Class. Quantum Grav. 7 (1990) 715-716. Printed in the UK

COMMENT

On the algebraic invariants of the four-dimensional Riemann tensor

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Received 13 November 1989

Abstract. In a recent comment Sneddon discussed the set of fourteen algebraic invariants of the Riemann curvature tensor in four dimensions. The focus was rectification of an error (in the form of lack of independence) in an earlier construction and the presentation of a corrected set suitable for application. Several authors who have worked on this problem were mentioned. The comment, however, did not mention the work of Narlikar and Karmarkar who presented a set of invariants well before the earliest work cited in the comment. The original publication by Narlikar and Karmarkar may not be readily available so we list and make a few comments on their set.

Recently, Sneddon (1986) discussed briefly the form in which several authors have presented the fourteen independent algebraic invariants (Haskins 1902)‡ of the Riemann curvature tensor in four dimensions. Among the authors mentioned were Géhéniau and Debever (1956a, b, c) with implied priority of publication. This accords with common usage in which these scalars are almost universally referred to as *the* 'Géhéniau-Debever' scalars. The fact is that Narlikar and Karmarkar (1948) published such a set substantially earlier.

The fourteen scalars according to Narlikar and Karmarkar are constructed from the following tensors. (The notation is theirs. They use C for the Weyl tensor. Several typographic errors in their paper are rectified here.)

$$A_{hijk} = C_{hipq} C_{rsjk} g^{pr} g^{qs} \tag{1}$$

$$B_{hijk} = C_{hipq} A_{rsjk} g^{pr} g^{qs} \tag{2}$$

$$D_{hijk} = B_{hijk} - \tfrac{1}{12} J_2 (g_{hj} g_{ik} - g_{hk} g_{ij}) - \tfrac{1}{4} J_1 C_{hijk} \tag{3}$$

$$\bar{D}_{hijk} = (J_3)^{-1/2} D_{hijk} \tag{4}$$

$$E_{hijk} = C_{hipq} D_{rsjk} g^{pr} g^{qs} \tag{5}$$

$$F_{hijk} = C_{hipq} E_{rsjk} g^{pr} g^{qs} \tag{6}$$

$$Q_\gamma{}^\mu = R_\alpha{}^\mu R_\gamma{}^\alpha . \tag{7}$$

The scalars follow.

$$I_1 = R_\mu{}^\mu \tag{8}$$

$$I_2 = R_\alpha{}^\mu R_\mu{}^\alpha \tag{9}$$

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‡ In this paper is determined the number of algebraic invariants as a function of the dimension of the space.

0264-9381/90/040715+02$03.50

$$I_3 = R_\alpha{}^\mu R_\beta{}^\alpha R_\mu{}^\beta \tag{10}$$

$$I_4 = R_\alpha{}^\mu R_\beta{}^\alpha R_\delta{}^\beta R_\mu{}^\delta \tag{11}$$

$$J_1 = A_{hijk} g^{hj} g^{ik} \tag{12}$$

$$J_2 = B_{hijk} g^{hj} g^{ik} \tag{13}$$

$$J_3 = E_{hijk} g^{hj} g^{ik} \tag{14}$$

$$J_4 = F_{hijk} g^{hj} g^{ik} \tag{15}$$

$$K_1 = C_{hijk} R^{hj} R^{ik} \tag{16}$$

$$K_2 = A_{hijk} R^{hj} R^{ik} \tag{17}$$

$$K_3 = \bar{D}_{hijk} R^{hj} R^{ik} \tag{18}$$

$$K_4 = C_{hijk} Q^{hj} Q^{ik} \tag{19}$$

$$K_5 = A_{hijk} Q^{hj} Q^{ik} \tag{20}$$

$$K_6 = D_{hijk} Q^{hj} Q^{ik}. \tag{21}$$

Some of these expressions may be substantially simplified because of the presence of the 'bivector' metric in the definition of D_{hijk} (equation (3)) and the fact that the Weyl tensor is completely traceless. These are J_3, J_4, K_3, and K_6. With $Y = (J_3)^{-1/2}$ in the latter two they may be written as

$$J_3 = A_{ab}{}^{cd} A_{cd}{}^{ab} - \tfrac{1}{2} J_1{}^2 \tag{22}$$

$$J_4 = A_{ab}{}^{cd} B_{cd}{}^{ab} - \tfrac{5}{12} J_1 J_2 \tag{23}$$

$$K_3 = Y(R^{ab} R^{cd} C_{acij} C_{bdkl} C^{ijkl} - \tfrac{1}{4} J_1 K_1 + \tfrac{1}{12} I_2 J_2 - \tfrac{1}{12} I_1{}^2 J_2) \tag{24}$$

$$K_6 = Q^{ac} Q^{bd} B_{abcd} - \tfrac{1}{4} J_1 K_1 + \tfrac{1}{12} I_4 J_2 - \tfrac{1}{12} I_2^2 J_2. \tag{25}$$

Tangentially to the main issue we note that of the four scalars which do not vanish identically in a vacuum space (J_i, $i = 1, 2, 3, 4$) only J_1 and J_2 appear in subsequent literature (Bergmann 1962†, Weinberg 1972‡). J_3 and J_4 are substantially more complicated than later presentations.

Substantial support for this work was provided by New York University in the form of computer time. The STENSOR program of Lars Hörnfeldt was utilised to accomplish the requisite algebraic manipulation. This help is gratefully acknowledged.

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† See p 231; scalars A^1 and A^3, though expressed in bivector formalism, correspond to J_1 and J_2 respectively.
‡ See p 146; the two scalars expressed without the permutation symbol correspond to J_1 and J_2 respectively.