

ALGEBRAIC INTEGERS AS SPECIAL VALUES OF MODULAR UNITS

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Abstract Let $\varphi(\tau) = \eta(\frac{1}{2}(\tau + 1))^2 / \sqrt{2\pi} \exp\{\frac{1}{4}\pi i\} \eta(\tau + 1)$, where $\eta(\tau)$ is the Dedekind eta function. We show that if τ_0 is an imaginary quadratic argument and m is an odd integer, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing \sqrt{m} . This is a generalization of a result of Berndt, Chan and Zhang. On the other hand, when K is an imaginary quadratic field and θ_K is an element of K with $\text{Im}(\theta_K) > 0$ which generates the ring of integers of K over \mathbb{Z} , we find a sufficient condition on m which ensures that $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ is a unit.

Keywords: Dedekind eta function; modular functions; automorphic functions

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1. Introduction

The *Dedekind eta function* $\eta(\tau)$ is defined to be the infinite product

$$\eta(\tau) = \sqrt{2\pi} e^{\pi i/4} q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \tau \in \mathfrak{H}, \quad (1.1)$$

where $q = e^{2\pi i\tau}$ with $i = \sqrt{-1}$ and $\mathfrak{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$. Define a function

$$\varphi(\tau) = \frac{1}{\sqrt{2\pi} e^{\pi i/4}} \frac{\eta((\tau + 1)/2)^2}{\eta(\tau + 1)} = \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n), \quad \tau \in \mathfrak{H}, \quad (1.2)$$

which is identical to Jacobi's $\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2}$, by Jacobi's triple product identity [1, p. 36]. Motivated by Ramanujan's evaluation of $\varphi(mi)/\varphi(i)$ for some positive integers m [10] which are algebraic numbers, Berndt *et al.* proved the following theorem.

Theorem 1.1 (Berndt *et al.* [2, Theorem 4.4]). *Let m and n be positive integers. If m is odd, then $\sqrt{2m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

In this paper we shall first revisit the theorem and improve it when m is odd, as follows.

Theorem 1.2. *Let m be a positive integer and let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. Then $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$. In particular, if m is odd, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing \sqrt{m} .*

For $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, the Siegel function $g_{(r_1, r_2)}(\tau)$ is defined by

$$g_{(r_1, r_2)}(\tau) = -q^{\mathbf{B}_2(r_1)/2} e^{\pi i r_2(r_1-1)} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^n q_z)(1 - q^n q_z^{-1}), \quad \tau \in \mathfrak{H}, \quad (1.3)$$

where $\mathbf{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial and $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. We shall express the function $\varphi(m\tau)/\varphi(\tau)$ as a product of certain eta-quotient and Siegel functions (Lemma 2.6 (i)). Also, we shall prove Theorem 1.2 in §3 by using integrality of Siegel functions over $\mathbb{Z}[j(\tau)]$ [6, §3], where

$$j(\tau) = \left(\frac{\eta(\tau)^{24} + 2^8 \eta(2\tau)^{24}}{\eta(\tau)^{16} \eta(2\tau)^8} \right)^3 = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \dots$$

is the well-known modular j -function [3, Theorem 12.17].

On the other hand, let K be an imaginary quadratic field with discriminant d_K , and define

$$\theta_K = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{for } d_K \equiv 0 \pmod{4}, \\ \frac{-1 + \sqrt{d_K}}{2} & \text{for } d_K \equiv 1 \pmod{4}, \end{cases} \quad (1.4)$$

which generates the ring of integers of K over \mathbb{Z} . Ramachandra showed in [9, §6] that if N ($N \geq 2$) is an integer with more than one prime ideal factor in K , then $g_{(0, 1/N)}(\theta_K)^{12N}$ is a unit. This fact, together with Shimura's Reciprocity Law (Proposition 4.6), enables us to prove the following theorem in §4.

Theorem 1.3. *If m ($m \geq 3$) is an odd integer whose prime factors split in K , then $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ is a unit.*

2. Arithmetic properties of Siegel functions

In this section we shall examine some arithmetic properties of Siegel functions. For the classical theory of modular functions, we refer the reader to [8, 11].

For each positive integer N , let $\zeta_N = e^{2\pi i/N}$ and let \mathcal{F}_N be the field of meromorphic modular functions of level N whose Fourier coefficients belong to the N th cyclotomic field $\mathbb{Q}(\zeta_N)$.

Proposition 2.1. *For each positive integer N , \mathcal{F}_N is a Galois extension of $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ whose Galois group is isomorphic to*

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} = G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\},$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

Here, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in G_N$$

acts on $\sum_{n=-\infty}^{\infty} c_n q^{n/N} \in \mathcal{F}_N$ by

$$\sum_{n=-\infty}^{\infty} c_n q^{n/N} \mapsto \sum_{n=-\infty}^{\infty} c_n^{\sigma_d} q^{n/N},$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ induced by $\zeta_N \mapsto \zeta_N^d$. Also, for an element $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ let $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ be a preimage of γ via the natural surjection $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$. Then γ acts on $h \in \mathcal{F}_N$ by composition

$$h \mapsto h \circ \gamma'$$

as a fractional linear transformation.

Proof. See [8, Chapter 6, Theorem 3]. □

Proposition 2.2. Let $(r_1, r_2) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for some integer $N \geq 2$.

- (i) $g_{(r_1, r_2)}(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$. Namely, $g_{(r_1, r_2)}(\tau)$ is a zero of a monic polynomial whose coefficients are in $\mathbb{Z}[j(\tau)]$.
- (ii) Suppose that (r_1, r_2) has the primitive denominator N (that is, N is the smallest positive integer such that $(Nr_1, Nr_2) \in \mathbb{Z}^2$). If N is composite (that is, N has at least two prime factors), then $g_{(r_1, r_2)}(\tau)^{-1}$ is also integral over $\mathbb{Z}[j(\tau)]$.
- (iii) $g_{(r_1, r_2)}(\tau)$ is holomorphic and has no zeros and poles on \mathfrak{H} . Furthermore, $g_{(r_1, r_2)}(\tau)$ (respectively, $g_{(r_1, r_2)}(\tau)^{12N/\mathrm{gcd}(6, N)}$) belongs to \mathcal{F}_{12N^2} (respectively, \mathcal{F}_N).

Proof.

- (i) See [6, § 3].
- (ii) See [7, Chapter 2, Theorems 2.2 (i)].
- (iii) See [7, Chapter 2, Theorem 1.2, and Chapter 3, Theorem 5.2]. □

Remark 2.3. Let $g(\tau)$ be an element of \mathcal{F}_N for some positive integer N . If both $g(\tau)$ and $g(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$ (respectively, $\mathbb{Z}[j(\tau)]$), then $g(\tau)$ is called a *modular unit* (respectively, *modular unit over \mathbb{Z}*) of level N . As is well known, $g(\tau)$ is a modular unit if and only if it has no zeros or poles on \mathfrak{H} (see [7, Chapter 2, § 2] or [6, § 2]). Hence, $g_{(r_1, r_2)}(\tau)$ is a modular unit for any $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, by (iii). Moreover, if (r_1, r_2) has a composite primitive denominator, then $g_{(r_1, r_2)}(\tau)$ is a modular unit over \mathbb{Z} , by (ii).

We recall the necessary transformation formulae of Siegel functions.

Proposition 2.4. *Let $r = (r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$.*

(i) *We have*

$$g_{-r}(\tau) = g_{(-r_1, -r_2)}(\tau) = -g_r(\tau).$$

(ii) *For*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned} g_r(\tau) \circ S &= \zeta_{12}^9 g_{rS}(\tau) = \zeta_{12}^9 g_{(r_2, -r_1)}(\tau), \\ g_r(\tau) \circ T &= \zeta_{12} g_{rT}(\tau) = \zeta_{12} g_{(r_1, r_1+r_2)}(\tau). \end{aligned}$$

Hence, we obtain that, for any $\gamma \in \text{SL}_2(\mathbb{Z})$,

$$g_r(\tau) \circ \gamma = \varepsilon g_{r\gamma}(\tau)$$

with ε a 12th root of unity (depending on γ).

(iii) *For $s = (s_1, s_2) \in \mathbb{Z}^2$ we have*

$$g_{r+s}(\tau) = g_{(r_1+s_1, r_2+s_2)}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 r_2 - s_2 r_1)} g_r(\tau).$$

(iv) *Let $r \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$ for some integer $N \geq 2$. Each element*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{in} \quad \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$$

acts on $g_r(\tau)^{12N/\text{gcd}(6,N)}$ by

$$(g_r(\tau)^{12N/\text{gcd}(6,N)})^\alpha = g_{r\alpha}(\tau)^{12N/\text{gcd}(6,N)} = g_{(r_1 a + r_2 c, r_1 b + r_2 d)}(\tau)^{12N/\text{gcd}(6,N)}.$$

Proof. (i)–(iii) See [6, Proposition 2.4].

(iv) See [7, Chapter 2, Proposition 1.3]. □

Remark 2.5. The expression $r\alpha$ in (iv) is well defined by (i) and (iii).

Lemma 2.6.

(i) *We can express $\varphi(\tau)$ as*

$$\varphi(\tau) = -\frac{1}{\sqrt{2\pi}} \eta(\tau) g_{(1/2, 1/2)}(\tau).$$

(ii) *We get*

$$g_{(0, 1/2)}(\tau) g_{(1/2, 0)}(\tau) g_{(1/2, 1/2)}(\tau) = 2e^{\pi i/4}.$$

(iii) If m ($m \geq 3$) is an odd integer, then we have the relation

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} = (-1)^{(m-1)/2} \prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau).$$

Proof. (i) By the definition (1.3) we have

$$\begin{aligned} g_{(1/2,1/2)}(\tau) &= -q^{\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1+q^{1/2}) \prod_{n=1}^{\infty} (1+q^{n+1/2})(1+q^{n-1/2}) \\ &= -e^{-\pi i/4} q^{-1/24} \prod_{n=1}^{\infty} (1+q^{n-1/2})^2. \end{aligned}$$

One can then obtain the assertion by the definition (1.1) of $\eta(\tau)$ and the infinite product expansion (1.2) of $\varphi(\tau)$.

(ii) It follows from the definition (1.3) that

$$\begin{aligned} g_{(0,1/2)}(\tau)g_{(1/2,0)}(\tau)g_{(1/2,1/2)}(\tau) &= -2e^{-3\pi i/4} \prod_{n=1}^{\infty} (1+q^n)^2(1-q^{n-1/2})^2(1+q^{n-1/2})^2 \\ &= 2e^{\pi i/4} \prod_{n=1}^{\infty} (1+q^n)^2(1-q^{2n-1})^2 \\ &= 2e^{\pi i/4} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)^2} \cdot \frac{(1-q^n)^2}{(1-q^{2n})^2} \\ &= 2e^{\pi i/4}. \end{aligned}$$

(iii) By the definition (1.3) we obtain

$$\begin{aligned} \frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} &= \frac{-q^{m\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1+q^{m/2}) \prod_{n=1}^{\infty} (1+q^{mn+m/2})(1+q^{mn-m/2})}{-q^{\mathbf{B}_2(1/2)/2} e^{-\pi i/4} (1+q^{1/2}) \prod_{n=1}^{\infty} (1+q^{n+1/2})(1+q^{n-1/2})} \\ &= q^{(1-m)/24} \prod_{n=1}^{\infty} \left(\frac{1+q^{m(n-1/2)}}{1+q^{n-1/2}} \right)^2 \end{aligned}$$

and

$$\begin{aligned} &\prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau) \\ &= \prod_{k=1}^{m-1} \left(-q^{\mathbf{B}_2(1/2)/2} \exp \left\{ \pi i \left(\frac{1}{2} + \frac{k}{m} \right) \left(-\frac{1}{2} \right) \right\} \right. \\ &\quad \left. \times (1+q^{1/2}\zeta_m^k) \prod_{n=1}^{\infty} (1+q^{n+1/2}\zeta_m^k)(1+q^{n-1/2}\zeta_m^{-k}) \right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{m-1} e^{\pi i(1-m)/2} q^{(1-m)/24} \prod_{k=1}^{m-1} \prod_{n=1}^{\infty} (1 + q^{n-1/2} \zeta_m^k)(1 + q^{n-1/2} \zeta_m^{-k}) \\
&= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \prod_{k=1}^{m-1} (1 + q^{n-1/2} \zeta_m^k)^2 \quad \text{because } m \text{ is odd} \\
&= (-1)^{(1-m)/2} q^{(1-m)/24} \prod_{n=1}^{\infty} \left(\frac{1 + q^{m(n-1/2)}}{1 + q^{n-1/2}} \right)^2
\end{aligned}$$

by the identity

$$\frac{1 + X^m}{1 + X} = \frac{1 - (-X)^m}{1 - (-X)} = \prod_{k=1}^{m-1} (1 - (-X) \zeta_m^k).$$

This proves (iii). □

3. Proof of Theorem 1.2

Let

$$\Delta(\tau) = \eta(\tau)^{24} = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad \tau \in \mathfrak{H},$$

be the modular discriminant function.

Proposition 3.1. *Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

- (i) $j(\tau_0)$ is an algebraic integer.
- (ii) Let a, b and d be integers with $ad > 0$ and $\gcd(a, b, d) = 1$. Then

$$\frac{a^{12} \Delta((a\tau_0 + b)/d)}{\Delta(\tau_0)}$$

is an algebraic integer dividing $(ad)^{12}$.

Proof.

- (i) See [8, Chapter 5, Theorem 4].
- (ii) See [8, Chapter 12, Theorem 4] or [4]. □

Remark 3.2. Case (ii) is the most important special case of the prime factorizations of $\Delta(\alpha\tau_0)/\Delta(\tau_0)$ ($\alpha \in M_2^+(\mathbb{Z})$); this was proved by Hasse for all factorizations.

Proposition 3.3. *Let m be a positive integer and $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

- (i) $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing \sqrt{m} .
- (ii) $2g_{(1/2, 1/2)}(m\tau_0)/g_{(1/2, 1/2)}(\tau_0)$ is an algebraic integer dividing 4. In particular, if m is odd, then $g_{(1/2, 1/2)}(m\tau_0)/g_{(1/2, 1/2)}(\tau_0)$ is a unit.

Proof. (i) Applying Proposition 3.1 (ii) with $(a, b, d) = (m, 0, 1)$, we see that

$$m^{12} \frac{\Delta(m\tau_0)}{\Delta(\tau_0)} = \left(\sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \right)^{24}$$

is an algebraic integer dividing m^{12} . We then get the assertion by taking the 24th root.

(ii) We obtain by Lemma 2.6 (ii) that

$$\begin{aligned} 2 \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} &= e^{-\pi i/4} g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0) g_{(1/2,1/2)}(\tau_0) \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} \\ &= e^{-\pi i/4} (g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)) g_{(1/2,1/2)}(m\tau_0). \end{aligned}$$

By Propositions 2.2 (i) and 3.1 (i) we know that the values $g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)$, $g_{(1/2,1/2)}(\tau_0)$, $g_{(0,1/2)}(m\tau_0) g_{(1/2,0)}(m\tau_0)$ and $g_{(1/2,1/2)}(m\tau_0)$ are algebraic integers. Moreover, since

$$\begin{aligned} (g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)) g_{(1/2,1/2)}(\tau_0) &= (g_{(0,1/2)}(m\tau_0) g_{(1/2,0)}(m\tau_0)) g_{(1/2,1/2)}(m\tau_0) \\ &= 2e^{\pi i/4} \end{aligned}$$

by Lemma 2.6 (ii), both $g_{(0,1/2)}(\tau_0) g_{(1/2,0)}(\tau_0)$ and $g_{(1/2,1/2)}(m\tau_0)$ are algebraic integers dividing 2. Hence, the value $2g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is an algebraic integer dividing $2 \cdot 2 = 4$.

Next, suppose that m ($m \geq 3$) is odd. Recall the relation

$$\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)} = (-1)^{(m-1)/2} \prod_{k=1}^{m-1} g_{(1/2,1/2+k/m)}(\tau)$$

given in Lemma 2.6 (iii). Since each vector

$$\left(\frac{1}{2}, \frac{1}{2} + \frac{k}{m} \right)$$

has a composite primitive denominator, $g_{(1/2,1/2+k/m)}(\tau)$ is a modular unit over \mathbb{Z} by Proposition 2.2 (ii); hence, so is $g_{(1/2,1/2)}(m\tau)/g_{(1/2,1/2)}(\tau)$. Therefore, the value $g_{(1/2,1/2)}(m\tau_0)/g_{(1/2,1/2)}(\tau_0)$ is a unit by Proposition 3.1 (i). \square

Now we are ready to prove Theorem 1.2. Let m be a positive integer and let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. By Lemma 2.6 (i) we have

$$2\sqrt{m} \frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot 2 \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)}.$$

Thus, it follows from Proposition 3.3 (i) and (ii) that $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$. Likewise, if m is odd, then

$$\sqrt{m} \frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m} \frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot \frac{g_{(1/2,1/2)}(m\tau_0)}{g_{(1/2,1/2)}(\tau_0)} \tag{3.1}$$

is an algebraic integer dividing \sqrt{m} . This completes the proof of Theorem 1.2.

On the other hand, when $\tau_0 = ni$ we are able to improve Theorem 1.1 as a corollary.

Corollary 3.4. *Let m and n be positive integers. If m is odd, then $\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing \sqrt{m} , while if m is even, then $2\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

Remark 3.5. Berndt *et al.* [2] used only the argument of Proposition 3.1 (ii) in order to achieve Theorem 1.1.

4. Proof of Theorem 1.3

Lemma 4.1. *Let m ($m \geq 2$) be an integer. Then we have the following identities.*

(i)

$$\prod_{\substack{a,b \in \mathbb{Z}, \\ 0 \leq a,b < m, (a,b) \neq (0,0)}} g_{(a/m,b/m)}(\tau)^{12m} = m^{12m}.$$

(ii)

$$\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau) = i^{m-1} \left(\sqrt{m} \frac{\eta(m\tau)}{\eta(\tau)} \right)^2.$$

Proof. (i) See [7, Example, p. 45].

(ii) We deduce from the definition (1.3) that

$$\begin{aligned} & \prod_{k=1}^{m-1} g_{(0,k/m)}(\tau) \\ &= \prod_{k=1}^{m-1} (-q^{\mathbf{B}_2(0)/2} \zeta_{2m}^{-k} (1 - \zeta_m^k) \prod_{n=1}^{\infty} (1 - q^n \zeta_m^k) (1 - q^n \zeta_m^{-k})) \\ &= i^{m-1} m q^{(m-1)/12} \prod_{n=1}^{\infty} \left(\frac{1 - q^{mn}}{1 - q^n} \right)^2 \\ & \quad \text{by the identity } \frac{1 - X^m}{1 - X} = 1 + X + \dots + X^{m-1} = \prod_{k=1}^{m-1} (1 - X \zeta_m^k) \\ &= i^{m-1} \left(\sqrt{m} \frac{\eta(m\tau)}{\eta(\tau)} \right)^2 \quad \text{by the definition (1.1).} \end{aligned}$$

□

Remark 4.2. Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. By Propositions 2.2 (i), 3.1 (i) and Lemma 4.1 (i) we see that $\prod_{k=1}^{m-1} g_{(0,k/m)}(\tau_0)$ is an algebraic integer dividing m . It then follows from Lemma 4.1 (ii) that $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing \sqrt{m} . This gives another proof of Proposition 3.3 (i) without using the usual argument of Hasse (namely, Proposition 3.1 (ii)).

From now on, we let K be an imaginary quadratic field and θ_K be as in (1.4). We denote by H_K and $K_{(N)}$ the Hilbert class field and the ray class field modulo N ($N \geq 1$) of K , respectively.

Proposition 4.3 (main theorem of complex multiplication).

$$K_{(N)} = K\mathcal{F}_N(\theta_K) = K(h(\theta_K) : h \in \mathcal{F}_N \text{ is defined and finite at } \theta_K).$$

Proof. See [8, Chapter 10, Corollary to Theorem 2] or [11, Chapter 6]. □

Proposition 4.4.

- (i) If N ($N \geq 2$) is an integer with more than one prime ideal factor in K , then $g_{(0,1/N)}(\theta_K)^{12N}$ is a unit in $K_{(N)}$.
- (ii) If m ($m \geq 3$) is an odd integer, then $(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$ is an algebraic integer in $K_{(48m^2)}$.

Proof. (i) See [9, §6].

(ii) We see that

$$\begin{aligned} \left(\frac{\sqrt{m}\varphi(m\tau)}{\varphi(\tau)}\right)^2 &= \left(\frac{\sqrt{m}\eta(m\tau)}{\eta(\tau)}\right)^2 \left(\frac{g_{(1/2,1/2)}(m\tau)}{g_{(1/2,1/2)}(\tau)}\right)^2 \quad \text{by Lemma 2.6 (i)} \\ &= (-1)^{(1-m)/2} \prod_{k=1}^{m-1} g_{(0,k/m)}(\tau)g_{(1/2,1/2+k/m)}(\tau)^2 \\ &\quad \text{by Lemmas 4.1 (ii) and 2.6 (iii).} \end{aligned} \tag{4.1}$$

And $(\sqrt{m}\varphi(m\tau)/\varphi(\tau))^2$ belongs to \mathcal{F}_{48m^2} by Proposition 2.2 (iii). Therefore,

$$(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$$

lies in $K_{(48m^2)}$ by Proposition 4.3, which is an algebraic integer by Theorem 1.2. □

Remark 4.5. In [5] Jung *et al.* showed that if K is an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, then the singular value $g_{(0,1/N)}(\theta_K)^{12N}$ in Proposition 4.4 (i) is in fact a primitive generator of $K_{(N)}$ over K , which is called a *Siegel–Ramachandra invariant* (see [7, Chapter 11, §1] or [9]).

On the other hand, we have the following explicit description of Shimura’s reciprocity law, due to Stevenhagen [12], which connects the class field theory with the theory of modular functions.

Proposition 4.6 (Shimura’s Reciprocity Law). *Let $\min(\theta_K, \mathbb{Q}) = X^2 + BX + C \in \mathbb{Z}[X]$. For every positive integer N the matrix group*

$$W_{K,N} = \left\{ \begin{pmatrix} t - Bs & -Cs \\ s & t \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : t, s \in \mathbb{Z}/N\mathbb{Z} \right\}$$

gives rise to the surjection

$$\begin{aligned} W_{K,N} &\rightarrow \text{Gal}(K_{(N)}/H_K), \\ \alpha &\mapsto (h(\theta_K) \mapsto h^\alpha(\theta_K)), \end{aligned}$$

where $h \in \mathcal{F}_N$ is defined and finite at θ_K . Its kernel is given by

$$\text{Ker}_{K,N} = \begin{cases} \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & \text{otherwise.} \end{cases}$$

Proof. See [12, §3]. □

Proposition 4.7. *If m ($m \geq 2$) is an integer whose prime factors split in K , then $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ is a unit.*

Proof. We get from Lemma 4.1 (ii) that

$$\left(\frac{\sqrt{m}\eta(m\theta_K)}{\eta(\theta_K)} \right)^{24m} = \prod_{k=1}^{m-1} g_{(0,k/m)}(\theta_K)^{12m}. \tag{4.2}$$

For each $1 \leq k \leq m - 1$, let us write

$$\frac{k}{m} = \frac{a}{b} \quad \text{with relatively prime positive integers } a \text{ and } b.$$

Note that b has more than one prime ideal factor in K , by the assumption on m . Thus, $g_{(0,1/b)}(\theta_K)^{12b}$ is a unit in $K_{(b)}$, by Proposition 4.4 (i). On the other hand, since

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in W_{K,b}/\text{Ker}_{K,b} \simeq \text{Gal}(K_{(b)}/H_K),$$

we deduce that

$$\begin{aligned} (g_{(0,1/b)}(\theta_K)^{12b})^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} &= (g_{(0,1/b)}(\tau)^{12b})^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}(\theta_K) && \text{by Proposition 4.6} \\ &= \left(g_{(0,1/b)\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}(\tau)^{12b} \right)(\theta_K) && \text{by Proposition 2.4 (iv)} \\ &= g_{(0,a/b)}(\theta_K)^{12b}, \end{aligned}$$

which is also a unit. Therefore, $\sqrt{m}\eta(m\theta_K)\eta(\theta_K)$ becomes a unit, by the relation (4.2). □

Now, we can prove Theorem 1.3. Let m ($m \geq 3$) be an odd integer whose prime factors split in K . Since both $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ and $g_{(1/2,1/2)}(m\theta_K)/g_{(1/2,1/2)}(\theta_K)$ are units by Propositions 4.7 and 3.3 (ii), the conclusion follows from (3.1) with $\tau_0 = \theta_K$.

Corollary 4.8. *Let m ($m \geq 3$) be an odd integer whose prime factors p satisfy $p \equiv 1 \pmod{4}$. Then $\sqrt{m}\varphi(mi)/\varphi(i)$ is a unit.*

Proof. If we take $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = i$. For each prime factor p of m , the fact that $p \equiv 1 \pmod{4}$ implies that p splits in K [3, Corollary 5.17]. Therefore, we get the assertion by applying Theorem 1.3. \square

We close this section by evaluating $\sqrt{m}\varphi(mi)/\varphi(i)$ when $m = 3$ and 5, explicitly.

Example 4.9. We shall first estimate $\sqrt{3}\varphi(3i)/\varphi(i)$. If $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = i$ and $H_K = K$ [3, Theorem 12.34]. By Proposition 4.6 we have

$$\begin{aligned} \text{Gal}(K_{(6)}/K) &\simeq W_{K,6}/\text{Ker}_{K,6} \\ &= \left\{ \alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \right\}. \end{aligned}$$

Since

$$\begin{aligned} x &= \left(\sqrt{3} \frac{\varphi(3i)}{\varphi(i)} \right)^{24} = g_{(0,1/3)}(i)^{12} g_{(0,2/3)}(i)^{12} g_{(1/2,5/6)}(i)^{24} g_{(1/2,7/6)}(i)^{24} \quad \text{by (4.1)} \\ &= g_{(0,1/3)}(i)^{24} g_{(1/2,1/6)}(i)^{48} \quad \text{by Proposition 2.4 (i) and (iii)} \\ &\approx 72954, \end{aligned}$$

x lies in $K_{(6)}$ by Propositions 2.2 (iii) and 4.3. Hence, its conjugates $x_k = x^{\alpha_k}$ ($1 \leq k \leq 4$) over K are

$$\begin{aligned} x_1 &= g_{(0,1/3)}(i)^{24} g_{(1/2,1/6)}(i)^{48}, \\ x_2 &= g_{(2/3,1/3)}(i)^{24} g_{(5/6,1/6)}(i)^{48}, \\ x_3 &= g_{(1/3,1/3)}(i)^{24} g_{(1/6,1/6)}(i)^{48}, \\ x_4 &= g_{(2/3,0)}(i)^{24} g_{(5/6,1/2)}(i)^{48} \end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). We claim that the minimal polynomial of x over K has integer coefficients. Indeed, since x is a real algebraic integer by definition (1.2) and Theorem 1.2, we have

$$[\mathbb{Q}(x) : \mathbb{Q}] = \frac{[K(x) : K] \cdot [K : \mathbb{Q}]}{[K(x) : \mathbb{Q}(x)]} = \frac{[K(x) : K] \cdot 2}{2} = [K(x) : K],$$

from which the claim follows. Thus, x is a zero of the polynomial

$$(X - x_1)(X - x_2)(X - x_3)(X - x_4) = (X^2 - 72954X + 729)^2$$

whose coefficients can be determined by numerical approximation with the aid of a computer. Therefore, we obtain

$$\sqrt{3} \frac{\varphi(3i)}{\varphi(i)} = \sqrt[24]{x} = \sqrt[24]{36\,477 + 21\,060\sqrt{3}} = \sqrt[4]{3 + 2\sqrt{3}}.$$

Example 4.10. Next, we consider $\sqrt{5}\varphi(\sqrt{5}i)/\varphi(i)$. Let $K = \mathbb{Q}(\sqrt{-1})$. By Proposition 4.6 we have

$$\begin{aligned} & \text{Gal}(K_{(10)}/K) \\ & \simeq W_{K,10}/\text{Ker}_{K,10} \\ & = \left\{ \alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & -6 \\ 6 & 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \right. \\ & \quad \left. \alpha_5 = \begin{pmatrix} 2 & -5 \\ 5 & 2 \end{pmatrix}, \alpha_6 = \begin{pmatrix} 2 & -7 \\ 7 & 2 \end{pmatrix}, \alpha_7 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \alpha_8 = \begin{pmatrix} 4 & -5 \\ 5 & 4 \end{pmatrix} \right\}. \end{aligned}$$

Since

$$\begin{aligned} x &= \left(\sqrt{5} \frac{\varphi(5i)}{\varphi(i)} \right)^{120} = g_{(0,1/5)}(i)^{120} g_{(0,2/5)}(i)^{120} g_{(1/2,1/10)}(i)^{240} g_{(1/2,3/10)}(i)^{240} \\ & \hspace{15em} \text{by Proposition 2.4 (i) and (iii)} \\ & \approx 41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002, \end{aligned}$$

x lies in $K_{(10)}$ by Propositions 2.2 (iii) and 4.3. Its conjugates $x_k = x^{\alpha_k}$ ($1 \leq k \leq 8$) over K are

$$\begin{aligned} x_1 &= x_5 = x_7 = x_8 = g_{(0,1/5)}(i)^{120} g_{(0,2/5)}(i)^{120} g_{(1/2,1/10)}(i)^{240} g_{(1/2,3/10)}(i)^{240}, \\ x_2 &= g_{(4/5,1/5)}(i)^{120} g_{(3/5,2/5)}(i)^{120} g_{(9/10,1/10)}(i)^{240} g_{(7/10,3/10)}(i)^{240}, \\ x_3 &= x_6 = g_{(1/5,1/5)}(i)^{120} g_{(2/5,2/5)}(i)^{120} g_{(1/10,1/10)}(i)^{240} g_{(3/10,3/10)}(i)^{240}, \\ x_4 &= g_{(3/5,2/5)}(i)^{120} g_{(1/5,4/5)}(i)^{120} g_{(3/10,7/10)}(i)^{240} g_{(9/10,1/10)}(i)^{240} \end{aligned}$$

with some multiplicity by Propositions 4.6 and 2.4 (iv). So x is a zero of the polynomial

$$(X^2 - 41\,473\,935\,220\,454\,921\,602\,871\,195\,774\,259\,272\,002X + 1)^4,$$

which shows that x is a unit. Therefore, we get

$$\begin{aligned} \sqrt{5} \frac{\varphi(5i)}{\varphi(i)} &= \sqrt[120]{x} \\ &= \sqrt[120]{20\,736\,967\,610\,227\,460\,801\,435\,597\,887\,129\,636\,001} \\ & \quad + 9\,273\,853\,844\,735\,993\,106\,095\,069\,260\,699\,853\,880\sqrt{5} \\ &= \sqrt[10]{682 + 305\sqrt{5}}. \end{aligned}$$

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