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ALGEBRAIC CONNECTIVITY OF GRAPHS*)

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1. INTRODUCTION

Let $G = (V, E)$ be a non-directed finite graph without loops and multiple edges. Having chosen a fixed ordering $w_1, w_2, \ldots, w_n$ of the set V, we can form a square n-rowed matrix $A(G)$ whose off-diagonal entries are $a_{ik} = a_{ki} = -1$ if $(w_i, w_k) \in E$ and $a_{ik} = 0$ otherwise and whose diagonal entries a_{ii} are equal to the valencies of the vertices w_i. This matrix $A(G)$, which is frequently used to enumerate the spanning trees of the graph G, is symmetric, singular (all the row sums are zero) and positive semidefinite ($A(G) = UU^T$ where U is the $(0, 1, -1)$ vertex-edge adjacency matrix of arbitrarily directed graph G). Let $n \geqq 2$ and $0 = \lambda_1 \leqq \lambda_2 = a(G) \leqq \lambda_3 \leqq \ldots \ldots \leqq \lambda_n$ be the eigenvalues of the matrix $A(G)$. From the Perron-Frobenius theorem applied to the matrix $(n - 1) I - A(G)$ it follows that $a(G)$ is zero if and only if the graph G is not connected. We shall call the second smallest eigenvalue $a(G)$ of the matrix $A(G)$ algebraic connectivity of the graph G. It is the purpose of this paper to find its relation to the usual vertex and edge connectivities.

We recall that many authors, e.g. A. J. HOFFMAN, M. DOOB, D. K. RAY-CHAUDHURI, J. J. SEIDEL have characterized graphs by means of the spectra of the $(0, 1)$ and $(0, 1, -1)$ adjacency matrices.

2. NOTATION AND CONVENTIONS

The notation introduced above is used throughout the present paper. All matrices and vectors are considered real. The transpose of a matrix M is denoted by M^T, the identity matrix by I, the vector $(1, 1, \ldots, 1)^T$ by e, the universal matrix ee^T by J, the cardinality of a set S by $|S|$.

For our purpose it is convenient to denote by W the set of all column vectors x such that $x^T x = 1$, $x^T e = 0$. Any square matrix M with all zero row sums has an

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eigenvalue 0 and a corresponding eigenvector is e. If M is positive semidefinite then the second smallest eigenvalue is equal to $\min_{x \in W} x^T M x$ by the well known Courant's theorem. We use that principle tacitly.

Further, let us mention two common concepts. Edge connectivity of the graph G, i.e. the minimal number of edges whose removal would result in losing connectivity of the graph G, is denoted by $e(G)$. Vertex connectivity which is defined analogously (vertices together with adjacent edges are removed) is denoted by $v(G)$. It is convenient to put $v(G) = n - 1$ for the complete graph G.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$. By $G_1 \times G_2$ we denote the graph $(V_1 \times V_2, E)$ such that $((u_1, u_2), (v_1, v_2)) \in E$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $(u_1, v_1) \in E_1$ and $u_2 = v_2$. Let $R = (r_{ij})$, S be matrices. Then by $R \times S$ the partitioned matrix $(r_{ij}S)$ is denoted.

3. PROPERTIES OF $a(G)$

3.1. *If G_1, G_2 are edge-disjoint graphs with the same set of vertices then $a(G_1) + a(G_2) \leqq a(G_1 \cup G_2)$.*

Proof. We have $A(G_1 \cup G_2) = A(G_1) + A(G_2)$. Thus $a(G_1 \cup G_2) = \min_{x \in W} (x^T A(G_1) x + x^T A(G_2) x) \geqq \min_{x \in W} x^T A(G_1) x + \min_{x \in W} x^T A(G_2) x = a(G_1) + a(G_2)$.

3.2. Corollary. *The function $a(G)$ is non-decreasing for graphs with the same set of vertices, i.e. $a(G_1) \leqq a(G_2)$ if $G_1 \subseteq G_2$ (and G_1, G_2 have the same set of vertices).*

3.3. *Let G be a graph, let G_1 arise from G by removing k vertices from G and all adjacent edges. Then*

$$a(G_1) \geqq a(G) - k\,. \tag{1}$$

Proof. Let G have n vertices and let G_1 arise from G by removing one vertex, say u_n. Define a new graph G' by completing in G all missing edges from u_n. Then

$$A(G') = \begin{pmatrix} A(G_1) + I, & -e^T \\ -e, & n - 1 \end{pmatrix}.$$

Let v be an eigenvector of $A(G_1)$ corresponding to the eigenvalue $a(G_1)$. Since

$$A(G') \begin{pmatrix} v \\ 0 \end{pmatrix} = [a(G_1) + 1] \begin{pmatrix} v \\ 0 \end{pmatrix},$$

$a(G_1) + 1$ is an eigenvalue of $A(G')$ different from zero, i.e.

$$a(G') \leqq a(G_1) + 1\,.$$

By 3.2, $a(G) \leqq a(G')$ which implies (1) for $k = 1$. The general case follows by induction.

3.4. *We have* $a(G_1 \times G_2) = \min(a(G_1), a(G_2))$.

Proof. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$. Order the set $V_1 \times V_2$ lexicographically. Then $A(G_1 \times G_2) = A(G_1) \times I_2 + I_1 \times A(G_2)$, I_j being the $|V_j|$-rowed identity matrix. By a well known result from the matrix theory [1] all eigenvalues of $A(G_1 \times \times G_2)$ are of the form $\mu + \nu$ where μ, ν resp. are eigenvalues of $A(G_1)$, $A(G_2)$ respectively. Hence the second smallest eigenvalue of $A(G_1 \times G_2)$ is either $a(G_1) + 0$ or $0 + a(G_2)$.

3.5. *Let* $G = (V, E)$, v_j *be the valency of the j-th vertex. Then*

$$a(G) \leqq [n/(n-1)] \min_i v_i \leqq 2|E|/(n-1)\,.$$

Proof. Since $n \min_i v_i \leqq \sum_i v_i = 2|E|$, the second inequality is true. The first is an immediate consequence of the following lemma.

Lemma. *Let* $M = (m_{ik})$ *be a symmetric positive semidefinite n by n matrix such that* $Me = 0$. *Then the second smallest eigenvalue* λ_2 *of M satisfies*

$$\lambda_2 \leqq [n/(n-1)] \min_i m_{ii}\,. \tag{2}$$

Proof. Observe that

$$\lambda_2 = \min_{x \in W} x^T M x\,. \tag{3}$$

Let us show that the matrix

$$\tilde{M} = M - \lambda_2(I - n^{-1}J)$$

is also positive semidefinite.

Let y be any vector in E_n. Then y can be written in the form $y = c_1 e + c_2 x$ where $x \in W$. Since $\tilde{M}e = 0$, it follows that

$$y^T \tilde{M} y = c_2^2 x^T \tilde{M} x = c_2^2(x^T M x - \lambda_2) \geqq 0$$

by (3). Thus the minimum diagonal entry of $\tilde{M}$ is nonnegative:

$$\min_i m_{ii} - \lambda_2(1 - n^{-1}) \geqq 0$$

and (2) is proved.

Remark. A matrix $M = (m_{ik})$ satisfying conditions of the preceding lemma has also the property that the numbers $\sqrt{m_{ii}}$ fulfil the polygonal inequality, i.e.

$$2 \max_i m_{ii}^{1/2} \leqq \sum_i m_{ii}^{1/2}\,.$$

This follows easily from the well known fact that M can be considered as the Gram matrix of a system of n vectors $u_1, \ldots, u_n$ with sum zero:

$$m_{ik} = (u_i, u_k) = u_i^T u_k\,, \quad \sum_i u_i = 0\,.$$

Then we have for the lengths

$$|u_i| \leqq \sum_{k \neq i} |u_k| \quad \text{for} \quad i = 1, \ldots, n\,,$$

or

$$2 \max_i |u_i| \leqq \sum_i |u_i|$$

and $|u_i| = (u_i, u_i)^{1/2} = m_{ii}^{1/2}$ yields the result.

If we apply (4) to the matrix $\tilde{M} = M - \lambda_2(I - (1/n)\,J)$, we obtain

$$2 \max_i (m_{ii} - [(n-1)/n]\,\lambda_2)^{1/2} \leqq \sum_i (m_{ii} - [(n-1)/n]\,\lambda_2)^{1/2}\,.$$

In terms of the graph G, we obtain

$$2 \max_i [nv_i - (n-1)\,a(G)]^{1/2} \leqq \sum_i [nv_i - (n-1)\,a(G)]^{1/2}\,.$$

For sake of completeness, we formulate the assertion

3.6. *For the complete graph K_n with n vertices $a(K_n) = n$.*

In the following theorem, we denote by $b(G)$, for a graph G with n vertices, the number

$$b(G) = n - a(\bar{G})$$

where $\bar{G}$ is the complementary graph to G.

3.7. *The function $b(G)$ has following properties:*

1° *$b(G)$ is the maximum eigenvalue of $A(G)$ or, equivalently*

$$b(G) = \max_{x \in W} x^T A(G)\, x\,; \tag{5}$$

we have thus

$$a(G) \leqq b(G)\,, \tag{6}$$

with equality if and only if G is a complete graph or a void graph (i.e. without edges);

2°
$$b(G) = \max_{i=1,\ldots,r} b(G_i)$$

if $G_1, \ldots, G_r$ are all components of G;

3° $b(G_1) \leqq b(G_2)$ if $G_1 \subseteq G_2$, i.e. $V_1 \subseteq V_2$ *and* $E_1 \subseteq E_2$ *where* $G_i = (V_i, E_i)$, $i = 1, 2$;

4° $$b(G_1 \cup G_2) \leqq b(G_1) + b(G_2) - a(G_1 \cap G_2)$$

where both graphs G_1 *and* G_2 *are considered to have the same set of vertices*;

5° $$[n/(n-1)] \max_i v_i(G) \leqq b(G) \leqq 2 \max_i v_i(G)$$

where $v_i(G)$ *means valency of the i-th vertex in* G.

Proof. If $\bar{G}$ is the complementary graph to G (with n vertices so that $\bar{G}$ has also n vertices) then

$$A(G) + A(\bar{G}) = nI - J\,.$$

Since

$$a(\bar{G}) = \min_{x \in W} x^T A(\bar{G})\, x$$

and

$$x^T(nI - J)\, x = n \quad \text{for} \quad x \in W,$$

we have

$$\max_{x \in W} x^T A(G)\, x = n - \min_{x \in W} x^T A(\bar{G})\, x = n - a(\bar{G}) = b(G)\,.$$

This implies (6). Let equality be attained in (6). Then $x^T A(G)\, x$ is constant on W. Taking first $x = [n(n-1)]^{-1/2}(ne_i - e)$ where e_i has all coordinates zero except the i-th equal to one, we obtain that all the diagonal entries in $A(G)$ are equal. Choosing then $x = 2^{-1/2}e_i - 2^{-1/2}e_k$ $(i \neq k)$, we obtain that also all off-diagonal entries of $A(G)$ are equal, thus all equal either to -1, or to zero. This proves 1°.

2° follows easily from 1° since $A(G)$ is the direct sum of $A(G_i)$ if G is not connected and G_i are components of G.

To prove 3° and 4°, we shall use (6). This implies 3° immediately while 4° follows from

$$A(G_1 \cup G_2) = A(G_1) + A(G_2) - A(G_1 \cap G_2)\,.$$

The right inequality in 5° follows from 1° and the well known fact that the maximum modulus of the eigenvalues is less than or equal to any norm. The maximum norm (i.e. with respect to the vector norm $\|x\| = \max_i |x_i|$) of $A(G)$ (known to be $\max_i \sum_k |a_{ik}|$) is $2 \max_i v_i(G)$ which yields this result. To prove the left inequality in 5°, let us apply 3.5 to the complementary graph $\bar{G}$. We obtain

$$a(\bar{G}) \leqq [n/(n-1)] \min_i v_i(\bar{G})\,,$$

which can be written as

$$n - b(G) \leqq [n/(n-1)]\,[n - 1 - \max_i v_i(G)]\,.$$

This implies the inequality and the proof is complete.

3.8. *We have*

$$a(G) \geqq 2 \min_i v_i(G) - n + 2 .$$

Proof. Follows immediately from the right inequality in 5° of 3.7 used for the complementary graph $\bar{G}$.

3.9. *Let G with n vertices contain an independent set of m vertices (i.e. no two of them joined by an edge of G). Then*

$$a(G) \leqq n - m .$$

Proof. If G contains an independent set of m vertices then $\bar{G}$ contains a complete subgraph K_m. Since $b(K_m) = m$, we have by 3° of 3.7 that

$$b(\bar{G}) \geqq m$$

so that $a(G) = n - b(\bar{G}) \leqq n - m$.

3.10. *If G is a graph with n vertices which is not complete then $a(G) \leqq n - 2$.*

Proof follows immediately from 3.9.

3.11. *If $K_{p,q}$ denotes the complete bipartite graph the parts of which contain p and q vertices then $a(K_{p,q}) = \min(p, q)$.*

Proof. Follows from 2° of 3.7 applied to the complement of $K_{p,q}$.

3.12. *Let $G = (V, E)$, let $V = V_1 \cup V_2$ be a decomposition of V, let G_i $(i = 1, 2)$ be the subgraph of G generated on V_i. Then*

$$a(G) \leqq \min(a(G_1) + |V_2|,\ a(G_2) + |V_1|) .$$

Proof. This is just a symmetric formulation of 3.3.

4. RELATIONS BETWEEN $a(G)$, $e(G)$ AND $v(G)$

4.1. *Let G be a non-complete graph. Then $a(G) \leqq v(G)$.*

Proof. Let $G = (V, E)$ and let V_1 be a vertex cut such that $V_2 = V - V_1 \neq \emptyset$. Since the subgraph G_2 generated by G on V_2 is not connected, we have by 3.12

$$a(G) \leqq |V_1| .$$

This implies the assertion.

4.2. *We have* $v(G) \leqq e(G)$.

Proof. This well known inequality is an easy consequence of the following theorem [3]:

Let w, w' be a pair of vertices of G. Then there exist $v(G)$ paths between w, w' in G, no two of them having any vertices in common (except w, w').

4.3. *Let* $C_1 = 2[\cos(\pi/n) - \cos(2\pi/n)]$, $C_2 = 2\cos(\pi/n)(1 - \cos(\pi/n))$ *and let* $q(G)$ *be the maximum vertex valency of the graph* G. *Then*

$$a(G) \geqq 2\, e(G)\,(1 - \cos(\pi/n))\,,$$

$$a(G) \geqq C_1\, e(G) - C_2\, q(G)\,,$$

the second bound being better if and only if $2\, e(G) > q(G)$.

Proof. Consider the eigenvalues $\sigma_1 \geqq \sigma_2 \geqq \ldots \geqq \sigma_n$ of the matrix $S = (s_{ij}) = I - q^{-1}(G)\, A(G)$. This matrix is symmetric and stochastic (i.e. its row sums are 1 and the entries are nonnegative so that $\sigma_1 = 1$). Denote by μ the "measure of irreducibility" of S, the number $\min\limits_{\emptyset \neq M \subsetneqq N} \sum\limits_{i \in M, k \notin M} s_{ik}$, where $N = \{1, 2, \ldots, n\}$. Then according to [2]

$$1 - \sigma_2 \geqq 2(1 - \cos(\pi/n))\, \mu\,,$$

$$1 - \sigma_2 \geqq 1 - 2(1 - \mu)\cos(\pi/n) - (2\mu - 1)\cos(2\pi/n)\,.$$

We have $\sigma_2 = 1 - a(G)/q(G)$, $\mu = e(G)/q(G)$, thus

$$a(G)/q(G) \geqq 2(1 - \cos(\pi/n))\, e(G)/q(G)\,,$$

$$a(G)/q(G) \geqq 1 - 2(1 - e(G)/q(G))\cos(\pi/n) - (2e(G)/q(G) - 1)\cos(2\pi/n)\,,$$

which implies the required inequalities.

The last assertion is easy to verify.

4.4. We have the following values for some types of graphs.

graph	$a(G)$	$e(G)$	$v(G)$	bound of 4.3
path	$2(1 - \cos(\pi/n))$	1	1	$2(1 - \cos(\pi/n))$
circuit	$2(1 - \cos(2\pi/n))$	2	2	$4\sin^2(\pi/n)$
star	1	1	1	$2(1 - \cos(\pi/n))$
complete g.	n	$n - 1$	$n - 1$	$2(n - 1)\sin^2(\pi/n)$
cube (m-dimensional)	2	m	m	$2m\sin^2(\pi/2m)$

Remark. After having finished this paper the author was informed that W. N. ANDERSON, Jr. and T. D. MORLEY had obtained some of these results in the paper Eigenvalues of the Laplacian of a graph, University of Maryland Technical Report TR-71-45, October 6, 1971.

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