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ALGEBRAIC KERNEL FUNCTIONS AND REPRESENTATION OF PLANAR DOMAINS

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Abstract. In this paper we study the non-degenerate n-connected canonical domains with $n > 1$ related to the conjecture of S. Bell in [4]. They are connected to the algebraic property of the Bergman kernel and the Szegő kernel. We characterize the non-degenerate doubly connected canonical domains.

1. Introduction

On a bounded planar domain, the Bergman kernel function and the Szegő kernel function play important role to reveal the properties of the holomorphic map between two domains. For example, on a simply connected planar domain, the Riemann mapping function is expressed in terms of the Szegő kernel function (see [6], [12]). The new discovery in [12] that the Szegő kernel is the solution to a Kerzman-Stein Fredholm integral equation of the second kind with C^∞ kernel and inhomogeneous term becomes a very effective way to represent the Szegő kernel numerically and so the Riemann mapping can be expressed explicitly via the Szegő kernel (see [13], [16]).

On the other hand, the classical kernel functions can be written by using conformal mapping since these kernel functions transform under biholomorphic mapping (see [2]). The Bergman kernel function associated to a simply connected domain is a rational combination of basic functions including Riemann mapping function and so is the Szegő kernel function. We also have the transformation formula for the Bergman

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kernel and the Szegő kernel under proper holomorphic mappings from a bounded finitely connected planar domain onto the unit disc (see [1], [10]). So we can get the properties of the classical kernel functions from the properties of the given holomorphic functions by help of those transformation formula and vice versa (see [3]). But the connection between the kernels on finitely connected domains and the kernels on the unit disc are much weaker than direct pull backs. So the transformation law of the kernels under proper mappings is not so powerful as the transformation law of the kernels under biholomorphic mappings.

In addition to the transformation formula for the kernels under proper mappings from a bounded finitely connected planar domain onto the unit disc, the Riemann surface was introduced to prove the result that the Bergman kernel function and the Szegő kernel function associated to a finitely connected domain are generated by finitely many basic functions (see [4]).

In [4] and [5], S. Bell posed the following problem while he was seeking the domain with algebraic Bergman kernel.

PROBLEM 1.1. Can every non-degenerate n-connected planar domain with $n > 1$ be mapped biholomorphically onto a domain of the form

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < r \right\}$$

with complex numbers a_k and b_k, and a positive r?

In this note we give the answer (by constructing a suitable Riemann surface). Also we give related open problems and solve them for $n = 2$.

2. Preliminaries

In this paper, a *non-degenerate n-connected planar domain* is a subdomain Ω of the Riemann sphere $\widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} - \Omega$ consists of exactly n connected components each of which contains more than one point.

Let Ω be a given non-degenerate n-connected planar domain with C^∞ smooth boundary $b\Omega$. Then by using the classical Riemann mapping theorem n times if necessary, we can assume that $b\Omega$ consists of exactly n smooth simple closed curves. Let $b\Omega$ consist of the n non-intersecting C^∞ simple closed curves γ_j with parametrization $z_j(t)$, $0 \le t \le 1$, $j = 1, \ldots, n$. Let T be the complex unit tangent function on $b\Omega$ defined by $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

The Bergman kernel is the kernel for the orthogonal projection of $L^2(\Omega)$ onto its subspace $H^2(\Omega)$ consisting of holomorphic functions. The Szegő kernel is the kernel for the orthogonal projection of $L^2(b\Omega)$ onto the Hardy space $H^2(b\Omega)$ in $L^2(b\Omega)$ consisting of L^2 boundary values of holomorphic functions. The Bergman kernel $B(z,w)$ and the Szegő kernel $S(z,w)$ are related by the identity

$$B(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{jk} H_j'(z)\overline{H_k'(w)},$$

where the function $H_j'(z)$ is the derivative of a multi-valued holomorphic function H_j which is obtained by analytically continuing around Ω a germ of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j, $j = 1,\ldots,n-1$.

Fix a point a in Ω, and let f_a be the Ahlfors map associated to the pair (Ω, a). Among all holomorphic functions h which map into the unit disc with $h(a) = 0$, the Ahlfors map f_a is the unique function which maximizes $|h'(a)|$ with $f_a'(a) > 0$. Here for the definition and properties of the Ahlfors maps, see [2]. In particular, f_a maps properly and holomorphically onto the unit disc. Moreover, f_a can be extended to a continuous map of $\overline{\Omega}$ onto the closed unit disc so that every component γ_j of $b\Omega$, where $j = 1,\ldots,n$, is mapped homeomorphically onto the unit circle.

The Ahlfors map can be expressed as the quotient of the Szegő kernel and the Garabedian kernel via

$$f_a(z) = \frac{S(z,a)}{L(z,a)}$$

for $z \in \Omega$. The Garabedian kernel $L(z,a)$ is the kernel for the orthogonal projection from $L^2(b\Omega)$ onto the orthogonal complement of $H^2(b\Omega)$ and is represented by

$$L(z,a) = \frac{1}{2\pi}\frac{1}{z-a} + H_a(z),$$

where H_a is holomorphic on a neighborhood of $\overline{\Omega}$. The Garabedian kernel $L(z,a)$ and the Szegő kernel $S(z,a)$ are related via the identity

$$S(a,z) = -iL(z,a)T(z)$$

for $a \in \Omega, z \in b\Omega$.

The Szegő kernel $S(z,a)$ has exactly $n-1$ zeroes $a_1, a_2, \ldots, a_{n-1}$ in Ω and $S(a,a) > 0$. The simple zero of f_a at a comes from the simple pole of $L(z,a)$ at a. Note that $f_a'(a) = 2\pi S(a,a)$. The n-to-one map f_a

must have $n-1$ zeroes besides the one at a and these zeroes coincide with the zeroes of $S(z,a)$ since $L(z,a)$ is nonvanishing.

The transformation formula for the Szegő kernel gives rise to a nice formula for the Riemann mapping function, i.e., if f_a is the Riemann mapping from a simply connected domain Ω onto the unit disc, then

$$f_a'(z) = 2\pi \frac{S(z,a)^2}{S(a,a)},$$

where $S(z,a)$ is the Szegő kernel associated to Ω (see [2], [12]).

3. Algebraic kernel functions

A holomorphic function $A(z,w)$ of two complex variables on an open set in $\mathbb{C} \times \mathbb{C}$ is algebraic if there is a holomorphic polynomial $P(a,z,w)$ of three complex variables such that A satisfies $P(A(z,w),z,w)=0$. It is well-known that a function $H(z,w)$, which is holomorphic in z and w on a product domain $\Omega_1 \times \Omega_2$, is algebraic if and only if, for each fixed b, the function $H(z,b)$ is algebraic in z, and for each fixed a, the function $H(a,w)$ is algebraic in w (see [7]). We say that the Bergman kernel function $B(z,w)$ associated to a domain Ω is algebraic if it can be written as $R(z,\overline{w})$, where R is a holomorphic algebraic function of two variables on $\{(z,\overline{w}) : (z,w) \in \Omega \times \Omega\}$. Because the Bergman kernel is hermitian, $B(z,w)$ is algebraic if and only if, for each point $b \in \Omega$, the function $B(z,b)$ is algebraic function of z.

Let Ω be a non-degenerate n-connected planar domain with smooth real analytic boundary and let U be the unit disc. Suppose that $f : \Omega \to U$ is a proper holomorphic map. It is well known that f extends holomorphically past the boundary of Ω and that f' is nonvanishing on $b\Omega$. S. Bell extended f to a meromorphic function on the double of Ω in the following way. Let $\widetilde{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widetilde{\Omega}$ that fixes $b\Omega$. Let $R(\Omega)$ denote the reflection of Ω in $\widetilde{\Omega}$ across the boundary. Since $f(z) = 1/\overline{f(z)}$ for $z \in b\Omega$ and $R(z) = z$ on $b\Omega$, it follows that

$$f(z) = 1/\overline{f(R(z))} \text{ for } z \in b\Omega.$$

The function on the left-hand side of this formula is holomorphic on Ω, the function on the right-hand side is meromorphic on $R(\Omega)$, and two functions extend continuously to $b\Omega$ from opposite sides and agree on $b\Omega$. Hence the function given by $f(z)$ on $\overline{\Omega}$ and $1/\overline{f(R(z))}$ on $R(\Omega)$ is a meromorphic extension of f on $\widetilde{\Omega}$. See [4], [5] for this meromorphic extension of f on $\widetilde{\Omega}$. Here we note that $\widetilde{\Omega}$ is a compact Riemann surface.

The above argument gives an idea to prove the following lemma which induces the fact that every non-degenerate n-connected planar domain Ω, where $n > 1$, is representable as $\Omega = \{z \in \mathbb{C} : |f(z)| < 1\}$ with a suitable rational function f of degree n.

LEMMA 3.1. *Let Ω be a non-degenerate n-connected planar domain. Let a be a point in Ω and let $f_a : \Omega \to U$ be the Ahlfors mapping from Ω to the unit disc U. There is a compact Riemann surface R (without boundary) of genus 0 and a holomorphic injection ι of Ω into R such that*

$$f_a \circ \iota^{-1}$$

can be extended to a meromorphic function, say F, on R.

The proof of Lemma 3.1, which is crucial for Theorem 3.4, is in [11], but for convenience we give it here.

Proof. Since there are only a finite number of zeros of f'_a, there is a positive constant ρ such that $\rho < 1$ and that

$$D = \{\zeta \in \mathbb{C} : \rho < |\zeta| < 1\},$$

is contained in $U - X$ where $X = \{f_a(z) \in U : f'_a(z) = 0\}$. Hence every component W_j, where $j = 1, \dots, n$, of $f_a^{-1}(D)$ is mapped biholomorphically onto D by the restriction $f_a|_{W_j}$, of f_a to W_j.

Now we construct a compact Riemann surface R by using the Ahlfors map f_a to attach discs to the exterior of Ω along each boundary curve. More precisely, we consider the disjoint union $\mathbf{R}$ of Ω and n copies V_j $(j = 1, \dots, n)$ of

$$V = \{\zeta \in \mathbb{C} : \rho < |\zeta|\} \cup \{\infty\}.$$

Identify every subdomain W_j of Ω with the subdomain D_j of V_j corresponding to D by the biholmorphic map corresponding to $f_a|_{W_j}$. Then the resulting set, which we denote by $R = \mathbf{R}/f_a$, has a natural complex structure induced from those on Ω and on every V_j, and hence is a Riemann surface. Here the natural inclusion map ι of Ω into R is a holomorphic injection, and using the complex coordinate ζ_j on the copy V_j corresponding to ζ on V, we have

$$f_a \circ \iota^{-1}(\zeta_j) = \zeta$$

on D_j by the definition.

Now, since topologically R is obtained from Ω by attaching a disc along each boundary curves of Ω, R is a simply connected compact

Riemann surface without boundary, and hence in particular, is of genus 0. Also we can extend $F = f_a \circ \iota^{-1}$ to a meromorphic function on the whole R by setting $F(\zeta_j) = \zeta$ and $F(\infty) = \infty$ on the whole V_j for every j. □

The following uniformization theorem (which is also called the generalized Riemann mapping theorem) is classical and well-known. As references, we cite, for instance, [8] and [9].

PROPOSITION 3.2 (Klein, Koebe and Poincaré). *Every simply connected Riemann surface is mapped biholomorphically onto one of*

- *the unit disc U,*
- *the complex plane $\mathbb{C}$, and*
- *the Riemann sphere $\widehat{\mathbb{C}}$.*

By using the above proposition, we get the following lemma.

LEMMA 3.3. *There is a biholomorphic map h of the above Riemann surface R onto the Riemann sphere $\widehat{\mathbb{C}}$, and hence $F \circ h^{-1}$ is a rational function.*

In [11], we get the following theorem by using Lemma 3.1 and Lemma 3.3.

THEOREM 3.4. *Every non-degenerate n-connected planar domain with $n > 1$ is mapped biholomorphically onto a domain $W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}$ defined by*

$$\left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

with suitable complex numbers a_k and b_k.

REMARK 3.5. It is well known that the reduced Teichmüller space $T(\Omega)$ of a non-degenerate n-connected planar domain Ω can be identified with the Fricke space of a Fuchsian model G of Ω (see [9]). Since G is a free real Möbius group with $n-1$ hyperbolic generators, $T(\Omega)$ is real $(3n-6)$-dimensional if $n > 2$.

The domain $W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}$ in Theorem 3.4 contains $2(n-1)$ complex, i.e. $4n-4$ real, parameters. Every representation of a domain in the above theorem is actually associated with an n-sheeted branched covering of the unit disc by Ω and so we need many more number of parameters in a representation in Theorem 3.4 than Teichmüller parameters for $T(\Omega)$.

Theorem 3.4 is considered as a natural generalization of the classical Riemann mapping theorem for simply connected planar domains. It has importance in the sense that every domain $W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}$ defined as in Theorem 3.4 has algebraic kernel functions. To be precise, the theorem in [4] is as follows.

PROPOSITION 3.6. *Let Ω be a non-degenerate n-connected planar domain with $n > 1$. The following conditions are equivalent.*

1. *The Bergman kernel associated to Ω is algebraic.*
2. *The Szegő kernel associated to Ω is algebraic.*
3. *There is a proper holomorphic mapping $f : \Omega \to U$ which is algebraic.*
4. *Every proper holomorphic mapping from Ω onto the unit disc U is algebraic.*

The function f defined by

$$f_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}(z) = z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k}$$

is a proper holomorphic mapping from $W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}$ onto the unit disc U which is algebraic. Hence the above proposition implies the following corollary.

COROLLARY 3.7. *Every non-degenerate n-connected planar domain with $n > 1$ is biholomorphic to a domain $W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}}$ with algebraic Bergman kernel and algebraic Szegő kernel.*

4. Open problems and an example

Now we pose the following natural problems for our canonical domains.

PROBLEM 4.1. Determine the locus $\mathbf{B}_n$ in $\mathbb{C}^{2n-2}$ of

$$(a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1})$$

such that the corresponding domain

$$W_{a_1,a_2,\ldots,a_{n-1},b_1,b_2,\ldots,b_{n-1}} = \left\{ z \in \mathbb{C} : \left| z + \sum_{k=1}^{n-1} \frac{a_k}{z - b_k} \right| < 1 \right\}$$

is a non-degenerate n-connected planar domain.

We call this locus $\mathbf{B}_n$ the *coefficient body for non-degenerate n-connected canonical domains.* Clearly, $\mathbf{B}_n \cap \Pi_j$ is empty for every j, where

$$\Pi_j = \{(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}) \in \mathbb{C}^{2n-2} : a_j = 0\}.$$

PROBLEM 4.2. Fix a point $(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1})$ in $\mathbf{B}_n$, and let

$$W = W_{a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}}$$

be the corresponding n-connected canonical domain. Determine the subset $E(W)$ of $\mathbf{B}_n$ consisting of all points which correspond to n-connected canonical domains biholomorphically equivalent to W.

We call $E(W)$ the *leaf* in $\mathbf{B}_n$ for W.

REMARK 4.3. For every such non-degenerate n-connected canonical domain W, the subset $E(W)$ is a non-empty proper subset of $\mathbf{B}_n$ by Theorem 3.4. Also note that $E(W)$ contains an element with $a_1 > 0$. Actually, if $E(W)$ contains an element with $a_1 = re^{i\theta} \notin \mathbb{R}$, where $r > 0$ and $\theta \in \mathbb{R}$, then by changing the variable z to $e^{i\theta/2}z$, we have such a point in $E(W)$ as desired.

Now we discuss the case that $n = 2$. It is well-known (cf. [5]) that

$$A(r) = \{z \in \mathbb{C} : |z + z^{-1}| < r\}$$

is a doubly connected domain with smooth real analytic boundary curves if $r > 2$ and the mapping

$$f_r(z) = \frac{1}{r}(z + z^{-1})$$

is a proper holomorphic map from $A(r)$ onto the unit disc which gives a 2-sheeted branched covering of U by $A(r)$. Moreover, since $f_r'(z) = (1/r)(1 - z^{-2})$, f_r also gives a 2-sheeted covering of the Riemann sphere $\hat{\mathbb{C}}$ by itself branched over $\pm 2/r$ for every positive r.

REMARK 4.4. It is well-known that two doubly connected domains are mutually biholomorphic if and only if the modulus of them are the same. Here the *modulus* of the doubly connected domain $\{1 < |z| < s\}$ is $\log s$ by definition. By using the standard results in [14], S. Bell observed in [5] that the modulus $m(r)$ of $A(r)$ is a continuous increasing function of r, which goes to zero as r approaches to 2 from above and which goes to ∞ as $r \to \infty$. So every non-degenerate doubly connected domain is biholomorphic to exactly one of $A(r)$ with $r > 2$.

First, we determine the coefficient body for the doubly connected canonical domains.

THEOREM 4.5. *For a complex number a, let a' be a complex number such that $(a')^2 = a$. Then*

$$\mathbf{B}_2 = \{(a,b) \in \mathbb{C}^2 : a \neq 0, |b+2a'| < 1, |b-2a'| < 1\}$$

(which is independent of the choice of a').

Proof. Take any (a,b) in $\mathbb{C}^2$ with $a \neq 0$, and consider the corresponding rational function

$$f_{a,b}(z) = z + \frac{a}{z-b}.$$

Then since the zeros of $f'_{a,b}$ are $b \pm a'$, we see that $f_{a,b}$ is a 2-sheeted covering of the Riemann sphere $\hat{\mathbb{C}}$ by itself branched over $f_{a,b}(b \pm a') = b \pm 2a'$, for every $a \neq 0$. Hence the following lemma, which is easy to see, implies the assertion. □

LEMMA 4.6. *The preimage $f_{a,b}^{-1}(U)$ of the unit disc U is*

1. *disconnected if and only if*

$$|b+2a'| \geq 1 \quad \text{and} \quad |b-2a'| \geq 1,$$

2. *a simply connected domain if and only if*

$$|b+2a'| \geq 1 > |b-2a'| \quad \text{or} \quad |b+2a'| < 1 \leq |b-2a'|,$$

and

3. *a doubly connected domain if and only if*

$$|b+2a'| < 1 \quad \text{and} \quad |b-2a'| < 1.$$

REMARK 4.7. For $(a,b) \in \mathbb{C}^2$ to be in $\mathbf{B}_2$, (a,b) should satisfy $|a| < 1/4, |b| < 1$.

Next to determine the leaves in $\mathbf{B}_2$, we show the following theorem.

THEOREM 4.8. *Fix $r > 2$ and a point (a,b) in $\mathbf{B}_2$. Then the corresponding domain $W_{a,b}$ is biholomorphic to $A(r)$ if and only if there is a biholomorphic map $T(z)$ of the unit disc U onto itself such that*

$$T(\{b \pm 2a'\}) = \{\pm 2/r\}.$$

Proof. First assume that $W_{a,b}$ is biholomorphic to $A(r)$. Then $W_{a,b}$ and $A(r)$ are mapped biholomorphically onto the same $R = \{z \in \mathbb{C} : \rho^{-1} < |z| < \rho\}$ with $\rho = e^{m(r)/2}$. Also it is known that for every holomorphic functions $\pi(z)$ on R which gives a 2-sheeted covering of R onto U, $\pi'(z)$ has two zeros which can be written as $\pm e^{i\theta}$ with a suitable real θ. (Actually, the sheet-interchange $I(z)$ of the covering $\pi : R \to U$ is a conformal automorphism of R which maps $\{|z| = \rho^{-1}\}$ onto $\{|z| = \rho\}$

and whose fixed points are exactly the zeros of $\pi'(z)$. And it is well-known that such a conformal automorphism $I(z)$ of R should be $e^{2i\theta}/z$ with a suitable real θ.)

Now fix a biholomorphic mappings $w_{a,b}(z)$ and $w_r(z)$, respectively, of $W_{a,b}$ and of $A(r)$ onto R. Then since $f_{a,b} \circ w_{a,b}^{-1}(z)$ and $f_r \circ w_r^{-1}(z)$ give 2-sheeted coverings of R onto U, we may assume that

$$w_{a,b}(b \pm a') = \pm 1, \qquad w_r(\pm 1) = \pm 1$$

by changing the variable z on R to $e^{i\theta}z$ with some suitable $\theta \in \mathbb{R}$ if necessary. Thus, if we set

$$g(z) = w_r^{-1} \circ w_{a,b}(z),$$

then $g(z)$ maps $W_{a,b}$ biholomorphically onto $A(r)$, and maps $b \pm a'$ to ± 1.

Recall that every $A(r)$ has the canonical biholomorphic involution

$$J(z) = \frac{1}{z},$$

which fixes $\{\pm 1\}$ pointwise, the image of which by f_r is $\{\pm 2/r\}$, and interchanges the sheets of the covering $f_r : A(r) \to U$. Hence

$$J_{a,b}(z) = g^{-1} \circ J \circ g(z)$$

is a biholomorphic involution of $W_{a,b}$, which fixes $\{b \pm a'\}$ pointwise, the image of which by $f_{a,b}$ is $\{b \pm 2a'\}$, and interchanges the sheets of the covering by $f_{a,b} : W_{a,b} \to U$. In particular, for every $z_0 \in W_{a,b}$, the preimage $\{z_0, J_{a,b}(z_0)\}$ of $f_{a,b}(z_0)$ by $f_{a,b}(z)$ is mapped by $g(z)$ onto $\{g(z_0), J(g(z_0))\}$, which is the preimage of $f_r(g(z_0))$ by $f_r(z)$.

Thus for every $\alpha \in U$, $g(z)$ maps the preimage $f_{a,b}^{-1}(\alpha)$ bijectively onto the preimage $f_r^{-1}(\beta)$ with some unique β with $|\beta| < 1$. This implies that $g(z)$ induces a bijection $T(z)$ of U onto itself, which is biholomorphic as is seen from the construction.

Next suppose that there is a biholomorphic map $T(z)$ of the unit disc U onto itself such that

$$T(\{b \pm 2a'\}) = \{\pm 2/r\}.$$

Then the following lemma shows the desired assertion. □

LEMMA 4.9. *The map $T(z)$ can be lifted to a biholomorphic map of $W_{a,b}$ onto $A(r)$.*

Though this is a well-known fact, we include a sketch of a proof for the sake of convenience. Recall that every $A(r)$ has the canonical anticonformal automorphism

$$\Pi(z) = \frac{1}{\bar{z}},$$

which fixes the unit circle S^1 pointwise, and the image $f_r(S^1)$ is the segment $L = [-2/r, 2/r]$.

Now cut U by L, then the preimage $f_r^{-1}(U - L)$ consists of two connected components, say $D_r^{\pm}$, each of which is biholomorphic to $U - L$ and bounded by two analytic Jordan curves. Similarly, cut U by $T^{-1}(L)$, then since $T^{-1}(L)$ is a circular arc connecting $b \pm 2a'$, the preimage $f_{a,b}^{-1}(U - T^{-1}(L))$ also consists of two connected components, say $D_{a,b}^{\pm}$, each of which is biholomorphic to $U - T^{-1}(L)$ and bounded by two analytic Jordan curves.

In particular, f_r^{-1} has single-valued branches $h_r^{\pm}$ which map $U - L$ biholomorphically onto $D_r^{\pm}$, respectively. Thus, on $D_{a,b}^{\pm}$ set

$$g^{\pm}(z) = h_r^{\pm} \circ T \circ f_{a,b}.$$

Then we can see that $g^{\pm}(z)$ has the same continuous boundary values on the common boundary of $D_{a,b}^{\pm}$. Thus the classical theorem of Panlevé implies that $g^{\pm}(z)$ determines a biholomorphic map of $W_{a,b}$ onto $A(r)$.

COROLLARY 4.10. *For every given* $r > 2$,

$$E(A(r)) = \left\{ (a,b) \in \mathbf{B}_2 : \left| \frac{4a'}{1 - \overline{(b+2a')}(b-2a')} \right| = \frac{4r}{4+r^2} \right\}.$$

In particular,

$$E(A(r)) \cap \{(a,0) \in \mathbb{C}^2\} = \{(a,0) \in \mathbb{C}^2 : |a| = r^{-2}\}.$$

Proof. Since

$$S(z) = -\frac{z - (2/r)}{1 - (2z/r)}$$

maps U biholomorphically onto U, $S(2/r) = 0$, and $S(-2/r) = 4r/(4 + r^2)$, there is a biholomorphic map $T(z)$ of the unit disc U onto itself such that

$$T(\{b \pm 2a'\}) = \{\pm 2/r\}$$

if and only if there is a biholomorphic map $\tilde{T}(z)$ of the unit disc U onto U such that

$$\tilde{T}(b + 2a') = 0, \qquad |\tilde{T}(b - 2a')| = 4r/(4 + r^2).$$

Thus using

$$\tilde{T}(z) = \frac{z - (b + 2a')}{1 - \overline{(b + 2a')}z}$$

we have the assertion.

In particular, if $(a, 0) \in E(A(r))$, then

$$\frac{|4a'|}{1 + 4|a'|^2} = \frac{4r}{4 + r^2}$$

implies that $|a'| = 1/r$ or $r/4$. Since $(a, 0) \in \mathbf{B}_2$, $|a'| < 1/2$ by Theorem 4.5. Hence $|a| = r^{-2}$ for $r > 2$ and the converse also holds. □

Finally, we give examples of a set which contains exactly one point of every leaf $E(A(r))$.

EXAMPLE 4.11. For every real positive a satisfying $0 < a < 1/4$, set

$$W_{a,0} = \{z \in \mathbb{C} : |z + \frac{a}{z}| < 1\}.$$

Note that $W_{a,0}$ becomes the disjoint union of two simply connected domains as a becomes bigger than $1/4$. By replacing z/r by z in the defining function $f(z) = (z + z^{-1})/r$ of $A(r)$, we get directly that each doubly connected domain is biholomorphic to $W_{a,0}$ with $a = r^{-2}$. It means that, in the family $\{W_{a,0}\}$ with $0 < a < 1/4$, there are no pair of mutually biholomorphic domains and the set $\{(a, 0) \in \mathbb{C}^2 : 0 < a < 1/4\}$ contains a point of every leaf $E(A(r))$.

Also for a real positive a,

$$W_{-a,0} = \{z \in \mathbb{C} : |z - \frac{a}{z}| < 1\}$$

is biholomorphic to $W_{a,0}$ by the map $z \to iz$. So we can say that, in the family $\{W_{-a,0}\}$ with $0 < a < 1/4$, there are no pair of mutually biholomorphic domains and the set $\{(-a, 0) \in \mathbb{C}^2 : 0 < a < 1/4\}$ contains a point of every $E(A(r))$.

More generally, for a real θ and a real positive a,

$$W_{e^{i\theta}a,0} = \{z \in \mathbb{C} : |z + \frac{e^{i\theta}a}{z}| < 1\}$$

is biholomorphic to $W_{a,0}$ by the map $z \to e^{i\theta/2}z$ by Remark 4.3. Hence in the family $\{W_{e^{i\theta}a,0}\}$ with $0 < a < 1/4$, there are no pair of mutually biholomorphic domains and the set $\{(e^{i\theta}a, 0) \in \mathbb{C}^2 : 0 < a < 1/4\}$ contains a point of every $E(A(r))$.

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