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ALGEBRAIC CYCLES ON ABELIAN VARIETIES WITH MANY REAL ENDOMORPHISMS

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1. **Main results.** Let A be an abelian variety of dimension g defined over $\boldsymbol{C}$. We denote by $\operatorname{End} A$ the endomorphism ring of A and put $\operatorname{End}^0 A = \operatorname{End} A \otimes \boldsymbol{Q}$. In the present paper we investigate algebraic cycles on an abelian variety with many real endomorphisms. More precisely, we consider an abelian variety such that $\operatorname{End}^0 A$ contains a product F of totally real fields with $[F: \boldsymbol{Q}] = \dim A$. Our main result is the following:

THEOREM (1.1). *Let A be as above. Suppose that no simple component of A (up to isogeny) is of* CM-*type of dimension greater than one. Then $\mathscr{B}^*(A)$ is generated by $\mathscr{B}^1(A)$. In particular, the Hodge conjecture holds for such A.*

Here we denote by $\mathscr{B}^*(A) = \bigoplus_{d=0}^{g} \mathscr{B}^d(A)$ the Hodge ring, where $\mathscr{B}^d(A) = H^{2d}(A, \boldsymbol{Q}) \cap H^{d,d}(A)$. As an application of this result, we have the following theorem on algebraic cycles on the jacobian variety $J_1(N)$ of the modular curve $X_1(N)$ (see §4 for the definition):

THEOREM (1.2). *$\mathscr{B}^*(J_1(N))$ is generated by $\mathscr{B}^1(J_1(N))$. In particular, the Hodge conjecture holds for $J_1(N)$.*

REMARK. After this paper was prepared, Professor Shioda informed the author that V. P. Murty obtained the above (1.2) independently ([7]).

2. **Preliminaries.** Here we recall some properties of the Hodge group of an abelian variety.

PROPOSITION (2.1) (Mumford [4]). *Let A be an abelian variety. Let $\operatorname{Hg}(A)$ denote the Hodge group of A. Then*

$$\operatorname{End}^0 A \cong \operatorname{End}_{\operatorname{Hg}(A)} H^1(A, \boldsymbol{Q}) , \qquad \mathscr{B}^d(A) = [H^{2d}(A, \boldsymbol{Q})]^{\operatorname{Hg}(A)} .$$

Here we use the following notations: For a group G and a G-module V we denote by $\operatorname{End}_G V$ the set of G-endomorphisms of V and we denote by $[V]^G$ the set of G-invariant elements in V.

PROPOSITION (2.2) (Tankeev [10, Lemma (1.4)]). *If the center of the*

$\boldsymbol{Q}$-algebra $\mathrm{End}^0 A$ *is a product of totally real fields, then the Hodge group* $\mathrm{Hg}(A)$ *is semi-simple.*

For the Hodge group and the Hodge ring of a product of elliptic curves the following theorems are known to hold:

THEOREM (2.3). *Let E be an elliptic curve. Then*

$$\mathscr{Lie}(\mathrm{Hg}(E)_C) \cong \begin{cases} \mathfrak{sl}_2 & \text{if } E \text{ has no complex multiplication}, \\ \boldsymbol{C} & \text{if } E \text{ has complex multiplications}. \end{cases}$$

THEOREM (2.4) (Imai [3]). *Let $E_1, \cdots, E_k$ be elliptic curves which are mutually non-isogenous. Then*

$$\mathrm{Hg}(E_1^{n_1} \times \cdots \times E_k^{n_k}) \cong \mathrm{Hg}(E_1) \times \cdots \times \mathrm{Hg}(E_k).$$

THEOREM (2.5) (Tate [11], Murasaki [6]). *For a power E^n of an elliptic curve E, the Hodge ring $\mathscr{B}^*(E^n)$ is generated by $\mathscr{B}^1(E^n)$.*

The following general proposition is frequently used when we compute the Hodge group of some product of abelian varieties:

PROPOSITION (2.6) (Ribet [8]). *Suppose that $\mathfrak{s}_1, \cdots, \mathfrak{s}_d$ are simple finite-dimensional Lie algebras and that $\mathfrak{u}$ is a subalgebra of the product $\mathfrak{s}_1 \times \cdots \times \mathfrak{s}_d$. Assume that whenever $1 \leqq i < j \leqq d$ the projection of $\mathfrak{u}$ to $\mathfrak{s}_i \times \mathfrak{s}_j$ is surjective. Assume also that the i-th projection maps $\mathfrak{u}$ onto $\mathfrak{s}_i$ for each i. Then $\mathfrak{u} = \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_d$.*

Next we note that abelian varieties satisfying the condition of Theorem (1.1) can be classified as follows:

THEOREM (2.7) (Giraud [2]). *Let A be an abelian variety which satisfies the condition of* (1.1), *and consider a decomposition of A into isotypic components up to isogeny. Then each isotypic component is one of the following*:

(1) *A_1^n, where A_1 is a simple abelian variety of type* I *under Albert's classification* (*Mumford* [5, §21, *Th.* 2]) *such that* $\mathrm{End}^0 A_1 \cong$ *a totally real field of degree g/n.*

(2) *A_2^n, where A_2 is a simple abelian variety of type* II *such that* $\mathrm{End}^0 A_2 \cong$ *a totally indefinite quaternion algebra over a totally real field of degree $g/2n$.*

(3) *E^g, where E is an elliptic curve of* CM-*type.*

3. Proof of Theorem (1.1). First we determine the Lie algebra of $\mathrm{Hg}(A)_C$ for $A = A_1^n$ (resp. $A = A_2^n$) appearing in the case (1) (resp. (2)) of (2.7). Put $\dim A_1 = g_1 = g/n$. Put $\rho(A) =$ rank of the Néron-Severi group of A. If A is of type (1),

$$\mathrm{End}^0 A \underset{Q}{\otimes} \boldsymbol{R} \cong M_n(\mathrm{End}^0 A_1) \underset{Q}{\otimes} \boldsymbol{R} \cong M_n(\boldsymbol{R}) \times \cdots \times M_n(\boldsymbol{R}) \quad (g_1 \text{ times}) .$$

Moreover we have

$$\rho(A) = n\rho(A_1) + (n(n-1)/2)\,\mathrm{rank}\,\mathrm{End}^0 A_1 = n(n+1)g_1/2 .$$

If A is type (2), then

$$\mathrm{End}^0 A \underset{Q}{\otimes} \boldsymbol{R} \cong M_n(M_2(\boldsymbol{R}) \times \cdots \times M_2(\boldsymbol{R})) \qquad (g_1/2 \text{ times})$$

$$\cong M_{2n}(\boldsymbol{R}) \times \cdots \times M_{2n}(\boldsymbol{R}) \qquad (g_1/2 \text{ times}) .$$

Moreover we have $\rho(A) = n(2n+1)g_1/2$. We denote by $\mathfrak{h}$ the Lie algebra of $\mathrm{Hg}(A)_C$, which is semi-simple by Proposition (2.2). Then in both cases by the isomorphism $\mathrm{End}^0 A \otimes_Q \boldsymbol{C} \cong \mathrm{End}_{\mathrm{Hg}(A)_C} H^1(A, \boldsymbol{C})$ (cf. (2.1)) and Schur's lemma we have a decomposition of the $\mathfrak{h}$-module $H^1(A, \boldsymbol{C})$:

$$H^1(A, \boldsymbol{C}) \cong (V_1 \oplus \cdots \oplus V_1) \oplus \cdots \oplus (V_k \oplus \cdots \oplus V_k) ,$$

where $k = g_1$ (resp. $g_1/2$) if A is of type (1) (resp. (2)) and V_i $(1 \leqq i \leqq k)$ are mutually non-isomorphic $\mathfrak{h}$-modules each of them occurring s times. Note that $s = n$ (resp. $2n$) if A is of type (1) (resp. (2)). We claim that $\dim_C V_i = 2$ for all i. Suppose on the contrary that there exists an i such that $\dim_C V_i \neq 2$. Then since $\sum \dim_C V_i = 2sk$ and one-dimensional $\mathfrak{h}$-modules are isomorphic, we see that there exists a unique j such that $\dim_C V_j = 1$. We may assume (renumbering V_i's, if necessary) that $\dim_C V_1 = 3$, $\dim_C V_2 = 1$, $\dim_C V_i = 2$ for $i \geqq 3$. Put $W_1 = V_1$ and $W_2 = \bigoplus$ {the other components}. Then

$$[\textstyle\bigwedge^2 V]^{\mathfrak{h}} \cong [\bigwedge^2 W_1]^{\mathfrak{h}} \oplus [\bigwedge^2 W_2]^{\mathfrak{h}} \oplus [W_1 \otimes W_2]^{\mathfrak{h}} .$$

If $s = 1$, then we get $[W_1 \otimes W_2]^{\mathfrak{h}} \cong \mathrm{Hom}_{\mathfrak{h}}(W_1^*, W_2) = 0$ since W_2 has no irreducible component of dimension three. Therefore

$$[\textstyle\bigwedge^2 V]^{\mathfrak{h}} \cong [\bigwedge^2 W_1]^{\mathfrak{h}} \oplus [\bigwedge^2 W_2]^{\mathfrak{h}} .$$

We denote by ω the element in $[\bigwedge^2 V]^{\mathfrak{h}}$ corresponding to the skew symmetric non-degenerate bilinear form on the $\mathfrak{h}$-module V. According to the above decomposition, ω can be written $\omega = \omega_1 + \omega_2$, where $\omega_1 \in [\bigwedge^2 W_1]^{\mathfrak{h}}$, $\omega_2 \in [\bigwedge^2 W_2]^{\mathfrak{h}}$. On the one hand we have $\bigwedge^g \omega \neq 0$ by the non-degeneracy of the bilinear form. On the other hand, $\bigwedge^g \omega = \sum_{i+j=g} \bigwedge^i \omega_1 \otimes \bigwedge^j \omega_2 = 0$, since $\bigwedge^i \omega_1 = 0$ for $i \geqq 2$ and $\bigwedge^j \omega_2 = 0$ for $j \geqq g-1$, a contradiction. Therefore $s \neq 1$. Then by a similar argument we get $V_1 \cong V_1^*$. This is possible only if $p_1(\mathfrak{h}) \cong \mathfrak{sl}_2$ and $V_1 \cong S^2(\boldsymbol{C}^2)$ (=the space of symmetric tensors of degree two over $\boldsymbol{C}^2$) by the representation theory of semi-simple Lie algebras (Bourbaki [1, Chaps. VII and VIII]). Here we denote by p_i the i-th projection: $\mathrm{End}\, V \to \mathrm{End}\, V_i$ $(1 \leqq i \leqq k)$.

As for V_i $(i \geqq 3)$, we see that $p_i(\mathfrak{h}) \cong \mathfrak{sl}_2$ and $V_i \cong \boldsymbol{C}^2$ (the natural representation). Then we are able to compute $\dim_C [\bigwedge^2 V]^{\mathfrak{h}}$ as follows:

$$\dim_C [\bigwedge^2 V]^{\mathfrak{h}} = \begin{cases} n(n+1)(g_1-2)/2 + n^2 & \text{in case (1)}, \\ n(2n+1)g_1/2 - 2n & \text{in case (2)}. \end{cases}$$

Since $\dim_C [\bigwedge^2 V]^{\mathfrak{h}} = \rho(A)$, this contradicts the above computation of $\rho(A)$. Thus we see that $\dim_C V_i = 2$ for all $i \geqq 1$, and $p_i(\mathfrak{h}) \cong \mathfrak{sl}_2$.

Now we claim that

$$\mathfrak{h} \cong \mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2 \qquad (k \text{ times}),$$

where the i-th component acts on $V_i \oplus \cdots \oplus V_i$ diagonally. To show this we use the following:

LEMMA (3.1). *Let $\mathfrak{h}$ be a semi-simple subalgebra of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ such that $p_i(\mathfrak{h}) = \mathfrak{sl}_2$ $(i = 1, 2)$, where p_i denotes the i-th projection. Then $\mathfrak{h}$ must be equal to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ or the graph of an automorphism of $\mathfrak{sl}_2$.*

PROOF OF (3.1). This is an easy consequence of "Goursat's lemma" (cf. [7, Lemma (5.2.1)]).

We apply this to $p_{ij}(\mathfrak{h}) \subset \mathfrak{sl}_2 \times \mathfrak{sl}_2$, where p_{ij} denotes the projection: $\operatorname{End} V \to \operatorname{End} V_i \times \operatorname{End} V_j$ $(1 \leqq i < j \leqq k)$. By the assumption, the $\mathfrak{h}$-modules V_i and V_j are not isomorphic, hence it follows from (3.1) that $p_{ij}(\mathfrak{h}) = \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Therefore the claim above follows from (2.6).

Now suppose that an abelian variety A satisfies the condition of Theorem (1.1). Then by (2.7),

$$A \underset{\text{isog.}}{\sim} A_1 \times A_2 \times \cdots \times A_s \times C_1^{m_1} \times \cdots \times C_t^{m_t},$$

where A_i $(1 \leqq i \leqq s)$ are of type (1) or (2) in (2.7) and C_j $(1 \leqq j \leqq t)$ are elliptic curves of CM-type with $C_j \not\sim C_k$ for $j \neq k$. Here we use the following lemmas which are proved easily:

LEMMA (3.2). *Let A be an abelian variety whose Hodge group is semi-simple and let B be an abelian variety of* CM-*type. Then* $\operatorname{Hg}(A \times B) \cong \operatorname{Hg}(A) \times \operatorname{Hg}(B)$.

LEMMA (3.3). *Let G, H be groups and let V (resp. W) be a G-module (resp. H-module). Then $[V \otimes W]^{G \times H} = [V]^G \otimes [W]^H$.*

Let $\mathfrak{h}$ be the Lie algebra of $\operatorname{Hg}(A)_C$. Then by the above argument and the lemmas, we see the representation of $\mathfrak{h}$ in the space $H^1(A, \boldsymbol{C})$ is equivalent to the representation of the Lie algebra of the Hodge group of some product of elliptic curves $E_1^{n_1} \times \cdots \times E_u^{n_u}$ $(E_i \not\sim E_j$ for $i \neq j)$. Therefore the proof is reduced to showing that the Hodge ring

$\mathscr{B}^*(E_1^{n_1} \times \cdots \times E_u^{n_u})$ is generated by $\mathscr{B}^1(E_1^{n_1} \times \cdots \times E_u^{n_u})$. But this follows immediately from (2.4), (2.5) and (3.3).

REMARK. In case (1) of (2.7), we have $\mathrm{Hg}\,(A_1^n) \cong \mathrm{Hg}\,(A_1) \cong \mathrm{SL}_2\,(F_1)$, where we denote $\mathrm{End}^0 A_1$ by F_1, and $V_t \cong H^1(A, \boldsymbol{Q}) \otimes_{F_1,\sigma} \boldsymbol{C}$ for some embedding σ of F_1 into $\boldsymbol{C}$. This fact was pointed out to us by the referee. Such a viewpoint will be investigated in our forthcoming paper on "stable non-degeneracy" of abelian varieties.

4. Proof of Theorem (1.2). For an arbitrary positive integer N, put

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z});\, c \equiv 0 \bmod (N)\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N);\, a \equiv 1 \bmod (N)\right\}.$$

We denote by $X_0(N)$ (resp. $X_1(N)$) the non-singular projective curve defined over $\boldsymbol{Q}$, which is associated to $\Gamma_0(N)$ (resp. $\Gamma_1(N)$). More precisely, the group $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) acts on the Poincaré half-plane $\mathfrak{H}$. We denote by $\mathfrak{H}^*$ the union of $\mathfrak{H}$ and the cusps of $\Gamma_0(N)$ (resp. $\Gamma_1(N)$). The quotient of $\mathfrak{H}^*$ by the action of $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) is a compact Riemann surface. It is known that the algebraic curve over $\boldsymbol{C}$ thus obtained is defined over $\boldsymbol{Q}$. We consider algebraic cycles on the jacobian variety $J_0(N)$ (resp. $J_1(N)$) of the curve $X_0(N)$ (resp. $X_1(N)$). We note that these abelian varieties satisfy the condition of Theorem (1.1) as is shown in [9]. Therefore we have Theorem (1.2). Moreover we have:

COROLLARY (4.1). *Let B be an abelian variety which is obtained as a quotient variety of $J_1(N)$. Then the Hodge ring $\mathscr{B}^*(B)$ is generated by $\mathscr{B}^1(B)$.*

PROOF. Each simple component of the abelian variety B is a simple component of $J_1(N)$ up to isogeny. Hence we have the assertion of the corollary by the same argument as that in the proof of Theorem (1.1).

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