Acta Math., 165 (1990), 189–227

Algebraic L^2 decay for Navier–Stokes flows in exterior domains

by

WOLFGANG BORCHERS and TETSURO MIYAKAWA

Universität Paderborn
Paderborn, West Germany

Hiroshima University
Hiroshima, Japan

1. Introduction

In this paper we deduce algebraic decay rates for the total kinetic energy of weak solutions of nonstationary Navier–Stokes equations in exterior domains $\Omega\subset\mathbf{R}^n$, $n\geqslant 3$:

$$\begin{aligned} &\frac{\partial v}{\partial t}+v\cdot\nabla v-\Delta v+\nabla p=0 \quad \text{in} \quad (0,\infty)\times\Omega \\ &\nabla\cdot v=0 \quad \text{in} \quad (0,\infty)\times\Omega \\ &v|_{\partial\Omega}=0; \quad v\to 0 \quad \text{as} \quad |x|\to\infty, \\ &v|_{t=0}=a. \end{aligned} \tag{NS}$$

Here $v=(v_1, \ldots, v_n)$ and p denote, respectively, unknown velocity and pressure, while $a=(a_1, \ldots, a_n)$ is a given initial velocity. By exterior domain we mean a connected open set Ω whose complement is the closure of the union of a finite number of bounded domains with C^∞ boundaries. For problem (NS) the existence of a weak solution in L^2 was first established by Hopf [16] for an arbitrary L^2-initial velocity. The uniqueness and the regularity of Hopf's weak solutions are still open questions.

The square of the L^2-norm of the fluid velocity v is proportional to the kinetic energy of the fluid under consideration; so in view of the presence of the viscosity term Δv and the no-slip boundary condition $v|_{\partial\Omega}=0$, it is reasonable to expect that the solution v would decay in L^2 as $t\to\infty$. However, it is in general not easy to deduce the expected L^2 decay property for the Navier–Stokes problem in unbounded domains. This L^2 decay problem was first raised by Leray [24] in the case of the Cauchy problem in $\mathbf{R}^3$ and then was affirmatively solved by Kato [20] for the Cauchy problem in $\mathbf{R}^3$ and $\mathbf{R}^4$ by using the fact that Leray's weak solutions become regular after a finite time.

In this paper we are interested in the L^2 decay property of weak solutions of the exterior problem (NS). Since we want to discuss also the case of space dimensions >4, in which the regularity after a finite time of weak solutions can no more be expected, we have to employ another approach different from that of [20]. Our approach adopted here is based on the Fourier analysis for closed linear operators in Banach spaces and extends those of Schonbek [33, 34], Kajikiya and Miyakawa [18], Borchers and Miyakawa [3] and Wiegner [43], all of which were developed in the case of entire spaces $\mathbf{R}^n$ and halfspaces $\mathbf{R}^n_+$, $n \geqslant 2$. This approach does not require the regularity of weak solutions and, moreover, provides apparently optimal decay rates.

To explain our approach, let us consider the linearized version of (NS), namely, the Stokes problem in exterior domains:

$$\begin{gathered} \frac{\partial v^0}{\partial t} - \Delta v^0 + \nabla p^0 = 0 \quad \text{in} \quad (0, \infty) \times \Omega \\ \nabla \cdot v^0 = 0 \quad \text{in} \quad (0, \infty) \times \Omega \\ v^0|_{\partial\Omega} = 0; \quad v^0 \to 0 \quad \text{as} \quad |x| \to \infty, \\ v^0|_{t=0} = a. \end{gathered} \tag{S}$$

It is known [4] that the map $a \to v^0(t)$, $t \geqslant 0$, defines a bounded analytic semigroup of class C_0 in each L^r space, $1<r<\infty$, of solenoidal vector fields. As in our previous work [3], we want to state our decay results in the form of the comparison of the decay rates of weak solutions v with those of functions v^0 corresponding to the same initial data as v. To do so, we need first analyze decay properties of $v^0(t)$ and then find an appropriate estimate on the nonlinear term $v \cdot \nabla v$ which ensures that the low-frequency components of $v \cdot \nabla v$ can be made as small as we please as $t \to \infty$. To this end we use as our basic tool the negative of the generator of the above-mentioned semigroup, namely, the Stokes operator $A = A_r$ in L^r spaces. Due to the boundedness and analyticity of the corresponding semigroup, the fractional powers of A_r are defined in the standard manner as in [21, 22, 26, 42]. Using the recent result of Giga and Sohr [13], which guarantees the existence of bounded pure imaginary powers of A_r, we apply the complex interpolation theory of Banach spaces to examine the domains of the fractional powers and thereby establish an embedding theorem of Sobolev type involving the fractional powers. This embedding theorem, stated in Section 4, enables us to analyze decay properties of functions $v^0(t)$ as well as to find a nice estimate on the nonlinear term $v \cdot \nabla v$. These results on v^0 and $v \cdot \nabla v$ combined with general calculation schemes as developed in [3, 18, 33, 34, 43] eventually yield the desired L^2 decay results for weak solutions of (NS).

As shown in Section 5, our estimate on $v \cdot \nabla v$ automatically gives a definite algebraic decay rate for its low-frequency components depending only on the space dimension n. This indicates that in general we cannot expect that our weak solutions themselves would decay more rapidly than the nonlinear term, even when the corresponding functions v^0 decay in L^2 exponentially.

In [25] Maremonti discussed L^2 decay problem for (NS) in three dimensions. Applying the energy integral method of Heywood [15], he proved that if a is in $L^r \cap L^2$ for some $1<r\leq 2$, then there is a weak solution which decays in L^2 like the corresponding solution v^0 of (S). This result does not reflect the presence of the nonlinear term, because, as will be shown in Section 2, in his case the nonlinear term decays more rapidly than the function v^0 and the decay property of his weak solutions is determined by that of v^0. Our results thus include that of [25] as a special case (see Theorems A and B in Section 2).

Using the boundedness of the semigroup $a \to v^0(t)$ in general L^r spaces, we can show (see Lemma 5.2) that any weak solutions decay in L^q-norms, $n/(n-1)\leq q<2$, if the corresponding initial data belong to $L^r \cap L^2$ for some $1<r\leq n/(n-1)$. This improves the same type of result of Galdi and Maremonti [10, 25] and implies in particular that the weak solutions treated in our Theorem A in Section 2 decay in L^q, $r\leq q\leq 2$, with explicit rates in case $r<q\leq 2$, if in addition $r<2n/(n+2)$; see Theorem C in Section 2.

Our main results are stated in Section 2. Sections 3 and 4 are devoted to the study of the Stokes operator A_r. Since in our case A_r has no bounded inverse, the study of fractional powers requires more careful arguments than in the case of bounded domains as treated in [12]. We use homogeneous Sobolev spaces to examine the domains of fractional powers by means of the complex interpolation theory, and prove that the functions ∇u and $A^{1/2}u$ have equivalent L^r-norms provided $1<r<n$. The same result is given in [13] for $1<r<n/2$ and $1<r\leq 2$. To extend the range of r to $1<r<n$, we consider the stationary Stokes problem with singular data and deduce a coercive estimate on L^r-Dirichlet norms, $1<r<n$, of solutions. The desired equivalence of ∇u and $A^{1/2}u$ in appropriate L^r spaces is then deduced through an interpolation argument, and this gives us an embedding theorem of Sobolev type for domains of fractional powers.

The above-mentioned estimate for the stationary Stokes system with singular data was first deduced by Cattabriga [6] in the case of three-dimensional bounded domains. We first extend Cattabriga's result to the case of general space dimensions and then apply the cut-off argument as developed in [4] in order to decompose our problem to the cases of entire spaces and bounded domains. This is carried out in Section 3.

The present work was initiated while the second author was visiting the University

of Paderborn in 1986–87. We wish to thank Professors R. Rautmann and H. Sohr at the University of Paderborn for a number of stimulating and helpful discussions and valuable suggestions.

2. Main results

We introduce some notation and definitions. Given a domain Ω of $\mathbf{R}^n$, we denote by $C_0^\infty(\Omega)$ the set of scalar, as well as vector, C^∞-functions with compact support in Ω. $C_{0,\sigma}^\infty(\Omega)$ is the set of solenoidal vector fields on Ω with components in $C_0^\infty(\Omega)$. For simplicity we use the same notation for denoting spaces of scalar and vector functions unless otherwise specified. $L^r(\Omega)$, $1\leqslant r\leqslant\infty$, is the usual Lebesgue space with norm $\|\cdot\|_r=\|\cdot\|_{r,\Omega}$; and for nonnegative integers k, $H^{k,r}(\Omega)$ denotes the L^r Sobolev space with norm $\|\cdot\|_{k,r}=\|\cdot\|_{k,r,\Omega}$. $H_0^{k,r}(\Omega)$ is the $H^{k,r}$-closure of $C_0^\infty(\Omega)$. When Ω is unbounded, we need also the homogeneous Sobolev space $\hat{H}_0^{k,r}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|\nabla^k u\|_r = \sum_{|\alpha|=k} \|\partial^\alpha u\|_r$$

where $\partial^\alpha=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$, $\partial_i=\partial/\partial x_i$ and $|\alpha|=\alpha_1+\dots+\alpha_n$ for any multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$ of nonnegative integers. The bracket $\langle\cdot,\cdot\rangle$ stands for the duality pairing between various Banach function spaces which extends the standard L^2-inner product for real-valued functions. $H^{-k,r}(\Omega)$, and $\hat{H}^{-k,r}(\Omega)$, $1<r<\infty$, denote the dual space of $H_0^{k,r'}(\Omega)$ and $\hat{H}_0^{k,r'}(\Omega)$, $r'=r/(r-1)$, respectively.

We now define the notion of *weak solution* of problem (NS). For an exterior domain Ω of $\mathbf{R}^n$, $n\geqslant 3$, we denote by $L_\sigma^r(\Omega)$, $1<r<\infty$, the L^r-closure of $C_{0,\sigma}^\infty(\Omega)$. Then we have the Helmholtz decomposition of L^r-vector fields:

$$\begin{aligned} L^r(\Omega) &= L_\sigma^r(\Omega)+G^r(\Omega) \quad \text{(direct sum)} \\ L_\sigma^r(\Omega) &= \{u\in L^r(\Omega);\ \nabla\cdot u=0,\ u\cdot\nu|_{\partial\Omega}=0\}; \qquad (2.1) \\ G^r(\Omega) &= \{\nabla p\in L^r(\Omega);\ p\in L_{\mathrm{loc}}^r(\bar{\Omega})\}, \end{aligned}$$

where $\nabla\cdot u$ is understood in the sense of distributions and the normal component $u\cdot\nu|_{\partial\Omega}$ of u is well defined in the dual space $W^{-1/r,r}(\partial\Omega)$ of the fractional Sobolev trace space $W^{1/r,r'}(\partial\Omega)=W^{1-1/r',r'}(\partial\Omega)$. Further we have ([28])

$$L_\sigma^r(\Omega)^* = L_\sigma^{r'}(\Omega); \quad G^r(\Omega) = L_\sigma^{r'}(\Omega)^\perp \qquad (2.2)$$

where * means the dual space and $^{\perp}$ the annihilator. The results (2.1) and (2.2) are proved in [28, 37] for three-dimensional exterior domains, but the proofs given [28] applies also to higher-dimensional case.

Let $a\in L^2_\sigma(\Omega)$. A function v in $L^\infty(0,\infty;\, L^2_\sigma(\Omega))\cap L^2(0,\infty;\, \hat{H}^{1,2}_0(\Omega))$ is called a *weak solution* of problem (NS) if v is continuous from $[0,\infty)$ to $L^2_\sigma(\Omega)$ in the weak topology, $v(0)=a$, and the identity

$$\langle v(t),\phi(t)\rangle+\int_s^t(\langle\nabla v,\nabla\phi\rangle+\langle v\cdot\nabla v,\phi\rangle)\,d\tau=\langle v(s),\phi(s)\rangle+\int_s^t\langle v,\phi'\rangle\,d\tau \tag{2.3}$$

holds for all $0\leq s\leq t<\infty$ and $\phi\in C^1([0,\infty);L^2_\sigma(\Omega))\cap C^0([0,\infty);H^{1,2}_0(\Omega)\cap L^n(\Omega))$. Here $\phi'=\partial\phi/\partial t$ and $\langle\nabla v,\,\nabla\phi\rangle=\Sigma_i\langle\partial_i v,\partial_i\phi\rangle$; the requirement that ϕ be in $L^n(\Omega)$ is necessary in order for the nonlinear term in (2.3) to be well defined. In the usual definitions of weak solution the function v is required only to be in $L^\infty_{\mathrm{loc}}([0,\infty);L^2_\sigma(\Omega))\cap L^2_{\mathrm{loc}}([0,\infty);\,\hat{H}^{1,2}_0(\Omega))$. However, since all the weak solutions constructed so far satisfy the energy inequality:

$$\|v(t)\|_2^2+2\int_0^t\|\nabla v\|_2^2\,d\tau\leq\|a\|_2^2$$

for all $t\geq 0$, we adopt our present definition. Since the weak continuity of v necessarily follows from (2.3), our definition of weak solution agrees with the usual ones (see [23, 27, 30, 35]).

We can now state our main results.

THEOREM A. *Let* $n\geq 3$, $a\in L^2_\sigma(\Omega)$ *and let* v^0 *be the solution of problem* (S) *with* $v^0(0)=a$.

(i) *There is a weak solution* v *of* (NS) *with the following properties:* (a) $\|v(t)\|_2\to 0$ *as* $t\to\infty$. (b) *If in addition* $\|v^0(t)\|_2=O(t^{-\alpha})$ *as* $t\to\infty$ *for some* $\alpha>0$, *then* $\|v(t)\|_2=O(t^{-\beta})$ *as* $t\to\infty$, *where* $\beta=\min(\alpha,\, n/4-\varepsilon)$ *and* ε *is an arbitrary number such that* $0<\varepsilon<1/4$. (c) *The function* $v(t)-v^0(t)$ *satisfies* $\|v(t)-v^0(t)\|_2=o(t^{-n/4+1/2})$ *as* $t\to\infty$. (d) *If in addition* $\|v^0(t)\|_2=O(t^{-\alpha})$ *as* $t\to\infty$ *for some* $\alpha>0$, *then* $\|v(t)-v^0(t)\|_2=O(t^{-\gamma})$ *as* $t\to\infty$, *where* $\gamma=n/4-1/2+\alpha$ *if* $\alpha<1/2$; *and* $0<\gamma<n/4$ *is arbitrary in case* $\alpha\geq 1/2$.

(ii) *If a weak solution* v *of* (NS) *satisfies the energy inequality of the following form:*

$$\|v(t)\|_2^2+2\int_s^t \|\nabla v\|_2^2\, d\tau \leqslant \|v(s)\|_2^2 \quad \textit{for} \quad s=0,\ a.e.\ s>0 \textit{ and all } t\geqslant s \tag{E}$$

then v possesses all the properties (a)–(d) *described in* (i).

Part (i) asserts the *existence* of a weak solution with properties (a)–(d) for any initial data $a\in L_\sigma^2(\Omega)$, while part (ii) asserts that *any* weak solutions satisfying the energy inequality (E) have properties (a)–(d). We note, however, that the existence of a weak solution satisfying (E) is known only when $n=3,4$ (see [20, 24, 29]), and, moreover, it seems impossible to deduce (E) for general weak solutions in case $n\geqslant 5$. It is also proved in [29] that the energy inequality (E) implies property (a). Our part (ii) is thus an improvement of the decay result established in [29].

Theorem A was first proved by Wiegner [43] for the Cauchy problem, with $\beta=\min(\alpha, (n+2)/4)$. The same result can be deduced also in the case of halfspaces if we use various estimates given in [3]. Contrary to these cases, our Theorem A provides slower decay rate: $\beta=\min(\alpha, n/4-\varepsilon)$. As will be shown in Sections 4 and 5, this is mainly because our embedding theorem for domains of fractional powers holds only for the exponents $1<r<n$.

When $a\in L^r(\Omega)\cap L_\sigma^2(\Omega)$ for some $1<r<2$, one can take $\alpha=(n/r-n/2)/2$ as shown in Section 4. Hence in this case $\beta=\alpha$ and we obtain the following, which is due to Maremonti [25] in case $n=3$.

THEOREM B. *If* $a\in L^r(\Omega)\cap L_\sigma^2(\Omega)$ *for some* $1<r<2$, *and* $n\geqslant 3$, *then there is a weak solution v of* (NS) *such that* $\|v(t)\|_2=O(t^{-\gamma})$ *as* $t\to\infty$, *where* $\gamma=(n/r-n/2)/2$. *The same holds for any weak solutions satisfying energy inequality* (E).

Our final result concerns the behavior of L^q-norms, $q<2$, of weak solutions. The following improves the same type of results of Galdi and Maremonti [10, 25].

THEOREM C. *If* $n\geqslant 3$ *and* $a\in L^r(\Omega)\cap L_\sigma^2(\Omega)$ *for some* $1<r\leqslant n/(n-1)$ *with* $r<2n/(n+2)$, *then the weak solution given in Theorem* A *lies in the space* $L^\infty(0,\infty; L^q(\Omega))$ *for all* $r\leqslant q\leqslant 2$; *and we have* $\|v(t)\|_q=o(t^{-\eta})$ *as* $t\to\infty$, *with* $\eta=(n/r-n/q)/2$ *provided* $q<2$.

Theorems A and C will be proved in Section 5, after preparing necessary material in Sections 3 and 4. In what follows we use the summation convention and C denotes constants which may vary from line to line.

3. The Stokes operator over an exterior domain

We first define the Stokes operator and discuss its basic properties. let $P=P_r$ be the bounded projection from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$, $1<r<\infty$, associated with the Helmholtz decomposition (2.1). The operator

$$Au=A_r u=-P_r\Delta u, \quad u\in D(A_r)=L^r_\sigma(\Omega)\cap H_0^{1,r}(\Omega)\cap H^{2,r}(\Omega) \tag{3.1}$$

is called the Stokes operator in $L^r_\sigma(\Omega)$. The equation $Au=Pf$ is equivalent to the stationary Stokes system:

$$\begin{aligned} &-\Delta u+\nabla p=f, \quad \nabla\cdot u=0 \quad \text{in } \Omega;\\ &u|_{\partial\Omega}=0; \quad u\to 0 \quad \text{as} \quad |x|\to\infty. \end{aligned} \tag{SS}$$

Since (SS) is elliptic in the sense of Douglis and Nirenberg, elliptic regularity theory as given in [1] implies that A_r is a densely defined closed linear operator in $L^r_\sigma(\Omega)$ and, for each $m=1, 2, \ldots, D(A_r^m)$ is contained in $H^{2m,r}(\Omega)$ with the graph-norm equivalent to $\|\cdot\|_{2m,r}$. The dual operators of P_r and A_r are given by

$$P_r^*=P_{r'}, \quad A_r^*=A_{r'}, \quad r'=r/(r-1) \quad \text{(see [9])}. \tag{3.2}$$

It is known [11, 37] that $-A_r$ generates an analytic semigroup $\{e^{-tA_r};\ t\geq 0\}$ of class C_0. In this paper, however, our subsequent argument is based on the following improvement of the results of [11, 37], which is due to [4] and [13]. In what follows the complexifications of various function spaces will be written with the same notation as the original real ones.

THEOREM 3.1. *If $n\geq 3$ and $1<r<\infty$, then for each $0<\varepsilon<\pi/2$ there is a constant $c_\varepsilon=c(\varepsilon, r, n, \Omega)$ so that for all $u\in L^r_\sigma(\Omega)$, $t\in\mathbf{R}$ and all complex numbers $\lambda\neq 0$ with $|\arg\lambda|\leq\pi-\varepsilon$, we have*

(i) $\|(\lambda+A_r)^{-1}u\|_r\leq c_\varepsilon|\lambda|^{-1}\|u\|_r$.

(ii) $\|\nabla^2(\lambda+A_r)^{-1}u\|_r\leq c_\varepsilon\|u\|_r$ *provided* $1<r<n/2$.

(iii) *The pure imaginary powers $(\lambda+A_r)^{it}$, $\lambda>0$, are defined as bounded linear operators on $L^r_\sigma(\Omega)$ satisfying the estimates*

$$\|(\lambda+A_r)^{it}u\|_r\leq c_\varepsilon e^{\varepsilon|t|}\|u\|_r.$$

Parts (i) and (ii) are proved in [4] and part (iii) in [13]. By (i) we can define the fractional powers A_r^α, $\alpha\geq 0$, as in [21, 22, 26, 42]. Part (iii) is proved in [13] only for $\lambda=0$;

but one can easily verify that the proof of [13] actually asserts our version (iii) stated above. Part (iii) enables us to study the domains of A_r^α with the aid of the complex interpolation theory. From (i) we can deduce

PROPOSITION 3.2. (i) *The analytic semigroup* $\{e^{-tA_r};\ t\geq 0\}$ *is bounded.*

(ii) *For each* $\alpha\geq 0$ *we have the estimate*

$$\|A^\alpha e^{-tA}u\|_r \leq Ct^{-\alpha}\|u\|_r, \quad u\in L^r_\sigma(\Omega),\ t>0. \tag{3.3}$$

(iii) *For each* $\alpha\geq 0$,

$$\|A^\alpha(\lambda+A)^{-\alpha}u\|_r \leq C\|u\|_r, \quad u\in L^r_\sigma(\Omega),\ \lambda>0. \tag{3.4}$$

(iv) *The operators* A_r^α, $\alpha\geq 0$, *are all injective.*

Proof. The boundedness of the semigroup and estimate (3.3) for integers $\alpha\geq 0$ are well known; see for instance the argument in [19, p. 491]. Application of the moment inequalities [22]:

$$\|A^\beta u\|_r \leq C\|A^\alpha u\|_r^\theta\,\|A^\gamma u\|_r^{1-\theta}, \quad 0\leq\alpha<\beta<\gamma\leq 1,\ \theta=(\gamma-\beta)/(\gamma-\alpha)$$

then yields (3.3) for general $\alpha\geq 0$. Estimate (3.4) follows from Theorem 3.1 (i) and [21, Proposition 6.3]; see also [26]. Now if $A_r u=0$, then elliptic regularity theory implies that $u\in L^q_\sigma(\Omega)$ for some $q>2$. Thus, assuming without loss of generality that the origin is outside $\bar{\Omega}$, we easily see that

$$\int_{\Omega\cap\{|x|\leq R\}} |u|^2|x|^{-n}dx = o(\log R) \quad \text{as} \quad R\to\infty.$$

Hence the uniqueness theorem of Chang and Finn [7, Theorem 6] implies that $u=0$. This shows that all integer powers of A_r are injective. If $A_r^{m+\alpha}u=0$ for some integer $m\geq 0$ and $0<\alpha<1$, then we obtain by (3.2),

$$0=\langle A_r^{m+\alpha}u, A_{r'}^{1-\alpha}\varphi\rangle = \langle u, A_{r'}^{m+1}\varphi\rangle \quad \text{for all} \quad \varphi\in D(A_{r'}^{m+1}).$$

This shows $u\in D(A_r^{m+1})$, $A_r^{m+1}u=0$ and therefore $u=0$. Thus, all powers A_r^α, $\alpha\geq 0$, are injective. The proof is complete.

By injectivity of A_r^α the map $u\to\|A^\alpha u\|_r$ defines a norm on $D(A_r^\alpha)$, so we can introduce the Banach space

$$D_r^\alpha = \text{the completion of } D(A_r^\alpha) \text{ in the norm } \|A^\alpha \cdot\|_r. \tag{3.5}$$

Our aim in this and the next sections is to characterize some of these spaces concretely in terms of the complex interpolation theory with the aid of Theorem 3.1 (iii). To do so, we begin with the following result of Bogovski [2] which shows existence of a continuous right-inverse for the divergence operator with zero boundary condition in a bounded domain.

PROPOSITION 3.3. *Let D be an n-dimensional bounded domain, $n \geq 2$, with locally Lipschitz boundary. Then there exists a linear operator S: $C_0^\infty(D) \to C_0^\infty(D)^n$ such that for all $f \in C_0^\infty(D)$,*

$$\|Sf\|_{m+1,r} \leq C\|f\|_{m,r}, \quad m = 0,1,2,\ldots, \quad 1 < r < \infty, \tag{3.6}$$

with C depending only on m, r and D; and

$$\nabla \cdot Sf = f \quad \textit{for all} \quad f \in C_0^\infty(D) \quad \textit{with} \quad \int_D f dx = 0. \tag{3.7}$$

Here $\|\cdot\|_{m,r}$ is the norm of $H^{m,r}(D)$.

From (3.6) it follows that S extends uniquely to a bounded operator from $H_0^{m,r}(D)$ to $H_0^{m+1,r}(D)^n$. We refer to [5] for a complete proof of Proposition 3.3 which is roughly described as follows: We first consider the case where each point in D is connected by a segment in D with a point of a fixed open ball B such that $\bar{B} \subset D$. The operator S is then expressed as

$$Sf(x) = \int_D G(x,y) f(y)\, dy, \quad G(x,y) = (x-y) \int_1^\infty h(y + t(x-y))\, t^{n-1}\, dt,$$

in terms of any fixed function $h \in C_0^\infty(B)$ such that $\int h dx = 1$, and the proof is carried out with the aid of the Calderon–Zygmund theory [40] on singular integrals. The general case is then treated by reducing the problem to the case stated above by means of a partition of unity. It is also shown in [5] that the method of proof illustrated above yields the following, which is important in the next section.

PROPOSITION 3.4. *The operator S restricted to $\{f \in C_0^\infty(D);\ \int_D f dx = 0\}$ extends uniquely to a bounded operator from $H^{-1,r}(D)$ to $L^r(D)^n$.*

We now prove an estimate on solutions of the stationary Stokes system (SS) with singular data which extends a result of Cattabriga [6] obtained in the case of bounded three-dimensional domains. In what follows the norm of the space $\hat{H}^{-m,r}(\Omega)$, $m>0$, $1<r<\infty$, is denoted by $|\cdot|_{-m,r}=|\cdot|_{-m,r,\Omega}$.

THEOREM 3.5. (i) *Let* $n\geq 3$, $1<r<n$, $u\in D(A_r)$, $p\in L^r(\Omega)$ *and* $f=-\Delta u+\nabla p$. *Then the estimate*

$$\|\nabla u\|_r+\|p\|_r\leq C|f|_{-1,r} \tag{3.8}$$

holds with C independent of u and p.

(ii) *If* $n\geq 2$, $p\in L^r(\Omega)$ *and* $1<r<\infty$, *then* $\nabla p\in \hat{H}^{-1,r}(\Omega)$ *and we have*

$$\|p\|_r\leq C|\nabla p|_{-1,r} \tag{3.9}$$

with C independent of p.

(iii) *If* $n\geq 2$ *and q is a distribution on* Ω *such that* $\nabla q\in \hat{H}^{-1,r}(\Omega)$ *for some* $1<r<\infty$, *then* $\nabla q=\nabla p$ *for some p in* $L^r(\Omega)$.

THEOREM 3.6. If $n\geq 3$ *and* $1<r<n$, *then we have the estimate*

$$\|\nabla u\|_r\leq C\sup_v|\langle\nabla u,\nabla v\rangle| \quad \textit{for} \quad u\in D(A_r), \tag{3.10}$$

where the supremum is taken over all $v\in C^\infty_{0,\sigma}(\Omega)$ *with* $\|\nabla v\|_{r'}=1$.

Remark. When Ω is bounded and $n=3$, estimate (3.8) is due to Cattabriga [6] and is valid for $1<r<\infty$. As shown below, this result of [6] is true in all dimensions $n\geq 2$. Kozono and Sohr [45] have also proved (3.8) and (3.10) for $n'<r<n$. Although the arguments in [45] are almost the same as ours, we give here the detailed proofs since our results cover a broader range $1<r<n$. In what follows $\hat{H}^{1,r}_{0,\sigma}(\Omega)$ denotes the $\hat{H}^{1,r}$-closure of $C^\infty_{0,\sigma}(\Omega)$.

Proof of Theorem 3.6. We deduce Theorem 3.6 from Theorem 3.5. For u in $D(A_r)$ we regard $g=-\Delta u$ as an element in $\hat{H}^{1,r'}_{0,\sigma}(\Omega)^*$, the norm of which we denote by $\|\cdot\|^*$. By the Hahn–Banach theorem one finds an $f\in\hat{H}^{-1,r}(\Omega)$ with $f=g$ on $\hat{H}^{1,r'}_{0,\sigma}(\Omega)$ and $|f|_{-1,r}=\|g\|^*$. By a theorem of De Rham [32, Theorem 17′], $f-g=\nabla p$ for some distribution p on Ω; and by Theorem 3.5 (iii) we may assume that $p\in L^r(\Omega)$. Applying (3.8) to $f=g+\nabla p=-\Delta u+\nabla p$ we find in particular that

$$\|\nabla u\|_r \leq C|f|_{-1,r} = C\|g\|^*.$$

By definition of the norm $\|\cdot\|^*$, this proves (3.10).

It remains now to prove Theorem 3.5. The proof will be carried out in several steps. We begin with the case of entire spaces $\mathbf{R}^n$, $n \geq 2$.

PROPOSITION 3.7. *Let* $n \geq 2$ and $1 < r < \infty$.

(i) If $p \in L^r(\mathbf{R}^n)$, *then* $\nabla p \in \hat{H}^{-1,r}(\mathbf{R}^n)$ *and the estimate*

$$\|p\|_{r,R^n} \leq C|\nabla p|_{-1,r,R^n}$$

holds with C independent of p.

(ii) *If q is a distribution on* $\mathbf{R}^n$ *with* $\nabla q \in \hat{H}^{-1,r}(\mathbf{R}^n)$ *for some r, then there is a (unique) function* $p \in L^r(\mathbf{R}^n)$ *with* $\nabla q = \nabla p$.

(iii) *If* $u \in \hat{H}^{1,r}_{0,\sigma}(\mathbf{R}^n)$, $p \in L^r(\mathbf{R}^n)$ *and* $f = -\Delta u + \nabla p$, *then the estimate*

$$\|\nabla u\|_{r,R^n} + \|p\|_{r,R^n} \leq C|f|_{-1,r,R^n} \tag{3.11}$$

holds with C independent of u and p.

Proof. (i) Since the reverse inequality is obvious, we may assume that p is in $C^1_0(\mathbf{R}^n)$. By an elementary calculation,

$$p(x) = c_n \int \frac{x-y}{|x-y|^n} \cdot (\nabla p)(y)\, dy \equiv K_j * (\partial_j p).$$

For $\phi \in C_0^\infty(\mathbf{R}^n)$ we have

$$|\langle p, \phi \rangle| = |\langle K_j * (\partial_j p), \phi \rangle| = |\langle (\partial_j p, K_j * \phi \rangle|.$$

Thus, if $K_j * \phi$ is in $\hat{H}^{1,r'}_0(\mathbf{R}^n)$, the Calderon–Zygmund theory [40] on singular integrals yields

$$|\langle (\partial_j p, K_j * \phi \rangle| \leq |\nabla p|_{-1,r,R^n} \|\nabla K * \phi\|_{r',R^n} \leq C|\nabla p|_{-1,r,R^n} \|\phi\|_{r',R^n}$$

and the proof of (i) is complete. We thus need only show that $K_j * \phi \in \hat{H}^{1,r'}_0(\mathbf{R}^n)$. Let $\zeta \in C_0^\infty(\mathbf{R}^n)$ be such that $0 \leq \zeta \leq 1$; $\zeta = 1$ if $|x| \leq 1$; $\zeta = 0$ if $|x| \geq 2$; and set $\zeta_N(x) = \zeta(x/N)$. Obviously $\zeta_N K_j * \phi \in C_0^\infty(\mathbf{R}^n)$. We write

$$\|\nabla(1-\zeta_N) K_j * \phi\|_{r',R^n} \leq \|(1-\zeta_N) \nabla K_j * \phi\|_{r',R^n} + \|(\nabla \zeta_N) K_j * \phi\|_{r',R^N} \equiv I_N + J_N.$$

Since $\nabla K_j * \phi \in L^{r'}(\mathbf{R}^n)$ by the Calderon–Zygmund theory, $I_N \to 0$ as $N \to \infty$. To handle J_N we fix $M>0$ so that $\operatorname{supp}\phi$ is contained in the ball of radius M centered at the origin. By an elementary calculation,

$$\begin{aligned}(J_N)^{r'} &\leq CN^{-r'} \int_{N\leq|x|\leq 2N} dx \left(\int_{|y|\leq M} |x-y|^{1-n} |\phi(y)|\, dy \right)^{r'} \\ &\leq CN^{-r'}(N-M)^{r'(1-n)} \int_{N\leq|x|\leq 2N} dx \left(\int_{|y|\leq M} |\phi(y)|\, dy \right)^{r'} \\ &\leq CN^{n(1-r')} \to 0 \quad \text{as} \quad N\to\infty,\end{aligned}$$

since $r'>1$. This proves that $K_j * \phi \in \hat{H}_0^{1,r'}(\mathbf{R}^n)$.

(ii) Let $\tilde{P}=\tilde{P}_r$ be the projection associated with the Helmholtz decomposition of $L^r(\mathbf{R}^n)$. Since $\tilde{P}u=u-\nabla p$, where p solves $\Delta p=\nabla\cdot u$ in $\mathbf{R}^n$,

$$(\tilde{P}u)_j = (\delta_{jk}+R_j R_k)\, u_k, \quad j=1,\ldots,n,$$

in terms of the Riesz transforms [40] $R=(R_1,\ldots,R_n)$ and Kronecker's symbol δ_{jk}. Thus we can directly decompose $\hat{H}^{-1,r}(\mathbf{R}^n)=R(\tilde{P})+N(\tilde{P})$, because the Riesz transforms are bounded linear operators in $\hat{H}^{-1,r}(\mathbf{R}^n)$. We write $\nabla q=u+\nabla p \in R(\tilde{P})+N(\tilde{P})$ with $p=-(-\Delta)^{-1/2}R\cdot g$ for some $g=(g_j)\in \hat{H}^{-1,r}(\mathbf{R}^n)$. As in the proof of (i), one can show the boundedness of the Riesz potential $(-\Delta)^{-1/2}$ from $L^{r'}(\mathbf{R}^n)$ to $\hat{H}_0^{1,r'}(\mathbf{R}^n)$; so by duality, it is bounded from $\hat{H}^{-1,r}(\mathbf{R}^n)$ to $L^r(\mathbf{R}^n)$. Hence $p\in L^r(\mathbf{R}^n)$. Since $\Delta(q-p)=\nabla\cdot u=0$, $\Delta(\nabla(q-p))=0$. Since $\nabla(q-p)\in \hat{H}^{-1,r}(\mathbf{R}^n)\subset H^{-1,r}(\mathbf{R}^n)$, elliptic regularity theory implies $\nabla(q-p)\in H^{1,r}(\mathbf{R}^n)$ and so $\nabla(q-p)=0$.

(iii) We first show that if p is a scalar function in $\hat{H}_0^{1,r}(\mathbf{R}^n)$, then

$$\|\nabla p\|_{r,R^n} \leq C|\Delta p|_{-1,r,R^n} \tag{3.12}$$

with C independent of p. By the Hahn–Banach theorem we can take $g=(g_j)$ from $L^r(\mathbf{R}^n)$ so that $-\Delta p=\nabla\cdot g$ and $|\Delta p|_{-1,r,R^n}=\|g\|_{r,R^n}$. We approximate g in L^r-norm by smooth and compactly supported g_m and set $p_m=(-\Delta)^{-1}\nabla\cdot g_m$, where $(-\Delta)^{-1}$ means the convolution with the standard fundamental solution of $-\Delta$. Then $\nabla p_m=R(R\cdot g_m)$ converges in L^r-norm to some $f\in L^r(\mathbf{R}^n)$; and by the Helmholtz decomposition, $f=\nabla q$ for some $q\in L^r_{\mathrm{loc}}(\mathbf{R}^n)$. But then, as $m\to\infty$,

$$-\Delta p_m = \nabla\cdot g_m \to -\Delta p = -\nabla\cdot f = -\Delta q$$

in the distribution topology, so $\Delta(q-p)=0$ and therefore $\Delta(\nabla(q-p))=0$. Since

$\nabla(q-p)\in L^r(\mathbf{R}^n)$, $\nabla q=\nabla p$. Estimate (3.12) follows from $\nabla p=\nabla q=R(R\cdot g)$ and L^r-boundedness of the Riesz transforms.

We can now prove (3.11). From the equation $-\Delta u=\tilde{P}f$, the boundedness of $\tilde{P}$ in $\hat{H}^{-1,r}(\mathbf{R}^n)$ and estimate (3.12) it follows that

$$\|\nabla u\|_{r,R^n}\leqslant C|\tilde{P}f|_{-1,r,R^n}\leqslant C|f|_{-1,r,R^n}. \tag{3.13}$$

Hence from (i) and the equation $\nabla p=f+\Delta u$ we obtain

$$\begin{aligned}\|p\|_{r,R^n}&\leqslant C|\nabla p|_{-1,r,R^n}\leqslant C(|f|_{-1,r,R^n}+|\Delta u|_{-1,r,R^n})\\&\leqslant C(|f|_{-1,r,R^n}+\|\nabla u\|_{r,R^n})\leqslant C|f|_{-1,r,R^n}.\end{aligned}$$

Combining this with (3.13) yields estimate (3.11). The proof is complete.

We next consider the case of bounded domains and extend the result of Cattabriga [6] to all dimensions $\geqslant 2$.

PROPOSITION 3.8. *Let D be a bounded domain with smooth boundary in* $\mathbf{R}^n$, $n\geqslant 2$, *and let* $1<r<\infty$.

(i) *If* $p\in L^r(D)$, *then* $\nabla p\in H^{-1,r}(D)$ *and*

$$\left\|p-\fint_D p\right\|_{r,D}\leqslant C\|\nabla p\|_{-1,r,D}$$

with C independent of p, where $\fint_D$ *means integration over D with respect to the normalized Lebesgue measure and* $\|\cdot\|_{-1,r,D}$ *is the norm of* $H^{-1,r}(D)$.

(ii) *If q is a distribution on D with* $\nabla q\in H^{-1,r}(D)$, *then* $\nabla q=\nabla p$ *for some* $p\in L^r(D)$.

(iii) *If* $u\in {}^{1,r}_{0,\sigma}(D)$ *and* $p\in L^r(D)$, *then* $f=-\Delta u+\nabla p\in H^{-1,r}(D)$ *and*

$$\|\nabla u\|_{r,D}+\left\|p-\fint_D p\right\|_{r,D}\leqslant C\|f\|_{-1,r,D} \tag{3.14}$$

with C independent of u and p, where $H^{1,r}_{0,\sigma}(D)$ *is the* $H^{1,r}$-*closure of* $C^\infty_{0,\sigma}(D)$.

Proof. (i) Proposition 3.3 implies that the divergence operator

$$\nabla\cdot: H^{1,r'}_0(D)\to L^{r'}(D)$$

has the closed range

$$R(\nabla\cdot)=\left\{f\in L^{r'}(D);\int_D f\,dx=0\right\}.$$

Hence (i) is obtained by duality and the closed range theorem [44].

(ii) Consider the gradient operator

$$\nabla: L^r(D) \to H^{-1,r}(D).$$

By the proof of (i) the range $R(\nabla)$ is closed and

$$R(\nabla) = N(\nabla\cdot)^{\perp} = \{u \in H_0^{1,r'}(D); \nabla\cdot u = 0\}^{\perp}.$$

It suffices therefore to show that $\langle \nabla q, u\rangle = 0$ for all $u \in N(\nabla\cdot)$. Take u_j from $C_0^\infty(D)$ so that $u_j \to u$ in $H^{1,r'}(D)$ and so $\nabla\cdot u_j \to \nabla\cdot u = 0$ in $L^{r'}(D)$. By Proposition 3.3 the functions $v_j = u_j - S(\nabla\cdot u_j)$ are in $C_{0,\sigma}^\infty(D)$ and satisfy

$$\begin{aligned} \|u - v_j\|_{1,r',D} &\leq \|u - u_j\|_{1,r',D} + \|S(\nabla\cdot u_j)\|_{1,r',D} \\ &\leq \|u - u_j\|_{1,r',D} + C\|\nabla\cdot u_j\|_{r',D} \\ &\to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

Since $\langle \nabla q, v_j\rangle = -\langle q, \nabla\cdot v_j\rangle = 0$, we obtain as $j \to \infty$

$$|\langle \nabla q, u\rangle| = |\langle \nabla q, u - v_j\rangle| \leq \|\nabla q\|_{-1,r,D}\|u - v_j\|_{1,r',D} \to 0.$$

This proves (ii).

(iii) Let $A = A_r$ be the Stokes operator in $L_\sigma^r(D)$. By Giga [12],

$$D(A_r^{1/2}) = H_0^{1,r}(D) \cap L_\sigma^r(D),$$

which equals $H_{0,\sigma}^{1,r}(D)$ in view of the proof of (ii), and we have the estimate

$$C^{-1}\|\nabla u\|_{r,D} \leq \|A^{1/2}u\|_{r,D} \leq C\|\nabla u\|_{r,D}, \quad u \in D(A_r^{1/2}). \tag{3.15}$$

Assume first that $u \in D(A_r)$ and $p \in H^{1,r}(D)$. Then $Au = P_D f$, where P_D is the projection associated with the Helmholtz decomposition of $L^r(D)$. Since $\{A_{r'}^{1/2}v; v \in C_{0,\sigma}^\infty(D)\}$ is dense in $L^{r'}(D)$, (3.15) yields

$$\begin{aligned} \|\nabla u\|_{r,D} \leq C\|A^{1/2}u\|_{r,D} &= C\sup\{|\langle A_r^{1/2}u, A_{r'}^{1/2}v\rangle| / \|A_{r'}^{1/2}v\|_{r',D};\ v \in C_{0,\sigma}^\infty(D)\} \\ &\leq C\sup\{|\langle P_D f, v\rangle| / \|\nabla v\|_{r',D};\ v \in C_{0,\sigma}^\infty(D)\}. \end{aligned}$$

Since $\langle P_D f, v\rangle = \langle f, v\rangle$ for $v \in C_{0,\sigma}^\infty(D)$, the last terms is estimated as

$$\leq C\sup\{|\langle f, w\rangle| / \|\nabla w\|_{r',D};\ w \in C_0^\infty(D)\} = C\|f\|_{-1,r,D}.$$

This, together with part (i), yields (3.14) in case $u\in D(A_r)$ and $p\in H^{1,r}(D)$. The general case is then treated through approximation. The proof is complete.

Proof of Theorem 3.5. (i) Take $\psi\in C_0^\infty(\mathbf{R}^n)$ with $\psi=1$ in a neighborhood of the complement of Ω, and let $u\in D(A_r)$, $p\in L^r(D)$, $1<r<\infty$. Choosing open balls B_1 and B so that

$$\Omega\cap(\operatorname{supp}\psi)\subset B_1\subset\bar{B}_1\subset B,$$

we decompose u as follows:

$$u=u_1+u_2;$$
$$u_1=\psi u-S(\nabla\psi\cdot u)\in D(A_{r,B\cap\Omega}),\quad u_2=(1-\psi)u+S(\nabla\psi\cdot u)\in D(A_{r,R^n}),$$

where S is the operator given in Proposition 3.3 with D a neighborhood of supp $\nabla\psi$ such that $\bar{D}$ is compact in $B\cap\Omega$. Since $S(\nabla\psi\cdot u)\in H_0^{3,r}(D)$ if $u\in D(A_r)$, we always understand that $S(\nabla\psi\cdot u)\in H_0^{3,r}(\mathbf{R}^n)$ by setting $S(\nabla\psi\cdot u)=0$ outside D. $A_{r,B\cap\Omega}$ and A_{r,R^n} denote the Stokes operator on $B\cap\Omega$ and $\mathbf{R}^n$, respectively. Now let $f=-\Delta u+\nabla p$; by direct calculation we have

$$f_1\equiv-\Delta u_1+\nabla(\psi p)=\psi f+p\nabla\psi-2\nabla\psi\cdot\nabla u-u\Delta\psi+\Delta S(\nabla\psi\cdot u). \tag{3.16}$$

Applying (3.14) with $D=B\cap\Omega$ yields

$$\begin{aligned}\|\nabla u_1\|_r+\|\psi p\|_r&\le C\Big(\|f_1\|_{-1,r,D}+\Big|\int_D\psi p\Big|\Big)\\&\le C\Big((\|\psi f\|_{-1,r,D}+\|\nabla\psi\cdot\nabla u\|_{-1,r,D}+\|u\Delta\psi\|_{-1,r,D}\\&\quad+\|\Delta S(\nabla\psi\cdot u)\|_{-1,r,D}+\|p\nabla\psi\|_{-1,r,D}+\Big|\int_D\psi p\Big|\Big)\\&\le C\Big(|\psi f|_{-1,r}+|\nabla\psi\cdot\nabla u|_{-1,r}+|u\Delta\psi|_{-1,r}+\|\nabla S(\nabla\psi\cdot u)\|_r\\&\quad+\|p\nabla\psi\|_{-1,r}+\Big|\int_D\psi p\Big|\Big),\end{aligned}\tag{3.17}$$

where $\|\cdot\|_{-1,r}$ is the norm of $H^{-1,r}(\Omega)$. We estimate the right-hand side as follows: Since supp $\nabla\psi\subset B\cap\Omega$ and since $\phi\in C_0^\infty(\Omega)$ vanishes on $\partial\Omega$, it follows from the Poincaré inequality that

$$|\langle \psi f, \phi \rangle| = |\langle f, \psi\phi \rangle| \leq |f|_{-1,r} \|\nabla(\psi\phi)\|_{r'}$$
$$\leq C|f|_{-1,r}(\|\nabla\phi\|_{r'} + \|\phi\|_{r',D}) \leq C|f|_{-1,r}\|\nabla\phi\|_{r'}$$

where $D = B \cap \Omega$, and C depends on ψ. Hence we have

$$|\psi f|_{-1,r} \leq C|f|_{-1,r} \quad (1 < r < \infty). \tag{3.18}$$

Next, Proposition 3.3 yields

$$\|\nabla S(\nabla\psi \cdot u)\|_r \leq C\|\nabla\psi \cdot u\|_r \leq C\|u\|_{r, \operatorname{supp} \nabla\psi} \quad (1 < r < \infty). \tag{3.19}$$

On the other hand, the Poincaré inequality yields

$$|\langle \nabla\psi \cdot \nabla u, \phi \rangle| = |\langle \nabla u, (\nabla\psi)\phi \rangle| = |\langle u, (\nabla^2\psi)\phi + \nabla\psi\nabla\phi \rangle|$$
$$\leq C\|u\|_{r, \operatorname{supp} \nabla\psi}(\|\phi\|_{r',D} + \|\nabla\phi\|_{r'}) \leq C\|u\|_{r, \operatorname{supp} \nabla\psi}\|\nabla\phi\|_{r'};$$
$$|\langle u\Delta\psi, \phi \rangle| \leq C\|u\|_{r, \operatorname{supp} \nabla\psi}\|\phi\|_{r',D} \leq C\|u\|_{r, \operatorname{supp} \nabla\psi}\|\nabla\phi\|_{r'}.$$

We thus have

$$|\nabla\psi \cdot \nabla u|_{-1,r} \leq C\|u\|_{r, \operatorname{supp} \nabla\psi}; \quad |u\Delta\psi|_{-1,r} \leq C\|u\|_{r, \operatorname{supp} \nabla\psi} \quad (1 < r < \infty). \tag{3.20}$$

From (3.17)–(3.20) we obtain

$$\begin{aligned} \|\nabla u_1\|_r + \|\psi p\|_r &\leq C\left(\|f_1\|_{-1,r,D} + \left|\int_D \psi p\right|\right) \\ &\leq C\left(|f|_{-1,r} + \|u\|_{r, \operatorname{supp} \nabla\psi} + \|p\nabla\psi\|_{-1,r} + \left|\int_D \psi p\right|\right). \end{aligned} \tag{3.21}$$

Consider now the function $f_2 = -\Delta u_2 + \nabla((1-\psi)p)$ on $\mathbf{R}^n$. By (3.11) we have

$$\|\nabla u_2\|_r + \|(1-\psi)p\|_r \leq C|f_2|_{-1,r,R^n}. \tag{3.22}$$

We first discuss the case where $n' < r < \infty$. Taking $\psi_2 \in C^\infty(\Omega)$ such that $\psi_2 = 1$ in a neighborhood of $\operatorname{supp}(1-\psi)$ and $\psi_2 = 0$ in a neighborhood of $\partial\Omega$, we find that, for $\phi \in C_0^\infty(\mathbf{R}^n)$,

$$\langle f_2, \phi \rangle = \langle f_2, \psi_2\phi \rangle = \langle f, \psi_2\phi \rangle - \langle f_1, \psi_2\phi \rangle. \tag{3.23}$$

Next, choose $\psi_1 \in C_0^\infty(B)$ such that $\psi_1 = 1$ in a neighborhood of B_1. Since f_1 vanishes outside B_1, we see that $\langle f_1, \psi_2\phi \rangle = \langle f_1, \psi_1\psi_2\phi \rangle$. So (3.23) gives

$$\langle f_2, \phi \rangle = \langle f, \psi_2 \phi \rangle - \langle f_1, \psi_1 \psi_2 \phi \rangle. \tag{3.24}$$

Applying Sobolev's inequality yields, with $1/(r')^* = 1/r' - 1/n$,

$$\begin{aligned}\|\nabla(\psi_2 \phi)\|_{r'} &\leq C(\|\nabla \phi\|_{r', R^n} + \|\phi \nabla \psi_2\|_{r', R^n}) \leq C(\|\nabla \phi\|_{r', R^n} + \|\phi\|_{(r')^*, R^n}) \\ &\leq C\|\nabla \phi\|_{r', R^n}; \\ \|\nabla(\psi_1 \psi_2 \phi)\|_{r', D} &\leq C(\|\nabla \phi\|_{r', R^n} + \|\phi \nabla(\psi_1 \psi_2)\|_{r', R^n}) \\ &\leq C(\|\nabla \phi\|_{r', R^n} + \|\phi\|_{(r')^*, R^n}) \leq C\|\nabla \phi\|_{r', R^n}.\end{aligned}$$

Thus (3.24) implies that

$$|f_2|_{-1, r, R^n} \leq C(|f|_{-1, r} + \|f_1\|_{-1, r, D}). \tag{3.25}$$

Combining (3.21), (3.22) and (3.25) gives

$$\|\nabla u\|_r + \|p\|_r \leq C\left(|f|_{-1, r} + \|u\|_{r, \operatorname{supp} \nabla \psi} + \|p \nabla \psi\|_{-1, r} + \left| ⨍_D \psi p \right|\right) \tag{3.26}$$

for $n' < r < \infty$, with C independent of u and p. To discuss the opposite case $1 < r \leq n'$, we need the following lemma.

LEMMA 3.9. *If $r \geq n$, the space $\hat{H}_0^{1,r}(\Omega)$ contains all the smooth functions which are constant for large $|x|$ and vanish in a neighborhood of $\partial\Omega$.*

Admitting this lemma for a moment, we continue the proof of Theorem 3.5. Let $1 < r \leq n'$. Since $\langle f_2, \phi \rangle = \langle \nabla u_2, \nabla \phi \rangle - \langle (1-\psi) p, \nabla \cdot \phi \rangle$, we may replace $\phi \in C_0^\infty(\mathbf{R}^n)$ by $\eta = \phi + c$, where c is a constant vector. We fix c so that

$$\int_{B_1} \eta \, dx = 0. \tag{3.27}$$

Using the functions ψ_1 and ψ_2 introduced above, we then obtain

$$\langle f_2, \phi \rangle = \langle f_2, \eta \rangle = \langle f, \psi_2 \eta \rangle - \langle f_1, \psi_1 \psi_2 \eta \rangle. \tag{3.28}$$

Using the Poincare inequality:

$$\|\eta\|_{r', B} \leq C\left(\|\nabla \phi\|_{r', B} + \left| \int_{B_1} \eta \, dx \right|\right)$$

and (3.27), we see that

$$\begin{aligned}\|\nabla(\psi_2\eta)\|_{r'} &\leq C(\|\nabla\phi\|_{r',R^n}+\|\eta\nabla\psi_2\|_{r',R^n})\\ &\leq C(\|\nabla\phi\|_{r',R^n}+\|\eta\|_{r',B})\leq C\|\nabla\phi\|_{r',R^n};\\ \|\nabla(\psi_1\psi_2\eta)\|_{r',D} &\leq C(\|\nabla\phi\|_{r',R^n}+|\eta\nabla(\psi_1\psi_2)\|_{r',D})\\ &\leq C(\|\nabla\phi\|_{r',R^n}+\|\eta\|_{r',B})\leq C\|\nabla\phi\|_{r',R^n}.\end{aligned}$$

Since $\psi_2\eta\in\hat{H}_0^{1,r'}(\Omega)$ by Lemma 3.9 (3.28) implies (3.25) and we obtain (3.26) for $1<r\leq n'$.

Now fix $1<r<n$ and suppose the estimate (3.8) is false; then there are sequences u_j and p_j with $\|\nabla u_j\|_r+\|p_j\|_r=1$ and $|f_j|_{-1,r}\to 0$, where $f_j=-\Delta u_j+\nabla p_j$. We may assume that $u_j\to u$ weakly in $\hat{H}_0^{1,r}(\Omega)$ and $p_j\to p$ weakly in $L^r(\Omega)$. Then, since $u_j\to u$ weakly in $L^{r^*}(\Omega)$, $1/r^*=1/r-1/n$, we obtain for $\phi\in D(A_{r'})\cap D(A_{(r^*)'})$,

$$\langle f_j,\phi\rangle=\langle\nabla u_j,\nabla\phi\rangle=\langle u_j,A_{(r^*)'}\phi\rangle\to\langle u,A_{(r^*)'}\phi\rangle=0.$$

Since $(r^*)'<r'$, $D(A_{r'})\cap D(A_{(r^*)'})$ is dense in $D(A_{(r^*)'})$ with respect to the graph-norm; so $\langle u,A_{(r^*)'}\phi\rangle=0$ for all $\phi\in D(A_{(r^*)'})$ and therefore $u\in D(A_{r^*}), A_{r^*}u=0$. Hence $u=0$. But then, $f_j\to-\Delta u+\nabla p=\nabla p=0$ in the distribution topology, and we get $p=0$ because $p\in L^r(\Omega)$ and Ω is an exterior domain. We have thus proved that $u_j\to 0$ weakly in $L^{r^*}(\Omega)$ and $p_j\to 0$ weakly in $L^r(\Omega)$. In particular, $u_j\to 0$ weakly in $\hat{H}_0^{1,r}(\Omega)$; and since $L^{r^*}\subset L^r$ on $\operatorname{supp}\nabla\psi$, it follows that u_j is bounded in $H^{1,r}$ in a neighborhood of $\operatorname{supp}\nabla\psi$. The Rellich–Kondrachov compactness theorem [8] now implies that

$$u_j\to 0\quad\text{in } L^r(\operatorname{supp}\nabla\psi)\quad\text{and}\quad p_j\nabla\psi\to 0\quad\text{in } H^{-1,r}(\Omega).$$

Since, clearly, $⨍_D\psi p_j\to 0$ by the definition of weak convergence, we deduce

$$|f_j|_{-1,r}+\|u\|_{r,\operatorname{supp}\nabla\psi}+\|p_j\nabla\psi\|_{-1,r}+\left|⨍_D\psi p_j\right|\to 0$$

and by (3.26), $\|\nabla u_j\|_r+\|p_j\|_r\to 0$: a contradiction. This proves (3.8).

(ii) Fix $1<r<\infty$ and suppose there is a sequence p_j such that $\|p_j\|_r=1$ and $|\nabla p_j|_{-1,r}\to 0$ as $j\to\infty$. We may assume that $p_j\to p$ weakly in $L^r(\Omega)$. For any bounded domain $D\subset\Omega$ the restriction map induces a bounded linear operator from $\hat{H}^{-1,r}(\Omega)$ to $H^{-1,r}(D)$; so Proposition 3.8 ensures the existence of constants $c_j=c_j(D)$ with $p_j-c_j\to 0$ in $L^r(D)$. Then, $c_j=(c_j-p_j)+p_j\to p$ weakly in $L^r(D)$ and so $p=\text{constant}=0$. We thus find that $p_j \to 0$ in $L^r(D)$ for any bounded $D \subset \Omega$. Now let ψ be the function exploited in the proof of (i). By Proposition 3.7 and the argument used in estimating f_2 in the proof of (i), we obtain

$$\begin{aligned}\|(1-\psi)(p_j-p_k)\|_{r,R^n} &\leqslant C|\nabla\{(1-\psi)(p_j-p_k)\}|_{-1,r,R^n}\\ &\leqslant C(|\nabla(p_j-p_k)|_{-1,r}+\|\nabla\{\psi(p_j-p_j)\|_{-1,r,B\cap\Omega})\\ &\leqslant C(|\nabla(p_j-p_k)|_{-1,r}+\|p_j-p_k\|_{r,B\cap\Omega})\to 0.\end{aligned}$$

Hence $p_j \to 0$ in $L^r(\Omega)$: a contradiction. This proves (ii).

(iii) We regard $(1-\psi)q$ as a distribution on $\mathbf{R}^n$. Since $q \in L^r(B\cap\Omega)$ by Proposition 3.8, we see as in the proof of (ii),

$$|\nabla\{(1-\psi)q\}|_{-1,r,R^n} \leqslant C(|\nabla q|_{-1,r}+\|q\|_{r,B\cap\Omega}) < +\infty.$$

Hence Proposition 3.7 (ii) ensures the existence of a function $p \in L^r(\mathbf{R}^n)$ such that $\nabla p = \nabla((1-\psi)q)$ in $\mathbf{R}^n$. Thus,

$$\nabla q = \nabla((1-\psi)q)+\nabla(\psi q) = \nabla(p+\psi q) \quad \text{in } \Omega$$

and the function $p+\psi q \in L^r(\Omega)$ is the desired one. The proof is complete.

Proof of Lemma 3.9. Take $\zeta \in C_0^\infty(\mathbf{R}^n)$ such that $\zeta=1$ for $|x|\leqslant 1$ and $\zeta=0$ for $|x|\geqslant 2$, and let $\zeta_N(x)=\zeta(x/N)$. For any u satisfying the assumption, we easily see that if $r>n$,

$$\|\nabla(u-u\zeta_N)\|_r = C\|\nabla\zeta_N\|_r \leqslant CN^{-1+n/r} \to 0 \quad \text{as} \quad N\to\infty$$

and this proves the result for $r>n$. In case $r=n$, $\|\nabla(u\zeta_N)\|_n$ is bounded, so the result follows from Mazur's theorem [44] if we take suitable convex combinations of the functions $u\zeta_N$. The proof is complete.

Remarks. The condition $r<n$ in Theorems 3.5 and 3.6 is optimal. Indeed, when $r\geqslant n$, it is known [5] that the smooth functions which are bounded near the infinity and vanish on $\partial\Omega$ belong to $\hat{H}_0^{1,r}(\Omega)$; consequently, the functions $u=c-W\varphi-V\psi$ composed of a constant vector c, a double layer potential $W\varphi$, and a single layer potential $V\psi$ belong to $\hat{H}_0^{1,r}(\Omega)$ and solve problem (SS) with $f=0$ and $u\to c$ as $|x|\to\infty$, together with some p, provided

$$(1/2+W)\varphi = c-V\psi \quad \text{on } \partial\Omega.$$

This last equation can be solved by the standard method presented for instance in [23]. Thus (3.8) and (3.10) are not valid for $r \geqslant n$.

Estimate of the form (3.10) was first deduced by Simader [36] in the case of the Dirichlet problem for the Laplacian in a bounded domain. Recently, Kozono and Sohr [45] have also proved (3.10) for $n'<r<n$. If $n'<r<n$, then $n'<r'<n$; so (3.8) and (3.10) are valid also for r', and this means that problem (SS) with $f \in \hat{H}^{-1,r}(\Omega)$ is always uniquely solvable in $\hat{H}_0^{1,r}(\Omega)$ provided that $n'<r<n$. For other types of estimates on (SS) we refer the reader to [39] and [45].

Estimate (3.8) is deduced from (3.10) via (3.9). Indeed, if $u \in D(A_r)$, $p \in L^r(\Omega)$ and $f=-\Delta u+\nabla p$, then for all $\phi \in C_{0,\sigma}^{\infty}(\Omega)$

$$|\langle \nabla u, \nabla \phi \rangle| = |\langle f, \phi \rangle| \leqslant |f|_{-1,r} \|\nabla \phi\|_{r'}$$

so (3.10) gives

$$\|\nabla u\|_r \leqslant C|f|_{-1,r} \quad (1<r<n). \tag{3.29}$$

From (3.9) and the equation $\nabla p = f+\Delta u$ it follows that if $1<r<n$,

$$\begin{aligned} \|p\|_r &\leqslant C|\nabla p|_{-1,r} \leqslant C(|f|_{-1,r}+|\Delta u|_{-1,r}) \\ &\leqslant C(|f|_{-1,r}+\|\nabla u\|_r) \leqslant C|f|_{-1,r}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30) we obtain (3.8).

Estimate (3.10) is essential in establishing in Section 4 an embedding result for domains of fractional powers of the Stokes operator.

4. Fractional powers of the Stokes operator and interpolation spaces

In this section we examine the domains of fractional powers of the Stokes operator and establish an embedding result of Sobolev type with the aid of the complex interpolation theory of Banach spaces. This is done by Giga [12] in the case where Ω is bounded and therefore the Stokes operator possesses the bounded inverse in each $L_\sigma^r(\Omega)$, $1<r<\infty$. In our case, however, the Stokes operator is not boundedly invertible and so we have to deal with our problem more carefully. The fractional powers of the Stokes operator in an exterior domain are studied also in the recent paper [13] of Giga and Sohr. We shall improve their interpolation result by applying Theorem 3.6. This improvement enables us to deduce in the next section apparently optimal decay rates for the L^2-norms of weak solutions of problem (NS).

First we define the homogeneous Sobolev space $\hat{H}_0^{s,r}(\mathbf{R}^n)$, $1<r<\infty$, of fractional order $s\geq 0$ to be the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\|\nabla^s u\|_{r,R^n} = \|F^{-1}|\xi|^s Fu\|_{r,R^n} \tag{4.1}$$

where F is the Fourier transformation and $|\xi|^s$ the multiplication operator in the phase space. When $s\geq 0$ is an integer, it follows from the Calderon–Zygmund theory [40] that $\hat{H}_0^{s,r}(\mathbf{R}^n)$ agrees with the one defined in Section 2. Since the multiplication by $|\xi|^{-s}$, $0<s<n$, corresponds to the convolution by the Riesz potentials, it follows from Sobolev's lemma [31, 40] that

$$\|u\|_{q,R^n} \leq C\|\nabla^s u\|_{r,R^n}, \quad u\in \hat{H}_0^{s,r}(\mathbf{R}^n) \quad \text{if} \quad 1/q = 1/r - s/n > 0. \tag{4.2}$$

We next recall a Sobolev type inequality which is valid for functions on exterior domains. For an exterior domain Ω in $\mathbf{R}^n$, $n\geq 3$, we denote by $C_{(0)}^m(\bar{\Omega})$, $m=0,1,2,\ldots$, the set of all restrictions to $\bar{\Omega}$ of functions in $C_0^m(\mathbf{R}^n)$. The following result is due to [4], [10] and [29].

LEMMA 4.1. *There is a constant C depending only on $n\geq 3$, $1<r<n$, and Ω such that, with* $1/r^* = 1/r - 1/n$,

$$\|u\|_{r^*} \leq C\|\nabla u\|_r, \quad \textit{for all} \quad u\in C_{(0)}^1(\bar{\Omega}). \tag{4.3}$$

Obviously, estimate (4.3) can be extended to a more general class of functions by taking completion.

We further recall a few basic notions in the complex interpolation theory of Banach spaces. Given an interpolation couple $\{X_0, X_1\}$ of complex Banach spaces, $F(X_0, X_1)$ denotes the space of all functions $f(z)$ defined to be continuous from the closed strip $\{0\leq \operatorname{Re} z\leq 1\}$ of the complex plane into X_0+X_1, analytic in the interior $\{0<\operatorname{Re} z<1\}$, and such that the maps: $t\to f(j+it)$, $j=0,1$, are bounded and continuous from $\mathbf{R}$ to X_j. Here i is the imaginary unit and X_0+X_1 is the Banach space $\{y=x_0+x_1; x_j\in X_j, j=0,1\}$ with norm

$$\|y\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1};\ y = x_0+x_1\}.$$

By the three-lines theorem $F=F(X_0, X_1)$ is a Banach space in the norm

$$|f|_F = \max\{\sup_t \|f(it)\|_{X_0}, \sup_t \|f(1+it)\|_{X_1}\}.$$

By $[X_0, X_1]_\theta$, $0 \leqslant \theta \leqslant 1$, we denote the complex interpolation space beteen X_0 and X_1 with norm

$$|u|_\theta = \inf\{|f|_F;\, f \in F(X_0, X_1),\, f(\theta) = u\}, \quad 0 < \theta < 1.$$

For basic facts in complex interpolation theory, we refer to [31] or [41]. If $s < n/r$, (4.2) shows that both $L^r(\mathbf{R}^n)$ and $\hat{H}_0^{s,r}(\mathbf{R}^n)$ are continuously embedded into $L^r(\mathbf{R}^n) + L^q(\mathbf{R}^n)$, $1/q = 1/r - s/n$, so $\{L^r(\mathbf{R}^n), \hat{H}_0^{s,r}(\mathbf{R}^n)\}$ is an interpolation couple (see [41]). Likewise, by letting $\lambda \to 0$ in Theorem 3.1 (ii) and applying (4.3), we see that $\{L_\sigma^r(\Omega), D_r^1\}$ is also an interpolation couple provided $2 < n/r$.

THEOREM 4.2. *If* $1 < r < \infty$ and $0 \leqslant \theta \leqslant 1$, *then with equivalent norms,*

$$[L^r(\mathbf{R}^n), \hat{H}_0^{s,r}(\mathbf{R}^n)]_\theta = \hat{H}_0^{\theta s, r}(\mathbf{R}^n) \quad \textit{for} \quad 0 \leqslant s < n/r; \tag{4.4}$$

$$[L_\sigma^r(\Omega), D_r^1]_\theta = D_r^\theta \quad \textit{for} \quad 2 < n/r. \tag{4.5}$$

Proof. (i) We may assume $0 < \theta < 1$ and $0 < s < n/r$, since otherwise the result is trivial. Let $\Lambda = (-\Delta)^{s/2} = F^{-1}|\xi|^s F$. Applying Michlin's multiplier theorem [41], we see that, as bounded operators in $L^r(\mathbf{R}^n)$,

$$\|\lambda(\lambda + \Lambda)^{-1}\| \leqslant M \quad \text{for all} \quad \lambda > 0, \tag{4.6}$$

$$\|\Lambda^\alpha(\lambda + \Lambda)^{-\alpha}\| \leqslant M_\alpha \quad (0 \leqslant \alpha \leqslant 1) \quad \text{for all} \quad \lambda > 0, \tag{4.7}$$

$$\|(\lambda + \Lambda)^{it}\| \leqslant M_\varepsilon e^{\varepsilon|t|} \quad (\varepsilon > 0) \quad \text{for all} \quad t \in \mathbf{R} \text{ and } \lambda > 0. \tag{4.8}$$

Let $w \in D(\Lambda)$ and consider the function $f(z) = e^{(z-\theta)^2}(\lambda + \Lambda)^{-(z-\theta)} w$, $\lambda > 0$, which is continuous for $0 \leqslant \operatorname{Re} z \leqslant 1$ and analytic for $0 < \operatorname{Re} z < 1$, with values in $L^r(\mathbf{R}^n)$. Since $f(it) \in L^r(\mathbf{R}^n)$, $f(1+it) \in D(\Lambda) \subset \hat{H}_0^{s,r}(\mathbf{R}^n)$ and $f(\theta) = w$, we obtain by (4.6)–(4.8),

$$\begin{aligned}
|\omega|_\theta &\leqslant \max\{\sup_t \|f(it)\|_{r,R^n}, \sup_t \|\Lambda f(1+it)\|_{r,R^n}\} \\
&\leqslant C \max\{\|(\lambda+\Lambda)^\theta w\|_{r,R^n}, \|\Lambda(\lambda+\Lambda)^{-1}(\lambda+\Lambda)^\theta w\|_{r,R^n}\} \\
&\leqslant C\|(\lambda+\Lambda)^\theta w\|_{r,R^n}.
\end{aligned}$$

Since the constant C is independent of $\lambda > 0$, letting $\lambda \to 0$ yields

$$|w|_\theta \leqslant C\|\Lambda^\theta w\|_{r,R^n}. \tag{4.9}$$

To show the converse, let $g(z)$ denote an arbitrary function expressed as a finite linear combination of functions of the form $\exp(\delta z^2+\gamma z)\,b$ with $\delta>0, \gamma\in\mathbf{R}$ and $b\in D(\Lambda)$. Since $(\lambda+\Lambda)^z g(z), \lambda>0$, is continuous in $0\leqslant\operatorname{Re} z\leqslant 1$ and analytic in $0<\operatorname{Re} z<1$, with values in $L^r(\mathbf{R}^n)$, we obtain

$$\|\Lambda^\theta w\|_{r,R^n}\leqslant C\|(\lambda+\Lambda)^\theta w\|_{r,R^n}\leqslant C\inf_{g(\theta)=w}\max_{j=0,1}\sup_t\|(\lambda+\Lambda)^{j+it}g(j+it)\|_{r,R^n}$$

by the three-lines theorem. Letting $\lambda\to 0$ and using (4.8) gives

$$\|\Lambda^\theta w\|_{r,R^n}\leqslant C\inf_{g(\theta)=w}\max_{j=0,1}\sup_t\|\Lambda^j g(j+it)\|_{r,R^n}.$$

Since $D(\Lambda)$ is dense in both of $L^r(\mathbf{R}^n)$ and $\hat{H}_0^{s,r}(\mathbf{R}^n)$, it follows from the argument in [41, Section 1.9] that

$$\|\Lambda^\theta w\|_{r,R^n}\leqslant C|w|_\theta. \tag{4.10}$$

By (4.9) and (4.10) we obtain (4.4).

(ii) To show (4.5) we have only to replace (4.6)–(4.8) by the estimates given in Theorem 3.1 and Proposition 3.2. The proof of (4.4) then applies with no change. The proof is complete.

The Riesz transforms $R=(R_1,\dots,R_n), R_j=F^{-1}(i\xi_j/|\xi|)F$, are bounded operators in $\hat{H}_0^{s,r}(\mathbf{R}^n)$, so the projection $\tilde{P}$ associated with the Helmholtz decomposition of $L^r(\mathbf{R}^n)$ defines the bounded projection from $\hat{H}_0^{s,r}(\mathbf{R}^n)$ onto the subspace $\hat{H}_\sigma^{s,r}(\mathbf{R}^n)$ of solenoidal vector fields. Since $\tilde{P}$ extends to a bounded projection on $L^r(\mathbf{R}^n)+\hat{H}_0^{s,r}(\mathbf{R}^n)$, Theorem 4.2 and a standard argument in the interpolation theory [41, Section 1.2.4] together yield

$$[L_\sigma^r(\mathbf{R}^n), \hat{H}_\sigma^{s,r}(\mathbf{R}^n)]_\theta=\hat{H}_\sigma^{\theta s,r}(\mathbf{R}^n),\quad 0\leqslant\theta\leqslant 1,\quad 0\leqslant s<n/r. \tag{4.11}$$

PROPOSITION 4.3. (i) *$\hat{H}_{0,\sigma}^{1,r}(\Omega)=\{v\in\hat{H}_0^{1,r}(\Omega);\ \nabla\cdot v=0\}$ for $1<r<\infty$.*

(ii) *$[L_\sigma^r(\Omega), D_r^1]_{1/2}=\hat{H}_{0,\sigma}^{1,r}(\Omega)$ if $1<r<n/2$.*

(iii)
$$[\hat{H}_{0,\sigma}^{1,r_0}(\Omega), \hat{H}_{0,\sigma}^{1,r_1}(\Omega)]_\theta=\hat{H}_{0,\sigma}^{1,r}(\Omega),$$

where $1<r_j<n$, $j=0,1$, $0\leqslant\theta\leqslant 1$, and $1/r=(1-\theta)/r_0+\theta/r_1$.

Proof. (i) For simplicity we write $X=\hat{H}_{0,\sigma}^{1,r}(\Omega)$ and Y the right-hand side of (i). Since X is closed in Y, it suffices to show that X is dense in Y. Let $f\in\hat{H}^{-1,r'}(\Omega), r'=r/(r-1)$, and suppose $f=0$ on X. By [32, Theorem 17′], $f=\nabla q$ for some distribution q on Ω. By

Theorem 3.5 (iii), we may assume that $q \in L^{r'}(\Omega)$. Now, given $v \in Y$, take a sequence $v_j \in C_0^\infty(\Omega)$ such that $\|\nabla(v_j - v)\|_r \to 0$. Since $\nabla \cdot v_j \to \nabla \cdot v = 0$ in L^r-norm, we obtain

$$\langle f, v \rangle = \lim_{j\to\infty} \langle f, v_j \rangle = \lim_{j\to\infty} \langle \nabla q, v_j \rangle = -\lim_{j\to\infty} \langle q, \nabla \cdot v_j \rangle = 0.$$

Hence $f=0$ on Y and the result follows from the Hahn–Banach theorem.

(ii) Let $D = \mathbf{R}^n \setminus \bar{\Omega}$ and let E and E_b denote, respectively, the extension operators:

$$E: C^2_{(0)}(\bar{\Omega}) \to C^2(\mathbf{R}^n); \quad E_b: C_2(\bar{D}) \to C^2(\mathbf{R}^n)$$

with the following properties.

(E1) $\operatorname{supp} E_b u$ $(u \in C^2(\bar{D}))$ is contained in a fixed open ball $B \supset \bar{D}$.

(E2) E_b extends uniquely to bounded operator: $H^{s,r}(D) \to H^{s,r}(\mathbf{R}^n)$, for all $1<r<\infty$ and $s=0,1,2$.

(E3) The operator E satisfies the estimate

$$\|\nabla^s E u\|_{r,R^n} \leqslant C(\|\nabla^s u\|_r + \|u\|_{r,\Omega\cap B}), \quad u \in C^2_{(0)}(\bar{\Omega}), \ s = 0,1,2. \tag{4.12}$$

These operators can be constructed in the standard manner via local maps since $\partial\Omega$ is smooth by assumption. If $1<r<n/2$, it follows from Lemma 4.1 and Hölder's inequality applied to the last term of (4.12) that

$$\|\nabla^s E u\|_{r,R^n} \leqslant C\|\nabla^s u\|_r, \quad u \in C^2_{(0)}(\bar{\Omega}), \ 1<r<n/2, \ s=0,1,2. \tag{4.13}$$

Hence, if $\hat{H}^{s,r}(\Omega)$, $s=0,1,2$, denotes the $\|\nabla^s \cdot\|_r$-completion of $C^2_{(0)}(\bar{\Omega})$, then (4.13) asserts that E is bounded from $\hat{H}^{s,r}(\Omega)$ to $\hat{H}^{s,r}_0(\mathbf{R}^n)$ for $1<r<n/2$, $s=0,1,2$. Now, letting $\lambda \to 0$ in Theorem 3.1 (ii) gives the estimate $\|\nabla^2 u\|_r \leqslant C\|Au\|_r$, $1<r<n/2$; so by Lemma 4.1, assertion (i) and the obvious estimate $\|Au\|_r \leqslant C\|\nabla^2 u\|_r$, we find that if $1<r<n/2$,

$$D_r^1 = L^{q_2}_\sigma(\Omega) \cap \hat{H}^{1,q_1}_{0,\sigma}(\Omega) \cap \hat{H}^{2,r}(\Omega), \quad 1/q_j = 1/r - j/n, \ j=1,2. \tag{4.14}$$

Hence by (4.13), $E: D_r^1 \to \hat{H}^{2,r}_0(\mathbf{R}^n)$ is bounded when $1<r<n/2$. It thus follows from (4.4) and (4.5) that $E: D_r^{1/2} \to \hat{H}^{1,r}_0(\mathbf{R}^n)$ is bounded, and we get

$$\|\nabla u\|_r \leqslant \|\nabla E u\|_{r,R^n} \leqslant C\|A^{1/2} u\|_r, \quad 1<r<n/2, \tag{4.15}$$

which shows the continuous embedding: $D_r^{1/2} \to \hat{H}^{1,r}_{0,\sigma}(\Omega)$ in view of assertion (i).

To prove the converse, we define the function Zu, $u \in C^\infty_{0,\sigma}(\mathbf{R}^n)$, by

$$Zu = \gamma_\Omega u - \gamma_{\Omega\cap B} E_b \gamma_D u + S(\nabla \cdot \gamma_{\Omega\cap B} E_b \gamma_D u), \quad u \in C^\infty_{0,\sigma}(\mathbf{R}^n), \tag{4.16}$$

where γ_X means restriction to X and S is the operator given in Proposition 3.3 with respect to the bounded domain $\Omega \cap B$. We regard the last two terms on the right-hand side as defined on Ω by setting $= 0$ outside B. Since

$$\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u = -\nabla \cdot (u - \gamma_{\Omega \cap B} E_b \gamma_D u) \quad \text{in} \quad \Omega \cap B,$$

since by the definition of E_b,

$$(\partial/\partial\nu)^j (u - \gamma_{\Omega B} E_b \gamma_D u)|_{\partial\Omega} = 0 \quad \text{for} \quad j = 0, 1,$$

where ν is the unit outward normal to $\partial\Omega$, and therefore since

$$\int_{\Omega \cap B} \nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u \, dx = \int_{\Omega \cap B} \nabla \cdot (\gamma_{\Omega \cap B} E_b \gamma_D u - u) \, dx = -\int_{\partial B} \nu \cdot u \, d\Sigma$$

$$= -\int_B \nabla \cdot u \, dx = 0 \quad (\nu = \text{the unit outward normal to } \partial B)$$

Proposition 3.3 shows that $\nabla \cdot Zu = 0$ in Ω, $S(\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u) \in H_0^{2,r}(\Omega \cap B)$, and

$$\begin{aligned} \|\nabla^2 S(\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u)\|_r &\leq C(\|\nabla^2 E_b \gamma_D u\|_{r,B} + \|\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u\|_{r,\Omega \cap B}) \\ &\leq C(\|\nabla^2 u\|_{r,D} + \|\nabla u\|_{r,D} + \|u\|_{r,D}) \\ &\leq C(\|\nabla^2 u\|_{r,R^n} + \|\nabla u\|_{q_1,R^n} + \|u\|_{q_2,R^n}) \\ &\leq C\|\nabla^2 u\|_{r,R^n} \quad (1/q_j = 1/r - j/n, \ j = 1, 2). \end{aligned}$$

Furthermore, by Proposition 3.4,

$$\begin{aligned} \|S(\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u)\|_r &= \|S(\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u)\|_{r,\Omega \cap B} \leq C\|\nabla \cdot \gamma_{\Omega \cap B} E_b \gamma_D u\|_{-1,r,\Omega \cap B} \\ &\leq C\|E_b \gamma_D u\|_{r,B} \leq C\|u\|_{r,R^n}. \end{aligned}$$

Since the term $\gamma_{\Omega \cap B} E_b \gamma_D u$ in (4.16) is similarly estimated, we see by (4.14) that the operator Z is bounded from $\hat{H}_\sigma^{2,r}(\mathbf{R}^n)$ to D_r^1 and from $L_\sigma^r(\mathbf{R}^n)$ to $L_\sigma^r(\Omega)$, respectively. Hence (4.5) and (4.11) together imply that Z is bounded from $\hat{H}_\sigma^{1,r}(\mathbf{R}^n)$ to $D_r^{1/2}$. Since $D_r^{1/2} \subset \hat{H}_{0,\sigma}^{1,r}(\Omega)$ and since $ZE_0 = I$ on $\hat{H}_{0,\sigma}^{1,r}(\Omega)$, where E_0 means the zero-extension of functions defined on Ω, we obtain for $u \in \hat{H}_{0,\sigma}^{1,r}(\Omega)$

$$\|A^{1/2} u\|_r = \|A^{1/2} Z E_0 u\|_r \leq C\|\nabla E_0 u\|_{r,R^n} = C\|\nabla u\|_r. \tag{4.17}$$

By (4.15) and (4.17) the proof of (ii) is complete.

(iii) By definition and Lemma 4.1, Z is bounded from $\hat{H}_\sigma^{1,r}(\mathbf{R}^n)$ to $\hat{H}_{0,\sigma}^{1,r}(\Omega)$ provided

$1<r<n$. Hence $Z\tilde{P}$ is bounded from $\hat{H}_0^{1,r}(\mathbf{R}^n)$ to $\hat{H}_{0,\sigma}^{1,r}(\Omega)$ if $1<r<n$. Here we shall use

$$\begin{gathered}[\hat{H}_0^{1,r_0}(\mathbf{R}^n), \hat{H}_0^{1,r_1}(\mathbf{R}^n)]_\theta = \hat{H}_0^{1,r}(\mathbf{R}^n), \quad 0 \leq \theta \leq 1, \\ \text{where} \quad 1<r_j<n,\ j=0,1 \text{ and } 1/r=(1-\theta)/r_0+\theta/r_1,\end{gathered} \tag{4.18}$$

postponing its proof until the end of this paragraph. We thus have

$$Z\tilde{P}\colon \hat{H}_0^{1,r}(\mathbf{R}^n) \to [\hat{H}_{0,\sigma}^{1,r_0}(\Omega), \hat{H}_{0,\sigma}^{1,r_1}(\Omega)]_\theta \quad \text{is bounded.}$$

Since $Z\tilde{P}E_0 u=u$ for $u \in \hat{H}_{0,\sigma}^{1,r}(\Omega)$, it follows that

$$|u|_\theta = |Z\tilde{P}E_0 u|_\theta \leq C\|\nabla E_0 u\|_{r,R^n} = C\|\nabla u\|_r. \tag{4.19}$$

Conversely, interpolating between the operators $\nabla\colon \hat{H}_{0,\sigma}^{1,r_j}(\Omega) \to L^{r_j}(\Omega)$, $j=0,1$, we see that $\nabla\colon [\hat{H}_{0,\sigma}^{1,r_0}(\Omega), \hat{H}_{0,\sigma}^{1,r_1}(\Omega)]_\theta \to L^r(\Omega)$ is bounded; hence

$$\|\nabla u\|_r \leq C|u|_\theta. \tag{4.20}$$

By (4.19) and (4.20) the proof of (iii) is complete.

It remains to prove (4.18). By Sobolev's inequality we see that if $1<r<n$, then $\nabla\colon \hat{H}_0^{1,r}(\mathbf{R}^n) \to L^r(\mathbf{R}^n)$ is bounded, injective, and the range $R(\nabla)$ is closed. We show that $R(\nabla)=R(I-\tilde{P}_r)=N(\tilde{P}_r)$. Since $R(\nabla) \subset R(I-\tilde{P}_r)$, we need only show that $R(\nabla)$ is dense in $R(I-\tilde{P}_r)$. By the Helmholtz decomposition and the property $\tilde{P}_r^*=\tilde{P}_{r'}$, $r'=r/(r-1)$, $1<r<\infty$, we easily see that $R(I-\tilde{P}_r)^*=R(I-\tilde{P}_{r'})$. Thus, if $\nabla g \in R(I-\tilde{P}_{r'})$ vanishes on $R(\nabla)$, then $\Delta g=0$, and so $\Delta(\nabla g)=0$. Hence $\nabla g=0$ and we get $R(\nabla)=R(I-\tilde{P}_r)$ by the Hahn–Banach theorem. Now we apply the complex interpolation to see that

$$\nabla\colon [\hat{H}_0^{1,r_0}(\mathbf{R}^n), \hat{H}_0^{1,r_1}(\mathbf{R}^n)]_\theta \to [R(I-\tilde{P}_{r_1}), R(I-\tilde{P}_{r_1})]_\theta$$

is a bounded bijection. Hence we have only to show that

$$[R(I-\tilde{P}_{r_0}), R(I-\tilde{P}_{r_1})]_\theta = R(I-\tilde{P}_r), \quad 1/r=(1-\theta)/r_0+\theta/r_1. \tag{4.21}$$

But, since $\tilde{P}$ is a bounded projection on each $L^r(\mathbf{R}^n)$, $1<r<\infty$, (4.21) follows from [41, Section 1.2.4, Theorem]. The proof is complete.

We are now ready to prove the following, which is our key result in this section.

THEOREM 4.4. (i) *If* $1<r<\infty$, *then the estimate*

$$\|A^{1/2}u\|_r \leq C\|\nabla u\|_r, \quad u\in D(A_r),$$

holds with C independent of u.

(ii) *If* $1<r<n$, *then we have*

$$\|\nabla u\|_r \leq C\|A^{1/2}u\|_r, \quad u\in D(A_r),$$

with C independent of u.

(iii) *If* $1<r<n$, *then* $D_r^{1/2}=\hat{H}^{1,r}_{0,\sigma}(\Omega)$.

Proof. By (4.15) and (4.17) both (i) and (ii) are valid for $1<r<n/2$. Also, in case $r=2$, both (i) and (ii) are obvious, since A_2 is the self-adjoint operator associated with the bilinear form $\langle\nabla u, \nabla v\rangle$ on $L^2_\sigma(\Omega)\cap H^{1,2}_0(\Omega)$. Now let $r_1=2$ and $1<r_0<n/2$ with $r_0<r_1$. By the above and estimate (3.4) with $\alpha=1/2$ the operator $\nabla(\lambda+A)^{-1/2}$ extends uniquely to bounded operators from $L^{r_j}_\sigma(\Omega)$ to $L^{r_j}(\Omega)$, $j=0, 1$, with operator-norms independent of $\lambda>0$. By interpolation, it thus follows that the same operator is bounded from $L^r_\sigma(\Omega)$ to $L^r(\Omega)$ for all $r_0\leq r\leq r_1$ with operator-norm independent of $\lambda>0$. This proves (ii) for $1<r\leq 2$. Now let $2<r<\infty$; since $R(A^{1/2}_{r'})$ is dense in $L^{r'}_\sigma(\Omega)$, it follows from (ii) with $r=r'<2$ that, for $u\in D(A_r)$,

$$\|A^{1/2}u\|_r = \sup_v |\langle A^{1/2}_r u, A^{1/2}_{r'} v\rangle| / \|A^{1/2}v\|_{r'} = \sup_v |\langle\nabla u, \nabla v\rangle| / \|A^{1/2}v\|_{r'}$$

$$\leq \|\nabla u\|_r \sup_v (\|\nabla v\|_{r'} / \|A^{1/2}v\|_{r'}) \leq C\|\nabla u\|_r.$$

We thus conclude that (i) holds for $1<r<n/2$ and $2\leq r<\infty$. Choosing $r_1=2$ and $1<r_0<n/2$ with $r_0<r_1$, and then interpolating between the operators $A^{1/2}$: $\hat{H}^{1,r_j}_{0,\sigma}(\Omega)\to L^{r_j}_\sigma(\Omega), j=0, 1$, we see by Proposition 4.3 (iii) that (i) holds also for $r_0\leq r\leq r_1=2$. The proof of (i) is complete. To finish the proof of (ii) we take an arbitrary $1<r<n$ and apply (3.10), as well as assertion (i) above with $r=r'$, obtaining

$$\|\nabla u\|_r \leq C\sup_v |\langle\nabla u, \nabla v\rangle| / \|\nabla v\|_{r'} = C\sup_v |\langle A^{1/2}_r u, A^{1/2}_{r'} v\rangle| / \|\nabla v\|_{r'}$$

$$\leq C\|A^{1/2}u\|_r \sup_v (\|A^{1/2}v\|_{r'} / \|\nabla v\|_{r'}) \leq C\|A^{1/2}u\|_r$$

for $u\in D(A_r)$. This proves (ii). (iii) is easily obtained from (i), (ii), Proposition 4.3 (i), and the fact that $D(A_r)$ is dense in $D(A^{1/2}_r)$.

Theorem 4.4 enables us to deduce an embedding theorem of Sobolev type for domains of fractional powers.

COROLLARY 4.5. *Let $n\geqslant 3$, $1<r<n$, $0\leqslant s<n/r$ and $1/q=1/r-s/n$. Then the estimate*

$$\|u\|_q \leqslant C\|A^{s/2}u\|_r, \quad u\in D(A_r^{s/2}) \tag{4.22}$$

holds with C independent of u.

Remark. Estimate (4.22) holds for $1<r<\infty$ in the case of entire and halfspaces provided only that $n\geqslant 2$ and $1/q=1/r-s/n>0$. For the entire spaces, this is easily seen from the well-known estimates on Riesz potentials [40]. For the case of halfspaces, we refer the reader to [3].

Proof of Corollary 4.5. First observe that $D_r^{1/2}\subset L^{r^*}(\Omega)$, $1/r^*-1/r-1/n$, by Theorem 4.4 and the Sobolev inequality, and therefore $\{L_\sigma^r(\Omega), D_r^{1/2}\}$ is an interpolation couple. The proof of Theorem 4.2 then applies to yield

$$[L_\sigma^r(\Omega), D_r^{1/2}]_\theta = D_r^{\theta/2} \quad (0\leqslant\theta\leqslant 1) \quad \text{if} \quad 1<r<n. \tag{4.23}$$

From (4.23) and the Riesz–Thorin theorem it follows that $D_r^{\theta/2}\subset L^q(\Omega)$ with continuous injection if $1<r<n$ and $1/q=1/r-\theta/n$. Now let $s=k+\theta$, where k is a nonnegative integer and $0\leqslant\theta<1$, and take m so large that $D(A_r^m)\subset D(A_q^{s/2})$, $1/q=1/r-s/n$, which is possible by the regularity theory for problem (SS) [1]. If we set

$$1/q_0 = 1/r-\theta/n \quad \text{and} \quad 1/q_j = 1/q_0-j/n, \quad j=0,1,\dots,k,$$

then, by assumption on r and s, we have $q=q_k$ and $1<q_j<n$ for $j=0,\dots,k-1$. It thus follows that

$$\|u\|_q \leqslant C\|A^{1/2}u\|_{q_{k-1}} \leqslant \dots \leqslant C\|A^{k/2}u\|_{q_0} \leqslant C\|A^{s/2}u\|_r$$

for $u\in D(A_r^m)$. The case of general $u\in D(A_r^{s/2})$ is treated through approximation. The proof is complete.

Corollary 4.5 is now applied to deduce the so-called L^p–L^q estimates for the semigroup $\{e^{-tA}; t\geqslant 0\}$.

COROLLARY 4.6. (i) *If* $1<q\le r<\infty$, *then the estimate*

$$\|e^{-tA}u\|_r \le Ct^{-(n/q-n/r)/2}\|u\|_q, \quad u\in L^q_\sigma(\Omega) \tag{4.24}$$

holds with C independent of u and $t>0$.

(ii) $\|e^{-tA}u\|_r \to 0$ *as* $t\to\infty$ *for all* $u\in L^r_\sigma(\Omega)$ *and* $1<r<\infty$.

(iii) *If* $1<q\le r<n$, *then*

$$\|\nabla e^{-tA}u\|_r \le Ct^{-1/2-(n/q-n/r)/2}\|u\|_q, \quad u\in L^q_\sigma(\Omega) \tag{4.25}$$

with C independent of u and $t>0$.

Proof. (i) Assume first that $1<q<n$ and take $0<s<n/q$ with $1/r_0\equiv 1/q-s/n<1/r$. Since $q<r<r_0$, Hölder's inequality and the boundedness of the semigroup yield

$$\|e^{-tA}u\|_r \le C\|e^{-tA}u\|^{\alpha}_{r_0}\|e^{-tA}u\|^{1-\alpha}_q \le C\|e^{-tA}u\|^{\alpha}_{r_0}\|u\|^{1-\alpha}_q,$$

$$\text{with} \quad \alpha=(1/q-1/r)/(1/q-1/r_0)=(n/q-n/r)/s.$$

By Corollary 4.5 and (3.2) we conclude that

$$\|e^{-tA}u\|_r \le C\|A^{s/2}e^{-tA}u\|^{\alpha}_q\|u\|^{1-\alpha}_q \le Ct^{-\alpha s/2}\|u\|_q,$$

which shows (4.24) for $1<q<n$. We next consider the case $n\le q\le r<\infty$. Take $1<r_0<n$; then the foregoing result and the boundedness of the semigroup together show that if we set $T=e^{-tA}$ for fixed $t>0$,

$$T: L^r_\sigma(\Omega)\to L^r_\sigma(\Omega) \quad \text{is bounded with bound} \le M; \quad \text{and}$$

$$T: L^{r_0}_\sigma(\Omega)\to L^r_\sigma(\Omega) \quad \text{is bounded with bound} \le Ct^{-(n/r_0-n/r)/2}.$$

Interpolating between these two cases gives the boundness of T from $L^q_\sigma(\Omega)$ to $L^r_\sigma(\Omega)$ with bound $\le Ct^{-(n/q-n/r)/2}$. The proof is complete.

(ii) If $u\in C^\infty_{0,\sigma}(\Omega)$, then $u\in L^q_\sigma(\Omega)$ for any $1<q<r$; so the result follows from (1). Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^r_\sigma(\Omega)$, the result follows in general case from the boundedness of the semigroup.

(iii) Since $1<r<n$, Theorem 4.4 and estimate (3.2) together yield

$$\|\nabla e^{-tA}u\|_r \le C\|A^{1/2}e^{-tA}u\|_r = C\|A^{1/2}e^{-tA/2}e^{-tA/2}u\|_r \le Ct^{-1/2}\|e^{-tA/2}u\|_r.$$

Applying (4.24) to the last term gives (4.25). The proof is complete.

Remarks. Iwashita [17] has recently proved (4.25) for $1<r\leq n$. In the case of halfspaces, (4.24) holds also for $q=1$ (resp. $r=\infty$) under an appropriate assumption on r (resp. q); see [3].

Proposition 4.3 (i) was first proved by Heywood [14] for $r=2$. Our proof of Proposition 3.8 (ii) indicates also that, for a bounded domain D,

$$H^{1,r}_{0,\sigma}(D)=\{u\in H^{1,r}_0(D);\ \nabla\cdot u=0\},\quad 1<r<\infty.$$

This result was also proved by Heywood [14] for $r=2$.

5. Proof of main results

We are now in a position to prove our main results, namely, Theorems A and C in Section 2. We begin by establishing the following, which is our key lemma in this section. Let

$$A_2=\int_0^\infty \lambda\, dE_\lambda$$

be the spectral decomposition of the nonnegative self-adjoint operator A_2.

LEMMA 5.1. *Let* $0<\varepsilon<1/4$ *and* $\theta+\varrho=1+2\varepsilon$ *with* $\theta\geq 0, \varrho\geq 0$. *Then there is a constant* $C=C(\varepsilon,\theta,\varrho,n,\Omega)$ *such that*

$$\|E_\lambda P(u\cdot\nabla)v\|_2\leq C\lambda^{n/4-\varepsilon}\|A^{\theta/2}u\|_2\,\|A^{\varrho/2}v\|_2 \tag{5.1}$$

for all $\lambda>0, u\in D(A_2^{\theta/2})$ *and* $v\in D(A_2^{\varrho/2})$.

Proof. Let $q=n/(1+2\varepsilon)$, $1/r=1/2-\theta/n$ and $1/s=1/2-\varrho/n$ so that $1/q+1/r+1/s=1$. Since $\nabla\cdot u=0$, an integration by parts and Hölder's inequality together yield

$$\|E_\lambda P(u\cdot\nabla)v\|_2=\sup_\varphi|\langle u\cdot\nabla E_\lambda\varphi, v\rangle|\leq\|u\|_r\|v\|_s\sup_\varphi\|\nabla E_\lambda\varphi\|_q$$

where the supremum is taken over all φ in the unit ball of $L^2_\sigma(\Omega)$. Since $2<q<n$, Theorem 4.4 and the fact that $E_\lambda\varphi\in D(A_2^\infty)\subset D(A_q^\infty)$ together imply that

$$\|\nabla E_\lambda\varphi\|_q\leq C\|A^{1/2}E_\lambda\varphi\|_q.$$

Since $1/q=1/2-\sigma/n$ with $\sigma=n/2-2\varepsilon-1$, Corollary 4.5 gives

$$\|A^{1/2}E_\lambda \varphi\|_q^2 \leqslant C\|A^{(\sigma+1)/2}E_\lambda \varphi\|_2^2 = C\int_0^\lambda \mu^{n/2-2\varepsilon} d_\mu\|E_\mu E_\lambda \varphi\|_2^2 \leqslant C\lambda^{n/2-2\varepsilon}.$$

Combining this with $\|u\|_r \leqslant C\|A^{\theta/2}u\|_2$ and $\|v\|_s \leqslant C\|A^{\varrho/2}v\|_2$ yields (5.1).

Remark. On entire and halfspaces estimate (5.1) takes the form

$$\|E_\lambda P(u\cdot\nabla)v\|_2 \leqslant C\lambda^{(n+2)/4}\|u\|_2\|v\|_2. \tag{5.2}$$

The proof is given in [3, 18]. The parameter λ will be identified with t^{-1} in deducing L^2-decay rates; thus our estimate (5.1) yields the rate $t^{\varepsilon-n/4}$ caused by the presence of the nonlinear term, while (5.2) gives $t^{-(n+2)/4}$ in the case of entire and halfspaces.

5.1. *Proof of Theorem* A. We give a detailed proof of assertion (ii) and then describe an outline of the proof of (i), since the proof of (i) is almost the same as, and in some sense easier than, that of (ii). Let v be a weak solution of (NS) with $v(0)=a$, satisfying the energy inequality:

$$\|v(t)\|_2^2+2\int_s^t \|\nabla v\|_2^2\,d\tau \leqslant \|v(s)\|_2^2 \quad \text{for} \quad s=0, \quad \text{a.e. } s>0; \quad \text{and all } t\geqslant s \tag{E}$$

and let $\lambda=\lambda(t)$ be any smooth positive function of $t>0$. From the estimate

$$\|\nabla v\|_2^2 = \|A^{1/2}v\|_2^2 = \int_0^\infty z\,d\|E_z v\|_2^2 \geqslant \int_\lambda^\infty z\,d\|E_z v\|_2^2 \geqslant \lambda(\|v\|_2^2-\|E_\lambda v\|_2^2)$$

and from (E) it follows that

$$\|v(t)\|_2^2+\int_s^t \lambda(\tau)\,\|v(\tau)\|_2^2\,dt \leqslant \|v(s)\|_2^2+\int_s^t \lambda(\tau)\,\|E_{\lambda(\tau)}v(\tau)\|_2^2\,dt. \tag{5.3}$$

The last integral in (5.3) is well defined in the sense of Lebesgue, since the function $f(\lambda,\tau)=\|E_\lambda v(\tau)\|_2$ is monotone in λ for each fixed τ and measurable in τ for each fixed λ. To estimate the term $\|E_{\lambda(\tau)}v(\tau)\|_2$ we go back to the definition of weak solution. In (2.3) we set $\phi(\tau)=e^{-(t-\tau)A}E_\lambda\varphi$, $\varphi\in L_\sigma^2(\Omega)$, which is legitimate since $E_\lambda\varphi\in L^n(\Omega)$, and obtain

$$\begin{aligned}\langle E_\lambda v(t),\varphi\rangle &= \langle E_\lambda e^{-(t-s)A}v(s),\varphi\rangle - \int_s^t \langle v\cdot\nabla v(\tau), e^{-(t-\tau)A}E_\lambda\varphi\rangle\,d\tau\\ &= \langle E_\lambda e^{-(t-s)A}v(s),\varphi\rangle + \int_s^t \langle v, v\cdot\nabla E_\lambda e^{-(t-\tau)A}\varphi\rangle\,d\tau\end{aligned}$$

for $t \geqslant s \geqslant 0$. The last term is estimated as in the proof of Lemma 5.1 and we obtain for $0<\varepsilon<1/4$,

$$\|E_\lambda v(t)\|_2 \leqslant \|E_\lambda e^{-(t-s)A} v(s)\|_2 + C\lambda^{n/4-\varepsilon} \int_s^t \|A^{1/2+\varepsilon/2} v\|_2^2 \, dt$$

$$\leqslant \|E_\lambda e^{-(t-s)A} v(s)\|_2 + C\lambda^{n/4-\varepsilon} \int_s^t \|A^{1/2} v\|_2^{1+2\varepsilon} \|v\|_2^{1-2\varepsilon} \, dt$$

$$\leqslant \|e^{-(t-s)A} v(s)\|_2 + C\lambda^{n/4-\varepsilon} \left(\int_s^t \|\nabla v\|_2^2 \, d\tau \right)^{(1+2\varepsilon)/2} \left(\int_s^t \|v\|_2^2 \, dt \right)^{(1-2\varepsilon)/2}.$$

Here we set $s=0$ and use the estimate $\int_0^\infty \|\nabla v\|_2^2 \, d\tau \leqslant \|a\|_2^2/2$ which follows from (E), to get

$$\|E_\lambda v(t)\|_2 \leqslant \|e^{-tA} a\|_2 + C\lambda^{n/4-\varepsilon} \left(\int_0^t \|v\|_2^2 \, d\tau \right)^{(1-2\varepsilon)/2}. \tag{5.4}$$

Substituting (5.4) with $\lambda = \lambda(t)$ into (5.3) yields the following inequality for $y(t) = \|v(t)\|_2^2$:

$$y(t) - g(t,s) + \int_s^t \lambda(\tau) y(\tau) \, dt \leqslant y(s) \quad \text{a.e. in} \quad s \in (0,t), \tag{5.5}$$

$$\text{with} \quad g(t,s) = 2 \int_s^t \left[\lambda(\tau) \| e^{-tA} a\|_2^2 + C\lambda^{n/2+1-2\varepsilon}(\tau) \left(\int_0^\tau \|v\|_2^2 \, d\sigma \right)^{1-2\varepsilon} \right] d\tau.$$

We now want to apply Gronwall's lemma to (5.5) with respect to s. Consider the function $h(s) = \int_s^t \lambda(\tau) y(\tau) \, dt$, which is a.e. differentiable in $(0,t)$ with $h' \in L^\infty(\delta, t)$ for small $\delta > 0$. From (5.5) we have

$$h'(\tau) = -\lambda(\tau) y(\tau) \leqslant -\lambda(\tau) [h(\tau) + y(t) - g(t,\tau)]. \tag{5.6}$$

Now let $H \geqslant 0$ be a function solving $H'(\tau) = \lambda(\tau) H(\tau)$. Multiplying (5.6) by H and then integrating over $[s,t]$ yields

$$(H(t) - H(s)) y(t) \leqslant H(s) h(s) + \int_s^t H'(\tau) g(t,\tau) \, d\tau,$$

since $h(t)=0$. Applying (5.5) to the right-hand side above and integrating by parts, we obtain, since $g(t,t)=0$,

$$H(t) y(t) \leqslant H(s) y(s) - \int_s^t H(\tau) g_\tau'(t,\tau) \, d\tau. \tag{5.7}$$

Now choose $\lambda(\tau) = m\tau^{-1}, m>0$, so that $H(\tau) = \tau^m$ and $H'(\tau) = m\tau^{m-1}$. Since (5.7) holds for a.e. $s \geqslant 0$ and since $y(s)$ is bounded, taking m sufficiently large we can pass to the limit $s \to 0$ in (5.7) to obtain

$$\|v(t)\|_2^2 \leqslant Ct^{-m} \int_0^t ms^{m-1} \|e^{-sA}a\|_2^2 \, ds + Ct^{-m} \int_0^t s^{m-n/2-1+2\varepsilon} \left(\int_0^s \|v\|_2^2 \, dt \right)^{1-2\varepsilon} ds. \quad (5.8)$$

Since $\|v(\tau)\|_2 \leqslant \|a\|_2$ as seen from energy inequality (E), the last term of (5.8) is $\leqslant Ct^{1-n/2}$; hence assertion (a) follows from the convergence $\|e^{-tA}a\|_2 \to 0$ $(t \to \infty)$. To prove (b), suppose that $\|e^{-sA}a\|_2 \leqslant Cs^{-\alpha}$ and $\|v(s)\|_2 \leqslant Cs^{-\beta_0}$; then (5.8) implies $\|v(t)\|_2 \leqslant Ct^{-\beta_1}$ with $\beta_1 = \min(\alpha, n/4 - 1/2 + \beta_0(1-2\varepsilon))$ so far as $\beta_0 < 1/2$. This shows (b) for $0 < \alpha < n/4 - 1/2$ and $\beta_0 = 0$. If $\alpha \geqslant n/4 - 1/2$, then the foregoing observation allows us to start with $\beta_0 = 1/4 - \varepsilon$ and, by definition of β_1, we obtain (b) for $\alpha < n/4 - 1/4$. When $\alpha \geqslant n/4 - 1/4$, we can take $\beta_0 = 1/2 - \varepsilon$, and thereby arrive at the conclusion in all cases.

We next prove assertion (c). Let $w(t) = v(t) - v^0(t)$ with $v^0(t) = e^{-tA}a$. Since $v^0(t)$ satisfies (E) with equality sign, direct calculation gives

$$\begin{aligned} \|w(t)\|_2^2 + 2\int_s^t \|\nabla w\|_2^2 \, d\tau &= \|v(t)\|_2^2 + \|v^0(t)\|_2^2 - 2\langle v(t), v^0(t)\rangle \\ &\quad + 2\int_s^t (\|\nabla v\|_2^2 + \|\nabla v^0\|_2^2 - 2\langle \nabla v, \nabla v^0\rangle) \, dt \\ &\leqslant \|v(s)\|_2^2 + \|v^0(s)\|_2^2 - 2\langle v(t), v^0(t)\rangle - 4\int_s^t \langle \nabla v, \nabla v^0\rangle \, d\tau \end{aligned} \quad (5.9)$$

for a.e. $s > 0$ and all $t \geqslant s$. We insert $\phi(\tau) = v^0(\tau)$ into (2.3) and get for $0 < s \leqslant t$,

$$\langle v(t), v^0(t)\rangle + 2\int_s^t \langle \nabla v, \nabla v^0\rangle \, dt + \int_s^t \langle v \cdot \nabla v, v^0\rangle \, d\tau = \langle v(s), v^0(s)\rangle$$

since $(v^0)' = -Av^0$. Using this to eliminate the last integral in (5.9) we have

$$\|w(t)\|_2^2 + 2\int_s^t \|\nabla w\|_2^2 \, d\tau \leqslant \|w(s)\|_2^2 + 2\int_s^t \langle v \cdot \nabla v, v^0\rangle \, d\tau \quad (5.10)$$

for a.e. $s > 0$ and all $t \geqslant s$. The integrand of the last term is estimated as

$$\begin{aligned} |\langle v \cdot \nabla v, v^0\rangle| &= |\langle w \cdot \nabla v^0, w\rangle + \langle v^0 \cdot \nabla v^0, w\rangle| \\ &\leqslant C\|A^{n/4-\varepsilon}v^0\|_2 (\|A^{\theta/2}w\|_2 \, \|A^{\varrho/2}w\|_2 + \|A^{\alpha/2}v^0\|_2 \, \|A^{\beta/2}w\|_2), \end{aligned}$$

where $0 < \varepsilon < 1/4$ and $\theta + \varrho = \alpha + \beta = 1 + 2\varepsilon$; recall the proof of (5.1). We set $\theta = \varrho = 1/2 + \varepsilon$, $\alpha = 2\varepsilon$, $\beta = 1$ and apply the moment inequality to get

$$|\langle v\cdot\nabla v, v^0\rangle| \leq C\|A^{n/4-\varepsilon}v^0\|_2(\|A^{1/2}w\|_2^{1+2\varepsilon}\|w\|_2^{1-2\varepsilon}+\|A^{\varepsilon}v^0\|_2\|A^{1/2}w\|_2)$$
$$\leq \frac{1}{2}\|\nabla w\|_2^2+C\|A^{n/4-\varepsilon}v^0\|_2^{2/(1-2\varepsilon)}\|w\|_2^2+C\|A^{n/4-\varepsilon}v^0\|_2^2\|A^{\varepsilon}v^0\|_2^2. \tag{5.11}$$

Since A_2 is nonnegative and self-adjoint, $\|A^{\gamma}v^0(\tau)\|_2 \leq C\tau^{-\gamma}$ and $\|A^{\gamma}v^0(\tau)\|_2 \leq C\tau^{-\gamma}\|v^0(\tau/2)\|_2$ for $\gamma \geq 0$; hence from (5.10) and (5.11) we find that

$$\|w(t)\|_2^2+\int_s^t\|\nabla w\|_2^2\,d\tau \leq \|w(s)\|_2^2+C\int_s^t[\tau^{-\gamma}\|w\|_2^2+\tau^{-n/2}\|v^0(\tau/2)\|_2^2]\,d\tau$$
$$\text{with}\quad \gamma=(n/2-2\varepsilon)/(1-2\varepsilon)=n/2+\delta,\quad \delta=\varepsilon(n-2)/(1-2\varepsilon), \tag{5.12}$$

for a.e. $s>0$ and all $t \geq s$. The remaining argument is nearly the same as in the proof of (a). We estimate $\|\nabla w\|_2=\|A^{1/2}w\|_2$ from below, using the spectral measure E_λ; use (2.3) as in the proof of (a) to obtain

$$\|E_\lambda w(t)\|_2 \leq C\lambda^{n/4-\varepsilon}\left(\int_0^t\|v\|_2^2\,d\tau\right)^{(1-2\varepsilon)/2};$$

and finally take $\lambda(\tau)=m\tau^{-1}$ with large m. This process leads us to

$$\|w(t)\|_2^2 \leq Ct^{-m}\int_0^t\left[s^{m-\gamma}\|w\|_2^2+s^{m-n/2}\|v^0\|_2^2+s^{m-n/2-1+2\varepsilon}\left(\int_0^s\|v\|_2^2\,d\tau\right)^{1-2\varepsilon}\right]ds$$
$$\leq Ct^{1-n/2}\left[t^{-1-\delta}\int_0^t\|w\|_2^2\,ds+t^{-1}\int_0^t\|v^0\|_2^2\,ds+\left(t^{-1}\int_0^t\|v\|_2^2\,ds\right)^{1-2\varepsilon}\right]. \tag{5.13}$$

Since $\|w(s)\|_2 \leq \|v(s)\|_2+\|v^0(s)\|_2 \to 0$ as $s\to\infty$ by assertion (a), this proves (c). Assertion (d) is easily deduced from (b) and (5.13). This completes the proof of (ii).

To prove (i), we use the approximate solutions v_k, $k=1,2,\ldots$, obtained by solving the integral equations:

$$v_k(t)=e^{-tA}a_k-\int_0^t e^{-(t-s)A}P(J_kv_k\cdot\nabla)v_k(s)\,ds, \tag{AP}$$

where $J_k=(I+k^{-1}A_2)^{-1-[n/4]}$, $a_k=J_ka$, and $[b]$ is the integral part of the real number b. Existence and uniqueness of a regular solution v_k of (AP) defined for all $t\geq 0$ and convergence of (a subsequence of) v_k are discussed in [3] and [29] along the idea of [38]. Since v_k satisfies (E) with equality sign and since $\|J_k\|\leq 1$ as bounded operators in $L^2_\sigma(\Omega)$, one can repeat all of the foregoing arguments to get the desired results for each function v_k. But, as readily seen from the foregoing arguments, all the estimates needed in the proof of (ii) are uniform in approximation parameter k, and so we obtain the existence of a function $v\in L^\infty(0,\infty;L^2_\sigma(\Omega))\cap L^2(0,\infty;\hat{H}^{1,2}_0(\Omega))$ with desired decay properties by passing to the limit $k\to\infty$; see [3, 18] for the details. This function v satisfies the identity:

$$h(t)\langle v(t),w\rangle+\int_s^t\langle\nabla v,\nabla w\rangle\,h\,d\tau+\int_s^t\langle v\cdot\nabla v,w\rangle\,h\,d\tau=\int_s^t\langle v,w\rangle\,h'd\tau+h(s)\,\langle v,(s),w\rangle \tag{2.3'}$$

for all $w\in H^{1,2}_0(\Omega)\cap L^n_\sigma(\Omega)$, $t\geq s\geq 0$, and $h\in C^1([s,t];\mathbf{R})$. This is verified as in [29, pp. 464–466]. Although in [29] only the case $n=3$, 4 is discussed, the argument given there applies to all dimensions $n\geq 3$ due to the requirement $w\in L^n(\Omega)$. That (2.3′) implies (2.3) is proved in [27, p. 638]; so the function v is the desired weak solution of (NS). The proof is complete.

5.2. *Proof of Theorem* C. We begin with the proof of the following

LEMMA 5.2. *If* $1<r\leq n'\leq q\leq 2$, $n'=n/(n-1)$, $n\geq 3$, *and* $a\in L^2_\sigma(\Omega)\cap L^r(\Omega)$, *then all weak solutions* v *of* (NS) *with* $v(0)=a$ *belong to* $L^\infty_{\mathrm{loc}}([0,\infty);L^r(\Omega))\cap L^\infty(0,\infty;L^q(\Omega))$. *Further we get* $\lim_{t\to\infty}\|v(t)\|_q=0$ *provided* $q<2$.[1]

Proof. We insert $\phi(\tau)=e^{-(t-\tau)A}\varphi$, $\varphi\in C^\infty_{0,\sigma}(\Omega)$, for (2.3) and obtain

$$\langle v(t),\varphi\rangle=\langle v(s),e^{-(t-s)A}\varphi\rangle-\int_s^t\langle v\cdot\nabla v(\tau),e^{-(t-\tau)A}\varphi\rangle\,d\tau \tag{5.14}$$

for all $t\geq s\geq 0$. By the Hölder and Sobolev inequalities we have

$$\begin{aligned}|\langle v\cdot\nabla v,e^{-(t-\tau)A}\varphi\rangle|&\leq C\|\varphi\|_{r'}\|v\|_{2r'/(r'-2)}\|\nabla v\|_2\\&\leq C\|\varphi\|_{r'}\|v\|_2^{1-n/r'}\|v\|_{2n/(n-2)}^{n/r'}\|\nabla v\|_2\\&\leq C\|\varphi\|_{r'}\|v\|_2^{1-n/r'}\|\nabla v\|_2^{1+n/r'}.\end{aligned} \tag{5.15}$$

(5.15) and (5.14) with $s=0$ yield

$$\|v(t)\|_r\leq C\left[\|a\|_r+\int_0^t\|v\|_2^{1-n/r'}\|\nabla v\|_2^{1+n/r'}\,d\tau\right]. \tag{5.16}$$

This shows that v is in $L^\infty_{\mathrm{loc}}([0,\infty);L^r(\Omega))$ and in particular, on taking $r=n'$, in $L^\infty(0,\infty;L^{n'}(\Omega))$. Hence $v\in L^\infty(0,\infty;L^q(\Omega))$ for $n'\leq q\leq 2$. Then (5.14) and (5.15) together imply

---

(1) The same result was obtained also by Professor H. Sohr (private communication).

$$\|v(t)\|_{n'} \leq C\left[\|e^{-(t-s)A}v(s)\|_{n'} + \int_s^\infty \|\nabla v\|_2^2\, d\tau\right]$$

for all $t \geq s \geq 0$, and therefore $\|v(t)\|_{n'} \to 0$ as $t \to \infty$ by Corollary 4.6 (ii). This implies for $n' \leq q < 2$,

$$\|v(t)\|_q \leq \|v(t)\|_{n'}^{(1/q-1/2)/(1/n'-1/2)} \|v(t)\|_2^{(1/n'-1/q)/(1/n'-1/2)} \to 0 \quad \text{as} \quad t \to \infty.$$

The proof is complete.

We now prove Theorem C. By Theorem B,

$$\|v(t)\|_2 \leq C(1+t)^{-\beta} \quad \text{with} \quad \beta = (n/r - n/2)/2 > 1/2,$$

and so $\int_0^\infty \|v\|_2^2\, ds$ is finite. Hence (5.16) yields

$$\|v(t)\|_r \leq C\left[\|a\|_r + \int_0^\infty \|v\|_2^{1-n/r'} \|\nabla v\|_2^{1+n/r'}\, d\tau\right]$$
$$\leq C\left[\|a\|_r + \left(\int_0^\infty \|v\|_2^2\, d\tau\right)^{1/2-n/2r'} \left(\int_0^\infty \|\nabla v\|_2^2\, d\tau\right)^{1/2+n/2r'}\right],$$

and therefore $v \in L^\infty(0, \infty; L^r(\Omega))$. (5.14) and (5.15) then imply, for $t \geq s \geq 0$,

$$\|v(t)\|_r \leq C\left[\|e^{-(t-s)A}v(s)\|_r + \left(\int_s^\infty \|\nabla v\|_2^2\, d\tau\right)^{1/2-n/2r'} \left(\int_s^\infty \|\nabla v\|_2^2\, d\tau\right)^{1/2+n/2r'}\right]$$

and we conclude that $\|v(t)\|_r \to 0$ as $t \to \infty$ by Corollary 4.6 (ii). Thus we obtain for $r \leq q < 2$,

$$\|v(t)\|_q \leq \|v(t)\|_r^{1-\alpha} \|v(t)\|_2^\alpha = o(t^{-\alpha\beta}), \quad \alpha = (1/r - 1/q)/(1/r - 1/2)$$

and $\alpha\beta = (n/r - n/q)/2$. This completes the proof of Theorem C.

Remarks. Lemma 5.2 asserts in particular that if an initial velocity is in $L^r(\Omega) \cap L^2_\sigma(\Omega)$ for some $1 < r \leq n'$, then all the corresponding weak solutions also belong to $L^r(\Omega)$ for a.e. $t > 0$. The converse to this statement is an open question.

As for the behavior of L^2-norms of general weak solutions, the following is known: *Given any $a \in L^2_\sigma(\Omega)$, every weak solution v with $v(0) = a$ satisfies*

$$\lim_{t\to\infty} \int_t^{t+1} \|v\|_2^2\, ds = 0. \tag{5.17}$$

See [27] for the proof. Our proof of Lemma 5.2 is in fact a simple modification of the proof of (5.17) given in [27].

Acknowledgements. The work of the first author is supported by Deutsche Forschungsgemeinschaft, that of the second by Alexander von Humboldt-Stiftung, in Federal Republic of Germany.

References

[1] AGMON, S., DOUGLIS, A. & NIRENBERG, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.*, 17 (1964), 35–92.

[2] BOGOVSKI, M. E., Solutions of the first boundary value problem for the equations of continuity of an incompressible medium. *Soviet Math. Dokl.*, 20 (1979), 1094–1098.

[3] BORCHERS, W. & MIYAKAWA, T., L^2 decay for the Navier–Stokes flow in half-spaces. *Math. Ann.*, 282 (1988), 139–155.

[4] BORCHERS, W. & SOHR, H., On the semigroup of the Stokes operator for exterior domains in L^q spaces. *Math. Z.*, 196 (1987), 415–425.

[5] — The equations div $u=f$ and rot $v=g$ with homogeneous Dirichlet boundary condition. *Hokkaido Math. J.*, 19 (1990), 67–87.

[6] CATTABRIGA, L., Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rend. Sem. Mat. Univ. Padova*, 31 (1961), 308–340.

[7] CHANG, I-DEE & FINN, R., On the solutions of a class of equations occurring in continuum mechanics with application to the Stokes paradox. *Arch. Rational Mech. Anal.*, 7 (1961), 388–401.

[8] FRIEDMAN, A., *Partial Differential Equations*. Holt, Rinehard & Winston, New York, 1969.

[9] FUJIWARA, D. & MORIMOTO, H., An L_r-theorem of the Helmholtz decomposition of vector fields. *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.*, 24 (1977), 685–700.

[10] GALDI, G. & MAREMONTI, P., Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier–Stokes equations in exterior domains. *Arch. Rational Mech. Anal.*, 94 (1986), 253–266.

[11] GIGA, Y., Analyticity of the semigroup generated by the Stokes operator in L_r spaces. *Math. Z.*, 178 (1981), 297–329.

[12] — Domains of fractional powers of the Stokes operator in L_r spaces. *Arch. Rational Mech. Anal.*, 89 (1985), 251–265.

[13] GIGA, Y. & SOHR, H., On the Stokes operator in exterior domains. *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.*, 36 (1989), 103–130.

[14] HEYWOOD, J. G., On uniqueness questions in the theory of viscous flow. *Acta Math.*, 138 (1976), 61–102.

[15] — The Navier–Stokes equations: On the existence, regularity and decay of solutions. *Indiana Univ. Math. J.*, 29 (1980), 639–681.

[16] HOPF, E., Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4 (1951), 213–231.

[17] IWASHITA, H., L_q–L_r estimates for solutions of nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problem in L_q spaces. *Math. Ann.*, 285 (1989), 265–288.

[18] KAJIKIYA, R. & MIYAKAWA, T., On L^2 decay of weak solutions of the Navier–Stokes equations in $\mathbf{R}^n$. *Math. Z.*, 192 (1986), 135–148.

[19] KATO, T., *Perturbation Theory for Linear Operators*. 2nd ed., Springer-Verlag, Berlin, 1976.

[20] — Strong L^p-solutions of the Navier–Stokes equation in $\mathbf{R}^m$, with applications to weak solutions. *Math. Z.*, 187 (1984), 471–480.

[21] KOMATSU, H., Fractional powers of operators. *Pacific J. Math.*, 19 (1966), 285–346.

[22] KREIN, S., *Linear Differential Equations in Banach Spaces*. Amer. Math. Soc., Providence, 1972.

[23] LADYZHENSKAYA, O. A., *The Mathematical Theory of Viscous Incompressible Flow*. Gordon & Breach, New York, 1969.

[24] LERAY, J., Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63 (1934), 193–248.

[25] MAREMONTI, P., On the asymptotic behavior of the L^2 norm of suitable weak solutions to the Navier–Stokes equations in three-dimensional exterior domains. *Comm. Math. Phys.*, 118 (1988), 385–400.

[26] MARTINEZ, C., SANZ, M. & MARCO, L., Fractional powers of operators. *J. Math. Soc. Japan*, 40 (1988), 331–347.

[27] MASUDA, K., Weak solutions of the Navier–Stokes equations. *Tôhoku Math. J.*, 36 (1984), 623–646.

[28] MIYAKAWA, T., On nonstationary solutions of the Navier–Stokes equations in an exterior domain. *Hiroshima Math. J.*, 12 (1982), 115–140.

[29] MIYAKAWA, T. & SOHR, H., On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier–Stokes equations in exterior domains. *Math. Z.*, 199 (1988), 455–478.

[30] PRODI, G., Un teorema di unicità per le equazioni di Navier–Stokes. *Annali di Mat.*, 48 (1959), 173–182.

[31] REED, M. & SIMON, B., *Methods of Modern Mathematical Physics, vol. II; Fourier analysis, self-adjointness*. Academic Press, New York, 1975.

[32] DE RHAM, G., *Differentiable Manifolds*. Springer-Verlag, Berlin, 1984.

[33] SCHONBEK, M.E., L^2 decay for weak solutions of the Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 88 (1985), 209–222.

[34] — Large time behaviour of solutions to the Navier–Stokes equations. *Comm. Partial Differential Equations*, 11 (1986), 733–763.

[35] SERRIN, J., The initial value problem for the Navier–Stokes equations, in *Nonlinear Problems*, R. Langer ed. The University of Wisconsin Press, Madison, 1963, pp. 69–98.

[36] SIMADER, C. G., *On Dirichlet's Boundary Value Problem*. Lecture Notes in Math., no. 268, Springer-Verlag, Berlin, 1972.

[37] SOLONNIKOV, V. A., Estimates for solutions of nonstationary Navier–Stokes equations. *J. Soviet Math.*, 8 (1977), 467–529.

[38] SOHR, H., VON WAHL, W. & WIEGNER, M., Zur Asymptotik der Gleichungen von Navier–Stokes. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. II*, 3 (1986), 45–59.

[39] SPECOVIUS, M., Die Stokes Gleichungen in Cantor Räumen und die Analytizität der Stokes-Halbgruppe in gewichteten L^p-Räumen. Dissertation, Paderborn, 1984.

[40] STEIN, E., *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1970.

[41] TRIEBEL, H., *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Publ. Co., Amsterdam, 1978.
[42] WESTPHAL, U., Ein Kalkül für gebrochene Potenzen infinitesimaler Erzeuger von Halbgruppen und Gruppen von Operatoren; Teil I: Halbgruppenerzeuger. *Compositio Math.*, 22 (1970), 67–103.
[43] WIEGNER, M., Decay results for weak solutions of the Navier–Stokes equations in $\mathbf{R}^n$. *J. London Math. Soc.*, 35 (1987), 303–313.
[44] YOSIDA, K., *Functional Analysis*. Springer-Verlag, Berlin, 1965.
[45] KOZONO, H. & SOHR, H., L^q-regularity theory for the Stokes operator in exterior domains. Preprint.

Received May 2, 1989