Advances in Discrete Tomography and Its Applications

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(Version of February 9, 2006)

Birkhäuser

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Contents

Algebraic Discrete Tomography

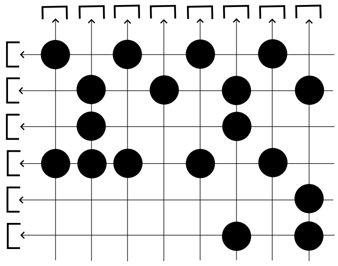
L. Hajdu and R. Tijdeman

Summary. In this chapter we present an algebraic theory of patterns which can be applied in discrete tomography for any dimension. We use that the difference of two such patterns yields a configuration with vanishing line sums. We show by introducing generating polynomials and applying elementary properties of polynomials that such so-called switching configurations form a linear space. We give a basis of this linear space in terms of the so-called switching atom, the smallest non-trivial switching configuration. We do so both in case that the material does not absorb light and absorbs light homogeneously. In the former case we also show that a configuration can be constructed with the same line sums as the original and with entries of about the same size, and we provide a formula for the number of linear dependencies between the line sums. In the final section we deal with the case that the transmitted light does not follow straight lines.

1 Introduction

One of the basic problems of discrete tomography is to reconstruct a function $f: A \to \{0,1\}$ where A is a finite subset of $\mathbb{Z}^n$ $(n \geq 2)$, if the sums of the function values (the so-called X-rays) along all the lines in a finite number of directions are given. A related problem on emission tomography is to reconstruct f if it represents (radio-active) material which is emitting radiation. If $f(\underline{i}) = 1$ for some $\underline{i} \in A$, then there is a unit of radiating material at $\underline{i}$, otherwise $f(\underline{i}) = 0$ and there is no such material at $\underline{i}$. The radiation is partially absorbed by the medium, such that its intensity is reduced by a factor β for each unit line segment in the given direction (with some real number $\beta \geq 1$).

As an illustration we include an example. In Figure 1 the row sums of f (the number of particles in each row, from top to bottom) are given by $[4,4,2,5,1,2]$, while the column sums (the number of particles in each column, from left to right) are $[2,3,2,1,2,3,2,3]$. Further, taking the line sums of f in the direction $(1,-1)$, i.e. the sums of elements lying on the same lines of slope -1, we get (from the bottom-left corner to the top-right corner)



$$f:\quad \begin{array}{cccccccc} 1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&0\\ 1&1&1&0&1&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&1 \end{array}$$

Fig. 1. The symbols • denote particles on a grid which are represented in the table f on the right by 1's. In the classical case the light is going horizontally and vertically, resulting in row and column sums. In the emission case the particles emit radiation which is partially absorbed by the material surrounding the particles. The intensity of the radiation is measured by detectors, denoted by [signs.

$[0, 0, 1, 1, 2, 3, 1, 3, 3, 2, 0, 2, 0]$. Finally, suppose that the particles emit radiation in the directions $(-1, 0)$ and $(0, 1)$. If β is the absorption coefficient in these directions, i.e. the absorption on a line segment of unit length is proportional with β, then the "absorption row sums " (measured at the detectors) from top to bottom are

$$[\beta^{-1} + \beta^{-3} + \beta^{-5} + \beta^{-7}, \beta^{-2} + \beta^{-4} + \beta^{-6} + \beta^{-8}, \beta^{-2} + \beta^{-6},$$
$$\beta^{-1} + \beta^{-2} + \beta^{-3} + \beta^{-5} + \beta^{-7}, \beta^{-8}, \beta^{-6} + \beta^{-8}]\ ,$$

and the "absorption column sums " from left to right are given by

$$[\beta^{-1} + \beta^{-4}, \beta^{-2} + \beta^{-3} + \beta^{-4}, \beta^{-1} + \beta^{-4}, \beta^{-2}, \beta^{-1} + \beta^{-4},$$
$$\beta^{-2} + \beta^{-3} + \beta^{-6}, \beta^{-1} + \beta^{-4}, \beta^{-2} + \beta^{-5} + \beta^{-6}]\ .$$

In the past decade considerable attention has been given to this type of problems, see e.g. [5, 6, 14, 15], and especially [18] for a historical overview. Many papers investigate the problem under which circumstances the line sums determine the original set uniquely, see e.g. [1, 7, 8, 10, 24] for the non-absorption and [19, 20] for the absorption case. However, in many cases there are more than one configuration yielding the same line sums. Observe that the "difference" of two configurations with equal line sums has zero line sums. Such a difference is called a switching configuration. In the case of row and column sums they were already studied by Ryser [22] in 1957. We refer to [16, 17] for the case of two general directions and for the investigation of so-called switching chains. Shliferstein and Chien [24] studied switching configurations in situations with more than two directions. Switching configurations play a role in solution methods of e.g. [1, 12, 16, 17, 19, 20, 24]. Already Ryser [22] showed in the case of row and column sums that every switching configuration can be composed of simple switching components $\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. An algebraic theory on their structure was developed by the authors [11, 13] based on switching components of minimal size, so-called switching atoms. In order to reconstruct the original itself, one can use additional known properties of the original object to favour some inverse images above the others, such as convexity (see e.g. [1]) or connectedness (see e.g. [2, 3, 12]). For an extensive study on the computational complexity of discrete tomographical problems see [9].

In this chapter we describe a general algebraic framework for switching configurations. We collect and at certain points generalize some of our previous results. We show that our method can be applied to more general problems than only the classical ones in discrete tomography. We mention that, though we focus on $\mathbb{Z}^n$ only, the results presented below can be generalized to any integral domain R such that $R[x_1, \ldots, x_n]$ is a unique factorization domain. We recommend the book of Lang [21] as a general reference for algebra.

To formulate the above problems in a precise way, we introduce some definitions and notation which we use throughout this chapter without any further reference. Let n be a positive integer. The j-th coordinate of a point $\underline{v} \in \mathbb{Z}^n$ will be denoted by v_j $(j = 1, \ldots, n)$, that is $\underline{v} = (v_1, \ldots, v_n)$. Let m_j $(j = 1, \ldots, n)$ denote positive integers, and put

$$A = \{\underline{i} \in \mathbb{Z}^n \colon 0 \leq i_j < m_j \text{ for } j = 1, \ldots, n\} .$$

Let d be a positive integer, and suppose that $\overset{k}{\sim}$ are equivalence relations on A for $k = 1, \ldots, d$. (For example, points are equivalent if they are on a line in some direction characterized by k.) Let $H_1^{(k)}, \ldots, H_{t_k}^{(k)}$ denote the equivalence classes of $\overset{k}{\sim}$. Finally, let $\varrho_k : \ A \to \mathbb{R}_{>0}$ be so-called weight functions for $k = 1, \ldots, d$, and set $\varrho = \sum_{k=1}^{d} \varrho_k$. Now the above mentioned problems can be formulated in the following more general way.

Problem 1. Let c_{kl} be given real numbers for $k = 1, \ldots, d$ and $l = 1, \ldots, t_k$. Construct a function $g : \ A \to \{0, 1\}$ (if it exists) such that

$$\sum_{\underline{i} \in H_l^{(k)}} g(\underline{i}) \varrho_k(\underline{i}) = c_{kl} \qquad (k = 1, \ldots d; \ l = 1, \ldots t_k) . \tag{1}$$

It is important to note that equation (1) is certainly underdetermined with respect to functions $g : \ A \to \mathbb{Z}$. Moreover, the same may be true for solutions $g : \ A \to \{0, 1\}$. For example, the function g given by

$$g: \quad \begin{matrix} 0&1&1&0&1&0&1&0 \\ 1&0&0&0&1&1&0&1 \\ 0&1&0&0&0&1&0&0 \\ 1&1&1&1&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1 \end{matrix}$$

has the same row and column sums as f from Figure 1. Consequently, $h := f - g$ has zero row and column sums. Vice versa, having a function $h: A \to \mathbb{Z}$ with zero line sums, the line sums of $g + h$ will coincide with those of g. It turns out that the study of switching configurations over $\mathbb{Z}$ is much simpler than that over $\{0, 1\}$. It is therefore important to note that the solutions to Problem 1 can be characterized as the solutions of the following optimization problem over $\mathbb{Z}$.

Problem 2. Construct a function $g: A \to \mathbb{Z}$ (if it exists) such that (1) holds, and

$$\sum_{\underline{i} \in A} g(\underline{i})^2 \varrho(\underline{i}) \quad \text{is minimal} .$$

Remark 1. If g is a solution to Problem 1, then g is a solution to Problem 2. To show this, let $f: A \to \mathbb{Z}$ be any other solution to (1). Then we have

$$\sum_{\underline{i} \in A} g(\underline{i})^2 \varrho(\underline{i}) = \sum_{\underline{i} \in A} g(\underline{i}) \varrho(\underline{i}) = \sum_{\underline{i} \in A} f(\underline{i}) \varrho(\underline{i}) \leq \sum_{\underline{i} \in A} f(\underline{i})^2 \varrho(\underline{i}) .$$

The idea used here, that a binary solution has small "length", has been used in several papers, see e.g. [2, 3, 12].

Remark 2. We also mention that when the equivalence relations $\stackrel{k}{\sim}$ mean that the corresponding points are on the same lines in given directions, and the weight functions ϱ_k are defined as certain powers of some real numbers $\beta_k \geq 1$ then in view of Remark 1, our problems just reduce to the classical problem of emission tomography with absorption. In particular, when $\beta_k = 1$ ($\varrho_k = 1$ for every k) we get back the classical problem on discrete tomography.

As we indicated, we will study the structure of the set of integral solutions of equation (1). It turns out that in case of line sums there exists a minimal configuration (the so-called switching atom) such that every integral solution of the homogenized equation (1) (i.e. with $c_{kl} = 0$) can be expressed as a linear combination of shifts of one of the switching atoms. For the case of row and column sums the switching atom is $\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. In this chapter we characterize and derive properties of switching configurations.

The structure of this chapter is as follows. In the next section we briefly outline the main principles of our method. In Section 3 we give a complete description of the set of integral solutions of (1) in case of the classical problem of discrete tomography, for arbitrary dimension (see Theorem 1). Theorem 2 shows that if Problem 2 admits a solution, then a relatively small solution can be found in polynomial time. In Section 4 we derive similar results for the case of emission tomography with absorption, also for any dimension n. Finally, in Section 5 we consider a new type of tomographical problems. Instead of lines, the X-rays (in $\mathbb{Z}^2$) are assumed to be parallel shifts of the graph of a function $G: \mathbb{Z} \to \mathbb{Z}$. It turns out that our machinery is applicable in this case, as well.

2 The main principles of the method

In this section we summarize the main principles of our approach. Our method relies on the following four fundamental observations.

1) If both functions $f, g: A \to \mathbb{Z}$ are solutions to equation (1), then the difference $h := f - g$ is a solution to (1) with $c_{kl} = 0$ for all k, l, that is to

$$\sum_{\underline{i} \in H_l^{(k)}} h(\underline{i}) \varrho_k(\underline{i}) = 0 \qquad (k = 1, \dots d;\ l = 1, \dots t_k)\ . \tag{2}$$

So to characterize the set of integral solutions of (1), it is sufficient to know one particular solution g together with all the solutions of (2).

2) Suppose that $H_1, \dots, H_t$ is a partition of A. Let $f: A \to \mathbb{Z}$ and $f_l: H_l \to \mathbb{Z}$ $(l = 1, \dots, t)$ be given functions and write $\chi_f(\underline{x}) = \sum_{\underline{i} \in A} f(\underline{i}) \underline{x}^{\underline{i}}$ for the generating polynomial of f. Suppose that $\chi_{f_l}(\underline{x}) = \sum_{\underline{i} \in H_l} f_l(\underline{i}) \underline{x}^{\underline{i}}$ vanishes for $l = 1, \dots, t$, and that $\chi_f(\underline{x}) = \sum_{l=1}^{t} \chi_{f_l}(\underline{x})$. Then $\chi_f(\underline{x})$ vanishes.

3) If $\chi_f(\underline{x})$ is divisible by polynomials $P_1(\underline{x}), \dots, P_s(\underline{x}) \in \mathbb{Z}[\underline{x}]$, then $\chi_f(\underline{x})$ is divisible by $\mathrm{lcm}(P_1(\underline{x}), \dots, P_s(\underline{x}))$ in $\mathbb{Z}[\underline{x}]$.

4) Let f be a solution to equation (2). Then in the cases investigated in this chapter we have $\chi_f(\underline{x}) = P(\underline{x}) Q(\underline{x})$, where P corresponds to a "minimal" solution M to (2), and Q indicates which combination of the translates of M yields f.

To illustrate how these principles work, we exhibit some examples.

Example 1 (row sums). Let $n = 2$, $A = \{(i, j) : 0 \le i < m_1,\ 0 \le j < m_2\}$ and $H_l = \{(i, l) : 0 \le i < m_1\}$ for $l = 0, \dots, m_2 - 1$. Let $f: A \to \mathbb{Z}$ be a given function. Define $f_l: H_l \to \mathbb{Z}$ for $l = 0, \dots, m_2 - 1$ by $f_l(i, l) = f(i, l)$ $(i = 0, \dots, m_1 - 1)$. Then

$$\chi_f(x, y) = \sum_{l=0}^{m_2 - 1} \chi_{f_l}(x, y) \text{ and } \chi_{f_l}(x, y) = y^l \sum_{l=0}^{m_2 - 1} f_l(i, l) x^i\ .$$

(i) Suppose $\sum_{i=0}^{m_1 - 1} f(i, l) = 0$ for $l = 0, \dots, m_2 - 1$, so we have vanishing row sums. Then

$$\chi_{f_l}(1, y) = y^l \sum_{i=0}^{m_1 - 1} f_l(i, l) = y^l \sum_{i=0}^{m_1 - 1} f(i, l) = 0 \ \text{ for } l = 0, \dots, m_2 - 1\ .$$

Hence

$$\chi_{f_l}(x, y) = \sum_{(i,j) \in H_l} f_l(i, j) x^i y^j = y^l \sum_{i=0}^{m_1 - 1} f_l(i, l) x^i$$

is divisible by $x-1$ for $l=0,\dots,m_2-1$. Thus $\chi_f(x,y)=\sum_{l=0}^{m_2-1}\chi_{f_l}(x,y)$ is divisible by $x-1$.

(ii) Let $\beta\in\mathbb{C}$, and suppose that $\sum_{i=0}^{m_1-1} f(i,l)\beta^i=0$ for $l=0,\dots,m_2-1$. Then

$$\chi_{f_l}(\beta,y)=y^l\sum_{i=0}^{m_1-1} f_l(i,l)\beta^i=y^l\sum_{i=0}^{m_1-1} f(i,l)\beta^i=0 \quad \text{for } l=0,\dots,m_2-1\ .$$

Hence $\chi_{f_l}(x,y)$ is divisible by $x-\beta$ over $\mathbb{C}$ for $l=0,\dots,m_2-1$. Then $\chi_f(x,y)$ is divisible by $x-\beta$ over $\mathbb{C}$. Since $\chi_f(x,y)\in\mathbb{Z}[x,y]$, this implies that $\chi_f=0$ if β is a transcendental number and that $\chi_f(x,y)$ is divisible by the minimal defining polynomial of β if it is an algebraic number.

Example 2 (column sums). Let n and A be as in Example 1, but now let $H_l=\{(l,j):0\le j<m_2\}$ for $l=0,\dots,m_1-1$. Let $f:\ A\to\mathbb{Z}$ be a given function. Define now $f_l:\ H_l\to\mathbb{Z}$ for $l=0,\dots,m_1-1$ by $f_l(l,j)=f(l,j)$ $(j=0,\dots,m_2-1)$. Then

$$\chi_f(x,y)=\sum_{l=0}^{m_1-1}\chi_{f_l}(x,y) \text{ and } \chi_{f_l}(x,y)=x^l\sum_{j=0}^{m_2-1} f_l(l,j)y^j\ .$$

If $\beta\in\mathbb{C}$ such that $\sum_{j=0}^{m_2-1} f(l,j)\beta^j=0$ for $l=0,\dots,m_1-1$, then $\chi_f=0$ if β is transcendental and $\chi_f(x,y)$ is divisible by the minimal defining polynomial of β if it is algebraic.

On combining Example 1 with β_1 and Example 2 with β_2 we obtain that if $\sum_{i=0}^{m_1-1} f(i,l)\beta_1^i=0$ for $l=0,\dots,m_2-1$ and $\sum_{j=0}^{m_2-1} f(l,j)\beta_2^j=0$ for $l=0,\dots,m_1-1$ then $\chi_f=0$ if β_1 or β_2 is transcendental and that otherwise χ_f is divisible by the product of the minimal defining polynomials $P_1(x,1)$ of β_1 and $P_2(1,y)$ of β_2 (as $P_1(x,1)$ and $P_2(1,y)$ are coprime).

Example 3 (line sums). Let n and A be as in Example 1 and $a,b\in\mathbb{Z}$. Without loss of generality we may assume that $a>0$. Suppose first that we have $b\le 0$. Put $H_l=\{(i,j):aj=bi+l\}$ for $l=0,\dots,m$ with $m=(m_1-1)b+(m_2-1)a$. Hence A is the disjoint union of the H_l. Define the functions $f_l:\ H_l\to\mathbb{Z}$ for the above values of l by $f_l(i,j)=f(i,j)$ $((i,j)\in H_l)$, where $f:\ A\to\mathbb{Z}$ is a given function. Then

$$\chi_f(x,y)=\sum_{l=0}^{m}\chi_{f_l}(x,y) \text{ where } \chi_{f_l}(x,y)=\sum_{(i,j)\in H_l} f_l(i,j)x^iy^j\ .$$

Let $\beta\in\mathbb{C}$, and suppose that $\sum_{(i,j)\in H_l} f_l(i,j)\beta^i=0$ for $l=0,\dots,m$. Then

$$\chi_{f_l}(x,y) = \sum_{(i,j)\in H_l} f_l(i,j)x^i y^{(bi+l)/a} = y^{l/a} \sum_{(i,j)\in H_l} f_l(i,j)(xy^{b/a})^i = 0$$

for $x = \beta y^{-b/a}$ and $l = 0,\ldots,m$. It follows that $\chi_f(\beta y^{-b/a}, y) \equiv 0$. Equivalently, $\chi_f(\beta y^{-b}, y^a) = 0$. We conclude that $\chi_f = 0$ if β is transcendental and that otherwise χ_f is divisible by the minimal defining polynomial of $x^{a/d} - \beta^{a/d}y^{-b/d}$ where $d = \gcd(a,b)$ if β is algebraic. Similarly we find in case $b > 0$ that χ_f is divisible by the minimal polynomial of $x^{a/d}y^{b/d} - \beta^{a/d}$.

Combine Example 1 with $\beta = \beta_1$ and Example 3 with $a = 1$, $b = -1$, $\beta = \beta_1^{\sqrt{2}}$. Suppose $\sum_{i=0}^{m_1-1} f(i,l)\beta_1^i = 0$ for $l = 0,\ldots,m_2-1$ and $\sum_{j=-i+l} f(i,j)\beta_1^{\sqrt{2}i} = 0$ for $l = 0,\ldots,m_1+m_2-2$. Then χ_f is divisible by both polynomials $x - \beta_1$ and $x - \beta_1^{\sqrt{2}}y$ over $\mathbb{C}$. By the theorem of Gelfond-Schneider we know that if $\beta_1 \neq 0, 1$, then $\beta_1^{\sqrt{2}}$ is transcendental if β_1 is algebraic. Hence either $\beta_1 = 0$ and χ_f is divisible by x, or $\beta_1 = 1$ and χ_f is divisible by $(x-1)(x-y)$, or $\chi_f = 0$.

Combine Example 3 with $a = 1$, $b = -1$, $\beta \neq 0$ arbitrary and Example 3 with $a = b = 1$, and β^{-1} in place of β. Suppose $\sum_{j=-i+l} f(i,j)\beta^i = 0$ for $l = 0,\ldots,m_1+m_2-2$ and $\sum_{j=i+l} f(i,j)\beta^{-i} = 0$ for $l = -m_1+1,\ldots,m_2-1$. Then χ_f is divisible by both polynomials $x-\beta y$ and $xy-\beta^{-1}$ over $\mathbb{C}$. Hence χ_f is identically zero if β is transcendental. If β is algebraic, then χ_f is divisible by the product of the minimal polynomials of $x-\beta y$ and $xy-\beta^{-1}$.

Finally, combine Example 3 with $a = 1$, $b = -1$, $\beta \neq 0$ arbitrary and Example 3 with $a = 1$, $b = -1$, β^{-1} in place of β. (The latter condition is equivalent with $a = -1$, $b = 1$, absorption coefficient β.) Suppose $\sum_{j=-i+l} f(i,j)\beta^i = \sum_{j=-i+l} f(i,j)\beta^{-i} = 0$ for $l = 0,\ldots,m_1+m_2-2$. Then $\chi_f = 0$ if β is transcendental. If β is algebraic then $\chi_f(x,y)$ is divisible by the minimal polynomial of $xy-\beta$, and, if the minimal polynomial of β is non-reciprocal, even by the product of the minimal polynomials of $x-\beta y$ and $x-\beta^{-1}y$.

3 Discrete tomography in nD

In [11] we developed a theory on switching configurations in case $n = 2$. In this section we generalize it to arbitrary n.

3.1 Some notation

Let $\underline{a} \in \mathbb{Z}^n$ with $\gcd(a_1,\ldots,a_n) = 1$, such that $\underline{a} \neq \underline{0}$, and for the smallest j with $a_j \neq 0$ we have $a_j > 0$. We call $\underline{a}$ a direction. By lines with direction

$\underline{a}$ we mean lines of the form $\underline{b} + t\underline{a}$ ($\underline{b} \in \mathbb{R}^n$, $t \in \mathbb{R}$) in $\mathbb{R}^n$. Let A be as in the Introduction. By the help of a direction $\underline{a}$ we can define an equivalence relation on A as follows. We call two elements of A equivalent if they are on the same line with direction $\underline{a}$. If $g: A \to \mathbb{Q}$ is a function, then the line sum of g along the line $T = \underline{b} + t\underline{a}$ is defined as $\sum_{\underline{i} \in A \cap T} g(\underline{i})$. Note that the line sums are in fact the "class sums" from (1), corresponding to the above defined equivalence.

We will work with polynomials $F \in \mathbb{Q}[x_1, \dots, x_n]$. For brevity we write $\underline{x} = (x_1, \dots, x_n)$ and $\underline{x}^{\underline{i}} = \prod_{j=1}^{n} x_j^{i_j}$ ($\underline{i} \in \mathbb{Z}^n$). The generating polynomial of a function $g: A \to \mathbb{Q}$ is defined as

$$\chi_g(\underline{x}) = \sum_{\underline{i} \in A} g(\underline{i}) \underline{x}^{\underline{i}} .$$

A set $S = \{\underline{a}_k\}_{k=1}^d$ of directions is called valid for A, if $\sum_{k=1}^{d} |a_{kj}| < m_j$ for any $j = 1, \dots, n$. Suppose that S is a valid set of directions for A. For $\underline{a} \in S$ put $f_{\underline{a}}(\underline{x}) = (\underline{x}^{\underline{a}} - 1) \prod_{a_j < 0} x_j^{-a_j}$ and set $F_S(\underline{x}) = \prod_{k=1}^{d} f_{\underline{a}_k}(\underline{x})$. Let

$$U = \{\underline{u} \colon 0 \le u_j < m_j - \sum_{k=1}^{d} |a_{kj}| \ (j = 1, \dots, n)\} .$$

For $\underline{u} \in U$ put $F_{(\underline{u};S)}(\underline{x}) = \underline{x}^{\underline{u}} F_S(\underline{x})$ and define the functions $M_{(\underline{u};S)}: A \to \mathbb{Z}$ by

$$M_{(\underline{u};S)}(\underline{i}) = \text{coeff}(\underline{x}^{\underline{i}}) \text{ in } F_{(\underline{u};S)}(\underline{x}) \text{ for } \underline{i} \in A .$$

The $M_{(\underline{u};S)}$'s are called the switching atoms corresponding to the direction set S. By the minimal corner of the switching atom $M_{(\underline{0};S)}$ we mean the element $\underline{i}^* \in A$ for which $M_{(\underline{0};S)}(\underline{i}^*) \neq 0$, but $M_{(\underline{0};S)}(\underline{i}) = 0$, whenever $\underline{i} \in A$ lexicographically precedes $\underline{i}^*$. That is, $\underline{i}^*$ is lexicographically the first element of A for which the function value of $M_{(\underline{0};S)}$ is non-zero. It follows from the definitions of $f_{\underline{a}}$ and F_S that

$$M_{(\underline{0};S)}(\underline{i}^*) = \pm 1 .$$

Since it corresponds with the minimal corner of $M_{(\underline{0};S)}$, for every $\underline{u} \in U$ we define the minimal corner of $M_{(\underline{u};S)}$ as $\underline{i}^* + \underline{u}$. Again, the minimal corner of $M_{(\underline{u};S)}$ is lexicographically the first element of A for which the function value of $M_{(\underline{u};S)}$ is non-zero, and we also have

$$M_{(\underline{u};S)}(\underline{i}^* + \underline{u}) = \pm 1 .$$

It is clear that a function g defined on A can be considered as a vector (a $\prod_{j=1}^{n} m_j$-tuple). If we want to emphasize this, we write $\mathbf{g}$ instead of g. We always assume that the entries of these vectors are arranged according to elements of A in lexicographical order. The length of $\mathbf{g}$ (or g) is $|g| = |\mathbf{g}| = \sqrt{\sum_{\underline{i}\in A} g(\underline{i})^2}$.

3.2 The structure of the switching configurations

Our main result shows that every switching configuration is a linear combination of translates of the switching atom $M_{(\underline{0};S)}$.

Theorem 1. *Let A be as before, $S = \{\underline{a}_k\}_{k=1}^d$ a valid set of directions for A, and let R be one of $\mathbb{Z}$ or $\mathbb{Q}$. Then any function $g:\ A \to R$ with zero line sums along the lines corresponding to S can be uniquely written in the form*

$$g = \sum_{\underline{u}\in U} c_{\underline{u}} M_{(\underline{u};S)}$$

with some $c_{\underline{u}} \in R$ $(\underline{u} \in U)$. Moreover, every such function g has zero line sums along the lines corresponding to S.

Remark 3. As one can easily see from the proofs, if S is not valid for A, then the only function having all its line sums zero is the identically zero function on A.

To prove the theorem, we need the following lemma.

Lemma 1. *Assume that $\underline{a}$ is a valid direction for A, and let R be one of $\mathbb{Z}$ or $\mathbb{Q}$. Then a function $g:\ A \to R$ has zero line sums along the lines with direction $\underline{a}$ if and only if $f_{\underline{a}}(\underline{x})$ divides $\chi_g(\underline{x})$ in $R[\underline{x}]$.*

Proof. We give the proof only when $a_j > 0$ $(j = 1, \ldots, n)$, the proof is similar in all the other cases. Put $B = \{\underline{b}:\ : \underline{b} \in A,\ \underline{b} - \underline{a} \notin A\}$, and for $\underline{b} \in B$ set $I_{\underline{b}} = \max\{t \in \mathbb{Z}\colon \underline{b} + t\underline{a} \in A\}$. Observe that we can write

$$\chi_g(\underline{x}) = \sum_{\underline{b}\in B} \sum_{t=0}^{I_{\underline{b}}} g(\underline{b} + t\underline{a}) \underline{x}^{\underline{b}+t\underline{a}} = \sum_{\underline{b}\in B} \underline{x}^{\underline{b}} \sum_{t=0}^{I_{\underline{b}}} g(\underline{b} + t\underline{a}) \underline{x}^{t\underline{a}} =$$

$$= (\underline{x}^{\underline{a}} - 1) \sum_{\underline{b}\in B} \underline{x}^{\underline{b}} \sum_{t=0}^{I_{\underline{b}}} g(\underline{b} + t\underline{a}) \sum_{s=0}^{t-1} \underline{x}^{s\underline{a}} + \sum_{\underline{b}\in B} \underline{x}^{\underline{b}} \sum_{t=0}^{I_{\underline{b}}} g(\underline{b} + t\underline{a})\ .$$

As $f_{\underline{a}}(\underline{x}) = \underline{x}^{\underline{a}} - 1$ and the line sums of g in the direction $\underline{a}$ are given by $\sum_{t=0}^{I_{\underline{b}}} g(\underline{b} + t\underline{a})$, the lemma follows. $\square$

Proof (of Theorem 1). By definition, for every $\underline{u} \in U$ the function $F_{(\underline{u};S)}$ is divisible by $f_{\underline{a}_k}$ for any k with $1 \leq k \leq d$. Hence by Lemma 1, $M_{(\underline{u};S)}$ has zero line sums along all the lines corresponding to S. This proves the second statement of Theorem 1.

Let now

$$H = \{f : \ A \to R \mid f \text{ has zero line sums corresponding to } S\} .$$

We first prove that the switching atoms generate H. Suppose that $g \in H$. Lemma 3 (from Section 4) implies that the polynomials $f_{\underline{a}_k}(\underline{x})$ are pairwise non-associated irreducible elements of the unique factorization domain $R[\underline{x}]$. Hence by Lemma 1 we obtain

$$F_S(\underline{x}) \mid \chi_g(\underline{x}) \quad \text{in} \quad R[\underline{x}] .$$

Hence there exists a polynomial $h(\underline{x}) = \sum_{\underline{u} \in U} c_{\underline{u}} \underline{x}^{\underline{u}}$ in $R[\underline{x}]$ such that $\chi_g(\underline{x}) = h(\underline{x}) F_S(\underline{x})$. We rewrite this equation as

$$\chi_g(\underline{x}) = \sum_{\underline{u} \in U} c_{\underline{u}} F_{(\underline{u};S)}(\underline{x}) .$$

Now by the definitions of $\chi_g(\underline{x})$ and the switching atoms $M_{(\underline{u};S)}$ we immediately obtain

$$g = \sum_{\underline{u} \in U} c_{\underline{u}} M_{(\underline{u};S)} ,$$

which proves that the functions $M_{(\underline{u};S)}$ generate H.

Suppose now that for some coefficients $l_{\underline{u}} \in R$ ($\underline{u} \in U$) we have

$$\sum_{\underline{u} \in U} l_{\underline{u}} M_{(\underline{u};S)}(\underline{i}) = 0 \text{ for all } \underline{i} \in A .$$

By the definitions of the switching atoms, at the minimal corner of $M_{(\underline{0};S)}$ all the other switching atoms vanish. This immediately implies $l_{\underline{0}} = 0$. Running through the switching atoms $M_{(\underline{u};S)}$ with $\underline{u} \in U$ in increasing lexicographical order, we conclude that all the coefficients $l_{\underline{u}}$ are zero. This shows that the switching atoms are linearly independent, which completes the proof of the theorem. $\square$

The following result is a consequence of Theorem 1.

Corollary 1. *Let A, S and R be as in Theorem 1. Let C be the set of those elements of A which are the minimal corners of the switching atoms. Then for any $f : \ A \to R$ and for any prescribed values from R for the elements of C, there exists a unique $g : \ A \to R$ having the prescribed values at the elements of C and having the same line sums as f along the lines corresponding to S.*

Proof. As every switching atom takes value ±1 at its minimal corner, we obtain that there are unique coefficients $c_{\underline{u}} \in R$ ($\underline{u} \in U$) such that

$$g := f + \sum_{\underline{u}\in U} c_{\underline{u}} M_{(\underline{u};S)}$$

has the prescribed values at the element of C. By the second statement of Theorem 1 the line sums of f and g corresponding to S coincide. $\square$

3.3 Existence of "small" solutions

We provide a polynomial-time algorithm for finding an approximation to f having the required line sums. We first compute a function $q: A \to \mathbb{Q}$ having the same line sums as f in the given directions by solving a system of linear equations. Subsequently we use the structure of switching configurations to find a function $g: A \to \mathbb{Z}$ which is not far from q and f. The general result is given in Theorem 2. It follows that in case when f has $\{0,1\}$ values the algorithm provides a solution $g: A \to \mathbb{Z}$ satisfying (1) with $|g(\underline{i})| \le 2^{d-1}+1$ on average, where d is the number of directions involved. The function obtained by replacing all function values of q which are greater than 1/2 by 1 and all others by 0 provides a good first approximation to f in practice. In [12] an algorithm is given, relying on this principle.

Theorem 2. *Let A, d and S be as in Theorem 1. Let all the line sums in the directions of S of some unknown function $f: A \to \mathbb{Z}$ be given. Then there exists an algorithm which is polynomial in $\max_{j=1,\dots,n}\{m_j\}$, providing a function $g: A \to \mathbb{Z}$ such that f and g have the same line sums corresponding to S, moreover*

$$|g| \le |f| + 2^{d-1}\sqrt{\prod_{j=1}^{n} m_j}\ . \tag{3}$$

Proof. Put $N_j = \sum_{k=1}^{d} |a_{kj}|$ for $j = 1,\dots,n$. First, compute some function $q: A \to \mathbb{Q}$ having the same line sums as f. It can be done by solving the system of linear equations provided by the line sums. This step is known to be polynomial in $\max_{j=1,\dots,n}\{m_j\}$ (see e.g. [4], p. 48). We construct a function $s: A \to \mathbb{Z}$ with the same line sums as f. We follow the procedure used in the second part of the proof of Theorem 1 and start with the minimal corner $\underline{i}^*$ of $M_{(\underline{0};S)}$. With an appropriate rational coefficient $r_{\underline{0}}$ with $|r_{\underline{0}}| \le 1/2$, the value $(q + r_{\underline{0}} M_{(\underline{0};S)})(\underline{i}^*)$ will be an integer. We now continue in increasing lexicographical order in $\underline{i}$ and choose coefficients $r_{\underline{i}}$ subject to $|r_{\underline{i}}| \le 1/2$ such that the value of $(q + \sum_{\underline{i}' \le \underline{i}} r_{\underline{i}'} M_{(\underline{i}';S)})(\underline{i})$ is an integer. (Here $\le$ under the $\sum$ refers to the lexicographical ordering.) Observe that the values at $\underline{i}'$ ($\underline{i}' < \underline{i}$) are not changed in the $\underline{i}$-th step. After executing this procedure for the whole set C of the minimal corners of the switching atoms, we obtain a function s having integer values on C. By a similar process (taking the switching atoms one-by-one, in increasing lexicographical order) we get that there exist integers $t_{\underline{u}}$ ($\underline{u} \in U$) such that the values of $f + \sum_{\underline{u}\in U} t_{\underline{u}} M_{(\underline{u};S)}$ and s coincide on C. As these functions have the same line sums corresponding to S, applying Corollary 1 with $R = \mathbb{Q}$, we conclude that they are equal, hence s takes integer values on the whole set A. Clearly, this construction of s needs only a polynomial number of steps in $\max_{j=1,\dots,n} \{m_j\}$.

Consider now all the functions as vectors ($\prod_{j=1}^{n} m_j$-tuples), and solve over $\mathbb{Q}$ the following system of linear equations

$$(\mathbf{s}, \mathbf{M}_{(\underline{v};S)}) = \sum_{\underline{u}\in U} c^*_{\underline{u}} (\mathbf{M}_{(\underline{u};S)}, \mathbf{M}_{(\underline{v};S)})$$

in $c^*_{\underline{u}}$, where $(.,.)$ denotes the inner product of vectors and $\underline{v}$ runs through the elements of U. As the switching atoms are linearly independent according to Theorem 1, this system of equations has a unique solution. This can be computed again in time which is polynomial in $\max_{j=1,\dots,n} \{m_j\}$. Put $\mathbf{g} = \mathbf{s} - \sum_{\underline{u}\in U} ||c^*_{\underline{u}}|| \mathbf{M}_{(\underline{u};S)}$, where $||\alpha||$ denotes the nearest integer to α. Observe that $\mathbf{s} - \sum_{\underline{u}\in U} c^*_{\underline{u}} \mathbf{M}_{(\underline{u};S)}$ is just the projection of $\mathbf{f}$ (but also of $\mathbf{q}$ and $\mathbf{s}$) onto the orthogonal complement of the linear subspace generated by the switching atoms. This implies

$$|\mathbf{g}| \le |\mathbf{f}| + \left| \sum_{\underline{u}\in U} (c^*_{\underline{u}} - ||c^*_{\underline{u}}||) \mathbf{M}_{(\underline{u};S)} \right| .$$

There are at most 2^d switching atoms which contribute to the value of any fixed point, each with a contribution at most $1/2$ in absolute value in the above equation. Thus we may conclude $|\mathbf{g}| \le |\mathbf{f}| + 2^{d-1}\sqrt{\prod_{j=1}^{n} m_j}$.

Finally, notice that all the steps of the above algorithm are polynomial in $\max_{j=1,\dots,n} \{m_j\}$. Thus the proof of Theorem 2 is complete. □

Remark 4. We mention that if we know that Problem 1 admits a solution, i.e. f has $\{0, 1\}$ values in the above theorem, then $|f| = \sqrt{\sum_{l=1}^{t_k} c_{kl}}$ (for any $k = 1, \dots, d$), whence we get $|g| \le (2^{d-1} + 1)\sqrt{\prod_{j=1}^{n} m_j}$. Moreover, as noted in the proof of Theorem 2 we can replace $|f|$ with $|q|$ (or with $|s|$) in the upper bound (3). Therefore an upper bound for $|g|$ can be given which only depends on the line sums and the directions.

3.4 Dependencies among the line sums

Obviously, the sum of all row sums of a function $f: A \to \mathbb{Z}$ coincides with the sum of all column sums of f. In this subsection we give a simple formula for the number of dependencies among the line sums corresponding to S.

Let A, S and $F_S(\underline{x})$ be as above, and write N_j for the degree of F_S in x_j $(j = 1, \ldots, n)$. Then by Theorem 1 the switching atoms form a basis of a module of dimension $\prod_{j=1}^{n}(m_j - N_j)$ over $\mathbb{Z}$. Suppose that L_S denotes the number of line sums for A corresponding to the directions in S, and let D_S denote the number of dependencies among these line sums. Then as the number of unknowns is $\prod_{j=1}^{n} m_j$, elementary linear algebra tells us that

$$D_S = L_S + \prod_{j=1}^{n}(m_j - N_j) - \prod_{j=1}^{n} m_j \; .$$

In particular, if $n = 2$ then there are $a_k m_2 + |b_k| m_1 - a_k |b_k|$ line sums belonging to a direction $(a_k, b_k) \in S$. Hence in this case as $a_k \geq 0$ we have

$$D_S = m_2 \sum_{k=1}^{d} a_k + m_1 \sum_{k=1}^{d} |b_k| - \sum_{k=1}^{d} a_k |b_k| +$$

$$+ \left(m_1 - \sum_{k=1}^{d} a_k \right) \left(m_2 - \sum_{k=1}^{d} |b_k| \right) - m_1 m_2 = \sum_{k=1}^{d} a_k \sum_{k=1}^{d} |b_k| - \sum_{k=1}^{d} a_k |b_k| \; .$$

4 Emission tomography with absorption

In this chapter we generalize the results from [13] which were presented for dimension 2 to the case of general dimension.

To model the physical background of emission tomography with absorption, consider a ray (such as light or X-ray) transmitting through homogeneous material. Let I_0 and I denote the initial and the detected intensities of the ray. Then

$$I = I_0 \cdot \mathrm{e}^{-\mu x} \; ,$$

where $\mu \geq 0$ denotes the absorption coefficient of the material, and x is the length of the path of the ray in the material. We put $\beta = \mathrm{e}^{\mu}$, and we call β the exponential absorption coefficient. We mention that as $\mu \geq 0$, we have $\beta \geq 1$.

Note that by the absorption we have to work with directed line sums which do not only depend on the line, but also on the direction of the radiation through that line.

We further assume that g represents (radio-active) material which is emitting radiation. If $g(\underline{i}) = 1$, then there is a unit of radiating material at $\underline{i}$, otherwise $g(\underline{i}) = 0$ and there is no such material at $\underline{i}$.

As we have absorption, we attach some absorption coefficient to each direction. Hence we slightly adjust our previous notation. Let d be a positive integer, and let $S = \{(\underline{a}_k, \beta_k) \ : \ k = 1, \dots, d\}$ be a set, where $\underline{a}_k \in \mathbb{Z}^n$ with $\gcd(a_{k1}, \dots, a_{kn}) = 1$ for $k = 1, \dots, d$, and for the real numbers β_k we have $\beta_k \geq 1$. For $k = 1, \dots, d$ put $B_k = \{\underline{b} \in A : \underline{b} + \underline{a}_k \notin A\}$, and for any $\underline{i} \in A$ let $s_{(\underline{i},k)}$ denote the integer for which $\underline{i} = \underline{b} - (s_{(\underline{i},k)} - 1)\underline{a}_k$ with some $\underline{b} \in B_k$. By the directed absorption line sum of g along the line $T = \underline{b} - t\underline{a}_k$ ($\underline{b} \in B_k, t \in \mathbb{Z}$) we mean

$$\sum_{\underline{i} \in T \cap A} g(\underline{i}) \beta_k^{-s_{(\underline{i},k)}} .$$

(Here there is a hidden assumption on the shape of the absorbing material, but this is irrelevant for the switching configurations.) In Figure 1 in the Introduction we illustrated how directed absorption line sums are interpreted.

Let $\underline{i}_1 \overset{k}{\sim} \underline{i}_2$ for $\underline{i}_1, \underline{i}_2 \in A$ and $k = 1, \dots, d$ if and only if $\underline{i}_1 - \underline{i}_2 = t\underline{a}_k$ for some $t \in \mathbb{Z}$, and write $H_1^{(k)}, \dots, H_{t_k}^{(k)}$ for the equivalence classes of $\overset{k}{\sim}$. Taking arbitrary real numbers c_{kl} ($k = 1, \dots, d; \ l = 1, \dots, t_k$), equation (1) is just given by

$$\sum_{\underline{i} \in H_l^{(k)}} g(\underline{i}) \beta_k^{-s_{(\underline{i},k)}} = c_{kl} \qquad (k = 1, \dots d; \ l = 1, \dots t_k) . \tag{4}$$

Thus in this case Problem 1 is the standard problem in emission tomography with absorption. (See also the DA2D(β) reconstruction problem in [19] for the two dimensional case.)

If the absorption is independent of the direction, then $\beta_k = \mathrm{e}^{\mu|\underline{a}|}$, since $|\underline{a}|$ is the distance between consecutive lattice points on the line $\underline{b} - t\underline{a}$. However, we prefer to leave the possibility open that the absorption coefficient depends on the direction in which the medium is passed. Our definition of $s_{(\underline{i},k)}$ makes it possible to distinguish between two opposite directions. Thus $\underline{b} - t\underline{a}$ and $\underline{b} - t(-\underline{a})$ represent the same line, but opposite directions.

Finally, we mention that in case when $\beta_k = 1$ ($k = 1, \dots, d$) the problem reduces to the classical problem of discrete tomography.

4.1 The structure of the switching configurations

In this section we give a full description of the set of solutions $g : \ A \to \mathbb{Z}$ to (4). First we consider the case when $c_{kl} = 0$ for all $k = 1, \dots, d$ and $l = 1, \dots, t_k$, that is when all the directed absorption line sums of g are zero. For this purpose we need some further notation.

First we note that if any of the β_k-s is transcendental, then f is uniquely determined by its directed absorption line sums in the corresponding direction $\underline{a}_k$. Hence from this point on we assume that all the exponential absorption coefficients are algebraic.

Let $\underline{a} \in \mathbb{Z}^n$ be a direction (i.e. $\gcd(a_1, \ldots, a_n) = 1$). Let β be a non-zero algebraic number of degree r, and let $P_\beta(z)$ be the defining polynomial of β having coprime integral coefficients. Put

$$f_{(\underline{a},\beta)}(\underline{x}) = P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j} .$$

Hence $f_{(\underline{a},\beta)}(\underline{x}) \in \mathbb{Z}[\underline{x}]$.

In the proof we shall make use of a fundamental correspondence between functions $g : A \to \mathbb{Z}$ and polynomials in n variables. Namely, to such a function g we attach the polynomial

$$\chi_g(\underline{x}) = \sum_{\underline{i} \in A} g(\underline{i}) \underline{x}^{\underline{i}} .$$

Then into direction $\underline{a}$ the line sums of g are the coefficients of $\chi_g(\underline{x})$ "modulo" $f_{(\underline{a},\beta)}$. The polynomials are pairwise coprime except for some well-described special cases, when they are conjugate. Therefore the polynomial F_S defined below represents the least common multiple of the polynomials $f_{(\underline{a}_k,\beta_k)}$. Let $S = \{(\underline{a}_k, \beta_k) \ : \ k = 1, \ldots, d\}$ be a set, where for each k, $\underline{a}_k$ is a direction and β_k is a real algebraic number with $\beta_k \geq 1$ of degree r_k. Two elements $(\underline{a}_k, \beta_k)$ and $(\underline{a}_c, \beta_c)$ of S are equivalent, if $\underline{a}_k = \underline{a}_c$ and β_k and β_c are algebraically conjugated elements, or $\underline{a}_k = -\underline{a}_c$ and β_k and $1/\beta_c$ are algebraically conjugated elements. Let S^* be a subset of S containing exactly one element of S from each class of this equivalence relation. Put

$$F_S(\underline{x}) = \prod_{(\underline{a}_k,\beta_k)\in S^*} f_{(\underline{a}_k,\beta_k)}(\underline{x}) .$$

We say that S is valid for A, if $N_j := \deg_{x_j}(F_S(\underline{x})) < m_j$ $(j = 1, \ldots, n)$. Put $U = \{\underline{u} \in \mathbb{Z}^n : 0 \leq u_j < m_j - N_j \ (j = 1, \ldots, n)\}$. For $\underline{u} \in U$ set $F_{(\underline{u};S)}(\underline{x}) = \underline{x}^{\underline{u}} F_S(\underline{x})$, and define the functions $M_{(\underline{u};S)} : \ A \to \mathbb{Z}$ by

$$M_{(\underline{u};S)}(\underline{i}) = \text{coeff}(\underline{x}^{\underline{i}}) \text{ in } F_{(\underline{u};S)}(\underline{x}) \text{ for } \underline{i} \in A .$$

The functions $M_{(\underline{u};S)}$ are called the switching atoms corresponding to the set S. By the minimal corner of the switching atom $M_{(\underline{0};S)}$ we mean the element $\underline{i}^*$ which is lexicographically the first element of A for which the function value of $M_{(\underline{0};S)}$ is non-zero. The minimal corner of $M_{(\underline{u};S)}$ is $\underline{i}^* + \underline{u}$.

Our main result in this section shows that switching configurations can be obtained as combinations of shifts of the switching atom $M_{(\underline{0};S)}$ also in the case of emission tomography.

Theorem 3. *Let A, S and $M_{(\underline{u};S)}$ be as above, with the assumption that S is valid for A. Then any function $g:\ A \to \mathbb{Z}$ with zero directed absorption line sums corresponding to the pairs $(\underline{a}_k, \beta_k)$ of S can be uniquely written in the form*

$$g = \sum_{\underline{u} \in U} c_{\underline{u}} M_{(\underline{u};S)}$$

with $c_{\underline{u}} \in \mathbb{Z}$ ($\underline{u} \in U$). Moreover, every such function g has zero directed absorption line sums corresponding to the elements of S.

Remark 5. Note that if S is not valid for A, then there is no non-trivial f having zero directed absorption line sums in the directions given by S. This fact simply follows from the proof of Theorem 3.

As an illustration, we give two examples (partly from [13]).

Example 4. First we consider a similar situation as Kuba and Nivat do in [19], however, in $\mathbb{Z}^3$. Let $S = \{((-1,0,0),\beta),((0,1,0),\beta),((0,0,1),\beta)\}$, where $\beta = (1+\sqrt{5})/2$. Then we have $P_\beta(z) = z^2 - z - 1$ and

$$f_{((-1,0,0),\beta)}(x_1,x_2,x_3) = -x_1^2 - x_1 + 1,\ \ f_{((0,1,0),\beta)}(x_1,x_2,x_3) = x_2^2 - x_2 - 1$$

and

$$f_{((0,0,1),\beta)}(x_1,x_2,x_3) = x_3^2 - x_3 - 1\ .$$

Thus we obtain

$$F_S(x_1,x_2,x_3) = (x_1^2x_2^2 - x_1^2x_2 - x_1^2 + x_1x_2^2 - x_1x_2 - x_1 - x_2^2 + x_2 + 1)(1 + x_3 - x_3^2)$$

and $N_1 = N_2 = N_3 = 2$. So if A is of type $m_1 \times m_2 \times m_3$ with $m_1, m_2, m_3 \geq 3$, then S is a valid set for A. Now $M_{(\underline{0};S)}$ is given by

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \end{array} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & -1 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \end{array}$$

where these tables represent the values of $M_{(\underline{0};S)}$ on the "slices" corresponding to the coefficients of $1, x_3, x_3^2$ in F_S, respectively. (All the other values are zero.) The switching atoms $M_{(\underline{u};S)}$ ($\underline{u} \in U$) form a basis of the set of functions $g:\ A \to \mathbb{Z}$ having zero line sums corresponding to the three elements of S.

Example 5. Now we consider an example for $n = 2$ where both opposite directions and different exponential absorption coefficients occur. Let

$$S = \{((-1,0),\beta),((1,0),\beta),((0,-1),\gamma),((0,1),\delta)\}$$

with $\beta = (1+\sqrt{5})/2$, $\gamma = 2+\sqrt{2}$ and $\delta = \gamma/2$. We obtain $P_\beta(z) = z^2 - z - 1$, $P_\gamma(z) = z^2 - 4z + 2$ and $P_\delta(z) = 2z^2 - 4z + 1$. We have

$$f_{((-1,0),\beta)}(x_1,x_2) = -x_1^2 - x_1 + 1, \quad f_{((1,0),\beta)}(x_1,x_2) = x_1^2 - x_1 - 1$$

and

$$f_{((0,-1),\gamma)}(x_1,x_2) = f_{((0,1),\delta)}(x_1,x_2) = 2x_2^2 - 4x_2 + 1 ,$$

as γ and $1/\delta$ are associated elements. We get

$$F_S(x_1,x_2) = -2x_1^4x_2^2 + 4x_1^4x_2 - x_1^4 + 6x_1^2x_2^2 - 12x_1^2x_2 + 3x_1^2 - 2x_2^2 + 4x_2 - 1$$

and $N_1 = 4$, $N_2 = 2$. So if A is of type $m_1 \times m_2$ with $m_1 \geq 5$ and $m_2 \geq 3$, then S is a valid set for A. Now $M_{(\underline{0};S)}$ is given by

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
-2 & 0 & 6 & 0 & -2 & 0 & \dots & 0 \\
4 & 0 & -12 & 0 & 4 & 0 & \dots & 0 \\
-1 & 0 & 3 & 0 & -1 & 0 & \dots & 0
\end{array}$$

and the switching atoms $M_{(\underline{u};S)}$ ($\underline{u} \in U$) form a basis of the set of functions $g:\ A \to \mathbb{Z}$ having zero line sums corresponding to the four elements of S.

To prove Theorem 3, we need several lemmas. To keep this exposition self-contained, we include their proofs. Lemma 2 shows the correspondence between zero line sums and division by polynomials. Note that line sums of functions $A \to L$ are defined in the obvious way.

Lemma 2. *Let A be as before, $\underline{a}$ a direction, and β a non-zero algebraic number. Let L be some field containing the splitting field of $P_\beta(z)$. Put*

$$\tilde{f}_{(\underline{a},\beta)}(\underline{x}) = (\underline{x}^{\underline{a}} - \beta) \prod_{a_j<0} x_j^{-a_j} .$$

Then a function $g:\ A \to L$ has zero line sums corresponding to the pair $(\underline{a},\beta)$ if and only if $\tilde{f}_{(\underline{a},\beta)}(\underline{x})$ divides $\chi_g(\underline{x})$ in $L[\underline{x}]$.

Proof. We prove the lemma only with $a_j > 0$ ($j = 1,\dots,n$), as the other cases can be treated similarly.

Put $B = \{\underline{b} \in A : \underline{b} + \underline{a} \notin A\}$ and let $I_{\underline{b}}$ be the number of the points of A on the line $\underline{b} - t\underline{a}$ ($\underline{b} \in B, t \in \mathbb{Z}$). Observe that we may write

$$\chi_g(\underline{x}) = \sum_{\underline{b}\in B} \sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - s\underline{a})\underline{x}^{\underline{b}-s\underline{a}} = \sum_{\underline{b}\in B} \underline{x}^{\underline{b}} \sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - s\underline{a})\underline{x}^{-s\underline{a}} .$$

If $\underline{x}^{\underline{a}} - \beta$ divides $\chi_g(\underline{x})$ in $L[\underline{x}]$, then after substituting $x_1 \leftarrow \beta^{1/a_1} \prod_{j=2}^{n} x_j^{a_j/a_1}$ the polynomial $\chi_g(\underline{x})$ becomes identically zero. This yields that $\sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - s\underline{a})\beta^{-s}$ vanishes for every $\underline{b} \in B$, hence g has zero absorption line sums corresponding to $(\underline{a}, \beta)$. This proves the 'if' part of the statement.

To prove the 'only if' part, suppose that all the line sums

$$\sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - s\underline{a})\beta^{-s-1} = \beta^{-I_{\underline{b}}} \sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - (I_{\underline{b}} - s - 1)\underline{a})\beta^{s} \quad (\underline{b} \in B)$$

of g corresponding to $(\underline{a}, \beta)$ vanish. This means that β is a root of the polynomial $Q_{\underline{b}}(z) := \sum_{s=0}^{I_{\underline{b}}-1} g(\underline{b} - (I_{\underline{b}} - s - 1)\underline{a})z^s$ for each $\underline{b} \in B$. Thus for every $\underline{b} \in B$ the polynomial $Q_{\underline{b}}(\underline{x}^{\underline{a}})$ is divisible by $\underline{x}^{\underline{a}} - \beta$ over L. Hence $\underline{x}^{\underline{a}} - \beta$ divides $\chi_g(\underline{x}) = \sum_{\underline{b} \in B} \underline{x}^{\underline{b}+(1-I_{\underline{b}})\underline{a}} Q_{\underline{b}}(\underline{x}^{\underline{a}})$ in $L[\underline{x}]$, and the lemma follows. □

Lemma 3. *Using the notation of Lemma 2, write r for the degree and $\beta^{(c)}$ $(1 \leq c \leq r)$ for the conjugates of β. Then the polynomials $\tilde{f}_{(\underline{a},\beta^{(c)})}(\underline{x})$ $(1 \leq c \leq r)$ defined in Lemma 2 are pairwise non-associated irreducible elements in $L[\underline{x}]$.*

Proof. As $\gcd(a_1, \ldots, a_n) = 1$, the irreducibility of these polynomials is a simple consequence of Corollary 2 of [23] p. 103. The statement that the polynomials are pairwise non-associated, is trivial. □

Corollary 2. *The polynomials $P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j}$ are irreducible in $\mathbb{Z}[\underline{x}]$.*

Proof. We prove the statement only for $a_j > 0$ $(j = 1, \ldots, n)$, the other cases are similar.

Let $\beta^{(c)}$ $(1 \leq c \leq r)$ be the conjugates of β, and let L be the splitting field of P_β over $\mathbb{Q}$. Then, in view of

$$P_\beta(\underline{x}^{\underline{a}}) = c_0 \prod_{c=1}^{r} (\underline{x}^{\underline{a}} - \beta^{(c)})$$

where c_0 is the leading coefficient of P_β, the statement immediately follows from Lemma 3. □

In the next lemma we show that the divisibility property of χ_g over L in Lemma 2 implies a stronger property over $\mathbb{Z}$.

Lemma 4. *Let $\underline{a}$ and β be as in Lemma 2. Using the previous notation, a function $g: A \to \mathbb{Z}$ has zero line sums corresponding to the pair $(\underline{a}, \beta)$ if and only if $P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j}$ divides $\chi_g(\underline{x})$ in $\mathbb{Z}[\underline{x}]$.*

Proof. The 'if' part of the statement easily follows from Lemma 2. We prove the 'only if' part only for $a_j > 0$ ($j = 1, \ldots, n$), the other cases can be handled similarly. In this case observe that by Lemma 2, $\underline{x}^{\underline{a}} - \beta$ divides $\chi_g(\underline{x})$ over any field L which contains the splitting field of $P_\beta(z)$. However, by conjugation, for every conjugate $\beta^{(c)}$ of β, $\underline{x}^{\underline{a}} - \beta^{(c)}$ also divides $\chi_g(\underline{x})$ in $L[\underline{x}]$. By Lemma 3 this assertion immediately implies the statement. □

It follows from Corollary 2 and the following Lemma 5 that the division polynomials in non-parallel directions are coprime, and in parallel directions are coprime or associated.

Lemma 5. *Let $\underline{a}, \underline{a}^*$ be directions, and β, β^* be non-zero algebraic numbers of degrees r and r^*, respectively. Then the polynomials $P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j}$ and $P_{\beta^*}(\underline{x}^{\underline{a}^*}) \prod_{a_j^*<0} x_j^{-r^*a_j^*}$ are associated in $\mathbb{Z}[\underline{x}]$ if and only if either $\underline{a} = \underline{a}^*$ and β and β^* are conjugated, or $\underline{a} = -\underline{a}^*$ and β and $1/\beta^*$ are conjugated.*

Proof. The 'if' part of the statement is trivial. Suppose that $P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j}$ and $P_{\beta^*}(\underline{x}^{\underline{a}^*}) \prod_{a_j^*<0} x_j^{-r^*a_j^*}$ are associated. Then the degrees of β and β^* must be equal, i.e. $r = r^*$. For $1 \le c \le r$ let $\beta^{(c)}$ and $\beta^{*(c)}$ be the conjugates of β and β^*, respectively. Let L be any field which contains the splitting fields of both P_β and P_{β^*}. Then we have the factorizations

$$P_\beta(\underline{x}^{\underline{a}}) \prod_{a_j<0} x_j^{-ra_j} = \prod_{c=1}^{r} \tilde{f}_{(\underline{a},\beta^{(c)})}(\underline{x})$$

and

$$P_{\beta^*}(\underline{x}^{\underline{a}^*}) \prod_{a_j^*<0} x_j^{-r^*a_j^*} = \prod_{c=1}^{r} \tilde{f}_{(\underline{a}^*,\beta^{*(c)})}(\underline{x})$$

in $L[\underline{x}]$, where the polynomials on the right hand sides are defined in Lemma 2. By our assumption and Lemma 3 we obtain that for each c_1 with $1 \le c_1 \le r$ there exists a c_2 also with $1 \le c_2 \le r$, such that $\tilde{f}_{(\underline{a},\beta^{(c_1)})}(\underline{x})$ and $\tilde{f}_{(\underline{a}^*,\beta^{*(c_2)})}(\underline{x})$ are associated elements in $L[\underline{x}]$. By comparing the exponents of x_j ($j = 1, \ldots, n$) in these polynomials, we get that $\underline{a} = \pm\underline{a}^*$ holds, and for the corresponding pairs (c_1, c_2), $\beta^{(c_1)} = \beta^{*(c_2)}$ or $\beta^{(c_1)}\beta^{*(c_2)} = 1$ is valid, respectively. This yields that $\{\beta^{(c)} : 1 \le c \le r\} = \{\beta^{*(c)} : 1 \le c \le r\}$ or $\{\beta^{(c)} : 1 \le c \le r\} = \{1/\beta^{*(c)} : 1 \le c \le r\}$, respectively, which establishes the 'only if' part of the statement. The proof of the lemma is now complete. □

Proof (of Theorem 3). By definition, for every $\underline{u} \in U$ the function $F_{(\underline{u};S)}$ is divisible by $f_{(\underline{a}_k,\beta_k)}$ for any k with $1 \le k \le d$. Hence by Lemma 2 $M_{(\underline{u};S)}$

has zero line sums corresponding to the pairs in S. This proves the second statement of the theorem.

Let

$$H = \{f : A \to \mathbb{Z} \mid f \text{ has zero absorption line sums for the elements of } S\} .$$

We first prove that the switching atoms $M_{(\underline{u};S)}$ ($\underline{u} \in U$) generate H. Combining Corollary 2 and Lemmas 4 and 5, for any $g \in H$ we obtain

$$F_S(\underline{x}) \mid \chi_g(\underline{x}) \text{ in } \mathbb{Z}[\underline{x}] .$$

Hence there exists a polynomial $Q(\underline{x}) = \sum_{\underline{u} \in U} c_{\underline{u}} \underline{x}^{\underline{u}}$ with $c_{\underline{u}} \in \mathbb{Z}$ ($\underline{u} \in U$) such that $Q(\underline{x})F_S(\underline{x}) = \chi_g(\underline{x})$. We rewrite this equation as

$$\chi_g(\underline{x}) = \sum_{\underline{u} \in U} c_{\underline{u}} F_{(\underline{u};S)}(\underline{x}) .$$

Now by the definitions of $\chi_g(\underline{x})$ and the switching atoms $M_{(\underline{u};S)}$ we immediately obtain

$$g = \sum_{\underline{u} \in U} c_{\underline{u}} M_{(\underline{u};S)} ,$$

which proves that the functions $M_{(\underline{u};S)}$ generate H.

Suppose now that for some coefficients $l_{\underline{u}} \in \mathbb{Z}$ we have

$$\sum_{\underline{u} \in U} l_{\underline{u}} M_{(\underline{u};S)}(\underline{i}) = 0 \text{ for } \underline{i} \in A . \tag{5}$$

By the definitions of the switching atoms, at the minimal corner of $M_{(\underline{0};S)}$ all the other switching atoms vanish. This immediately implies $l_{\underline{0}} = 0$. Considering now $M_{(\underline{u};S)}$ with $\underline{u} \in U$ in increasing lexicographical order, we conclude that all the coefficients $l_{\underline{u}}$ are zero in (5). This shows that the switching atoms are linearly independent, which completes the proof of the theorem. □

Remark 6. Similarly as in case of the classical problem of discrete tomography in Section 3, it would be possible to provide an algorithm that produces a "small" integral solution to (1) in case of emission tomography. We omit the details.

5 Tomography on curves

In this section we illustrate that our method is rather flexible in the sense that variations to other sums than line sums are possible. In this more general case there do not exist translation invariant switching atoms. However, our polynomial method allows us to construct non-trivial configurations with vanishing sums and characterize such configurations in Theorems 4 and 5.

We shall illustrate the method in two dimensions by examples where sums are taken over sets of the shape $H_k = \{(i,j) \in A : a_k j = b_k G(i) + t\}$ where $G : \mathbb{Z} \to \mathbb{Z}$, $t \in \mathbb{Z}$ and the (a_k, b_k) are distinct pairs of co-prime integers for $k = 1, \ldots, d$. The basic idea is that to the given function $g : A \to \mathbb{Z}$ we adjoin the "generating" polynomial $\sum_{(i,j)\in A} g(i,j) x^{G(i)} y^j$ (instead of $\sum_{(i,j)\in A} g(i,j) x^i y^j$). Since $a_k j = b_k G(i) + t$ the exponent pairs $(G(i), j)$ for $(i,j) \in H_k$ are on the lines $a_k y = b_k x + t$. So the sums over H_k turn into line sums and we can apply the preceding theory. Doing so we find switching atoms. The problem is to return to the original situation, where there is no linear structure. However, by constructing polynomials with exponents of prescribed form which are multiples of the switching atom polynomial, we are able to construct configurations with vanishing sums for all given H_k. We give two examples.

Example 6 (broken line sums). We consider the situation where light (or X-ray) entering from the left along the halfline $ay = bx + t$ $(x \le 0)$ is broken when reaching the y-axis and continues along the halfline $ay = cbx + t$ $(x > 0)$, where c is a given integer.

To describe this case, we slightly need to adjust our previous settings. Let m_1, m_2 be positive integers and n_1 a negative integer. Put

$$A = \{(i,j) \in \mathbb{Z}^2 : n_1 \le i < m_1, 0 \le j < m_2\} ,$$

and let a_k, b_k $(k = 1, \ldots, d)$ and c be non-zero integers with $\gcd(a_k, b_k) = 1$ and $a_k \ge 0$ $(k = 1, \ldots, d)$. Set

$$T_{kt} = \{(i,j) \in \mathbb{Z}^2 : i \le 0,\ a_k j = b_k i + t\} \cup \{(i,j) \in \mathbb{Z}^2 : i > 0,\ a_k j = c b_k i + t\}$$

for $k = 1, \ldots, d$ and $t \in \mathbb{Z}$. Let $(i_1, j_1) \overset{k}{\sim} (i_2, j_2)$ for $(i_1, j_1), (i_2, j_2) \in A$ and $k = 1, \ldots, d$ if and only if these points belong to the same set T_{kt} for some integer t. Write $H_1^{(k)}, \ldots, H_{t_k}^{(k)}$ for the equivalence classes of $\overset{k}{\sim}$ on A. These classes are in fact the intersections of the broken lines T_{kt} with A. By the broken line sums corresponding to (a_k, b_k) of a given function $g : A \to \mathbb{Z}$ we mean the expressions

$$c_{kl} := \sum_{(i,j)\in H_l^{(k)}} g(i,j) \quad for\ k = 1, \ldots, d;\ l = 1, \ldots, t_k . \tag{6}$$

Note that (6) is a special case of equation (1), with unit weights $\varrho_k = 1$ $(k = 1, \ldots, d)$.

With the above modifications we can apply our machinery to the broken line case as well. First we introduce some further notation.

Let $S = \{(a_k, b_k)\}_{k=1}^d$ with (a_k, b_k) as above, and write $N_1 = \sum_{k=1}^d a_k$ and $N_2 = \sum_{k=1}^d |b_k|$. We say that S is valid for A, if $N_1 < m_1 - n_1$ and $N_2 < m_2$.

For $k = 1, \ldots, d$ put

$$f_k(x,y) = \begin{cases} x^{a_k}y^{b_k} - 1, & \text{if } b_k \geq 0 \\ x^{a_k} - y^{-b_k}, & \text{if } b_k < 0 \ , \end{cases}$$

and set $F_S(x,y) = \prod_{k=1}^{d} f_k(x,y)$.

In view of the broken lines, we define

$$\chi_g(x,y) = x^{-n_1}\left(\sum_{i=n_1}^{0}\sum_{j=0}^{m_2-1} g(i,j)x^i y^j + \sum_{i=1}^{m_1-1}\sum_{j=0}^{m_2-1} g(i,j)x^{ci}y^j\right) .$$

as the "generating" polynomial of $g: \ A \to \mathbb{Z}$. Note that the factor x^{-n_1} is introduced only to keep the exposition inside $\mathbb{Z}[x,y]$.

For the solutions of (6) we have the following

Theorem 4. *Let A and S be as above, with the assumption that S is valid for A. Then a function $g: \ A \to \mathbb{Z}$ has zero broken line sums corresponding to S if and only if $\chi_g(x,y)$ is divisible by $F_S(x,y)$ in $\mathbb{Z}[x,y]$.*

Proof. Let $g: \ A \to \mathbb{Z}$ be an arbitrary function and let $(a,b) \in S$. For simplicity we assume that $b \geq 0$, the case when $b < 0$ is similar. Observe that we can write

$$\chi_g(x,y) = x^{-n_1}\sum_{t\in\mathbb{Z}}\left(\sum_{i=n_1}^{0}\sum_{\substack{aj=bi+t\\ 0\leq j<m_2}} g(i,j)x^i y^j + \sum_{i=1}^{m_1-1}\sum_{\substack{aj=cbi+t\\ 0\leq j<m_2}} g(i,j)x^{ci}y^j\right) =$$

$$x^{-n_1}\sum_{b\in\mathbb{Z}} y^{t/a}\left(\sum_{i=n_1}^{0}\sum_{\substack{aj=bi+t\\ 0\leq j<m_2}} g(i,j)(xy^{b/a})^i + \sum_{i=1}^{m_1-1}\sum_{\substack{aj=cbi+t\\ 0\leq j<m_2}} g(i,j)(xy^{b/a})^{ci}\right) .$$

Now just as previously (see e.g. the proof of Theorem 1) we obtain that g has zero broken line sums corresponding to $(a,b) \in S$ if and only if $x^a y^b - 1$ divides $\chi_g(x,y)$ in $\mathbb{Z}[x,y]$. Observing that the polynomials $f_k(x,y)$ $(k = 1, \ldots, d)$ are pairwise coprime (in fact prime) elements of $\mathbb{Z}[x,y]$, the theorem follows. $\square$

We illustrate the above theory by the example when $S = \{(1,1),(3,1)\}$ and $c = 2$. In this case the broken line sums are calculated in accordance with Figure 2. Moreover, we have

$$F_S(x,y) = (xy-1)(x^3y-1) = x^4y^2 - x^3y - xy + 1 .$$

Theorem 4 gives that $g: \ A \to \mathbb{Z}$ has zero broken line sums corresponding to S if and only if F_S divides χ_g over $\mathbb{Z}$. Hence to present a non-trivial example, we

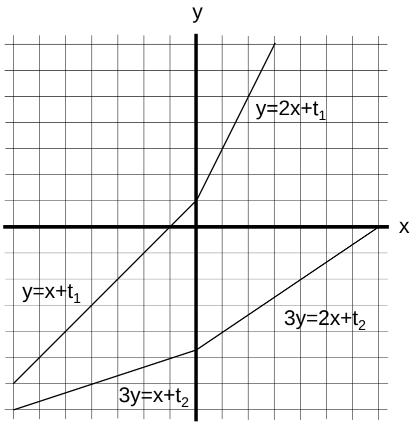


Fig. 2. Broken lines corresponding to $S = \{(1,1),(3,1)\}$ and $c = 2$.

should find a non-zero multiple of F_S in which all the exponents of x greater than some non-negative integer are even. For switching configurations entirely contained in $\{(x,y) : x \leq 0\}$ or in $\{(x,y) : x > 0\}$ the theory of Section 3 applies. Suppose we want a switching configuration with "minimal corner" at $(-3,0)$. Then all exponents of x in χ_g greater than 3 should be odd. We have

$$(xy+1)F_S(x,y) = x^5y^3 - x^3y - x^2y^2 + 1 = x^3(x^2y^3 - y - x^{-1}y^2 - x^{-3}) \ .$$

Hence if $n_1 \leq -3$, $m_1 \geq 3$ and $m_2 \geq 4$ then the function $g : \ A \to \mathbb{Z}$ represented by

$$\begin{array}{ccccccccccc}
0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 & & & & & & \uparrow & & & &
\end{array}$$

has zero broken line sums along the corresponding broken lines. Here $\uparrow$ indicates the y-axis.

Example 7 (parabola sums). We consider the situation when the X-rays (or light) pass along parabolas $ay = bx^2 + t$ $(x \geq 0)$.

Let A be as before, and let a_k, b_k be coprime non-zero integers with $a_k \geq 0$ $(k = 1, \ldots, d)$. Let $(i_1, j_1) \overset{k}{\sim} (i_2, j_2)$ for $(i_1, j_1), (i_2, j_2) \in A$ and $k = 1, \ldots, d$ if and only if $b_k(i_1^2 - i_2^2) = a_k(j_1 - j_2)$ (i.e. for some integer t_k we have $b_k i_1^2 = a_k j_1 - t_k$ and $b_k i_2^2 = a_k j_2 - t_k$, that is, these points lay on the same vertical translate of the graph of the function $a_k y = b_k x^2$). Further, write $H_1^{(k)}, \ldots, H_{t_k}^{(k)}$ for the equivalence classes of $\overset{k}{\sim}$ on A. Let a function $g : \ A \to \mathbb{Z}$ be given. By the parabola sums of g corresponding to (a_k, b_k) we mean the expressions

$$c_{kl} := \sum_{(i,j)\in H_l^{(k)}} g(i,j) \quad \text{for } k=1,\dots,d;\ l=1,\dots,t_k\ . \tag{7}$$

Obviously, (7) is a special case of equation (1) with $\varrho_k = 1$ $(k=1,\dots,d)$.

As it will turn out, with the modifications indicated above we can apply our previous results to this case. We need, however, some notation. Let S, N_1, N_2, $f_k(x,y)$ and $F_S(x,y)$ be defined as in case of broken lines.

We choose

$$\chi_g(x,y) = \sum_{(i,j)\in A} g(i,j)x^{i^2}y^j$$

as the "generating" polynomial of $g:\ A\to\mathbb{Z}$.

For the solutions of (7) we have the following

Theorem 5. *Let A and S be as above, with the assumption that S is valid for A. Then a function $g:\ A\to\mathbb{Z}$ has zero parabola sums corresponding to S if and only if $\chi_g(x,y)$ is divisible by $F_S(x,y)$ in $\mathbb{Z}[x,y]$.*

Proof. Let $g:\ A\to\mathbb{Z}$ be an arbitrary function and let $(a,b)\in S$. For simplicity we assume that $b\geq 0$, the case when $b<0$ is similar. Observe that we can write

$$\chi_g(x,y) = \sum_{t\in\mathbb{Z}} \sum_{\substack{aj=bi^2+t\\ (i,j)\in A}} g(i,j)x^{i^2}y^j = \sum_{t\in\mathbb{Z}} y^{t/a} \sum_{\substack{aj=bi^2+t\\ (i,j)\in A}} g(i,j)(xy^{b/a})^{i^2}\ .$$

Now similarly as e.g. in the proof of Theorem 1, one can easily verify that g has zero parabola sums corresponding to $(a,b)\in S$ if and only if x^ay^b-1 divides $\chi_g(x,y)$ in $\mathbb{Z}[x,y]$. As the polynomials $f_k(x,y)$ $(k=1,\dots,d)$ are pairwise coprime elements of $\mathbb{Z}[x,y]$, the theorem follows. □

We illustrate the example by analyzing two particular cases. We start with $S=\{(1,1),(1,2)\}$, i.e. the parabolas are given by $y=x^2+t_1$ and $y=2x^2+t_2$, respectively. In this case we have

$$F_S(x,y) = (xy-1)(xy^2-1) = x^2y^3 - xy^2 - xy + 1\ .$$

Theorem 5 gives that $g:\ A\to\mathbb{Z}$ has zero parabola sums corresponding to S if and only if F_S divides χ_g over $\mathbb{Z}$. The problem, however, is to find some non-zero multiple of F_S such that all the exponents of x are squares. Suppose we want a switching configuration with "minimal corner" at the origin. One can readily verify that

$$(x^2y^4+xy^3+xy^2+y^2+y+1)F_S(x,y) = x^4y^7-xy^4-xy^3-xy^2-xy+y^2+y+1\ .$$

Thus if $m_1\geq 2$ and $m_2\geq 8$ then the function $g:\ A\to\mathbb{Z}$ represented by

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & -1 & 0 & 0 & \ldots & 0 \\
0 & -1 & 0 & 0 & \ldots & 0 \\
1 & -1 & 0 & 0 & \ldots & 0 \\
1 & -1 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
\uparrow & & & & &
\end{array}
$$

provides a non-trivial configuration having zero parabola sums along the parabolas $y = x^2 + t_1$ and $y = 2x^2 + t_1$ for any $t_1, t_2 \in \mathbb{Z}$.

Finally, we consider $S = \{(1,1),(1,2),(1,3)\}$, i.e. we have three parabolas given by $y = x^2 + t_1$, $y = 2x^2 + t_2$ and $y = 3x^2 + t_3$, respectively. Now we have

$$
\begin{aligned}
F_S(x,y) = (xy-1)(xy^2-1)(xy^3-1) = \\
= x^3y^6 - x^2y^5 - x^2y^4 - x^2y^3 + xy^3 + xy^2 + xy - 1 \ .
\end{aligned}
$$

By Theorem 5 we know that $g: \ A \to \mathbb{Z}$ has zero parabola sums corresponding to S if and only if F_S divides χ_g over $\mathbb{Z}$. The problem is again to find some non-zero multiple of F_S in which all the exponents of x are squares. One can easily check that the polynomial

$$
\begin{aligned}
&(y^{26}+y^{25}+2y^{24}+y^{23}+y^{22})x^9-(y^{21}+y^{20}+2y^{19}+2y^{18}+3y^{17}+3y^{16}+4y^{15}+ \\
&+4y^{14}+4y^{13}+3y^{12}+3y^{11}+2y^{10}+2y^9+y^8+y^7)x^4+(y^{15}+2y^{14}+4y^{13}+ \\
&+6y^{12}+8y^{11}+9y^{10}+10y^9+10y^8+10y^7+9y^6+8y^5+6y^4+4y^3+2y^2+y)x-(y^{12}+ \\
&+2y^{11}+4y^{10}+5y^9+7y^8+7y^7+8y^6+7y^5+7y^4+5y^3+4y^2+2y+1)
\end{aligned}
$$

is a multiple of F_S in $\mathbb{Z}[x,y]$. Hence we obtain a non-trivial $g: \ A \to \mathbb{Z}$ having zero parabola sums along the three parabolas by replacing x^9 with x^3 and x^4 with x^2 and making the corresponding table.

Acknowledgments

Research supported in part by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences, and by the OTKA grants F043090, F034981 and T042985.

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