ALGEBRAIC INDEPENDENCE RESULTS RELATED TO PATTERN SEQUENCES IN DISTINCT $\langle q, r \rangle$-NUMERATION SYSTEMS

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Abstract. In this paper, we prove the algebraic independence over $C(z)$ of the generating functions of pattern sequences defined in distinct $\langle q, r \rangle$-numeration systems. Our result asserts that any nontrivial linear combination over C of pattern sequences chosen from distinct $\langle q, r \rangle$-numeration systems can not be a linear recurrence sequence. As an application, we give a linear independence over C of the pattern sequences.

1. Introduction and main results. Let $q \geq 2$ and r be fixed integers with $r \in \{0, 1, \ldots, q - 2\}$. Any positive integer n is uniquely expressed as

$$n = \sum_{i=0}^{k} a_i q^i, \quad a_i \in \Sigma_{q,r}, \quad a_k > 0,$$

for an integer $k \geq 0$, where $\Sigma_{q,r} := \{-r, 1 - r, \ldots, 0, 1, \ldots, q - 1 - r\} \supset \{0, 1\}$. The set of all finite nonempty strings of elements in $\Sigma_{q,r}$ is denoted by $\Sigma_{q,r}^*$. Then the string of $\langle q, r \rangle$-digits

$$(n)_{q,r} := a_k \cdots a_1 a_0 \in \Sigma_{q,r}^*$$

is called the $\langle q, r \rangle$-expansion of n. The $\langle q, 0 \rangle$-expansion is the ordinary q-ary expansion. These numeration systems are called $\langle q, r \rangle$-numeration systems.

For $w \in \Sigma_{q,r}^*$, we write $w^k = ww \cdots w$ (k times). In particular w^0 denotes the empty word. If $w \neq 0^l$ for any $l \geq 1$, we say that w is a *nonzero pattern*. For a nonzero pattern $w \in \Sigma_{q,r}^*$, we define $e_{q,r}(w; 0) = 0$ and $e_{q,r}(w; n)$ ($n \geq 1$) to be the number of (possibly overlapping) occurrences of w in the $\langle q, r \rangle$-expansion of n. Here, in evaluating $e_{q,r}(w; n)$ for $n \geq 1$, we suppose that the $\langle q, r \rangle$-expansion of n has an arbitrary long string of zeros on the left. The resulting sequence $\{e_{q,r}(w; n)\}_{n \geq 0}$ is called the *pattern sequence* for the pattern w in the $\langle q, r \rangle$-numeration system (cf. Allouche and Shallit [1]).

Define the generating function of the pattern sequence for $w \in \Sigma_{q,r}^*$ by

$$f(w; z) := \sum_{n \geq 0} e_{q,r}(w; n) z^n.$$

For any fixed integer $q \geq 2$ and any patterns $w_1, \ldots, w_m \in \Sigma_{q,0}^*$, Uchida [10] gave necessary and sufficient conditions for the generating functions $f(w_1; z), \ldots, f(w_m; z)$ to be

algebraically dependent over $\boldsymbol{C}(z)$. Shiokawa and the author [7, 8] generalized this results to any fixed $\langle q, r \rangle$-numeration systems and studied the linear relations between the pattern sequences. Recently, the author [9] proved that the generating functions of pattern sequences defined in distinct q-ary number systems are algebraically independent over $\boldsymbol{C}(z)$.

In this paper, we prove the algebraic independence over $\boldsymbol{C}(z)$ of the generating functions of pattern sequences defined in distinct $\langle q, r \rangle$-numeration systems. The tools to prove the following main theorem are also similar to those in [9], however we need some different techniques.

THEOREM 1.1. *Let* $w_{q,r} \in \Sigma_{q,r}^*$ $(q = 2, 3, \ldots, \ r = 0, 1, \ldots, q - 2)$ *be any nonzero patterns. Then the functions*

$$(1) \qquad f_{q,r}(z) = \sum_{n \geq 0} e_{q,r}(w_{q,r}; n) z^n \quad (q = 2, 3, \ldots, \ r = 0, 1, \ldots, q - 2)$$

are algebraically independent over $\boldsymbol{C}(z)$. *Furthermore, the values* $f_{q,r}(\alpha)$ $(q = 2, 3, \ldots,$ $r = 0, 1, \ldots, q - 2)$ *are algebraically independent over* $\boldsymbol{Q}$ *for any algebraic number* α *with* $0 < |\alpha| < 1$.

Applying Theorem 1.1, we see that any nontrivial linear combination over $\boldsymbol{C}$ of the functions (1) cannot be a rational function, namely,

$$\sum_{q=2}^{m} \sum_{r=0}^{q-2} c_{q,r} f_{q,r}(z) \notin \boldsymbol{C}(z)$$

for an arbitrary integer $m \geq 2$ and $c_{q,r} \in \boldsymbol{C}$ not all zero. Hence we have the following corollary.

COROLLARY 1.2. *Let* $w_{q,r} \in \Sigma_{q,r}^*$ $(q = 2, 3, \ldots, \ r = 0, 1, \ldots, q - 2)$ *be any nonzero patterns and* $c_{q,r} \in \boldsymbol{C}$ *not all zero. Then the linear combination of the pattern sequences*

$$\left\{ \sum_{q=2}^{m} \sum_{r=0}^{q-2} c_{q,r} e_{q,r}(w_{q,r}; n) \right\}_{n \geq 0}$$

cannot be a linear recurrence sequence. In particular, the sequences $\{e_{q,r}(w_{q,r}; n)\}_{n \geq 0}$ $(q = 2, 3, \ldots, \ r = 0, 1, \ldots, q - 2)$ *are linearly independent over* $\boldsymbol{C}$.

EXAMPLE 1.3. Let $w = a_k \cdots a_0$ be a nonzero pattern with $a_i \in \{0, 1\}$. Then $e_{2,0}(w; n)$ and $e_{3,1}(w; n)$ indicate the number of occurrences of w in the binary and the balanced ternary expansions of n, respectively, and any nontrivial linear combination over $\boldsymbol{C}$

$$\{c_1 e_{2,0}(w; n) + c_2 e_{3,1}(w; n)\}_{n \geq 0}$$

cannot be a linear recurrence sequence.

EXAMPLE 1.4.   For any integer $q \geq 2$, the pattern sequences

$$\{e_{2,0}(1; n)\}_{n \geq 0}, \ \{e_{3,0}(1; n)\}_{n \geq 0}, \ \{e_{3,1}(1; n)\}_{n \geq 0}, \ldots,$$

$$\{e_{q,0}(1; n)\}_{n \geq 0}, \ldots, \{e_{q,q-2}(1; n)\}_{n \geq 0},$$

which are defined by the number of "1" appearing in each $\langle q, r \rangle$-numeration systems of n, are linearly independent over $\boldsymbol{C}$.

2.   Some properties of pattern sequences.   In this section, we show as Proposition 2.1 a non-periodicity for the linear combination of the pattern sequences defined in distinct $\langle q^i, r_i \rangle$-numeration systems ($i = 1, \ldots, m$). Proposition 2.1 will be used in the proof of Theorem 1.1 (see Section 4).

PROPOSITION 2.1.   *Let $q \geq 2$ be a fixed integer, $m \geq 1$ be any integer, and $w_{q^i, r_i} \in \Sigma^*_{q^i, r_i}$ ($i = 1, 2, \ldots, m$, $r_i \in \{0, 1, \ldots, q^i - 2\}$) be arbitrary nonzero patterns. Then for any $c_i \in \boldsymbol{C}$ ($i = 1, \ldots, m$), not all zero, the sequence*

$$\{c_1 e_{q, r_1}(w_{q, r_1}; n) + c_2 e_{q^2, r_2}(w_{q^2, r_2}; n) + \cdots + c_m e_{q^m, r_m}(w_{q^m, r_m}; n)\}_{n \geq 0}$$

cannot be purely periodic with a period q^l ($l \geq 1$).

To prove Proposition 2.1, we need some lemmas. In what follows, let $q \geq 2$ and $r \in \{0, 1, \ldots, q - 2\}$ be fixed integers. For any nonzero pattern $w = b_{l-1} b_{l-2} \cdots b_0 \in \Sigma^*_{q,r}$ with $b_i \in \Sigma_{q,r}$, let $|w|$ denote the length l and put

$$\nu(w) := \begin{cases} \displaystyle\sum_{k=0}^{l-1} b_k q^k & \text{if } \displaystyle\sum_{k=0}^{l-1} b_k q^k > 0, \\ q^l + \displaystyle\sum_{k=0}^{l-1} b_k q^k & \text{otherwise}. \end{cases}$$

By definition, $0 < \nu(w) < q^l$. The following Lemma 2.2 is a generalization of [9, Lemma 1] to the case of $\langle q, r \rangle$-numeration systems with $r = 1, 2, \ldots, q - 2$.

LEMMA 2.2.   *Let $m \geq 1$ and $d \geq 0$ be integers. Then for any nonzero pattern $w \in \Sigma^*_{q^m, r}$ with $w \neq 0^l 1$ ($l \geq 0$), we have*

(2) $$e_{q^m, r}(w; \nu(w)q^d) = \begin{cases} 1 & \text{if } d \equiv 0 \pmod{m}, \\ 0 & \text{otherwise}. \end{cases}$$

PROOF.   In the case of $r = 0$, the equality (2) holds for all nonzero patterns $w \in \Sigma^*_{q^m, 0}$ (cf. [9, Lemma 1]). Let $r \geq 1$ and

(3) $$w = 0^{l_1} a_k \cdots a_0 0^{l_2}, \quad a_j \in \Sigma_{q^m, r}, \quad a_k, a_0 \neq 0, \quad l_1, l_2 \geq 0.$$

Then we have

$$(\nu(w))_{q^m, r} = \begin{cases} a_k \cdots a_0 0^{l_2} & \text{if } a_k > 0, \\ 10^{l_1} a_k \cdots a_0 0^{l_2} & \text{otherwise}. \end{cases}$$

If d is divisible by m, noting that $a_k, a_0 \neq 0$, we see that the pattern w appears just once in the $\langle q^m, r \rangle$-expansion of $\nu(w)q^d$. Hence the equality (2) holds for this case. In what follows, we

assume that d is not divisible by m and put $d = mh + u$, where $h \geq 0$ and u ($1 \leq u \leq m - 1$) are integers.

Here we prove (2) with assuming $a_k > 0$. The case $a_k < 0$ is proved similarly. Let

(4) $\qquad (\nu(w)q^u)_{q^m,r} = b_s b_{s-1} \cdots b_0 0^{l_2} \in \Sigma^*_{q^m,r}, \quad b_j \in \Sigma_{q^m,r}, \quad b_s > 0$.

Then we have $(\nu(w)q^d)_{q^m,r} = (\nu(w)q^u(q^m)^h)_{q^m,r} = b_s b_{s-1} \cdots b_0 0^{l_2+h}$ with $s = k$ or $k+1$, since

$$k + 1 + l_2 = |(\nu(w))_{q^m,r}| \leq |(\nu(w)q^u)_{q^m,r}| \leq |(\nu(w)q^m)_{q^m,r}| = k + 2 + l_2.$$

Suppose on the contrary that $e_{q^m,r}(w; \nu(w)q^d) \geq 1$, that is, the pattern w appears at least once in the expansion (4). If $s = k$, then $w = 0^{l_1} b_k b_{k-1} \cdots b_0 0^{l_2}$, which implies $\nu(w) = \nu(w)q^u$ from (4). This is impossible, and hence $s = k + 1$. We distinguish the two cases.

Case I. $l_2 = 0$.   If $l_1 \geq 1$, noting that $b_{k+1} \neq 0$, we see that the pattern $w = 0^{l_1} a_k \cdots a_0$ coincides with $0^{l_1} b_{k+1} b_k \cdots b_1$. Hence we obtain by (4)

$$\nu(w)q^m + b_0 = \nu(b_{k+1} \cdots b_1)q^m + b_0 = \nu(b_{k+1} \cdots b_1 b_0) = \nu(w)q^u,$$

so that

$$0 < \nu(w) = \frac{-b_0}{q^m - q^u} \leq \frac{q^m - 2}{q^m - q^u} < 2,$$

which implies $\nu(w) = 1$, namely $w = 0^{l_1}1$. By the assumption, this is a contradiction.

Suppose that $l_1 = 0$. Then the pattern w must be of the form either $w = b_{k+1} \cdots b_1$ or $w = b_k \cdots b_0$. In a similar way to the case $l_1 \geq 1$, we can show $w \neq b_{k+1} \cdots b_1$. If $w = b_k \cdots b_0$, then

$$\nu(w) + b_{k+1}(q^m)^{k+1} = \nu(b_k \cdots b_0) + b_{k+1}(q^m)^{k+1} = \nu(b_{k+1} \cdots b_1 b_0) = \nu(w)q^u,$$

so that $b_{k+1}q^{m(k+1)} = \nu(w)(q^u - 1)$. Since $u \geq 1$, the positive integers $q^{m(k+1)}$ and $q^u - 1$ are coprime, and hence $\nu(w) \equiv 0 \pmod{q^{m(k+1)}}$. This implies $a_j = 0$ for all $j = 0, \ldots, k$ in (3), a contradiction.

Case II. $l_2 \geq 1$.   In the case of $l_1 \geq 1$, the pattern w must be the form $w = 0^{l_1} b_{k+1} \cdots b_1 b_0 0^{l_2-1}$ with $b_0 = 0$. Hence we have by (4)

$$\nu(w) = \nu(b_{k+1} \cdots b_1 b_0 0^{l_2-1}) = \nu(w)q^{u-m},$$

a contradiction.

If $l_1 = 0$, then $w = b_{k+1} \cdots b_1 b_0 0^{l_2-1}$ ($b_0 = 0$) or $w = b_k \cdots b_0 0^{l_2}$. By the same way as above, we can deduce $w \neq b_{k+1} \cdots b_1 b_0 0^{l_2-1}$ ($b_0 = 0$). Furthermore, we see that $w \neq b_k \cdots b_0 0^{l_2}$ by a similar way to the proof of Case I. $\qquad \square$

For the patterns $w = 0^l 1 \in \Sigma^*_{q^m,r}$ ($l \geq 0$), we prepare the following lemma;

LEMMA 2.3.   *Let $w \in \Sigma^*_{q^m,r}$ be of the form $w = 0^l 1$ for some $l \geq 0$. Then for any integer $k \geq 1$, we have*

$$e_{q^m,r}(w; \nu(w^k)q^d) = \begin{cases} k & \text{if } d \equiv 0 \pmod{m}, \\ 0 \text{ or } 1 & \text{otherwise}. \end{cases}$$

PROOF.   The assertion is trivial for the case that d is divisible by m. Suppose that d is not divisible by m and put $d = mh + u$, where $h \geq 0$ and u ($1 \leq u \leq m - 1$) are integers. If $q^u \in \Sigma_{q^m,r}$, we have

$$(\nu(w^k)q^d)_{q^m,r} = (\nu(w^k)q^u(q^m)^h)_{q^m,r} = q^u(0^l q^u)^{k-1}0^h \,,$$

so that $e_{q^m,r}(w; \nu(w^k)q^d) = 0$. If $q^u \notin \Sigma_{q^m,r}$, then $b := -(q^m - q^u) \in \Sigma_{q^m,r}$ and $(q^u)_{q^m,r} = 1b \in \Sigma^*_{q^m,r}$. Noting that $b < 0$, we have for $w = 0^l 1$

$$(\nu(w^k)q^d)_{q^m,r} = \begin{cases} 1(b+1)^{k-1}b0^h & \text{if } l = 0 \,, \\ 1b(0^{l-1}1b)^{k-1}0^h & \text{otherwise} \,, \end{cases}$$

where $b, b + 1 \in \Sigma_{q^m,r} \setminus \{1\}$. Hence $e_{q^m,r}(w; \nu(w^k)q^d) = 1$ and the lemma is proved.   □

Let $m \geq 2$ be an integer. Define

(5) $$S_{m-1} := \{(k_1, k_2, \ldots, k_{m-1}) \in \mathbf{Z}^{m-1} \,; \, 0 \leq k_j \leq j - 1\}$$

and its subset

(6) $$S_{m-1,n} := \{(k_1, k_2, \ldots, k_{m-1}) \in S_{m-1} \,; \, k_1 + \cdots + k_{m-1} = n\}$$

for each $n = 0, 1, \ldots, (m-1)(m-2)/2$. Note that $S_{m-1} = \bigcup_{n \geq 0} S_{m-1,n}$.

LEMMA 2.4 (cf. [9, Lemma 2]).   *For any integer $m \geq 2$, there exist integers d_1 and d_2 with $0 \leq d_1 < d_2 \leq m - 1$ such that*

$$\sum_{\substack{n \geq 0 \\ n \equiv d_1 \,(\mathrm{mod}\, m)}} \sharp S_{m-1,n} \neq \sum_{\substack{n \geq 0 \\ n \equiv d_2 \,(\mathrm{mod}\, m)}} \sharp S_{m-1,n} \,,$$

where $\sharp S_{m-1,n}$ is the number of elements in $S_{m-1,n}$.

PROOF OF PROPOSITION 2.1.   We prove the lemma by induction on m. The assertion is trivial for $m = 1$. Indeed, for any pattern $w \in \Sigma^*_{q,r}$ and the integer $n_j := \nu(w^j) > 0$, we have $e_{q,r}(w; n_j) \to \infty$ as $j \to \infty$. Let $m \geq 2$ and assume the assertion for lower m. Suppose on the contrary that there exist $c_i \in \mathbf{C}$ not all zero such that the sequence

(7) $$\{c_1 e_{q,r_1}(w_{q,r_1}; n) + c_2 e_{q^2,r_2}(w_{q^2,r_2}; n) + \cdots + c_m e_{q^m,r_m}(w_{q^m,r_m}; n)\}_{n \geq 0}$$

is purely periodic with a period q^l for some $l \geq 1$. We may assume $c_m \neq 0$. Let d_1 and d_2 ($0 \leq d_1 < d_2 \leq m - 1$) be integers as in Lemma 2.4 and

$$N_j = \sum_{k_1=0}^{0} \sum_{k_2=0}^{1} \cdots \sum_{k_{m-1}=0}^{m-2} q^{k_1+\cdots+k_{m-1}-d_j+DLl(1+k_1+mk_2+\cdots+m^{m-2}k_{m-1})} \,, \quad j = 1, 2 \,,$$

where $D := \mathrm{lcm}(1, 2, \ldots, m - 1)$ and L is a sufficiently large integer. Since $N_1, N_2 \equiv 0$ (mod q^l) and the sequence (7) is periodic with the period q^l, we get

(8) $$\sum_{i=1}^{m} c_i e_{q^i,r_i}(w_{q^i,r_i}; aN_1) = \sum_{i=1}^{m} c_i e_{q^i,r_i}(w_{q^i,r_i}; aN_2)$$

for any integer $a > 0$. On the other hand, in a similar way to the proof of [9, Theorem 1], we can prove

$$(9) \qquad e_{q^i, r_i}(w_{q^i, r_i}; aN_1) = e_{q^i, r_i}(w_{q^i, r_i}; aN_2), \quad i = 1, 2, \ldots, m-1.$$

Hence, combining (8) and (9), we obtain for any integer $a > 0$

$$e_{q^m, r_m}(w_{q^m, r_m}; aN_1) - e_{q^m, r_m}(w_{q^m, r_m}; aN_2) = 0.$$

In what follows, we show the existence of the integer $a > 0$ satisfying

$$(10) \qquad e_{q^m, r_m}(w_{q^m, r_m}; aN_1) \neq e_{q^m, r_m}(w_{q^m, r_m}; aN_2)$$

and deduce a contradiction. We distinguish two cases.

Case I. $w_{q^m, r_m} \neq 0^l 1$ for any $l \geq 0$.   For the integer $a = \nu(w_{q^m, r_m})$, we have by Lemma 2.2

$$
\begin{aligned}
e_{q^m, r_m}(w_{q^m, r_m}; aN_j) &= \sum_{k_1=0}^{0} \cdots \sum_{k_{m-1}=0}^{m-2} e_{q^m, r_m}(w_{q^m, r_m}; aq^{k_1+\cdots+k_{m-1}+m-d_j+DL}) \\
&= \sharp\left\{(k_1, \ldots, k_{m-1}) \in S_{m-1}; \sum_{i=1}^{m-1} k_i \equiv d_j \pmod{m}\right\} \\
&= \sum_{\substack{n \geq 0 \\ n \equiv d_j \pmod m}} \sharp S_{m-1, n}, \quad j = 1, 2,
\end{aligned}
$$

where S_{m-1} and $S_{m-1,n}$ are the finite sets defined by (5) and (6), respectively. Hence we obtain (10) by Lemma 2.4.

Case II. $w_{q^m, r_m} = 0^l 1$ for some $l \geq 0$.   We may assume

$$\sum_{\substack{n \geq 0 \\ n \equiv d_1 \pmod m}} \sharp S_{m-1, n} < \sum_{\substack{n \geq 0 \\ n \equiv d_2 \pmod m}} \sharp S_{m-1, n}.$$

Let $a = \nu(w_{q^m, r_m}^{m!})$. Then we have

$$e_{q^m, r_m}(w_{q^m, r_m}; aN_1) = m! \sum_{\substack{n \geq 0 \\ n \equiv d_1 \pmod m}} \sharp S_{m-1, n} + \sum_{\substack{n \geq 0 \\ n \not\equiv d_1 \pmod m}} \delta_n \sharp S_{m-1, n},$$

where $\delta_n := e_{q^m, r_m}(w_{q^m, r_m}; aq^{n-d_1+DL})$. By Lemma 2.3, $\delta_n = 0$ or 1 for each $n \geq 0$. Hence, noting that

$$\sum_{\substack{n \geq 0 \\ n \not\equiv d_1 \pmod m}} \sharp S_{m-1, n} \leq \sum_{n \geq 0} \sharp S_{m-1, n} = \sharp S_{m-1} < m!,$$

we obtain the inequalities

$$e_{q^m, r_m}(w_{q^m, r_m}; aN_1) < m! \sum_{\substack{n \geq 0 \\ n \equiv d_2 \pmod m}} \sharp S_{m-1, n} \leq e_{q^m, r_m}(w_{q^m, r_m}; aN_2),$$

which implies (10). Therefore Proposition 2.1 is proved. $\square$

3. Lemmas for the proof of Theorem 1.1. We prepare some lemmas for proving Theorem 1.1. In the next section, we show Theorem 1.1 by using these lemmas together with Proposition 2.1.

LEMMA 3.1 (Nishioka [6, Corollary]). *Let K be an algebraic number field and $d_1, \ldots, d_t \geq 2$ be integers with $\log d_i / \log d_j \notin \mathbf{Q}$ if $i \neq j$. Suppose that $f_{i,j}(z) \in K[[z]]$ $(i = 1, \ldots, t, \ j = 1, \ldots, m(i))$ satisfy the functional equations*

$$f_{i,j}(z) = f_{i,j}(z^{d_i}) + b_{i,j}(z), \quad b_{i,j}(z) \in K(z).$$

If for each i, $f_{i,1}(z), \ldots, f_{i,m(i)}(z)$ are algebraically independent over $K(z)$ and all $f_{i,j}(z)$ converge at an algebraic number $z = \alpha$ with $0 < |\alpha| < 1$, then the values

$$f_{i,j}(\alpha) \quad (i = 1, \ldots, t, \ j = 1, \ldots, m(i))$$

are algebraically independent over $\mathbf{Q}$.

LEMMA 3.2 (Kubota [2], Loxton and van der Poorten [3]; see also Nishioka [5, Theorem 3.2.1]). *Let $d \geq 2$ be an integer. Suppose that $g_1(z), \ldots, g_m(z) \in \mathbf{C}[[z]]$ are algebraically dependent over $\mathbf{C}(z)$ and satisfy the functional equations*

$$(11) \qquad g_i(z^d) = g_i(z) + a_i(z) \quad a_i(z) \in \mathbf{C}(z), \quad i = 1, 2, \ldots, m.$$

Then there exist constants $c_1, \ldots, c_m \in \mathbf{C}$ not all zero such that

$$c_1 g_1(z) + c_2 g_2(z) + \cdots + c_m g_m(z) \in \mathbf{C}(z).$$

The following lemma is a generalization of [7, Lemma 3], which is the case $L = d^2 - 1$ of the lemma.

LEMMA 3.3. *Let $d \geq 2$, $l \geq 1$, and $L \geq 1$ be arbitrary integers. Suppose that $c(z) \in \mathbf{C}(z)$ satisfies the functional equation*

$$(12) \qquad c(z^d) = c(z) + \frac{1 - z^L}{1 - z^{d^l L}} a(z), \quad a(z) \in \mathbf{C}[z].$$

Then there exists $b(z) \in \mathbf{C}[z]$ such that

$$(13) \qquad c(z) = \frac{1 - z^L}{1 - z^{d^{l-1}L}} b(z)$$

with $\deg b(z) \leq -L + d^{-1} \max\{d^l L, \ L + \deg a(z)\}$. Here we understand $\deg 0 = -\infty$.

PROOF. Similarly to the proof of [7, Lemma 3], we see that there exists $h(z) \in \mathbf{C}[z]$ such that $c(z) = h(z)/(1 - z^{d^{l-1}L})$ and

$$h(z^d) = (1 + z^{d^{l-1}L} + z^{2d^{l-1}L} + \cdots + z^{(d-1)d^{l-1}L})h(z) + (1 - z^L)a(z).$$

Let ζ be an arbitrary L-th root of unity. Substituting $z = \zeta^{d^i}$ $(i = 1, 2, \ldots)$ into the above identity, we have $h(\zeta^{d^{i+1}}) = dh(\zeta^{d^i})$, so that $h(\zeta^{d^i}) = d^i h(\zeta)$ for every $i \geq 1$. If $h(\zeta) \neq 0$, then $|h(\zeta^{d^i})| \to \infty$ as $i \to \infty$, a contradiction. Hence $b(z) = h(z)/(1 - z^L) \in \mathbf{C}[z]$ and

we obtain the form (13). Next we estimate the degree of $b(z)$. Substituting $c(z)$ in (13) to the equation (12) and multiplying the both sides by $(1 - z^{d^l L})/(1 - z^L)$, we have

$$\frac{1 - z^{d^l L}}{1 - z^{d^{l-1}L}} b(z) = \frac{1 - z^{dL}}{1 - z^L} b(z^d) - a(z).$$

If the degree of the first term of the right-hand side is not greater than that of the left-hand side, we get $\deg b(z) \le (d^{l-1} - 1)L$. Otherwise, the degree of the first term coincides with $\deg a(z)$, so that $\deg b(z) = -L + (\deg a(z) + L)/d$. The lemma is proved.   □

4. Proof of Theorem 1.1.   Let N be the set of positive integers and M be a subset of N defined by

$$M = \{q \in N; \ q \ne a^n \quad \text{for any} \quad a, n \in N, n \ge 2\}.$$

Then $\log q_1 / \log q_2 \notin Q$ for distinct integers $q_1, q_2 \in M$ and

$$N \setminus \{1\} = \bigcup_{q \in M} \{q, q^2, \dots\} \quad \text{(disjoint union)}.$$

We can choose distinct integers $q_1, \dots, q_t \in M$ such that the set of nonzero patterns

$$w_{i,j,r} \in \Sigma^*_{q_i^j, r} \quad (i = 1, 2, \dots, t, \ \ j = 1, 2, \dots, m(i), \ \ r = 0, 1, \dots, q_i^j - 2)$$

includes the nonzero patterns given in Theorem 1.1. Let

(14) $$f_{i,j,r}(z) = \sum_{n \ge 0} e_{q_i^j, r}(w_{i,j,r}; n) z^n$$

$(i = 1, \dots, t, j = 1, \dots, m_i, r = 0, \dots, q_i^j - 2)$. By [7, Lemma 5], the functions (14) satisfy Mahler type functional equations

(15) $$f_{i,j,r}(z) = \frac{1 - z^{q_i^j}}{z^r (1 - z)} f_{i,j,r}(z^{q_i^j}) + \frac{z^{v(w_{i,j,r})}}{1 - z^{q_i^{j|w_{i,j,r}|}}}.$$

We transform the functions $f_{i,j,r}(z)$ into the functions which satisfy the functional equation (11). Let L and $u_{i,j,r}$ be positive integers defined by $L = \prod_{i=1}^t \prod_{j=1}^{m(i)} (q_i^j - 1)$ and $u_{i,j,r} = rL/(q_i^j - 1)$, respectively. Multiplying the both sides of (15) by $z^r(1 - z)$ and substituting z^L for z, we get

$$z^{rL}(1 - z^L) f_{i,j,r}(z^L) = (1 - z^{q_i^j L}) f_{i,j,r}(z^{q_i^j L}) + \frac{z^{(r + v(w_{i,j,r}))L}(1 - z^L)}{1 - z^{q_i^{j|w_{i,j,r}|}L}}.$$

Noting that $rL = -u_{i,j,r} + q_i^j u_{i,j,r}$, we obtain the functional equation

(16) $$F_{i,j,r}(z) = F_{i,j,r}(z^{q_i^j}) + \frac{z^{v(w_{i,j,r})L - u_{i,j,r}}(1 - z^L)}{1 - z^{q_i^{j|w_{i,j,r}|}L}},$$

where

(17) $$F_{i,j,r}(z) := z^{-u_{i,j,r}}(1 - z^L) f_{i,j,r}(z^L) \in Q[[z]].$$

Using (16) repeatedly, we have

$$(18) \qquad F_{i,j,r}(z) = F_{i,j,r}(z^{q_i^{D_i}}) + \sum_{k=0}^{D_i/j-1} z^{q_i^{kj}(v(w_{i,j,r})L - u_{i,j,r})} \frac{1 - z^{q_i^{kj}L}}{1 - z^{q_i^{j|w_{i,j,r}|+kj}L}},$$

where $D_i = \mathrm{lcm}(1, 2, \ldots, m(i))$.

PROOF OF THEOREM 1.1. Suppose on the contrary that the first assertion of Theorem 1.1 does not holds, namely, the functions $f_{i,j,r}(z)$ $(i = 1, \ldots, t, \; j = 1, \ldots, m(i), \; r = 0, \ldots, q^i - 2)$ defined by (14) are algebraically dependent over $C(z)$. Then so are the functions $F_{i,j,r}(z)$, and hence the values $F_{i,j,r}(\alpha)$ $(i = 1, \ldots, t, \; j = 1, \ldots, m(i), \; r = 0, \ldots, q^i - 2)$ are algebraically dependent over Q for any algebraic number α with $0 < |\alpha| < 1$ (cf. Nesterenko [4, Lemma 2.3]). Therefore by Lemma 3.1 together with (16), the functions $F_{i,j,r}(z)$ $(j = 1, \ldots, m(i), \; r = 0, \ldots, q^i - 2)$ are algebraically dependent over $Q(z)$ for some fixed i $(1 \le i \le t)$.

Thus, for a fixed integer $q \ge 2$ we may assume that the functions

$$F_{j,r}(z) = z^{-u_{j,r}} (1 - z^L) \sum_{n \ge 0} e_{q^j,r}(w_{j,r}; n) z^{Ln}, \quad w_{j,r} \in \Sigma_{q^j,r}^*,$$

$(j = 1, \ldots, m, \; r = 0, 1, \ldots, q^j - 2)$ are algebraically dependent over $Q(z)$, where $u_{j,r} := rL/(q^j - 1)$. By (18)

$$(19) \qquad F_{j,r}(z) = F_{j,r}(z^{q^D}) + \sum_{k=0}^{D/j-1} z^{q^{kj}(Lv(w_{j,r}) - u_{j,r})} \frac{1 - z^{q^{kj}L}}{1 - z^{q^{j|w_{j,r}|+kj}L}},$$

with $D := \mathrm{lcm}\{1, 2, \ldots, m\}$, and by Lemma 3.2 there exist constants $c_{j,r} \in C$ not all zero such that

$$R(z) := \sum_{j=1}^{m} \sum_{r=0}^{q^j-2} c_{j,r} F_{j,r}(z) \in C(z).$$

Without loss of generality, we may assume $c_{m,r_m} \ne 0$ for some r_m $(0 \le r_m \le q^m - 2)$. Using (19) we obtain

$$R(z) = R(z^{q^D}) + \sum_{j=1}^{m} \sum_{r=0}^{q^j-2} \sum_{k=0}^{D/j-1} c_{j,r} z^{q^{kj}(Lv(w_{j,r}) - u_{j,r})} \frac{1 - z^{q^{kj}L}}{1 - z^{q^{|w_{j,r}|j+kj}L}}$$

$$= R(z^{q^D}) + \frac{1 - z^L}{1 - z^{q^{Dl}L}} a(z),$$

where $l := \max\{|w_{j,r}|; 1 \le j \le m, \; 0 \le r \le q^j - 2\}$ and

$$a(z) = \sum_{j=1}^{m} \sum_{r=0}^{q^j-2} \sum_{k=0}^{D/j-1} c_{j,r} z^{q^{kj}(Lv(w_{j,r}) - u_{j,r})} \frac{1 - z^{q^{kj}L}}{1 - z^L} \frac{1 - z^{q^{Dl}L}}{1 - z^{q^{|w_{j,r}|j+kj}L}}.$$

Since $|w_{j,r}|j + kj \leq (|w_{j,r}| + D/j - 1)j \leq Dl$, we get $a(z) \in \mathbf{C}[z]$ with $\deg a(z) \leq (q^{Dl} - 1)L$ if $a(z) \neq 0$. Hence, applying Lemma 3.3 with $d = q^D$ and $c(z) = R(z) \in \mathbf{C}(z)$, we see that there exists $b(z) \in \mathbf{C}[z]$ such that

$$(20) \qquad R(z) = \frac{1 - z^L}{1 - z^{q^{D(l-1)}L}} b(z),$$

where $\deg b(z) \leq (q^{D(l-1)} - 1)L$ if $b(z) \neq 0$. By the expression (20) and the definition of $R(z)$, we have

$$b(z) = \frac{1 - z^{q^{D(l-1)}L}}{1 - z^L} \sum_{j=1}^{m} \sum_{r=0}^{q^j - 2} c_{j,r} F_{j,r}(z)$$

$$= (1 - z^{q^{D(l-1)}L}) \sum_{n \geq 0} \sum_{j=1}^{m} \sum_{r=0}^{q^j - 2} c_{j,r} e_{q^j,r}(w_{j,r}; n) z^{nL - u_{j,r}}$$

$$(21) \qquad = Q(z) + \sum_{n \geq 0} \sum_{j=1}^{m} \sum_{r=0}^{q^j - 2} (d_{j,r}(n + q^{D(l-1)}) - d_{j,r}(n)) z^{nL + q^{D(l-1)}L - u_{j,r}},$$

where $d_{j,r}(n) := c_{j,r} e_{q^j,r}(w_{j,r}; n)$ and

$$Q(z) := \sum_{n=0}^{q^{D(l-1)}-1} \sum_{j=1}^{m} \sum_{r=0}^{q^j - 2} d_{j,r}(n) z^{nL - u_{j,r}} \in \mathbf{C}[z].$$

It is easily seen that $\deg Q(z) \leq (q^{D(l-1)} - 1)L$ if $Q(z) \neq 0$. Hence, noting that the exponents of the power series of the right-hand side in (21) are greater than $(q^{D(l-1)} - 1)L$ for every n, j, and r, we obtain $b(z) = Q(z)$ and

$$(22) \qquad \sum_{n \geq 0} \sum_{j=1}^{m} \sum_{r=0}^{q^j - 2} (d_{j,r}(n + q^{D(l-1)}) - d_{j,r}(n)) z^{nL + q^{D(l-1)}L - u_{j,r}} = 0.$$

Define the finite sets

$$T = \{(j, r) \; ; \; 1 \leq j \leq m, \; 0 \leq r \leq q^j - 2\},$$

$$T_\beta = \{(j, r) \in T \; ; \; r/(q^j - 1) = \beta\}.$$

Then there exist distinct rational numbers $\beta_1, \ldots, \beta_k$ such that $T = \bigcup_{i=1}^{k} T_{\beta_i}$. In the power series expansion of the left-hand side in (22), if the exponents of z for $(n, j, r) = (n_1, j_1, r_1)$ and for (n_2, j_2, r_2) are the same, namely

$$n_1 L + q^{D(l-1)}L - u_{j_1,r_1} = n_2 L + q^{D(l-1)}L - u_{j_2,r_2},$$

then we have

$$|n_1 - n_2| = \left| \frac{r_1}{q^{j_1} - 1} - \frac{r_2}{q^{j_2} - 1} \right| < 1,$$

so that $n_1 = n_2$ and $(j_1, r_{j_1}), (j_2, r_{j_2}) \in T_{\beta_i}$ for some i. Hence we obtain

$$\sum_{(j,r)\in T_{\beta_i}} (d_{j,r}(n + q^{D(l-1)}) - d_{j,r}(n)) = 0, \quad n \geq 0$$

for each $i = 1, 2, \ldots, k$, that is, the sequence

$$\left\{ \sum_{(j,r)\in T_{\beta_i}} d_{j,r}(n) \right\}_{n\geq 0} = \left\{ \sum_{(j,r)\in T_{\beta_i}} c_{j,r} e_{q^j, r}(w_{j,r}; n) \right\}_{n\geq 0}$$

is purely periodic with a period $q^{D(l-1)}$. In particular, noting that $c_{m,r_m} \neq 0$, we see that there exist constants $c_j \in C$ not all zero and r_j $(j = 1, \ldots, m, \ r_j \in \{0, \ldots, q^j - 2\})$ such that

$$\left\{ \sum_{j=1}^{m} c_j e_{q^j, r_j}(w_{j,r_j}; n) \right\}_{n\geq 0}$$

is purely periodic with a period $q^{D(l-1)}$. This contradicts Proposition 2.1, and therefore the functions $f_{i,j,r}(z)$ $(i = 1, \ldots, t, \ j = 1, \ldots, m(i), \ r = 0, \ldots, q^i - 2)$ defined by (14) are algebraically independent over $C(z)$.

The algebraic independence of the values of the functions $f_{i,j,r}(z)$ follows immediately. Indeed, if $f_{i,j,r}(\beta)$ are algebraically dependent over Q for some algebraic number β with $0 < |\beta| < 1$, then by (17) so are the values $F_{i,j,r}(\alpha)$ for $\alpha = \beta^{1/L}$. This contradicts the algebraic independence of the F's as we have already proved above, and the proof of Theorem 1.1 is completed. □

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