}(\alpha,\beta,\gamma,\delta)$. *The particular values of* $\alpha,\beta,\gamma,\delta$ *are determined as usual by the traces of the local monodromies about* 0*,* 1*,* ∞*,* t*.*

This result is then applied to regular local systems of special types. From the hypergeometric local systems of arithmetic Fuchsian triangle groups we obtain a finite list of topological types corresponding to algebraic Painlevé VI solutions in Corollary 4.6. In case of planar and spherical triangles we find infinitely many such topological types in Corollary 4.7 and Corollary 4.8 respectively.

Remark 1. We provide a significant amount of background material on the theories of isomonodromic deformation of Fuchsian ordinary differential equations, elliptic surfaces and their moduli, and Hurwitz spaces parametrizing families of rational functions/covers of $\mathbb{P}^1$. The expository coverage of these classical topics is not uniform, however; we have placed an emphasis on the differential equations so that the paper can be more easily followed by its expected audience of geometers. In addition, references to good expository papers/monographs in each of these fields are included throughout.

Remark 2. This paper is a generalization of a portion of the author's Harvard thesis [13], where it was used to illustrate the variation of the string-theoretic "mirror map" in families. Although the results and methods are clearly of independent mathematical interest, the reader interested in a "mirror map perspective" should refer to the short note [16].

2. Isomonodromic deformation equations

In this section we introduce the classical theory of isomonodromic deformation of Fuchsian systems, with an emphasis on its modern reformulation in terms of flat connections.

We begin with a quick review of Fuchsian differential systems and flat connections on the projective line. The second subsection presents both local and global theories of isomonodromic deformation.

2.1 Fuchsian systems and flat connections

In this subsection we describe Fuchsian systems and equations, their monodromy representations, and their interpretation as flat connections and local systems. A good reference is the survey paper by Varadarajan [53].

2.1.1 Fuchsian systems and equations

Consider the first order differential system

$$\frac{dy}{dx} = A(x)y, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$$

and assume that it has *singularities* $a_1, \ldots, a_m$, i.e., that $A(x)$ is holomorphic in $S := \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \ldots, a_m\}$. The system is called *Fuchsian at* a_i (and a_i is a *Fuchsian singularity* of the system) if $A(x)$ has a pole at a_i of order at most one. The system is *Fuchsian* if it is Fuchsian at all the a_i.

Suppose now that all $a_i \neq \infty$. Then

$$A(x) = \sum_{i=1}^{m} \frac{1}{x - a_i} B_i + B(x),$$

where B is holomorphic on $\mathbb{C}$. When is such a system Fuchsian at ∞? By rewriting the system in terms of the new independent variable $t = 1/x$, we find

$$dy/dt = (D_1(t) - D_2(t))y,$$

where

$$D_1(t) = -\frac{1}{t} \sum_{i=1}^{m} \frac{B_i}{(1 - a_i t)}, \ D_2(t) = \frac{1}{t^2} B(1/t).$$

The matrix function D_1 has at worst a first order pole at $t = 0$, so the system is Fuchsian at $t = 0$ if and only if $(1/t^2)B(1/t)$ has a first order pole there. Thus $B(\infty) = 0$, and since B is holomorphic on $\mathbb{C}$, $B = 0$ everywhere. So systems which are Fuchsian on $\mathbb{P}^1(\mathbb{C})$ are just those systems with A of the form

$$A(x) = \sum_{i=1}^{m} \frac{1}{x - a_i} B_i.$$

There are no singularities at ∞ if and only if the residue of D_1 at $t = 0$ is zero, i.e.,

$$\sum_{i=1}^{m} B_i = 0.$$

An equation

$$y^{(p)} + q_1(x)y^{(p-1)} + \cdots + q_p(x)y = 0$$

is *Fuchsian at a point* a if its coefficients $q_1(x), \ldots, q_p(x)$ are holomorphic in some punctured neighborhood of this point and

$$q_i(x) = \frac{r_i(x)}{(x-a)^i}, \; i = 1, \ldots, p,$$

where $r_1(x), \ldots, r_p(x)$ are functions holomorphic at a. The equation is *Fuchsian* if all $q_i(x)$ are holomorphic on $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \ldots, a_m\}$ and the equation is Fuchsian at points $a_1, \ldots, a_m$.

2.1.2 Monodromy representation

Instead of the vector equation above it is sometimes preferable to consider the matrix equation

$$\frac{dY}{dx} = A(x)Y,$$

where Y is a $p \times p$ matrix. The columns of Y consist of p vectors, a set of solutions $y_1, \ldots, y_p$ to the vector system. We shall assume always that they are linearly independent, i.e., $Y \in \mathrm{GL}(p, \mathbb{C})$. In this case we say that they form a *fundamental system of solutions*.

Now let $\pi : \widetilde{S} \to S$ denote the universal covering surface for the domain S. The solutions y and Y are holomorphic functions on $\widetilde{S}$. Let G be the group of deck transformations of the covering π. If $\sigma \in G$, and $y(\widetilde{x}), Y(\widetilde{x})$ are solutions to our system, then so are

$$y \circ \sigma, \; Y \circ \sigma.$$

One invertible solution to the matrix form of the system can be obtained from another just by multiplying the latter on the right by some constant matrix, so

$$Y = (Y \circ \sigma)\chi(\sigma).$$

We call $\chi : G \to \mathrm{GL}(p, \mathbb{C})$ the *monodromy representation*.

As usual, the monodromy representation is well-defined up to conjugacy in $\mathrm{GL}(p, \mathbb{C})$. We call this conjugacy class the *monodromy* of the Fuchsian system.

For $p > 1$ a Fuchsian equation of pth order with singularities $a_1, \ldots, a_m$ contains fewer parameters than the set of classes of conjugate $\mathrm{GL}(p, \mathbb{C})$-representations. In fact, the difference between the number of parameters of the space of conjugacy classes of representations and the number of regular singular points in a Fuchsian equation with that monodromy representation is

$$\frac{1}{2}(m-2)p(p-1) + 1 - p \tag{2.1}$$

which is positive for all $m > 3, p > 1$ (or $m = 3, p > 2$) [1]. Thus in general the construction of a Fuchsian equation with a given monodromy requires the appearance of additional *apparent singularities.* These are not ramification points for the solutions of the equation, but are regular singular points of the equation nonetheless. In particular the monodromy about an apparent singularity is the identity I_p.

2.1.3 Flat connections and local systems

Let S be a topological space (typically a scheme over $\mathbb{C}$). Let $\mathcal{E}$ denote a quasicoherent sheaf of $\mathcal{O}_S$-modules. A *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear homomorphism

$$\nabla : \mathcal{E} \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{E} = \Omega^1_S(\mathcal{E})$$

satisfying the *Leibnitz identity*

$$\nabla(gs) = dg \otimes s + g\nabla s,$$

with g and s local sections of sheaves $\mathcal{O}_S$ and $\mathcal{E}$ respectively.

Given a connection $\nabla = \nabla_0 : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$, it can be extended to a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla_i : \Omega^i_S \otimes_{\mathcal{O}_S} \mathcal{E} \to \Omega^{i+1}_S \otimes_{\mathcal{O}_S} \mathcal{E}$$

by setting

$$\nabla_i(\omega \otimes s) = d\omega \otimes s + (-1)^i \omega \wedge \nabla_0(s).$$

The $\mathbb{C}$-linear homomorphism obtained by composition

$$R = \nabla_1 \circ \nabla_0 : \mathcal{E} \to \Omega^2_S(\mathcal{E})$$

is the *curvature* of the connection ∇ on $\mathcal{E}$, and is naturally a section of the sheaf

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \Omega^2_S(\mathcal{E})) \simeq \Omega^2_S(\mathcal{E}nd_{\mathbb{C}}(\mathcal{E})).$$

A connection ∇ is *flat* (or *integrable*) if $R = \nabla_1 \circ \nabla_0 = 0$.

Let S be a connected and locally connected topological space. A locally constant sheaf of vector spaces E on S is called a *local system* on S.

Theorem 2.1 ([32], (2.5.2)). *The fundamental group $\pi_1(S, x_0)$, $x_0 \in S$, acts on the fiber $E_{x_0} \simeq \mathbb{C}^p$ of the locally free sheaf E, and the functor $E \mapsto E_{x_0}$ yields an equivalence of the category of local systems on S with the category of $\mathbb{C}$-valued representations of the group $\pi_1(S, x_0)$.*

Let E be a local system on a complex manifold S. Consider a locally free sheaf $\mathcal{E}$ of holomorphic sections of E, $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$. There is a *canonical connection* ∇ on $\mathcal{E}$, whose sheaf of horizontal sections is simply E, i.e., $\mathrm{Ker}\nabla = E$, defined by

$$\nabla(g\, s) = dg\, s,$$

where g and s are sections of $\mathcal{O}_S$ and E respectively.

The canonical connection is clearly flat; conversely we have:

Theorem 2.2 (Deligne, Theorem (2.5.4) in [32]). *Let ∇ be a connection on a locally free sheaf $\mathcal{E}$ over connected S. Set*

$$E = \mathrm{Ker}\nabla = \{s \in \mathcal{E} \mid \nabla s = 0\}.$$

If ∇ is an integrable connection, then E is a local system on S and $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$.

As a consequence we see that the functor taking a local system E to the sheaf $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$ with the canonical connection, and that taking a locally free sheaf $\mathcal{E}$ with a flat connection ∇ to $E = \mathrm{Ker}\nabla$, define an equivalence of the category of local systems E and that of locally free sheaves with flat connections $(\mathcal{E}, \nabla)$.

2.2 Isomonodromic deformations

In this subsection we describe the theory of isomonodromic deformation of Fuchsian systems. Already from the local theory it is clear that isomonodromy can be described as an integrability condition. By reformulating this theory after Malgrange in the language of flat connections with logarithmic poles one obtains a universal description of isomonodromic deformations. Explicit formulas for these integrability equations were given in special cases by Schlesinger and in the simplest case by Painlevé (the Painlevé VI equation).

2.2.1 Local theory

Consider a *family* of Fuchsian systems

$$\frac{dy}{dx} = \left(\sum_{i=1}^{m} \frac{B_i(a)}{x - a_i} \right) y, \quad \sum_{i=1}^{m} B_i(a) = 0$$

depending holomorphically on the parameter $a = (a_1, \ldots, a_m)$ in a small disk $D(a^0)$ about $a^0 = (a_1^0, \ldots, a_m^0)$.

Such a family of Fuchsian systems is called *isomonodromic*, or an *isomonodromic deformation* of the initial Fuchsian system with $a = a^0$, if for each fixed a the corresponding system has the same monodromy as that with $a = a^0$, with respect to the homotopy classes of loops g_i^a and g_i^0 respectively. Thus for each value of a there exists a fundamental matrix $Y(x, a)$ of the corresponding system, such that $Y(x, a)$ has the same monodromy matrices, with respect to the g_i^a, for all $a \in D(a^0)$. We call such a family of matrices an *isomonodromic family of matrices.*

Proposition 2.3 ([5], Prop. 1)**.** *For any isomonodromic family there exists an isomonodromic family of matrices analytic simultaneously in x and a on*

$$S := \mathbb{P}^1(\mathbb{C}) \times D(a^0) \setminus \cup_{i=1}^{m} \{(x - a_i) = 0\}.$$

By analyticity, the monodromy of $Y(x, a)$ depends only on the homotopy classes of paths starting at (x_0, a^0). The isomonodromy condition implies that the monodromy matrices, as functions of the initial point (x_0, a^0) for fixed x_0, are locally constant with respect to a^0. By definition of the monodromy of a linear system of ordinary differential equations, the monodromy matrices are locally constant with respect to x_0 for each fixed a^0. Thus $Y(x, a)$ defines a monodromy representation

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{a_1^0, \ldots, a_m^0\}, (x_0, a^0)) \to GL(p, \mathbb{C}).$$

By construction, the matrix differential one-form

$$\omega = \frac{dY(x, a)}{Y(x, a)}$$

is single valued and thus can be considered as a form on S. For all $g \in \pi_1(S, (x_0, a^0))$,

$$g^*\omega = dg^*Y(x,a)g^*Y^{-1}(x,a) = \omega.$$

The Pfaffian system

$$dy = \omega y$$

on S is completely integrable, and for each fixed a it agrees with the Fuchsian system. Thus:

Theorem 2.4 ([5], Theorem 2)**.** *The family of Fuchsian systems is isomonodromic if and only if there exists a matrix differential one-form ω on S such that*

(1) *The one form can be expressed as*

$$\omega = \sum_{i=1}^{m} \frac{B_i(a)}{x - a_i} dx$$

for each fixed $a \in D(a^0)$

(2) $d\omega = \omega \wedge \omega$.

2.2.2 Meromorphic connections with logarithmic poles

Consider a complex manifold X, and a smooth codimension one submanifold Y. Let E be a rank n holomorphic vector bundle over X, and (E, ∇) a rank n holomorphic vector bundle on $X \backslash Y$ with flat connection on its sheaf of holomorphic sections.

The connection ∇ is called *meromorphic over* Y if there exists for each $y \in Y$ a neighborhood U of y such that $E|_U$ is trivial and the connection form of ∇ with respect to a basis of trivializing sections of $E|_U$ is meromorphic on U.

The connection ∇ has a *logarithmic pole along* Y if, in a system of local coordinates $(t_1, \ldots, t_r)$, with Y defined by $\{t_1 = 0\}$, we have the connection form

$$\Omega = A_1 \frac{dt_1}{t_1} + A_2 dt_2 + \cdots + A_r dt_r,$$

where all the A_i are holomorphic.

The simplest case to consider involves

$$X = \mathbb{P}^1(\mathbb{C}) \text{ and } Y = \{a_1^0, \ldots, a_{m-1}^0, \infty\}.$$

Let x be the usual function identifying $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. Initial data will consist of a Fuchsian differential equation over X with regular singularities over Y, i.e., a holomorphic vector bundle E^0 over X with a flat connection ∇^0 of $E|_{X \setminus Y}$ meromorphic over Y.

2.2.3 Isomonodromic deformation of flat connections

We now wish to consider *global* versions of the local isomonodromic deformations in Section 2.2.1.

Assume that the *deformation space*, the parameter space of the monodromy preserving deformation, is a connected complex variety T. The motion of the poles under deformation is specified by a set of holomorphic *deformation functions*

$$a_i : T \to \mathbb{C},\ 1 \leq i \leq m-1.$$

We fix a base point $t_0 \in T$ such that $a_i(t_0) = a_i^0$ for all i. Throughout we will always assume that the deformation functions satisfy

$$a_i(t) \neq a_j(t) \text{ for all } t \in T,\ i \neq j.$$

If X is taken to be $\mathbb{P}^1(\mathbb{C}) \times T$, then it is natural to introduce the smooth codimension one submanifold

$$Y = Y_1 \cup \cdots \cup Y_{m-1} \cup Y_\infty,$$

where

$$Y_i = \{(x,t) | (x,t) \in X, x = a_i(t)\}$$

and

$$Y_\infty = \{(\infty,t) | (\infty,t) \in X\}.$$

Following Malgrange [35] and Helminck [23] we introduce global isomonodromic deformations of flat connections on $\mathbb{P}^1$. An *isomonodromic deformation*

$$(E, \nabla) \text{ of the pair } (E^0, \nabla^0)$$

with deformation space T, deformation functions $\{a_i\}$, and base point $t_0 \in T$ is given by the data of

(1) a holomorphic vector bundle E over $X = \mathbb{P}^1 \times T$ of rank n

(2) an integrable (flat) connection ∇ of $E|_{X \setminus Y}$, meromorphic over Y, such that the restriction of (E, ∇) to $\mathbb{P}^1 \times \{t_0\}$ is isomorphic to (E^0, ∇^0).

The injection $i : \mathbb{P}^1 \to \mathbb{P}^1 \times T$ taking $x \mapsto (x, t_0)$ induces a natural map

$$i^* : \pi_1(\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\}) \to \pi_1(X \setminus Y).$$

Given an isomonodromic deformation (E, ∇) of (E^0, ∇^0), the monodromy representation ρ of $\pi_1(\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\})$ corresponding to (E^0, ∇^0) factors through i^*. Consider the projection $p_2 : X \setminus Y \to T$, $p_2(x, t) = t$, with fiber over t equal to

$$\mathbb{C} \setminus \{a_1(t), \ldots, a_{m-1}(t)\}.$$

The fiber bundle $(X \setminus Y, T, p_2)$ yields the long exact sequence

$$\pi_2(T) \to \pi_1(\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\}) \xrightarrow{i^*} \pi_1(X \setminus Y) \to \pi_1(T) \to 1.$$

Example 2.5. An example of a deformation space with

$$\pi_2(T) = \pi_1(T) = 1$$

is given by $T = \widetilde{Z}$, the universal cover of the space

$$Z = \mathbb{C}^{m-1} \setminus \cup_{i \neq j} D_{ij},$$

where $D_{ij} = \{(x_k) \in \mathbb{C}^{m-1} \mid x_i = x_j\}$. The $a_i : \widetilde{Z} \to \mathbb{C}$ can be taken to be the composition of the natural projection $\pi : \widetilde{Z} \to Z$ with the projection onto the ith coordinate. The long exact homotopy sequence shows $\pi_k(\widetilde{Z}) = \pi_k(Z)$ $(k \geq 2)$, and Fox and Neuwirth [21] show $\pi_k(Z) = 1$ in the same range. In particular, there are thus no topological obstructions.

2.2.4 Universal isomonodromic deformation space

By the long exact homotopy sequence above we know that morphism

$$i^* : \pi_1(\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\}) \to \pi_1(\widetilde{Z} \setminus Y)$$

is an isomorphism. Thus the p-dimensional representation of

$$\pi_1(\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\})$$

corresponding to the pair

$$(E^0|_{\mathbb{P}^1 \setminus \{a_1^0, \ldots, a_{m-1}^0, \infty\}}, \nabla^0)$$

determines a p-dimensional representation of $\pi_1(\widetilde{Z}\backslash Y)$, yielding a vector bundle with connection $(\widetilde{E}, \widetilde{\nabla})$ on $\widetilde{Z} \setminus Y$. It is possible to extend $\widetilde{E}$ to $\widetilde{Z}$ and to show that $\widetilde{\nabla}$ is meromorphic over Y, by a local analysis about each Y_i, $i = 1, \ldots, m-1, \infty$. This yields the following theorem characterizing the *universal isomonodromic deformation space.*

Theorem 2.6 ([35, 23])**.** *For each pair (E^0, ∇^0) there exists an isomonodromic deformation $(\widetilde{E}, \widetilde{\nabla})$ of (E^0, ∇^0) with $\widetilde{Z}$ as deformation space and the $\{a_i\}$ as deformation functions.*

An immediate corollary of this result allows one to construct isomonodromic deformations for other deformation spaces:

By our condition that the poles should never collide, we have an analytic map $a : T \to Z$ given by $a(t) = (a_1(t), \ldots, a_{m-1}(t))$. We say that the morphism a can be *lifted to* $\widetilde{Z}$ if there is a holomorphic map $\widetilde{a} : T \to \widetilde{Z}$ such that $a = \pi \circ \widetilde{a}$. If a lifts to $\widetilde{Z}$, then the pullback $(\widetilde{a}^*(\widetilde{E}), \widetilde{a}^*(\widetilde{\nabla}))$ to T of the isomonodromic deformation $(\widetilde{E}, \widetilde{\nabla})$ over $\widetilde{Z}$ is an isomonodromic deformation of (E^0, ∇^0).

Corollary 2.7. *There exists an isomonodromic deformation of (E^0, ∇^0) with T as deformation space and $\{a_i\}$ as deformation functions.*

Equivalently, this pullback construction defines a solution to the corresponding equation describing the flatness condition for $\widetilde{a}^*(\widetilde{\nabla})$, i.e., to the associated *isomonodromic deformation equation.*

2.3 Explicit isomonodromic deformation equations

To make things concrete, we will consider here a special case studied by Schlesinger. Consider the isomonodromic deformation of the initial Fuchsian system

$$\frac{dy}{dx} = \left(\sum_{i=1}^{m} \frac{B_i^0}{x - a_i^0} \right) y, \;\; B_i^0 = B_i(a^0)$$

given by the (Schlesinger) one-form

$$\omega_S = \sum_{i=1}^{m} \frac{B_i(a)}{x - a_i} d(x - a_i).$$

The condition that $d\omega_S = \omega_S \wedge \omega_S$ is in fact equivalent to

$$dB_i(a) = - \sum_{j \neq i, j=1}^{m} \frac{[B_i(a), B_j(a)]}{a_i - a_j} d(a_i - a_j),$$

called the *Schlesinger equation*.

Now in order to derive the simplest possible isomonodromic deformation equation, consider the still more special case in which there are exactly four regular singular points in the Fuchsian system being deformed. These may be normalized to lie at positions $0, 1, \infty, t \in \mathbb{P}^1$.

Let

$$A(x) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}$$

represent our Fuchsian system, with

$$A_i = \begin{pmatrix} a_i^{11}(t) & a_i^{12}(t) \\ a_i^{21}(t) & a_i^{22}(t) \end{pmatrix}, \ (i = 0, 1, t).$$

Denote

$$z_i = -a_i^{22}, \ \theta_i = a_i^{11} + a_i^{22},$$

$$u = \frac{a_0^{12}}{a_0^{22}}, \ v = \frac{a_1^{12}}{a_1^{22}}, \ w = \frac{a_t^{12}}{a_t^{22}}.$$

Also set

$$A_\infty = -(A_0 + A_1 + A_t) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

so

$$\kappa_1 + \kappa_2 = -(\theta_0 + \theta_1 + \theta_t), \ \kappa_1 - \kappa_2 = \theta_\infty.$$

We implicitly define λ, which will be the position of the apparent singularity (see Lemma 2.9), by

$$u = \frac{k\lambda}{tz_0}, \ v = -\frac{k(\lambda-1)}{(t-1)z_1}, \ w = \frac{k(\lambda-t)}{t(t-1)z_t}$$

and

$$\frac{d}{dt}\log k = (\theta_\infty - 1)\frac{\lambda - t}{t(t-1)}.$$

By rewriting the Schlesinger equation as a nonlinear second order ordinary differential equation, we obtain the *Painlevé VI equation* PVI(α, β, γ, δ)

(2.2)

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right)$$

where

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_t^2).$$

What is the relationship between an isomonodromic deformation of the Fuchsian system

$$\frac{dW}{dx} = A(x)W, \quad W = (w_1, w_2)^t$$

and that of its associated Fuchsian ordinary differential equation? To begin with, suppose that $a_{12}(x)$ is not identically zero. Then $w_1(x)$ satisfies

$$\frac{d^2w_1}{dx^2} + p_1(x)\frac{dw_1}{dx} + p_2(x)w_1 = 0$$

with

$$p_1(x) = -\frac{d}{dx}\log(a_{12}(x)) - (a_{11}(x) + a_{22}(x))$$

$$p_2(x) = a_{11}(x)a_{22}(x) - a_{12}(x)a_{21}(x) - \frac{da_{11}(x)}{dx} + a_{11}(x)\frac{d}{dx}\log(a_{12}(x)).$$

Lemma 2.8. *If $p = 2$ and $m = 4$, there can be at most one apparent regular singular point in each equation in an isomonodromic family.*

Proof. In the case of a general Fuchsian system of rank $p = 2$ with $m = 4$ regular singular points (normalized to lie at 0, 1, ∞, and t say) by Equation (2.1) there will be but a single additional regular singular point at $\lambda \in \mathbb{C} \setminus \{0, 1, \infty, t\}$. q.e.d.

In an isomonodromic deformation with respect to the parameter t, the function $\lambda(t)$ will play a distinguished role.

Lemma 2.9. *The function $\lambda(t)$ above parametrizes the position of the unique apparent regular singular point in each equation in an isomonodromic family.*

Proof. The zeros of $a_{12}(x)$ and poles of all $a_{ij}(x)$ could contribute apparent regular singular points of the ordinary differential equation. A direct computation shows that aside from 0, 1, and ∞ the only other factors in the denominators of $p_1(x)$ and $p_2(x)$ are among those of

$$\begin{aligned} &a_0^{12}(t)(x-1)(x-t) + a_1^{12}(t)x(x-t) + a_t^{12}(t)x(x-1) \\ &= a_0^{12}(t)t(1-x) + a_1^{12}(t)x(1-t). \end{aligned}$$

We know by Lemma 2.8 that all but one factor in this expression will cancel with terms in the numerator. Suppose $(x-\rho)$, for some $\rho \neq 0, 1, \infty, t$, divides $(a_0^{12}(t)t(1-x)+a_1^{12}(t)x(1-t))$. But $x-\lambda$ is given by

$$x-\lambda = \frac{1}{k}(a_0^{12}(t)t(1-x)+a_1^{12}(t)x(1-t))$$

where we recall that $k(t) = ((1-t)a_1^{12}(t) - ta_0^{12}(t))$. Thus the only additional regular singularity is $\lambda(t)$. q.e.d.

For a complete, recent introduction to the Schlesinger and Painlevé VI equations, see the survey by Mahoux [34].

3. Geometric isomonodromic deformations

In this section we introduce *geometric isomonodromic deformations*.

In the first subsection, we begin by discussing algebraic solutions to the Painlevé VI equation and their modular curve interpretations. Next we recall the Gauss-Manin connection on the Hodge bundle over the base of a family of varieties. Using the Gauss-Manin connection for a family over a $\mathbb{P}^1$-fibered base we finally define geometric isomonodromic deformations.

The second subsection is dedicated to making this notion concrete in the simplest nontrivial case. We consider multiparameter families of elliptic curves, reinterpreted as families of elliptic surfaces over $\mathbb{P}^1$. From each “combinatorial stratum” in a moduli space of elliptic surfaces we recover a geometric isomonodromic deformation. In particular, the geometric solutions to Painlevé VI arising in this way are completely classified.

3.1 Algebraic and geometric isomonodromic deformations

We begin this subsection by considering algebraic solutions to isomonodromic deformation equations, with particular attention to solutions of the Painlevé VI equation. After introducing the Gauss-Manin connection associated with a family of complex manifolds, the natural generalization of the Picard-Fuchs equation, we define *geometric* solutions to isomonodromic deformation equations.

3.1.1 Algebraic isomonodromic deformations

We call an algebraic solution to an isomonodromic deformation equation an *algebraic isomonodromic deformation.* The simplest examples of algebraic solutions to isomonodromic deformation equations are those for the Painlevé VI equations.

Recall from Lemma 2.9 that, as an isomonodromic deformation equation for second order Fuchsian equations, a solution $\lambda(t)$ to Painlevé VI relates the position $\lambda \in \mathbb{P}^1(\mathbb{C})$ of the extra apparent singularity to the position $t \in \mathbb{P}^1(\mathbb{C})$ of the fourth regular singular point of the original Fuchsian system (the other three regular singular points being normalized to lie at the points $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$). As the position of $t \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ parametrizes the base of the isomonodromic deformation, we see that a solution to the isomonodromic deformation equation Painlevé VI will be naturally defined on the universal cover of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ (the upper half plane $\mathcal{H}$). Moreover, an *algebraic* solution to Painlevé VI will define a finite unramified cover of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

Let X be a compact Riemann surface. A holomorphic mapping $\beta : X \to \mathbb{P}^1(\mathbb{C})$ is called a *Belyi function* if the set of critical values of β lies in $\{0, 1, \infty\}$. The pair (X, β) is known as a *Belyi pair.* In particular, to each Belyi function there is associated a subgroup $\Gamma \subset \Gamma(2)$ of finite index such that $X \simeq \Gamma \backslash \mathcal{H}^*$ is exhibited as a modular curve [47, p. 71].

In this language we can now restate our observations about algebraic solutions to Painlevé VI:

> An algebraic solution to Painlevé VI defines a Belyi pair (X, β). The Riemann surface X is the cover defined by the algebraic solution. The Belyi function β is given by the covering map $t(\lambda) : X \to \mathbb{P}^1(\mathbb{C})$.

We call a Belyi pair (X, β) arising in this way from an algebraic solution to Painlevé VI a *Painlevé-Belyi pair*, and the function $\beta = t(\lambda)$ a *Painlevé-Belyi function.*

Not every Belyi function is a Painlevé-Belyi function. Since the algebraic solution $\lambda(t)$ satisfies the Painlevé VI (non-linear, second order, ordinary) differential equation, its inverse $t(\lambda)$ must satisfy another differential equation, related to Painlevé VI via the chain rule:

Lemma 3.1. *A Belyi function $t(\lambda)$ is a Painlevé-Belyi function if* *and only if it satisfies*

$$\frac{d^2t}{d\lambda^2} + \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\frac{dt}{d\lambda} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\left(\frac{dt}{d\lambda}\right)^2$$

$$+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right)\left(\frac{dt}{d\lambda}\right)^3 = 0,$$

for some complex constants α, β, γ, *and* δ.

Remark 3. As there are several equivalent characterizations of Belyi pairs, it is natural to ask if our differential equations characterization of Painlevé-Belyi pairs can be replaced by a more natural one in each context. The well-known interpretation in terms of "curves definable over $\overline{\mathbb{Q}}$" begs an algebraic formulation of the Painlevé-Belyi property. One might also expect a graph-theoretic description of "Painlevé dessins d'enfants" [45].

Remark 4. In [52], Tod showed by a twistor construction that ASD Bianchi IX Einstein metrics correspond to solutions of the Painlevé VI equation PVI(1/8, −1/8, 1/8, 3/8). Before the complete solution of this PVI equation was found in terms of elliptic functions, Hitchin [25] discovered a family of algebraic solutions coming from Poncelet polygons, and hence modular curves. Four of these he could write out explicitly, as the corresponding modular curves were rational. Maszczyk, Mason, and Woodhouse [37] extended the twistor correspondence to show that more generally ASD Bianchi IX metrics (not necessarily Einstein) correspond to solutions of (arbitrary) Painlevé VI equations. From the above we see that algebraic Painlevé VI solutions determine ASD Bianchi IX metrics coming from modular curves.

3.1.2 Gauss-Manin connection

We briefly recall here a natural algebraic connection associated with geometric fibrations, the *Gauss-Manin connection*. More details can be found in [3, §2], for example.

Let T be a topological space. We call a locally constant sheaf $\mathcal{V}$ (of sets, groups, vector spaces, etc.) on T a *local system*. Now suppose that $f : X \to T$ is a fibration, defining a family of *smooth* projective varieties $X_t = f^{-1}(t)$. By the theorem of direct images [9, Théorème 10.6.] the abelian sheaf $\mathbf{R}^k f_* \mathbb{C}$ is a local system on T, the local system with fiber over t the cohomology group $H^k(X_t, \mathbb{C})$

The *Gauss-Manin connection* on the cohomology bundle $\mathcal{H}^k(X/T) = \mathbf{R}^k f_*(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_T$ is the unique (holomorphic) connection whose flat sections are those of $\mathbf{R}^k f_*(\mathbb{C})$, i.e., for a section e

$$\nabla_{GM}(e) = 0 \Leftrightarrow e \in \mathbf{R}^k f_*(\mathbb{C}).$$

In addition to its flatness (integrability), there are two more basic facts about the Gauss-Manin connection of which we will make essential use:

(1) The Gauss-Manin connection is *algebraic*.

(2) The Gauss-Manin connection is *regular with logarithmic poles*.

These were observed by Katz and Oda [41, 29, 30] who constructed the Gauss-Manin connection from the Koszul complex for the complex

$$\Omega^{\cdot}_{X/T}(\log D)$$

of *differential forms, regular on* $T \setminus D$*, with logarithmic poles along* D.

In its differential equation form over a one dimensional T, the Gauss-Manin connection is just the Picard-Fuchs differential equation described in the Introduction.

3.1.3 Geometric isomonodromic deformations

Now we will consider the Gauss-Manin connection on a family of varieties over a $\mathbb{P}^1$ fibered base, defining a *geometric* class of isomonodromic deformations.

Let $X \to B$ be a flat family of algebraic varieties, with discriminant locus $D \subset B$ assumed to have normal crossings. Let the corresponding Gauss-Manin connection ∇ on $B \setminus D$ be given. Suppose that in addition there is a fibration of B by rational curves, the generic member of which intersects D transversely. Thus the base S of the induced fibration on $B \setminus D$ parametrizes a family of punctured projective lines $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1(s), \ldots, a_m(s)\}$. The number m of punctures is constant away from the locus $\Sigma \subset S$ over which the discriminant loci intersect or become singular. Let $U \subset S$ be a simply connected open subset of $S \setminus \Sigma$. Then ∇, U, and the $a_i(s)$ $(i = 1, \ldots, m)$ satisfy the conditions of Corollary 2.7, and the flat connection ∇ defines a solution to the associated isomonodromic deformation equation.

We call such isomonodromic deformations coming from the Gauss-Manin connection *geometric isomonodromic deformations*, or *geometric solutions* to the corresponding isomonodromic deformation equation.

The simplest nontrivial examples of geometric isomonodromic deformations arise from certain two parameter families of elliptic curves over a rational surface base, with discriminant locus consisting of four components. This can also be viewed as a one parameter family of Weierstrass elliptic surfaces over $\mathbb{P}^1$ with four singular fibers in each surface. We will determine *all* such examples in the next section (Theorem 3.13).

The well-known Katz-Oda "algebraic" construction of the Gauss-Manin connection implies that

> Geometric isomonodromic deformations are algebraic isomonodromic deformations.

We will see explicit examples of this phenomenon in the next subsection.

Remark 5. It seems plausible that the theory of geometric isomonodromic deformations could be reformulated using the language of differential algebraic geometry over function fields. Note, however, that it most certainly is not equivalent to the *external Gauss-Manin connection* of Buium. That theory strongly depends on having *isotrivial* fibrations, in stark contrast with our own ([6]).

Constructions analogous to ours do appear in the local theory of deformation of isolated hypersurface singularities ([42], "Hamiltonian system") and in the theory of degenerations of discriminantal configurations, where they are related to R-matrices, quantum groups, and the Knizhnik-Zamolodchikov equations ([54, 20]).

3.2 Geometric isomonodromic deformations from elliptic surface moduli

Here we investigate the geometric solutions to isomonodromic deformation equations coming from families of elliptic curves over a $\mathbb{P}^1$-fibered base, i.e., from moduli of Weierstrass elliptic surfaces over $\mathbb{P}^1$. Combinatorial strata, in which the number and Kodaira types of singular fibers remain the same but their positions vary, correspond naturally to geometric isomonodromic deformations. Finally, we determine all such geometric Painlevé VI solutions and tabulate the results.

3.2.1 Weierstrass elliptic fibrations over $\mathbb{P}^1$ and their moduli

Let $\pi : X \to C$ be a flat proper map from a reduced irreducible $\mathbb{C}$-scheme X to a complete, smooth curve C, such that every geometric fiber is an irreducible curve of arithmetic genus one, i.e., each fiber is one of

(1) an elliptic curve, or

(2) a rational curve with a node, or

(3) a rational curve with a cusp.

The total space X is normal, and the generic fiber of π is smooth. Assume further that a section $s : C \to X$ is given, not passing through the nodes or cusps of the fibers. Call the collection of such data $(\pi : X \to C, s)$ a *Weierstrass fibration* over C. We may resolve the singularities of X to obtain an elliptic surface with section $\overline{\pi} : \overline{X} \to C$, called the *induced elliptic surface*.

In fact there is a *canonical form* for such a Weierstrass fibration, exhibiting X as a divisor in a $\mathbb{P}^2$-bundle over the base curve C.

Theorem 3.2 ([38], Theorem (2.1)). *Let Σ denote the given section of π, i.e., $\Sigma = s(C)$, a divisor on X which is taken isomorphically onto C by π. Let $\mathcal{L} = \pi_*[\mathcal{O}_X(\Sigma)/\mathcal{O}_X]$. Suppose that the general fiber of π is smooth. Then $\mathcal{L}$ is invertible and X is isomorphic to the closed subscheme of $\mathbb{P} = \mathbb{P}(\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_Y)$ defined by*

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3,$$

where

$$g_2 \in \Gamma(C, \mathcal{L}^{\otimes -4}),\ \ g_3 \in \Gamma(C, \mathcal{L}^{\otimes -6}),$$

and $[x, y, z]$ is the global coordinate system of $\mathbb{P}$ relative to $(\mathcal{L}^{\otimes 2}, \mathcal{L}^{\otimes 3}, \mathcal{O}_C)$. Moreover the pair (g_2, g_3) is unique up to isomorphism, and the discriminant

$$g_2^3 - 27 g_3^2 \in \Gamma(C, \mathcal{L}^{\otimes -12})$$

vanishes at a point $q \in C$ precisely when the fiber X_q is singular.

Let $\mathcal{J}$ denote the composition of the period morphism $\omega_1/\omega_0 : C \to \mathcal{H}$ and the morphism $J : \mathcal{H} \to \mathbb{P}^1$ extending the classical modular function, i.e., $\mathcal{J} = J \circ \omega_1/\omega_0$. This is Kodaira's *functional invariant* of $\pi : X \to C$ from the Introduction.

Kodaira has classified the singular fiber types which can arise in Weierstrass fibered elliptic surfaces.

Theorem 3.3 ([33]). *The singular fibers which appear in a smooth minimal elliptic surface fall into "types":* I_n $(n \geq 0)$, II, III, IV, I_n^* $(n \geq 0)$, IV^*, III^*, *and* II^*. *Let* I_0 *denote a smooth elliptic fiber. The fiber of type* I_1 *is a rational curve with a single node. More generally, fibers of type* I_n *consist of a* n*-cycle of intersecting rational curves for* $n \geq 1$. *A fiber of type* II *is just a rational curve with a single cusp. Type* III *fibers consist of two rational curves with a single point of tangency. Fibers of type* IV *consist of three rational components intersecting at a single point. There are also fibers of types* I_n^*, $n \geq 0$, IV^*, III^*, *and* II^*, *whose dual intersection graphs, minus in each case a multiplicity one component, correspond to those graphs of Dynkin types* D_{n+4}, E_6, E_7, *and* E_8 *respectively.*

We now recall how the Kodaira fiber types correlate with the ramification behavior of the $\mathfrak{J}$-map.

Lemma 3.4 ([39], Lemma IV.4.1). *Let* $F = X_q$ *be the fiber of* π *over* $q \in C$, *and let* $\nu_q(\mathfrak{J})$ *be the multiplicity of the functional invariant at* q.

(1) *If* F *has type* II, IV, IV^*, *or* II^*, *then* $\mathfrak{J}(q) = 0$. *Conversely, suppose that* $\mathfrak{J}(q) = 0$. *Then:*
   - F *has type* I_0 *or* I_0^* *if and only if* $\nu_q(\mathfrak{J}) \equiv 0 \mod 3$.
   - F *has type* II *or* IV^* *if and only if* $\nu_q(\mathfrak{J}) \equiv 1 \mod 3$.
   - F *has type* IV *or* II^* *if and only if* $\nu_q(\mathfrak{J}) \equiv 2 \mod 3$.

(2) *If* F *has type* III *or* III^*, *then* $\mathfrak{J}(q) = 1$. *Conversely, suppose that* $\mathfrak{J}(q) = 1$. *Then:*
   - F *has type* I_0 *or* I_0^* *if and only if* $\nu_q(\mathfrak{J}) \equiv 0 \mod 2$.
   - F *has type* III *or* III^* *if and only if* $\nu_q(\mathfrak{J}) \equiv 1 \mod 2$.

(3) F *has type* I_n *or* I_n^* *with* $n \geq 1$ *if and only if* $\mathfrak{J}$ *has a pole at* q *of order* n.

As one might expect from the explicitness of the Weierstrass description of elliptic fibrations with section over curves, it is indeed possible to construct a geometric invariant theory moduli space of such surfaces. For convenience here we will describe the theory for surfaces over $\mathbb{P}^1$, as originally formulated by Miranda, but the generalization to higher genus was completed shortly thereafter by Seiler.

Proposition 3.5 ([38], Proposition (2.7)). *Let $T_n = V_{4n} \oplus V_{6n}$ be the open set of pairs of forms (g_2, g_3) satisfying:*

(1) *The discriminant $4g_2^3 + 27g_3^2 \in V_{12n}$ is not identically zero, and vanishes at $q \in \mathbb{P}^1$ if and only if the fiber of the minimal elliptic surface over q is singular.*

(2) *For every $q \in \mathbb{P}^1$, either $\nu_q(g_2) \leq 3$ or $\nu_q(g_3) \leq 5$.*

Given the Weierstrass fibration X, the pair (g_2, g_3) is unique up to isomorphism: The multiplicative group $\mathbb{C}^*$ acts on T_n sending

$$(g_2, g_3) \mapsto (\lambda^{4n} g_2, \lambda^{6n} g_3),\ \ \lambda \in \mathbb{C}^*.$$

Also the group $\mathrm{SL}(2, \mathbb{C}) = \mathrm{SL}(V)$ acts on T_n via the appropriate symmetric products, $V_k = \mathrm{Sym}^k(V)$. These commuting actions thus define an action of $\mathbb{C}^* \times \mathrm{SL}(V)$ on T_n. Two pairs of forms in T_n give rise to isomorphic Weierstrass fibrations if and only if they are in the same orbit of this action.

Corollary 3.6 ([38], Corollary (2.8)). *The set of isomorphism classes of minimal elliptic surfaces over $\mathbb{P}^1$ with $\deg L = n$ and fixed section is in one to one correspondence with the set of orbits $T_n/\mathbb{C}^* \times \mathrm{SL}(V)$.*

Miranda has further studied the stability (in the sense of geometric invariant theory) of this action and has established:

Theorem 3.7 ([38]).

(1) *The pair (g_2, g_3) is properly stable if and only if the induced elliptic surface $\overline{X}$ has only reduced fibers.*

(2) *The pair (g_2, g_3) is strictly semi-stable if and only if $\overline{X}$ has a fiber of type I_n^*.*

(3) *The pair (g_2, g_3) is unstable if and only if $\overline{X}$ has a fiber of type II^*, III^*, or IV^*.*

As stability properties of such elliptic surfaces are entirely a property of the types of singular fibers which they contain, it is natural to consider the subloci of the GIT moduli space with a specified set of singular fibers and no others. Seiler has shown:

Theorem 3.8 ([46], Satz 5.4). *Let Z be an irreducible component of the global GIT moduli scheme, and let $Z(T)$ be the subvariety of surfaces with a singular fiber of type T. Then:*

(1) *$Z(T)$ is locally closed.*

(2) *The codimension $N(T)$ of $Z(T)$ is given by the following table:*

T	I_n	II	III	IV	I_n^*	IV*	III*	II*
$N(T)$	$n-1$	1	2	3	4 + n	6	7	8

.

(3) *The subscheme $Z(T_1, \ldots, T_r)$ consisting of the basic elliptic surfaces X/C with r points $p_1, \ldots, p_r \in C$ and singular fibers over p_i of type T_i, is locally closed and has codimension $N(T_1) + \cdots + N(T_r)$.*

We call such a locus $Z(T_1, \ldots, T_r)$ a *combinatorial stratum* associated with the collection of singular fibers $(T_1, \ldots, T_r)$. Each kind of stratum is characterized by its collection of "combinatorial data", i.e., the list of types and number of occurences of each Kodaira singular fiber.

As a simple application of the fact that the functional and homological invariants (global monodromy data) determine a Weierstrass fibration uniquely, the Riemann-Hurwitz formula, and Lemma 3.4 we have:

Theorem 3.9 ([4], Prop. 2.10 and p. 295). *The dimension of a combinatorial stratum (without I_0^* fibers) consisting of i_m (respectively i_m^*, ii, ii^*, iii, iii^*, iv, iv^*) singular fibers of* I_m *(respectively* I_m^*, II, II*, III, III*, IV, IV*) *type, equals the extra ramification of the $\mathfrak{J}$-map, i.e.,*

$$2g - 2 + \frac{1}{6}\left[6\sum_{m\geq 1}(i_m + i_m^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - d\right].$$

3.2.2 Geometric isomonodromic deformations from combinatorial strata

We begin by describing the Picard-Fuchs differential equations for a Weierstrass elliptic curve fibration. We recall the Griffiths-Dwork approach, which starts with a first order regular system from which the second order equation can be determined. This method is computationally convenient if all one wants is the Picard-Fuchs equation in terms of the Weierstrass form, but it will not be so useful for our later purposes. Ultimately, we will only care about properties of the projective normal form of the Picard-Fuchs differential equation. For that reason we also present Stiller's approach by "pulling back by $\mathcal{J}$" which suffices to determine this.

Theorem 3.10 ([43], p. 304)**.** *The Griffiths-Dwork approach to computing the Picard-Fuchs equation of a Weierstrass elliptic surface yields*

$$\frac{d}{dz}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \begin{bmatrix} \frac{-1}{12}\frac{d\Delta/dz}{\Delta} & \frac{3\delta}{2\Delta} \\ \frac{-g_2\delta}{8\Delta} & \frac{1}{12}\frac{d\Delta/dz}{\Delta}\end{bmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

where

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2$$

$$\delta(z) = 3g_3(z)\frac{dg_2(z)}{dz} - 2g_2(z)\frac{dg_3(z)}{dz}$$

and

$$\eta_1 = \int_\gamma \frac{dx}{y},\ \eta_2 = \int_\gamma \frac{xdx}{y}.$$

Now we turn to Stiller's "pulling back by the $\mathcal{J}$-map" approach.

Recall from [57] that the *uniformizing differential equation* for the arithmetic triangle group $\mathrm{PSL}(2,\mathbb{Z})$ is

$$\Lambda f = \frac{d^2 f}{dx^2} + \frac{36x^2 - 41x + 32}{144x^2(x-1)^2} f = 0,$$

the projective normal form of the Picard-Fuchs equation of the family $\mathcal{E}$ of Example 1.2. Note that this is also the projective normal form of the hypergeometric differential equation ${}_2F_1(1/12, 1/12, 2/3)$.

Denote by $\Lambda_{\mathcal{J}}$ the projective normal form of the pullback equation $\mathcal{J}^*(\Lambda)$. By an explicit computation using Theorem 3.10 one can show that

Proposition 3.11. *Given a Weierstrass fibration with functional invariant $\mathcal{J}$, the projective normal form of the Picard-Fuchs differential equation and the equation $\Lambda_{\mathcal{J}}$ are identical.*

Actually, this can also be seen indirectly through Stiller's classification of *K-equations* [50].

We now can recast the results of Lemma 3.4 above as describing the singular fiber types associated to various regular singular points of the projective normal form of Picard-Fuchs $\Lambda_{\mathcal{J}}$, using Theorem 4.3.

Proposition 3.12. *A regular singular point of* $\Lambda_{\mathcal{J}}$ *corresponds to a fiber of type*

- I_n *or* I_n^*, $n \geq 1 \Leftrightarrow$ *the characteristic exponents are equal;*
- I_0 *or* $\mathrm{I}_0^* \Leftrightarrow$ *the difference in characteristic exponents is a nonnegative integer;*
- II *or* $\mathrm{IV}^* \Leftrightarrow$ *the difference in characteristic exponents is* $1/3$ *modulo integers;*
- III *or* $\mathrm{III}^* \Leftrightarrow$ *the difference in characteristic exponents is* $1/2$ *modulo integers;*
- IV *or* $\mathrm{II}^* \Leftrightarrow$ *the difference in characteristic exponents is* $2/3$ *modulo integers.*

The Gauss-Manin connection on the family of elliptic curves over the product of $\mathbb{P}^1$ and the combinatorial stratum now gives us the necessary data to define a geometric isomonodromic deformation.

Remark 6. In fact, even combinatorial strata in the *unstable* locus may yield geometric isomonodromic deformations in this way.

Theorem 3.13. *All of the geometric solutions to the Painlevé* VI *equation coming from moduli of Weierstrass elliptic surfaces are listed in Tables* 2 *and* 3*. Weierstrass equations for the five relevant families appear in Table* 1*.*

Proof. Euler characteristic constraints tell us that we need only consider rational and K3 surface Weierstrass fibrations, else the number of singular fibers would have to be greater than the requisite number (four) of regular singular points in the Gauss-Manin system to obtain PVI as isomonodromic deformation equation by Lemma 2.8. Also, the functional invariant must not be constant.

But all such examples were explicitly constructed in a paper of Herfurtner, where it is also noted that all the possible K3 surface examples are related by quadratic twists to rational ones [24]. Quadratic twists of the Weierstrass equation do not affect the projective normal form of the Picard-Fuchs differential equation, and thus do not alter any geometric isomonodomic deformations from such strata.

Family	Sing. fiber types	Weierstrass presentation
1	$\mathrm{I}_1, \mathrm{I}_1, \mathrm{II}, \mathrm{IV}^*$	$g_2(a,s) = 3(s-1)(s-a^2)^3$ $g_3(a,s) = (s-1)(s-a^2)^4(s+a)$
2	$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_2^*$	$g_2(a,s) = 12s^2(s^2+as+1)$ $g_3(a,s) = 4s^3(2s^3+3as^2+3as+2)$
3	$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_3^*$	$g_2(a,s) = 12s^2(s^2+2as+1)$ $g_3(a,s) = 4s^3(2s^3+3(a^2+1)s^2$ $+6as+2)$
4	$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_1, \mathrm{III}^*$	$g_2(a,s) = 3s^3(s+a)$ $g_3(a,s) = s^5(s+1)$
5	$\mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_2, \mathrm{IV}^*$	$g_2(a,s) = 3s^3(s+2a)$ $g_3(a,s) = s^4(s^2+3as+1)$

Table 1: The five families.

Of the 50 core examples which Herfurtner catalogues, 7 have I_0^* fibers and thus by a quadratic twist may be reduced to examples with three singular fibers, which cannot possess geometric isomonodromic deformations. There are also 38 "isolated" examples with four singular fibers, which do not lie in families. Of the 50 examples, the 5 remaining families (see Table 1) satisfy the "extra ramification of $\mathcal{J} = 1$" criterion from Theorem 3.9. Here s represents the parameter on the base $\mathbb{P}^1$ of each elliptic surface, and a is the deformation parameter (in the coordinates of [24]). Hence each of these defines a geometric isomonodromic deformation of the corresponding projective normal form Picard-Fuchs equation.

In order to get explicit expressions for the associated geometric Painlevé VI solutions we need to choose coordinates on the base $\mathbb{P}^1$ such that three of the singular fibers lie over $\{0, 1, \infty\}$; the fourth then lies over t, and the extra ramification point is at λ. The $\mathrm{SL}(2, \mathbb{C})$-action from Proposition 3.5 defines the action of the geometric symmetry group, isomorphic to S_4, exchanging the positions of the singular fibers at 0, 1, ∞, and t. The Painlevé VI solution in each case is given by the (algebraic) relation between the position of the single apparent singularity λ and the "mobile" singular fiber at t (Lemma 2.9). All of this data is tabulated in Tables 2 and 3, where we see how the multiplicities of occurrence of the algebraic relations correspond to the different fixed point behavior under the action of the geometric symmetry group S_4.

q.e.d.

Sol'n.	Polynomial equation (λ = apparent, t = free)
1A	$\lambda^2 - t$
1B	$\lambda^2 - 2\lambda + t$
1C	$\lambda^2 - 2\lambda t + t$
2A	$\lambda^2 - t$
2B	$\lambda^2 - 2\lambda + t$
2C	$\lambda^2 - 2\lambda t + t$
3A	$\lambda^4 - 6\lambda^2 t + 4\lambda t + 4\lambda t^2 - 3t^2$
3B	$3\lambda^4 - 4\lambda^3 - 4\lambda^3 t + 6\lambda^2 t - t^2$
3C	$\lambda^4 - 4\lambda^3 + 6t\lambda^2 - 4t^2\lambda + t^2$
3D	$\lambda^4 - 4t\lambda^3 + 6t\lambda^2 - 4t\lambda + t^2$
4A	$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t$
4B	$\lambda^4 - 2t\lambda^3 + 2t^2\lambda - t^3$
4C	$\lambda^4(t^2 - t + 1) - 2\lambda^3 t(t+1) + 6t^2\lambda^2 - 2\lambda t^2(t+1) + t^3$
4D	$\lambda^4 - 2\lambda^3 + 2t\lambda - t$
5A	$-2\lambda^3 + 3t\lambda^2 + 3\lambda^2 - 6t\lambda + t^2 + t$
5B	$\lambda^3 - 3\lambda^2 + 3t\lambda - 2t^2 + t$
5C	$\lambda^3 - 3t\lambda^2 + 3t\lambda + t^2 - 2t$
5D	$2\lambda^3 - 3t\lambda^2 + t^2$
5E	$\lambda^3 - 3t\lambda + t^2$
5F	$\lambda^3 - 3t\lambda^2 + 3t\lambda - t^2$
5G	$\lambda^3(2-t) - 3t\lambda^2 + 3t^2\lambda - t^2$
5H	$\lambda^3(t+1) - 6t\lambda^2 + 3t(t+1)\lambda - 2t^2$
5I	$(1-2t)\lambda^3 + 3t\lambda^2 - 3t\lambda + t^2$
5J	$\lambda^3 - 3\lambda^2 + 3t\lambda - t$
5K	$\lambda^3 - 3t\lambda + 2t$
5L	$2\lambda^3 - 3\lambda^2 + t$

Table 2: Polynomial equations for the geometric solutions.

Remark 7. There is a classification theorem due to Dubrovin-Mazzocco which covers the algebraic solutions described in Tables 2 and 3. After correcting Equation B_3 in [18] to read

$$y = \frac{(2-s)(s+1)(s^2-3)^2}{((2+s)(5s^4-10s^2+9))} \tag{3.1}$$

$$x = \frac{(2-s)^2(1+s)}{((2+s)^2(1-s))}, \tag{3.2}$$

Stab(Sol'n.)	Sol'n.	PVI(α, β, γ, δ)	Stab(PVI)
D_4	1A	(0, 0, 1/18, 4/9) (1/18, - 1/18, 0, 1/2)	V_4
	1B	(1/18, 0, 1/18, 1/2) (0, -1/18, 0, 4/9)	
	1C	(1/18, 0, 0, 4/9) (0, -1/18, 1/18, 1/2)	
D_4	2A 2B 2C	(0, 0, 0, 1/2)	S_4
S_3	3A 3B 3C 3D	(0, 0, 0, 1/2)	S_4
S_3	4A 4B 4C 4D	(0, -1/18, 0, 1/2) (0, 0, 1/18, 1/2) (1/18, 0, 0, 1/2) (0, 0, 0, 4/9)	S_3
S_2	5A 5B 5C	(0, -1/18, 0, 1/2)	S_3
	5D 5E 5F	(0, 0, 1/18, 1/2)	
	5G 5H 5I	(1/18, 0, 0, 1/2)	
	5J 5K 5L	(0, 0, 0, 4/9)	

Table 3: Solution-equation correspondences and geometric symmetry group stabilizers.

the transformations in their §1 can then be applied to relate, for example, our solution 4C to their Equation B_3, the "Cubic-Octahedral" case [26].

Remark 8. Our methods are not limited to the Painlevé VI equations. In [17] we extend this result to obtain explicit algebraic solutions to certain multiparameter isomonodromic deformation equations (Garnier systems) from elliptic surface moduli.

Remark 9. There is a natural map from the joint moduli space of Fuchsian projective connections with specified local monodromies on curves of higher genus to the moduli space of representations of the fundamental group of the curves with punctures at the singular points of the connections. The fibers of this map are monodromy-preserving families of projective connections/curves. There is a natural symplectic structure on the space of representations. Pulling back this symplectic structure, taking the null-foliation, and picking coordinates in moduli we obtain an explicit isomonodromic deformation equation in the higher genus setting [27, 31]. Geometric isomonodromic deformations are then defined as before, replacing the $\mathbb{P}^1$-fibrations with fibrations by curves of higher genus. This time combinatorial strata in the Seiler moduli spaces for Weierstrass elliptic fibrations over curves of higher genus yield algebraic solutions to these more complicated isomonodromic deformation equations.

Remark 10. Elliptic surfaces over $\mathbb{P}^1$ and the Painlevé VI equation also make an appearance in work of Manin involving the non-homogeneous Picard-Fuchs equation, elliptic μ-equations, and his coordinate independent, time-dependent Hamiltonian formulation for the Painlevé VI equation [36]. However, his purpose, construction, and results are of a quite different nature than our own.

4. Algebraic isomonodromic deformations by "pullback"

In this section we generalize our construction of algebro-geometric isomonodromic deformations from Weierstrass elliptic fibrations over $\mathbb{P}^1$ to construct algebraic isomonodromic deformations by "pulling back" a fixed rank two regular local system via a family of rational functions. The actual algebraic expressions for these solutions correspond to equations for the special Hurwitz spaces parametrizing these rational functions.

In the first subsection we begin by studying the behavior of the regular singular points of second order Fuchsian ordinary differential equations under "pullback". Next we introduce some elementary candidates for the *fixed* rank two local system to be used: Gauss hypergeometric differential equations associated with Fuchsian triangles.

The second subsection begins with a general discussion of Hurwitz curves and methods for deriving explicit algebraic expressions both for them and for the families of rational functions which they parametrize. Finally we characterize, given the fixed rank two local system, the topological conditions on a one-parameter family of rational functions for the local system to pull back to an algebraic solution to a Painlevé VI equation.

4.1 "Pulling back" Fuchsian ordinary differential equations

We begin by examining the effect of "pulling back" a Fuchsian ordinary differential equation on the regular singular points. Afterwards, we introduce hypergeometric differential equations associated with Fuchsian triangles.

4.1.1 Regular and apparent singularities under "pullback"

Suppose then that we pullback a second order Fuchsian equation $L_2 f = 0$, with

$$L_2 = \frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x),$$

by a rational function $x = \mathcal{R}(z)$.

Proposition 4.1. *The regular singular points of the pulled back equation $\mathcal{R}^*(L_2)$ lie among the inverse image points of the regular singular points of the original equation L_2 and the extra ramification points of the rational function $\mathcal{R}(z)$.*

Proof. This is a straightforward computation:

$$\frac{df}{dx} = \frac{df}{dz}\Big/\frac{d\mathcal{R}}{dz}$$

and

$$\frac{d^2 f}{dx^2} = \left(\frac{d\mathcal{R}}{dz}\right)^{-2}\left(\frac{d^2 f}{dz^2} - \frac{df}{dz}\left(\frac{d^2\mathcal{R}}{dz^2}\Big/\frac{d\mathcal{R}}{dz}\right)\right)$$

so that the pullback $\mathcal{R}^*(L_2) = 0$ of the equation $L_2 f = 0$ takes the form

$$\frac{d^2 f}{dz^2} + \frac{df}{dz}\left(P_1(\mathcal{R}(z))\frac{d\mathcal{R}}{dz} - \left(\frac{d^2\mathcal{R}}{dz^2}\Big/\frac{d\mathcal{R}}{dz}\right)\right) + f\left(P_2(\mathcal{R}(z))\left(\frac{d\mathcal{R}}{dz}\right)^2\right) = 0.$$

Thus the only poles in the coefficients can occur at points where $P_1(\mathcal{R}(z))$ has one, or where $P_2(\mathcal{R}(z))$ has one, or at other places where $d\mathcal{R}/dz = 0$ (i.e., the *extra* ramification points of the $\mathcal{R}$ map). q.e.d.

Lemma 4.2. *The characteristic exponents of a point in the inverse image as above are exactly those of the point being pulled back times the order of ramification. The characteristic exponents at an extra ramification point differ by an integer.*

Proof. For simplicity assume that the singular point of L_2 under consideration is at $x = 0$. In general (in the absence of ramification) there are $\deg \mathcal{R}(z)$ points in the inverse image of 0, each with the same characteristic exponents as has L_2 at 0. If ramification of $\mathcal{R}(z)$ occurs at an inverse image point z_0, then $\mathcal{R}(z)$ has a multiple root there with multiplicity equal to the order of ramification. Thus the characteristic exponents at z_0 are those of L_2 at 0 times this order of ramification. At an *extra* ramification point z_1, i.e., a point in the inverse image of an ordinary point of L_2 (characteristic exponents $0, 1$), the characteristic exponents are thus 0 and the ramification index of $\mathcal{R}(z)$ at z_1. q.e.d.

Now regular singular points which are extra ramification points have characteristic exponent difference an integer ≥ 2 (if the difference were equal to 1 this would be an ordinary point [57, p. 23]). Also, by Lemma 4.2, the characteristic exponent difference is rescaled by the degree of the rational function; if this difference is zero it remains so. Finally, we know that the characteristic exponent difference in case of a generic regular singular point with characteristic exponent difference $1/b$ is the ramification degree r divided by b. The reduced form of this fraction has numerator $r/\gcd(r, b)$. Thus we have

Theorem 4.3. *Characterization of types of regular singular points of a pullback of a second order Fuchsian differential equation without apparent singularities by a rational function:*

- *The extra ramification points of the rational function are all apparent singularities.*
- *The points in the inverse image of logarithmic singularities with equal characteristic exponents are logarithmic singularities with equal characteristic exponents.*
- *For a regular singular point in the inverse image of a generic regular singular point with characteristic exponent difference $1/b$: if*

the characteristic exponent difference is integral (i.e., if b divides the ramification index), then the regular singular point is apparent; if the characteristic exponent difference is not integral (i.e., if b does not divide the ramification index), then the singular point is generic.

4.1.2 Hypergeometric differential equations and triangle groups

The *Gauss hypergeometric differential equation* ${}_2F_1(a,b,c)$ $(a,b,c \in \mathbb{R})$ is the second order Fuchsian ordinary differential equation

$$x(1-x)\frac{d^2u}{dx^2} + \{c-(a+b+1)x\}\frac{du}{dx} - abu = 0$$

with characteristic exponents $(0, 1-c)$ at $x = 0$, $(0, c-a-b)$ at $x = 1$, and (a, b) at $x = \infty$. The differences between the two characteristic exponents are invariants of the equation under projective equivalence.

The orders of local monodromies about $x = 0, 1, \infty$ are $b_0 = 1/|1-c|, b_1 = 1/|c-a-b|, b_\infty = 1/|a-b| \in \mathbb{Z}^+ \cup \{\infty\}$.

Restrict attention to the upper half plane $\mathcal{H} = \{x \in \mathbb{C} \mid \Im x > 0\}$. Consider two linearly independent solutions restricted to $(\infty, 0)$, $(0, 1)$, and $(1, \infty)$. The image of $\mathcal{H}$ under the projective solution is a *Schwarz triangle* — a curvilinear triangle bounded by circular arcs and angles $\pi|1-c|$, $\pi|c-a-b|$, and $\pi|a-b|$ at the images of 0, 1, ∞ respectively. We will frequently refer to such triangles just by their associated triples (b_0, b_1, b_∞). The triangle (b_0, b_1, b_∞) falls into one of three natural classes as the quantity

$$\mathcal{B} = \frac{1}{b_0} + \frac{1}{b_1} + \frac{1}{b_\infty}$$

is < 1, $= 1$, or > 1.

In the first case ($\mathcal{B} < 1$) the monodromy group is a *triangular subgroup* of $\mathrm{PSL}(2, \mathbb{R})$. There are infinitely many possibilities for such triples. Among these is the triangle $(2, 3, \infty)$ which corresponds to the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This triangular Fuchsian group is *arithmetic* in the sense that it is commensurable with the unit group of a quaternion algebra. In fact, we have:

Theorem 4.4 ([51], Table (1))**.** *There are finitely many such arithmetic triangles (grouped below by isomorphism class of corresponding quaternion algebras):*

I	$(2,3,\infty),(2,4,\infty),(2,6,\infty),(2,\infty,\infty),(3,3,\infty),(3,\infty,\infty),$ $(4,4,\infty),(6,6,\infty),(\infty,\infty,\infty)$
II	$(2,4,6),(2,6,6),(3,4,4),(3,6,6)$
III	$(2,3,8),(2,4,8),(2,6,8),(2,8,8),(3,3,4),(3,8,8),(4,4,4),$ $(4,6,6),(4,8,8)$
IV	$(2,3,12),(2,6,12),(3,3,6),(3,4,12),(3,12,12),(6,6,6)$
V	$(2,4,12),(2,12,12),(4,4,6),(6,12,12)$
VI	$(2,4,5),(2,4,10),(2,5,5),(2,10,10),(4,4,5),(5,10,10)$
VII	$(2,5,6),(3,5,5)$
VIII	$(2,3,10),(2,5,10),(3,3,5),(5,5,5)$
IX	$(3,4,6)$
X	$(2,3,7),(2,3,14),(2,4,7),(2,7,7),(2,7,14),(3,3,7),(7,7,7)$
XI	$(2,3,9),(2,3,18),(2,9,18),(3,3,9),(3,6,18),(9,9,9)$
XII	$(2,4,18),(2,18,18),(4,4,9),(9,18,18)$
XIII	$(2,3,16),(2,8,16),(3,3,8),(4,16,16),(8,8,8)$
XIV	$(2,5,20),(5,5,10)$
XV	$(2,3,24),(2,12,24),(3,3,12),(3,8,24),(6,24,24),(12,12,12)$
XVI	$(2,5,30),(5,5,15)$
XVII	$(2,3,30),(2,15,30),(3,3,15),(3,10,30),(15,15,15)$
XVIII	$(2,5,8),(4,5,5)$
XIX	$(2,3,11)$

The second case ($\mathcal{B}=1$) corresponds to the *planar* triangles, i.e., those uniformized by $\mathbb{C}$. There are only four possibilities:

$$(2,2,\infty),(2,3,6),(2,4,4),(3,3,3).$$

The third case ($\mathcal{B}>1$) consists of the *spherical* triangles, which fall naturally into four families. These are the *dihedral* triangles $(2,2,n)$, $n\geq 2$, the *tetrahedral* triangle $(2,3,3)$, the *octahedral* triangle $(2,3,4)$, and the *icosahedral* triangle $(2,3,5)$.

4.2 Algebraic Painlevé VI solutions by "pullback"

After introducing Hurwitz spaces for families of covers, we derive a topological characterization of those covers which, together with a fixed second order Fuchsian ordinary differential equation without apparent singularities, define algebraic isomonodromic deformations by "pullback" and algebraic solutions to a Painlevé VI equation.

4.2.1 Hurwitz spaces for families of covers

Consider the moduli spaces $H^0_{n,B}$ for irreducible covers of $\mathbb{P}^1$ with fixed degree $n > 1$ and branching locus consisting of a preassigned finite subset $B \subset \mathbb{P}^1$ and one variable additional point $p \in \mathbb{P}^1 \setminus B$ (i.e., at r points, $r = |B| + 1$). A pair of such n-sheeted branched covers $f_1 : X_1 \to \mathbb{P}^1$, $f_2 : X_2 \to \mathbb{P}^1$ are isomorphic if there is an isomorphism $g : X_1 \to X_2$ such that $f_1 = f_2 \circ g$. The natural map $\pi^0_{n,B} : H^0_{n,B} \to \mathbb{P}^1$, taking f to p in the complement of B in the branch locus of f, makes $H^0_{n,B}$ into a finite cover of $\mathbb{P}^1 \setminus B$. Any irreducible component of the unique nonsingular complete curve $H_{n,B}$ with branched covering map $\pi_{n,B} : H_{n,B} \to \mathbb{P}^1$ extending $\pi^0_{n,B}$, we call a *Hurwitz curve* [10].

In fact, these components are naturally classified by specifying additional ramification/monodromy data over/about the branch locus in $\mathbb{P}^1$. We attach to each point in the branch locus a conjugacy class of elements of S_n: $\mathcal{C} = (C_1, \ldots, C_r)$, each conjugacy class C_i being specified by a partition of n as usual. Consider all vectors of permutations $\zeta = (\zeta_1, \ldots, \zeta_r)$, up to global conjugacy, such that $\zeta_i \in C_i$ and $\zeta_r \circ \cdots \circ \zeta_1 = 1$. It is sometimes convenient to assume further that the $\zeta_1, \ldots, \zeta_r$ generate a transitive subgroup $G \subset S_n$ ("Galois cover" condition); this will not be necessary for our applications.

Let $X_{0,r} = (\mathbb{P}^1)^r \setminus \Delta_r$ denote the naive configuration space, where Δ_r is the discriminant variety. There is a natural map $X_{0,r+1} \to X_{0,r}$ given by "forgetting" the position of the $(r+1)$st point on $\mathbb{P}^1$. The homotopy exact sequence associated with this map induces an action of the fundamental group of $X_{0,r}$ on the fundamental group of $\mathbb{P}^1$ minus r points, up to inner automorphisms. The group $\mathcal{B}_{0,r} = \pi_1(X_{0,r}, (p_1, \ldots, p_r))$ is called the *group of colored braids with r plaits*. There is a natural correspondence between coverings of $\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}$ up to topological isomorphism and finite index subgroups of $\pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}, b)$ up to inner automorphism. Now the r-tuple $\{\zeta_1, \ldots, \zeta_r\}$ characterizes the topological isomorphism class of the covering, so we get an action of $\mathcal{B}_{0,r}$ on the sets of $\{\zeta_1, \ldots, \zeta_r\}$. This is the action that describes the monodromy of the Hurwitz cover $\pi^0_{n,B}$.

Let's make this more precise in the case $r = 4$, $B = \{0, 1, \infty\}$: Assume for convenience that p_1, p_2, p_3, p_4 lie on the real axis. There are then three distinguished generators $t_{1,2}, t_{2,3}, t_{3,4} \in \mathcal{B}_{0,4}$, where $t_{i,i+1}(0) = t_{i,i+1}(1) = (p_1, p_2, p_3, p_4)$ and the p_i and p_{i+1} turn clockwise along a circle of diameter $[p_i, p_{i+1}]$, $i = 1, 2, 3$. The action of the $t_{i,i+1}$ on 4-vectors $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is given by

$$\begin{array}{rcl} t_{1,2}(\zeta) & = & ({}^{\zeta_2}\zeta_1, {}^{\zeta_2\zeta_1}\zeta_2, \zeta_3, \zeta_4) \\ t_{2,3}(\zeta) & = & (\zeta_1, {}^{\zeta_3}\zeta_2, {}^{\zeta_3\zeta_2}\zeta_3, \zeta_4) \\ t_{3,4}(\zeta) & = & (\zeta_1, \zeta_2, {}^{\zeta_4}\zeta_3, {}^{\zeta_4\zeta_3}\zeta_3), \end{array}$$

where ${}^{\sigma}\tau = \sigma\tau\sigma^{-1}$.

There is also a natural map $\rho : X_{0,4} \to M_{0,4}$ which maps a quadruple (x, y, z, t) to its cross-ratio

$$\rho(x, y, z, t) = [x, y, z, t] := \frac{t - x}{y - x}\frac{y - z}{t - z}.$$

By taking the section

$$\sigma : \mathbb{P}^1 \setminus \{0, 1, \infty\} = M_{0,4} \to X_{0,4} = (\mathbb{P}^1)^4 \setminus \Delta_4$$

of ρ defined by $\sigma(w) = (0, 1, \infty, w)$, our Hurwitz cover restricts to the Hurwitz curve. Moreover, the monodromies about 0, 1, and ∞ are easily obtained in terms of the $t_{i,i+1}$ ([7]).

Now we're ready to briefly describe a method, due to Couveignes, for computing algebraic expressions for Hurwitz curves (see [8] for details). Assume that our Hurwitz curve parametrizes *genus zero* covers of the four-punctured sphere. For simplicity we will also assume that $r = 4$, but the method still works for general r. Fix $\mathcal{C} = (C_1, C_2, C_3, C_4)$, list all $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ as above, and compute the monodromy action about $0, 1, \infty$ on the list of ζ's. Pick a point on the Hurwitz curve lying over one of $0, 1, \infty$ and at which the Hurwitz cover is ramified (it is best to pick a point with as much ramification as possible). Such a point q on the Hurwitz curve corresponds to a member of our family of covers with ramification only over $B = \{0, 1, \infty\}$. One can think of this as a "confluent limit" of covers whose branch locus consists of four points $\{0, 1, \infty, p\}$, in the limit as the fourth "mobile" point p flows towards B. Now define a local coordinate μ nearby this point q on the Hurwitz curve. All the coordinates which appear in the algebraic model for the Hurwitz curve are Laurent series in this μ.

Next pick a coordinate on the four-punctured sphere (i.e., choose $0, 1, \infty$) and on its genus zero covering spaces, chosen so that as μ tends to zero none of these collide (*admissible* families of points in the sense of [7]). Label two additional points t and λ on the covers, lying over the branch locus. At $\mu = 0$, the rational function describing the cover can be computed from the monodromy data alone, by a method dating back to Fricke and Klein (see [2] for a modern presentation). For $\mu \neq 0$ we again use the monodromy/ramification data derived from the $t_{i,i+1}$ action to present an ansatz for the μ-dependent rational function describing the covers. This suffices to obtain first order approximations for all the coeficients arising in our algebraic model. By Hensel's lemma, computation of the higher order approximations reduces to linear algebra, with developments in the variable μ. To determine an algebraic relation between t and λ, we need only know an *a priori* bound on the possible degrees of an algebraic relation. In [8] such a bound is determined (essentially the product of the degree n of the covers in our family and the degree of the Hurwitz cover $= \#\zeta$'s); this procedure is also worked out in detail there for a simple, but nontrivial, example.

Remark 11. It is worth noting that, at least under the hypothesis that the covers are Galois, a great deal is known regarding the *field of definition* of for the components of Hurwitz curves. In particular, if $H(\mathcal{C})$ is the union of the components that correspond to the $\mathcal{B}_{0,r}$-orbits, then $H(\mathcal{C})$ is defined over $\mathbb{Q}^{ab}$. Moreover, if $\mathcal{C}$ is *rational*, then $\mathcal{H}(\mathcal{C})$ is even defined over $\mathbb{Q}$ [55]. Good general references on the construction of Hurwitz spaces and questions of field of definition of their components are the article by Emsalem [19] and Chapter 10 of the book by Völklein [56].

4.2.2 Topological characterization of "pullback" algebraic Painlevé VI solutions

To begin with let us mention two motivations for this approach to finding algebraic Painlevé VI solutions.

In the first place, we discovered while studying the geometric solutions to Painlevé VI arising from families of Weierstrass elliptic fibrations over $\mathbb{P}^1$ that it is sufficient to compute the projective normal form of the Picard-Fuchs differential equation for each Weierstrass fibration. This reduces to computing Kodaira's functional invariant for such families, and this in turn is simply determined by the coefficients of the Weierstrass equation.

A natural generalization would be to consider families of K3 surface fibrations over $\mathbb{P}^1$. In fact, the Picard-Fuchs equation of such a fibration will have order equal to $22 = b_2(\mathrm{K3})$ minus the Picard number (= Néron-Severi rank) of the generic K3 surface fiber. Since K3 surfaces with Picard number 20 do not form nonisotrivial continuous families, we cannot directly obtain any second order Picard-Fuchs equations. However, if we consider fibrations with generic fiber of Picard number 19, there *is* a natural way to construct algebraic isomonodromic deformations from the Picard-Fuchs equations. These equations are of order three, but they are *symmetric squares* of second order Fuchsian ordinary differential equations. It is natural then to consider the projective normal form of this second order "square root" equation [13, 14, 15]. Moreover, such rank 19 lattice polarized K3 surfaces have a one dimensional moduli space which, like the J-line for elliptic curve moduli, admits a presentation as an arithmetic quotient of the upper half plane, i.e., a quotient of the upper half plane by an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ [12]. If the base of our nonisotrivial rank 19 K3 surface fibration is $\mathbb{P}^1$, then we know also that the corresponding arithmetic subgroup/moduli curve has genus zero.

Once again, one can define a rational function from the base of the K3 surface fibration to the moduli space, a *generalized functional invariant*, and this rational function determines the second order Fuchsian ordinary differential equation (again by "pulling back" the natural differential equation on the moduli space and taking projective normal form). There is one problem with such an approach, however: No obvious analogue of "Weierstrass form" exists for K3 surfaces, and hence we have no analogous clean description of the generalized functional invariant to use in practical computations. In addition, there is at present no theory for describing moduli of rank 19 lattice polarized K3 surface fibered threefolds over $\mathbb{P}^1$. This suggests that the right generalization of our approach to geometric Painlevé VI solutions from the elliptic curve fibration case to the K3 surface fibration case should make essential use of the *existence* of the generalized functional invariant, but use other methods to determine explicit formulas for these rational functions (and consequently the associated geometric Painlevé VI solutions).

The second motivation comes from the relationship between Belyi pairs and Hurwitz curves. We saw in Section 3.1.1 that algebraic solutions to Painlevé VI equations define a particular sort of Belyi pair (a *Painlevé-Belyi pair*), with the λ-parametrized curve X forming a finite cover of $\mathbb{P}^1_t$ ramified only over $t = 0, 1, \infty$. A result of Diaz, Donagi, and Harbater [10] shows that "every curve over $\overline{\mathbb{Q}}$ is a Hurwitz space". This correspondence is far from canonical, however, and in general many moduli problems will yield the same Belyi pair. This does raise the question: Is there an interpretation whereby our geometric Painlevé VI solutions from Weierstrass elliptic fibrations over $\mathbb{P}^1$ would actually be equations for Hurwitz curves?

The answer is a strong affirmative; in fact, there is a completely natural such interpretation. Each point on the the curve described by the geometric solution to Painlevé VI corresponds to a *rational function* branching over $B = \{0, 1, \infty\}$ and an additional point p, with exactly four (nonidentity monodromy) ramification points over B and one (identity monodromy) ramification point over p, i.e., the *functional invariant* of a Weierstrass elliptic surface over $\mathbb{P}^1$ with four singular fibers over the discriminant locus, and lying in a one dimensional combinatorial stratum. The quadruples of conjugacy classes $(C_0, C_1, C_\infty, C_p)$ for the five families from Table 1 (viewed as a vector of partitions of n) are

1: $([1,1],[2],[1,1],[2])$
2: $([3,3],[2,2,2],[2,2,1,1],[2,1,1,1,1])$
3: $([3,3],[2,2,2],[3,1,1,1],[2,1,1,1,1])$
4: $([3],[2,1],[1,1,1],[2,1])$
5: $([3,1],[2,2],[2,1,1],[2,1,1])$,

with $n = 2, 6, 6, 3$, and 4, respectively. Furthermore, by applying Couveigne's method from Section 4.2.1 to these quadruples, one can recover families of functional invariants (rational functions for the families of covers) and the corresponding geometric Painlevé VI solutions (equations for the Hurwitz curves).

We will now see how to put all this together to obtain a topological characterization of the algebraic Painlevé VI solutions which arise from "pulling back" rank two regular local systems on $\mathbb{P}^1$ minus several points.

Let us begin by establishing some notation:

$n = \#$ sheets in the covers in our family.

$m = \#$ points $p_1, \ldots, p_m$ over which the local system $\mathcal{L}$ has nonidentity local monodromies of finite orders $a_1, \ldots, a_m$ respectively ($a_i \geq 2$).

We assume that the second order Fuchsian ordinary differential equation on $\mathbb{P}^1$ associated with the regular local system $\mathcal{L}$ has *no apparent singularities*, i.e., all regular singular points have nonidentity local monodromies.

$s = \#$ points $q_1, \ldots, q_s$ over which $\mathcal{L}$ has ∞ order local monodromies.

$l =$ ramification order of the unique ramification point λ *not* lying over $\{p_1, \ldots, p_m, q_1, \ldots, q_s\}$ ($2 \leq l \leq n$).

Note that there are two assumptions built in here. The first is that there is exactly one point in $\mathbb{P}^1 \setminus \{p_1, \ldots, p_m, q_1, \ldots, q_s\}$ over which the covers ramify ($m + s = r$ in the notation of the previous subsection); this is our "variable" branching point. The second is that there is a unique ramification point in the fiber over this branching point; this will correspond to the unique apparent singularity in the pulled back local system.

$\beta_i = \#$ unramified points over p_i.

$\gamma_i = \#$ ramified points over p_i.

$\tau_i = \beta_i + \gamma_i = \#$ points over p_i.

$\mu_i = \#$ points lying over p_i with ramification a multiple of a_i by a positive integer.

$d_k = \#$ points over q_k.

$r_j =$ list of ramification orders of points over p_j (if $1 \le j \le m$) or q_{j-m} (if $m+1 \le j \le m+s$), expressed as a partition of n.

The hypothesis that our covers have genus zero implies, by direct application of Riemann-Hurwitz,

$$2n - 2 = \sum_{i=1}^{m} (n - \tau_i) + \sum_{k=1}^{s} (n - d_k) + (l - 1).$$

Furthermore, the condition that we have precisely four non-apparent regular singular points in $\overline{\pi^*(\mathcal{L})}$ becomes

$$\sum_{i=1}^{m} (\tau_i - \mu_i) + \sum_{k=1}^{s} d_k = 4.$$

Since n is finite, the local monodromy at a point of the projective normalized pulled back local system $\overline{\pi^*(\mathcal{L})}$ is of infinite order if and only if the point lies over one at which $\mathcal{L}$ has infinite order local monodromy. Of course, $1 \le d_k \le n$, $m \ge 0$, $1 \le \tau_i \le n$, $0 \le \beta_i \le n$, $0 \le \gamma_i \le \lfloor n/2 \rfloor$, $0 \le \mu_i \le \lfloor n/a_i \rfloor$. We know that there can be at most four infinite order monodromy points in each $\pi^*(\mathcal{L})$, so in particular $0 \le s \le 4$.

We call the collection $(n; r_1, \ldots, r_{m+s}; l)$ the *topological type* of the family of covers.

Theorem 4.5. *Let $\mathcal{L}$ be a rank 2 regular local system on*

$$\mathbb{P}^1 \setminus \{p_1, \ldots, p_m, q_1, \ldots, q_s\}$$

with local monodromies of orders $a_1 \ldots, a_m, \infty, \ldots, \infty$. Assume that the associated rank two Fuchsian ordinary differential equation on $\mathbb{P}^1$ has *no singular points outside of* $B = \{p_1, \dots, p_m, q_1, \dots, q_s\}$. *Consider a cover of topological type* $(n; r_1, \dots, r_{m+s}; l)$ *such that*

$$\sum_{i=1}^{m}(\tau_i - \mu_i) + \sum_{k=1}^{s} d_k = 4$$

and

$$\sum_{i=1}^{m} \mu_i = (m+s-2)n + l - 3$$

defines a one parameter family of genus zero covers

$$\pi_t : \mathbb{P}^1_z \to \mathbb{P}^1_x$$

where $t \in \mathbb{P}^1_z \setminus \{0, 1, \infty\}$,

$$\pi_t(0), \pi_t(1), \pi_t(\infty) \in \{p_1, \dots, p_m, q_1, \dots, q_s\},$$

and $\pi_t(\lambda) \in \mathbb{P}^1 \setminus B$. *Then the projective normalized pulled back family of local systems* $\overline{\pi_t^*(\mathcal{L})}$ *forms an isomonodromic family with five regular singular points, exactly one apparent (at* $\lambda(t)$*), and hence determines an algebraic solution* $\lambda(t)$ *to the Painlevé* VI *equation* PVI $(\alpha, \beta, \gamma, \delta)$. *The particular values of* $\alpha, \beta, \gamma, \delta$ *are determined as usual by the traces of the local monodromies about* $0, 1, \infty, t$, *i.e., here they are specified by the local monodromies of* $\mathcal{L}$ *about* $p_1, \dots, p_m, q_1, \dots, q_s$ *and the ramification data of (any)* $\pi_t, t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Proof. The proof is now immediate. The condition that

$$\sum_{i=1}^{m} \mu_i = (m+s-2)n + l - 3$$

entails the genus zero condition for our covers in light of

$$\sum_{i=1}^{m}(\tau_i - \mu_i) + \sum_{k=1}^{s} d_k = 4.$$

By our hypotheses on $\mathcal{L}$, $\{p_1, \dots, p_m, q_1, \dots, q_s\}$, the topological type

$$(n; r_1, \dots, r_{m+s}; l),$$

and our coordinate choices, we know that each $\overline{\pi_t^*(\mathcal{L})}$ corresponds to a second order Fuchsian ordinary differential equation with one apparent regular singular point at λ, and precisely four non-apparent regular singular points at $\{0, 1, \infty, t\}$. The local monodromies about these points do not vary with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By Lemma 2.9, we thus know that λ as a function of t determines a solution to a Painlevé VI equation as described. q.e.d.

A direct application of this criterion to the natural hypergeometric local systems associated to triangles yields the following three corollaries:

Corollary 4.6. *The following is the complete list of topological types corresponding to algebraic Painlevé* VI *solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

$(2; [2], [1, 1], [1, 1]; 2)$	$(2, \square, \square)$
$(3; [2, 1], [3], [1, 1, 1]; 2)$	$(2, 3, \square)$
$(4; [2, 2], [3, 1], [2, 1, 1]; 2)$	$(2, 3, \square)$
$(4; [2, 2], [4], [1, 1, 1, 1]; 2)$	$(2, 4, \square)$
$(6; [2, 2, 2], [3, 3], [2, 2, 1, 1]; 2)$	$(2, 3, \square)$
$(6; [2, 2, 2], [3, 3], [3, 1, 1, 1]; 2)$	$(2, 3, \square)$
$(10; [2, \ldots, 2], [3, 3, 3, 1], [7, 1, 1, 1]; 2)$	$(2, 3, 7)$
$(12; [2, \ldots, 2], [3, 3, 3, 3], [7, 2, 1, 1, 1]; 2)$	$(2, 3, 7)$
$(12; [2, \ldots, 2], [3, 3, 3, 3], [8, 1, 1, 1, 1]; 2)$	$(2, 3, 8)$
$(18; [2, \ldots, 2], [3, \ldots, 3], [7, 7, 1, 1, 1, 1]; 2)$	$(2, 3, 7)$

Here $\square$ *represents any of the possible entries as listed in Theorem* 4.4.

Note that in the case of the arithmetic triangle group $\mathrm{PSL}(2, \mathbb{Z})$, with triangle $(2, 3, \infty)$, as expected we recover from this list the topological types of the Kodaira functional inveriants of our five families. In this corollary, the restriction to arithmetic Fuchsian triangle groups is for convenience only — we just wanted a finite set of triangle groups in $\mathrm{PSL}(2, \mathbb{R})$ to which to apply our criterion, and in this case they yielded a finite list of topological types. By contrast, for some triangles one can explicitly construct infinite lists of allowable topological types (unlike the previous result, the proofs of these corollaries do not produce an exhaustive list of types, merely an infinite one):

Corollary 4.7. *There are infinitely many topological types corresponding to algebraic Painlevé* VI *solutions arising by pullback from each triangle uniformized by* $\mathbb{C}$*, except for* $(3, 3, 3)$ *which has none.*

Proof. By direct construction we find the following topological types for: $(2,2,\infty)$,

$$(2k;[2,\ldots,2],[2,\ldots,2,1,1],[k,k];2);$$

$(2,3,6)$,

$$(6k;[2,\ldots,2],[3,\ldots,3],[6,\ldots,6,2,2,1,1];2);$$

and $(2,4,4)$,

$$(4k;[2,\ldots,2],[4,\ldots,4],[4,\ldots,4,1,1,1,1];2).$$

However, for $(3,3,3)$ the condition that $\mu_1+\mu_2+\mu_3 = n-1$ can never be satisfied and still leave enough points over B to correspond to $0,1,\infty,t$.

q.e.d.

Corollary 4.8. *There are infinitely many topological types corresponding to algebraic Painlevé* VI *solutions arising by pullback from each dihedral, tetrahedral, octahedral, and icosahedral spherical triangle.*

Proof. Again by direct construction we find for: $(2,2,n)$, $n \geq 2$,

$$(2(k+n);[2,\ldots,2],[2,\ldots,2,1,1],[2n-1,2k+1];2), \text{ when } (2k+1,n)=1;$$

$(2,3,3)$,

$$(6k;[2,\ldots,2],[3,\ldots,3],[3,\ldots,3,2,2,1,1];2);$$

$(2,3,4)$,

$$(12k;[2,\ldots,2],[3,\ldots,3],[4,\ldots,4,4k+1,1,1,1];2);$$

and $(2,3,5)$,

$$(30k;[2,\ldots,2],[3,\ldots,3],[5,\ldots,5,5k+2,1,1,1];2).$$

q.e.d.

We leave to the interested reader the task of applying the method of Couveignes to compute the associated families of rational functions and algebraic solutions for these topological types.

Note that we do not claim that all these solutions with distinct topological types correspond to distinct algebraic solutions. As we saw with the first and second elliptic surface families in Table 1, very different looking topological types may yield the same algebraic equations for their Belyi pair/Hurwitz curve. On the other hand, this may be a useful tool for those hoping to classify all algebraic Painlevé VI solutions, insofar as it offers the capacity to construct a large supply of potentially new solutions, with great control over the local monodromies/$(\alpha, \beta, \gamma, \delta)$ values.

Remark 12. One can also pose the corresponding "inverse problem": Given a topological type, for which local systems $\mathcal{L}$ (or at least for which $m + s = r$-tuples of monodromy orders) will the topological type yield an algebraic Painlevé VI solution by pulling back $\mathcal{L}$? In each particular instance this reduces to a (sometimes quite subtle) diophantine problem. For example, if we make the additional assumption that the $\gamma_i = \mu_i$, then we obtain a "generalized optic formula/Egyptian fraction" problem, whose solution in monodromy orders with gcd $= 1$ was described almost a century ago by Sós ([11, 48, 49]), coupled to some auxiliary congruence conditions on the monodromy orders. Even for $n = 5, r = 3$, and ignoring the extra congruence conditions, it can be notoriously difficult just to establish the existence of a solution to such diophantine problems (see the Conjecture of Erdös and Straus in Chapter 30 of [40]).

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Department of Mathematics
The Pennsylvania State University

Present address: Department of Mathematics
Columbia University, New York, NY 10027