Journal of Geometry

CrossMark

Algebraic CMC hypersurfaces of order 3 in Euclidean spaces

Oscar Perdomo and Vladimir G. Tkachev

Abstract. Understanding and finding of general algebraic constant mean curvature surfaces in the Euclidean spaces is a hard open problem. The basic examples are the standard spheres and the round cylinders, all defined by a polynomial of degree 2. In this paper, we prove that there are no algebraic hypersurfaces of degree 3 in $\mathbb{R}^n$, $n \geq 3$, with nonzero constant mean curvature.

1. Introduction

There are plenty of examples of algebraic *zero* mean curvature (i.e. minimal) hypersurfaces: the classical Enneper degree 9 and Henneberg degree 15 minimal surfaces [7] in $\mathbb{R}^3$, the minimal cones of degree 2 over the Clifford tori $\mathbb{S}^k \times \mathbb{S}^m$ in $\mathbb{R}^{n+m+1}$ and isoprametric cones of degrees $2, 3, 4, 6$ in $\mathbb{R}^n$, to name only few; see also [1,5,8,11–13] and references therein.

On the other hand, the problem of finding nonzero constant mean curvature (CMC) algebraic hypersurfaces in the Euclidean space is a hard open problem. The standard spheres $\mathbb{S}^n$ and the round cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ are the only algebraic hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ with constant ($\neq 0$) mean curvature known so far. All these hypersurfaces are given by a polynomial equation of degree 2.

Understanding of general algebraic CMC surfaces in the Euclidean spaces is a basic question that has very little progress. For the three dimensional space $\mathbb{R}^3$, the first author showed in [9] that there are no degree 3 algebraic CMC surfaces. Also, for surfaces in the Euclidean space $\mathbb{R}^3$, Do Carmo and Barbosa proved [2] that if $M = f^{-1}(0)$ with f a polynomial and, ∇f never vanishes

on M, then M cannot have CMC unless M is a plane, a cylinder or a sphere. On the other hand, very little is known on the classification of all immersed algebraic surfaces in $\mathbb{R}^3$ or the classification of algebraic CMC hypersurfaces in $\mathbb{R}^n$; see, however, a recent classification of semialgebraic CMC surfaces in $\mathbb{R}^3$ with isolated singularities in [10].

In this paper we will establish that the class of CMC hypersurfaces in $\mathbb{R}^n$ given by a polynomial equation of degree 3 is empty. In [9] it was proven that if $\phi : M \to \mathbb{R}^n$ is an immersion with constant mean curvature $H \neq 0$ and $\phi(M) = f^{-1}(0)$ where f is an irreducible polynomial, and, for at least one point $x_0 \in \phi(M)$, $\nabla f(x_0)$ does not vanishes, then

$$4(n-1)^2 H^2 |\nabla f|^6 - \left(2|\nabla f|^2 \Delta f - \nabla f^t \, \nabla |\nabla f|^2\right)^2 = pf \qquad (1)$$

with p a polynomial.

We will say that an irreducible polynomial f defines an *algebraic CMC hypersurface* if f satisfies condition (1) with some $H \neq 0$.

Notice that $f^{-1}(0)$ may have singular points. If x_0 is a regular point of f then (1) implies that $f^{-1}(0)$ is a CMC hypersurface in a neighbourhood of x_0. With this definition in mind we show that there are not algebraic CMC hypersurfaces in the Euclidean n-dimensional space of degree 3.

Theorem 1.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ is an irreducible polynomial of degree three, then, the zero level set of f cannot be an algebraic CMC hypersurface. That is, f cannot satisfy Eq. (1) with $H \neq 0$.*

Some further remarks are worth making at this point. Recall that

$$\Delta_p f := |\nabla f|^2 \Delta f + \frac{p-2}{2} \nabla f^t \, \nabla |\nabla f|^2 = 0 \qquad (2)$$

is called the p-Laplace equation. Then for $H = 0$, the CMC Eq. (1) is closely related to the 1-Laplace equation, which appears very naturally in the context of minimal cones, see section 6 in [6]. The assumption $H \neq 0$ is crucial, because otherwise, when $n > 3$, there exists irreducible algebraic minimal $(H = 0)$ hypersurfaces of any arbitrarily higher degree, see for instance [13].

On the other hand, it is interesting to compare Theorem 1.1 with a similar situation for polynomial solutions of the general p-Laplacian equation, $p \neq 2$. It follows from [3] and the recent results in [4,14] that there are no homogeneous polynomial solutions to (2) in $\mathbb{R}^n$ of degree $d = 2, 3, 4$ for any $n \geq 2$, of degree 5 in $\mathbb{R}^3$ and also of any degree in $\mathbb{R}^2$.

2. Proof of the main result

Before proving the Theorem 1.1, let us prove some lemmas.

Lemma 2.1. *If $g : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree 3 and $|\nabla g|$ vanishes anytime g vanishes, then $g = l^3$ where l is a linear function.*

Proof. Lemma 2.5 in [13] states that if g is a cubic irreducible polynomial then $g^{-1}(0)$ contains at least one regular point. Therefore, we conclude that our g cannot be irreducible and then, for some polynomial u of degree 2, we have that $g = uv$ where v is linear. Since ∇v never vanishes and $\nabla g = u\nabla v + v\nabla u$ we conclude that $u(x) = 0$ whenever $v(x) = 0$. From the real nullstellensatz theorem we conclude that $u = wv$ for some polynomial w. Thus $g = wv^2$. Applying the same argument that we did above, using the polynomial w instead of the polynomial v, we conclude that v^2 must be a multiple of w. Therefore the lemma follows. $\qquad\square$

Proof of Theorem 1.1. Without loss of generality we can assume that the origin is in an element in the hypersurface, in other words, we assume that $f(0, \ldots, 0) = 0$. Let us write

$$f = f_3 + f_2 + f_1 \quad \text{and} \quad p = p_9 + \cdots + p_1 + p_0$$

where $f_i = f_i(x)$ and $p_i = p_i(x)$ are homogeneous polynomials of degree i. Comparing the degree 12 homogeneous part in both sides of equation (1), we conclude that

$$\widetilde{H}^2|\nabla f_3|^6 = p_9 f_3, \qquad \text{where } \widetilde{H} = 2(n-1)H \neq 0. \tag{3}$$

Using Lemma 2.1, we conclude that $f_3 = l^3$ where l is linear. Since $f(0) = 0$, then $l(0) = 0$. By doing a rotation and a dilation of the coordinates, if necessary, we can assume that $l = x_1$. For notation sake, let us denote the coordinates in $\mathbb{R}^n$ as $x = x_1$ and $y = (y_1, \ldots, y_{n-1})^t = (x_2, \ldots, x_n)^t$. With this notation we have the for the homogenous parts in $f = f_3 + f_2 + f_1$:

$$\begin{aligned} f_3 &= x^3 \\ f_2 &= y^t A y + k_0 x^2 + y^t r\, x \\ f_1 &= k_1 x + y^t s, \end{aligned} \tag{4}$$

where A is a symmetric $(n-1) \times (n-1)$ matrix, $r = (r_1, \ldots, r_{n-1})^t$ and $s = (s_1, \ldots, s_{n-1})^t$.

At this moment we would like to point out that to finish the theorem it is enough to show that the matrix A is zero. The reason for this is that if A is the zero matrix and we relabel the axis so that the vectors r and s are in the plane spanned by the vectors $(1, 0, \ldots, 0)^t$ and $(0, 1, 0, \ldots, 0)^t$, them, the function f would be a function that depends only on the three variables x, y_1 and y_2 and, by [9] we already know that there are not algebraic CMC surfaces in $\mathbb{R}^3$.

Let us continue the proof by showing that the matrix A must be the zero matrix. A direct computation yields the following decomposition into homogeneous parts:

$$|\nabla f|^2 = h_0 + h_1 + h_2 + h_3 + h_4, \tag{5}$$

where h_k is a homogeneous polynomial of degree k given explicitly by

$$\begin{aligned} h_0 &= k_1^2 + |s|^2 \\ h_1 &= 4s^t A y + 4k_0 k_1 x + 2k_1(r^t y) + 2x\, r^t s \end{aligned}$$

$$h_2 = 4x\, r^t Ay + 4|Ay|^2 + 4k_0 x\, r^t y + (r^t y)^2 + x^2(4k_0^2 + 6k_1 + |r|^2)$$
$$h_3 = 12k_0 x^3 + 6x^2\, r^t y$$
$$h4 = 9x^4$$

Using this representation, we obtain

$$\Delta_1 f := 2|\nabla f|\Delta f - \nabla f^t \nabla |\nabla f|^2$$
$$= 4|\nabla f|^2 \operatorname{Tr} A + 16(k_0 + 3x)(s^t Ay + |Ay|^2) - 8r^t Ay\left(k_1 + r^t y - 3x^2\right)$$
$$- 8x\, s^t Ar - 4x^2\, r^t Ar - 16x\, r^t A^2 y - 16s^t A^2 y - 4s^t A.s - 16y^t A^3 y$$
$$- 4x(k_0 x + k_1 + r^t y)|r|^2 + 4(k_0 + 3x)|s|^2 - 4r^t s\left(k_1 + r^t y - 3x^2\right),$$

which implies that

$$\Delta_1(f) \equiv 4\operatorname{Tr} A|\nabla f|^2 \mod \mathrm{Pol}_3,$$

where we follow an agreement to write

$$A \equiv B \mod \mathrm{Pol}_k$$

if $A - B$ is a polynomial of degree k or less.

It also follows from (5) that

$$|\nabla f|^4 \equiv 81\, x^8 \mod \mathrm{Pol}_7$$

Thus, using the made observations, equation (1) becomes

$$(p_0 + p_1 + \cdots + p_9)(f_1 + f_2 + f_3) \equiv \widetilde{H}^2|\nabla f|^6 - 6^4(\operatorname{Tr} A)^2 x^8 \mod \mathrm{Pol}_7. \quad (6)$$

Our next goal is by using the decomposition for $|\nabla f|^2$ in terms of the h_i's and and the expression for $(\Delta_1 f)^2 - 4(n-1)^2 H^2|\nabla f|^6$ in terms of the $|\nabla f|^2$ up to order 8, deduce that the matrix A should be the zero matrix. To this end, we consider (6) as a polynomial identity with respect to a variable x over the ring $\mathbb{R}[y]$.

A key observation is that since $h_4 = Q_0 x^4$ and $h_3 = Q_2 x^2$, where Q_i is a homogeneous polynomial of degree i, one immediately obtains from (5) the following homogenous decomposition:

$$\widetilde{H}^2|\nabla f|^6 - 6^4(\operatorname{Tr} A)^2 x^8 = L_0 x^{12} + L_1 x^{10} + L_2 x^8$$
$$+ L_3 x^6 + L_4 x^4 \mod \mathrm{Pol}_7. \quad (7)$$

Here L_i are homogeneous polynomials of degree i. In particular,

$$L_0 = 3^6 \widetilde{H}^2 \neq 0. \quad (8)$$

Next, identifying the homogeneous parts of degrees k, $8 \leq k \leq 12$, in both sides of (6) one obtains respectively

$$p_9 f_3 = L_0 x^{12}$$
$$p_8 f_3 + p_9 f_2 = L_1 x^{10}$$
$$p_7 f_3 + p_8 f_2 + p_9 f_1 = L_2 x^8$$

$$p_6 f_3 + p_7 f_2 + p_8 f_1 = L_3 x^6$$
$$p_5 f_3 + p_6 f_2 + p_7 f_1 = L_4 x^4. \tag{9}$$

The first two equations yield

$$p_9 = L_0 x^9$$

and

$$p_8 = S_2 x^6,$$

where $S_2 = L_1 x^2 - L_0 f_2$. Arguing similarly, one easily finds that

$$p_7 = S_4 x^3,$$
$$p_6 = (L_3 - S_2 f_1) x^3 - S_4 f_2,$$

where $S_4 = L_2 x^2 - L_0 f_1 S x^3 - S_2 f_2$. Thus, evaluating the last identity of (9) for $x = 0$ and taking into account that $f_3(0) = p_7(0) = 0$ we obtain

$$p_6(0) f_2(0) = 0.$$

Since

$$p_6(0) = -S_4(0) f_2(0) = S_2(0) f_2^2(0) = -L_0 f_2^3(0),$$

it follows by (8) that $f_2(0) = 0$, therefore using (4) we obtain

$$f_2(0) = y^t A y = 0 \text{ for any } y \in \mathbb{R}^{n-1}.$$

Since A is symmetric, we have $A = 0$. As explained before, after sowing that A vanishes we have that f essentially depends on three variable and therefore it cannot define an algebraic hypersurface with constant mean curvature. The theorem is proved. $\square$

Acknowledgements

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Hoppe, J., Linardopoulos, G., Turgut, O.T.: *New minimal hypersurfaces in* $\mathbb{R}^{(k+1)(2k+1)}$ *and* $\mathbb{S}^{2k^2+3k}$. Math. Nachr. **290**(17–18), 2874–2878 (2017)

[2] Lucas, J., Barbosa, M., do Carmo, M.P.: *On regular algebraic surfaces of* $\mathbb{R}^3$ *with constant mean curvature.* J. Differ. Geom. **102**(2), 173–178 (2016)

[3] Lewis, J.L.: *Smoothness of certain degenerate elliptic equations.* Proc. Am. Math. Soc. **80**(2), 259–265 (1980)

[4] Lewis, J.L., Vogel, A.: *On p-Laplace polynomial solutions.* J. Anal. **24**(1), 143–166 (2016)

[5] López, R.: *Surfaces with constant mean curvature in Euclidean space.* Int. Electron. J. Geom. **3**(2), 67–101 (2010)

[6] Nadirashvili, N., Tkachev, V.G., Vlăduţ, S.: Nonlinear Elliptic Equations and Nonassociative Algebras, vol. 200. Mathematical Surveys and MonographsAMS, Providence (2014)

[7] Nitsche, J.C.C.: Lectures on Minimal Surfaces, vol. 1. Cambridge University Press, Cambridge (1989). Introduction, fundamentals, geometry and basic boundary value problems. Translated from the German by Jerry M. Feinberg, With a German foreword

[8] Odehnal, B.: *On algebraic minimal surfaces.* KoG **20**(20), 61–78 (2017)

[9] Perdomo, O.M.: *Algebraic constant mean curvature surfaces in Euclidean space.* Houst. J. Math. **39**(1), 127–136 (2013)

[10] Sampaio, J.E.: *Semialgebraic cmc surfaces in* $\mathbb{R}^3$ *with singularities.* https://bird.bcamath.org/handle/20.500.11824/844 (2018) (**preprint**)

[11] Small, A.: *Algebraic minimal surfaces in* $\mathbf{R}^4$. Math. Scand. **94**(1), 109–124 (2004)

[12] Small, A.: *On algebraic minimal surfaces in* $\mathbb{R}^3$ *deriving from charge 2 monopole spectral curves.* Int. J. Math. **16**(2), 173–180 (2005)

[13] Tkachev, V.: *Minimal cubic cones via Clifford algebras.* Complex Anal. Oper. Theory **4**(3), 685–700 (2010)

[14] Tkachev, V.G.: *On the non-vanishing property for real analytic solutions of the p-Laplace equation.* Proc. Am. Math. Soc. **144**(6), 2375–2382 (2016)

Oscar Perdomo
Department of Mathematics
Central Connecticut State University
New Britain
CT 06050
USA
e-mail: `perdomoosm@ccsu.edu`

Vladimir G. Tkachev
Department of Mathematics
Linköping University
58183 Linköping
Sweden
e-mail: `vladimir.tkatjev@liu.se`