Algebraic model structures

Emily Riehl

University of Chicago

http://www.math.uchicago.edu/~eriehl

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Outline

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Functorial weak factorization systems

Definition

A functorial weak factorization system (wfs) $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$:

- $\mathcal{L} \boxslash \mathcal{R}$:

$$\begin{tikzcd} \cdot \arrow[r, "a"] \arrow[d, "\mathcal{L} \ni l"'] & \cdot \arrow[d, "r \in \mathcal{R}"] \\ \cdot \arrow[r, "b"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Furthermore $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

- There exists a functorial factorization $\vec{E} : \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{3}}$:

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ Ef \arrow[r, "E(u,v)"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$.

Algebraic perspective

$L, R : \mathcal{M}^{\mathbf{2}} \to \mathcal{M}^{\mathbf{2}}$ are pointed endofunctors with $\vec{\epsilon}\colon L \Rightarrow 1$, $\vec{\eta}\colon 1 \Rightarrow R$:

$$\vec{\epsilon}_f = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"'] & \cdot \end{tikzcd} \quad \text{and} \quad \vec{\eta}_g = \begin{tikzcd} \cdot \arrow[r, "Lg"] \arrow[d, "g"'] & \cdot \arrow[d, "Rg"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

Algebraic left maps

$f \in \mathcal{L}$ iff $\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed, "s"'] & \cdot \end{tikzcd}$ iff (f, s) is a $(L, \vec{\epsilon})$-coalgebra.

Algebraic right maps

$g \in \mathcal{R}$ iff $\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rg"'] \arrow[ur, dashed, "t"] & \cdot \end{tikzcd}$ iff (g, t) is a $(R, \vec{\eta})$-algebra.

Algebraic lifts

Recall

$$f \in \mathcal{L} \quad \text{iff} \quad \begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed, "s"'] & \cdot \end{tikzcd} \qquad\qquad g \in \mathcal{R} \quad \text{iff} \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rg"'] \arrow[ur, dashed, "t"] & \cdot \end{tikzcd}$$

Constructing lifts

Given a coalgebra (f, s) and an algebra (g, t), any lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \text{has a solution} \quad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ \cdot \arrow[r, dashed, "E(u,v)"] \arrow[d, "Rf"'] & \cdot \arrow[u, dashed, "t"] \arrow[d, "Rg"] \\ \cdot \arrow[r, "v"'] \arrow[u, dashed, "s"'] & \cdot \end{tikzcd}$$

Natural weak factorization systems

Definition (Grandis, Tholen)

A natural weak factorization system (nwfs) $(\mathbb{L}, \mathbb{R})$ on a category $\mathcal{M}$:

- a comonad $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ and a monad $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$

such that

- $(L, \vec{\epsilon})$ and $(R, \vec{\eta})$ come from a functorial factorization $\vec{E}$
- the canonical map $LR \Rightarrow RL$ is a distributive law.

Its underlying wfs is $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$, the retract closures of the $\mathbb{L}$-coalgebras and $\mathbb{R}$-algebras.

Cofibrantly generated nwfs

Let $\mathcal{J}$ be a small category over $\mathcal{M}^{\mathbf{2}}$.

Theorem (Garner)

If $\mathcal{M}$ permits the small object argument, then $\mathcal{J}$ generates a nwfs $(\mathbb{L}, \mathbb{R})$ such that

- *(free) There exists a canonical functor $\lambda : \mathcal{J} \to \mathbb{L}\text{-}\mathbf{coalg}$ over $\mathcal{M}^{\mathbf{2}}$, universal among morphisms of nwfs.*
- *(algebraically-free) There is a canonical isomorphism $\mathbb{R}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$.*

Outline

Algebraic model structures

Recall a model structure on a bicomplete category $\mathcal{M}$ is $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ s.t.:

- $\mathcal{W}$ satisfies the 2-of-3 property
- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are wfs

Definition (R.)

An algebraic model structure on $(\mathcal{M}, \mathcal{W})$ consists of a pair of nwfs $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ on $\mathcal{M}$ together with a morphism of nwfs

$$\xi \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$$

called the comparison map such that the underlying wfs of $(\mathbb{C}_t, \mathbb{F})$ and $(\mathbb{C}, \mathbb{F}_t)$ give the trivial cofibrations, fibrations, cofibrations, and trivial fibrations, respectively, of a model structure on $\mathcal{M}$, with weak equivalences $\mathcal{W}$.

NB: By the universal property of Garner's small object argument, any cofibrantly generated model structure can be algebraicized.

The comparison map

The comparison map $\xi\colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$

- consists of natural arrows ξ_f satisfying

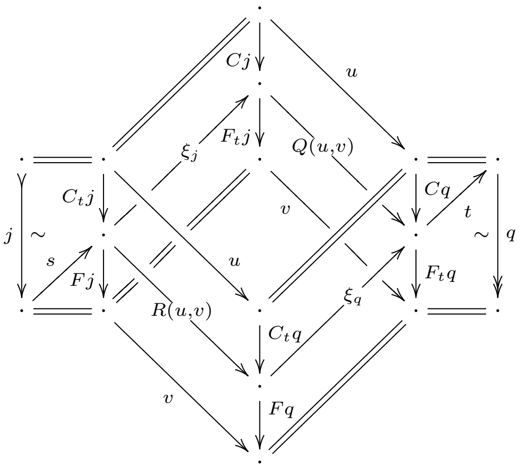
$$\begin{tikzcd} & \text{dom } f \arrow[dl, "C_t f"'] \arrow[dr, "C f"] & \\ Rf \arrow[rr, "\xi_f"] \arrow[dr, "F f"'] & & Qf \arrow[dl, "F_t f"] \\ & \text{cod } f & \end{tikzcd}$$

- induces functors

$$\xi_*\colon \mathbb{C}_t\textbf{-coalg} \to \mathbb{C}\textbf{-coalg} \quad \text{and} \quad \xi^*\colon \mathbb{F}_t\textbf{-alg} \to \mathbb{F}\textbf{-alg},$$

which provide an algebraic way to regard a trivial cofibration (trivial fibration) as a cofibration (fibration).

Both ways of lifting an algebraic trivial cofibration $(j, s) \in \mathbb{C}_t$-**coalg** against an algebraic trivial fibration $(q, t) \in \mathbb{F}_t$-**alg** are the same!



Algebraically fibrant-cofibrant objects

Any algebraic model structure induces a fibrant replacement monad $\mathbb{R}$ and a cofibrant replacement comonad $\mathbb{Q}$ on $\mathcal{M}$ together with $\chi : RQ \Rightarrow QR$.

$$\begin{tikzcd}
 & \emptyset \arrow[dl, tail] \arrow[dr, tail] & \\
QX \arrow[rr, dashed, "Q\eta_X"] \arrow[dd, tail, "\eta_{QX}"', "\sim"] \arrow[dr, two heads, "\epsilon_X", "\sim"'] & & QRX \arrow[dd, two heads, "\epsilon_{RX}", "\sim"'] \\
 & X \arrow[dr, tail, "\eta_X", "\sim"'] & \\
RQX \arrow[rr, dashed, "R\epsilon_X"'] \arrow[dr, two heads] & & RX \arrow[dl, two heads, "p"] \\
 & * &
\end{tikzcd}
\qquad
\begin{tikzcd}
QX \arrow[r, "Q\eta_X"] \arrow[d, tail, "\eta_{QX}"', "\sim"] & QRX \arrow[d, two heads, "\epsilon_{RX}", "\sim"'] \\
RQX \arrow[r, "R\epsilon_X"'] \arrow[ur, dashed, "\chi_X"] & RX
\end{tikzcd}$$

Theorem (R.)

The comonad Q lifts to $\mathbb{R}$-$\mathbf{alg}$ the category of algebraically fibrant objects and the monad R lifts to $\mathbb{Q}$-$\mathbf{coalg}$. Their algebras are isomorphic and give a category of algebraically bifibrant objects.

Other interesting features

Theorem (R.)

Lack's trivial model structure on the 2-category $\mathbf{Cat}^{\mathcal{A}}$ is a cofibrantly generated algebraic model structure, even though it is not cofibrantly generated in the classical sense.

Theorem (Garner, R., Shulman)

Given any algebraic model structure generated by $\mathcal{J} \hookrightarrow \mathcal{I}$ such that the cofibrations are monomorphisms, the components of the comparison map ξ are $\mathbb{C}$-coalgebras.

Outline

Passing algebraic model structures across an adjunction

Many ordinary model structures are constructed using a theorem due to Kan, which we extend to algebraic model structures:

Theorem (R.)

Let $\mathcal{M}$ have an algebraic model structure, generated by $\mathcal{J}$ and $\mathcal{I}$ and with weak equivalences $\mathcal{W}_{\mathcal{M}}$. Let $T\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ be an adjunction. Suppose $\mathcal{K}$ permits the small object argument and also that

($\star$) S maps arrows underlying the left class of the nwfs generated by $T\mathcal{J}$ into $\mathcal{W}_{\mathcal{M}}$.

Then $T\mathcal{J}$ and $T\mathcal{I}$ generate an algebraic model structure on $\mathcal{K}$ with $\mathcal{W}_{\mathcal{K}} = S^{-1}(\mathcal{W}_{\mathcal{M}})$.

NB: When a nwfs $(\mathbb{C}, \mathbb{F})$ is cofibrantly generated, all fibrations are algebraic: i.e., the class $\mathcal{F}$ underlying $\mathbb{F}\text{-}\mathbf{alg} \cong \mathcal{J}^{\boxslash}$ is retract closed.

About the adjunction

Consider an adjunction $T\colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}\colon S$ where $\mathcal{J}$ generates a nwfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$ and $T\mathcal{J}$ generates a nwfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$.

Theorem (R.)

Then S lifts to a functor
$$\begin{tikzcd} \mathbb{R}\textbf{-alg} \arrow[r, dashed, "\tilde{S}"] \arrow[d, "U"'] & \mathbb{F}\textbf{-alg} \arrow[d, "U"] \\ \mathcal{K}^{\mathbf{2}} \arrow[r, "S"] & \mathcal{M}^{\mathbf{2}} \end{tikzcd} \ldots$$

…and T lifts to a functor
$$\begin{tikzcd} \mathbb{C}\textbf{-coalg} \arrow[r, dashed, "\tilde{T}"] \arrow[d, "U"'] & \mathbb{L}\textbf{-coalg} \arrow[d, "U"] \\ \mathcal{M}^{\mathbf{2}} \arrow[r, "T"] & \mathcal{K}^{\mathbf{2}} \end{tikzcd} .$$

Adjunctions of nwfs

Definition

An adjunction of nwfs $(T, S, \gamma, \rho) : (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$ consists of a nwfs $(\mathbb{C}, \mathbb{F})$ on $\mathcal{M}$ and a nwfs $(\mathbb{L}, \mathbb{R})$ on $\mathcal{K}$, an adjunction $T \colon \mathcal{M} \underset{\leftarrow}{\overset{\rightarrow}{\perp}} \mathcal{K} \colon S$, and lifts $\tilde{T} : \mathbb{C}\textbf{-coalg} \to \mathbb{L}\textbf{-coalg}$ and $\tilde{S} : \mathbb{R}\textbf{-alg} \to \mathbb{F}\textbf{-alg}$ such that the natural transformations γ and ρ characterizing these lifts are mates.

$$\begin{tikzcd}
\mathcal{M}^{\mathbf{2}} \arrow[r, "Q"] \arrow[d, shift right=1ex, "T"'] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[d, shift right=1ex, "T"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{K}^{\mathbf{2}} \arrow[r, "E"'] \arrow[u, shift right=1ex, "S"'] & \mathcal{K} \arrow[u, shift right=1ex, "S"']
\end{tikzcd}$$

NB: An adjunction of nwfs over over $1 \dashv 1$ is exactly a morphism of nwfs.

Theorem (R.)

When $\mathcal{J}$ generates $(\mathbb{C}, \mathbb{F})$ and $T\mathcal{J}$ generates $(\mathbb{L}, \mathbb{R})$ with $T \dashv S$, there is a canonical adjunction of nwfs $(T, S, \gamma, \rho) \colon (\mathbb{C}, \mathbb{F}) \to (\mathbb{L}, \mathbb{R})$.

Algebraic Quillen adjunctions

Let $\mathcal{M}$ have an algebraic model structure $\xi^{\mathcal{M}} \colon (\mathbb{C}_t, \mathbb{F}) \to (\mathbb{C}, \mathbb{F}_t)$ and let $\mathcal{K}$ have an algebraic model structure $\xi^{\mathcal{K}} \colon (\mathbb{L}_t, \mathbb{R}) \to (\mathbb{L}, \mathbb{R}_t)$.

Definition (R.)

An adjunction $T \colon \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K} \colon S$ is an algebraic Quillen adjunction if there exist natural transformations γ_t, γ, ρ_t, and ρ determining five adjunctions of nwfs

$$\begin{tikzcd}
(\mathbb{C}_t, \mathbb{F}) \arrow[r, "{(T,S,\gamma_t,\rho)}"] \arrow[d, "{(1,1,\xi^{\mathcal{M}},\xi^{\mathcal{M}})}"'] \arrow[dr, "{(T,S,\gamma\cdot T\xi^{\mathcal{M}},S\xi^{\mathcal{K}}\cdot\rho)}" description] & (\mathbb{L}_t, \mathbb{R}) \arrow[d, "{(1,1,\xi^{\mathcal{K}},\xi^{\mathcal{K}})}"] \\
(\mathbb{C}, \mathbb{F}_t) \arrow[r, "{(T,S,\gamma,\rho_t)}"'] & (\mathbb{L}, \mathbb{R}_t)
\end{tikzcd}$$

such that both triangles commute.

Naturality in an algebraic Quillen adjunction

The naturality condition says that the lifts commute:

$$\begin{tikzcd}
\mathbb{R}_t\text{-}\mathbf{alg} \arrow[rr, "\tilde{S}_t"] \arrow[dr, "(\xi^{\mathcal{K}})^*"] \arrow[dd] & & \mathbb{F}_t\text{-}\mathbf{alg} \arrow[dr, "(\xi^{\mathcal{M}})^*"] \arrow[dd] & \\
& \mathbb{R}\text{-}\mathbf{alg} \arrow[rr, "\tilde{S}"] \arrow[dl] & & \mathbb{F}\text{-}\mathbf{alg} \arrow[dl] \\
\mathcal{K}^{\mathbf{2}} \arrow[rr, "S"'] & & \mathcal{M}^{\mathbf{2}} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{C}_t\text{-}\mathbf{coalg} \arrow[rr, "\tilde{T}_t"] \arrow[dr, "(\xi^{\mathcal{M}})_*"] \arrow[dd] & & \mathbb{L}_t\text{-}\mathbf{coalg} \arrow[dr, "(\xi^{\mathcal{K}})_*"] \arrow[dd] & \\
& \mathbb{C}\text{-}\mathbf{coalg} \arrow[rr, "\tilde{T}"] \arrow[dl] & & \mathbb{L}\text{-}\mathbf{coalg} \arrow[dl] \\
\mathcal{M}^{\mathbf{2}} \arrow[rr, "T"'] & & \mathcal{K}^{\mathbf{2}} &
\end{tikzcd}$$

Theorem (R.)

For any algebraic model structure on $\mathcal{K}$ constructed by passing a cofibrantly generated algebraic model structure on $\mathcal{M}$ across an adjunction, the adjunction is canonically an algebraic Quillen adjunction.

Toward the proof of naturality

To prove the preceding theorem, we need this result.

Corollary (R.)

The two canonical ways of assigning $\mathbb{L}_t$-coalgebra structures to the generators $T\mathcal{J}$ are the same, i.e.,

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\lambda^{\mathcal{M}}"] \arrow[d] & \mathbb{C}_t\textbf{-coalg} \arrow[d, "\tilde{T}"] \\ T\mathcal{J} \arrow[r, "\lambda^{\mathcal{K}}"] & \mathbb{L}_t\textbf{-coalg} \end{tikzcd}$$

commutes.

Goal: Understand change of base along left adjoints of specified adjunctions in Garner's small object argument.

Garner's small object argument

Given a category $\mathcal{M}$ that permits the small object argument, Garner's construction produces a reflection of any small category $\mathcal{J}$ over $\mathcal{M}^{\mathbf{2}}$ along the so-called "semantics" functor

$$\begin{array}{ccc} \mathbf{NWFS}(\mathcal{M}) & \xrightarrow{\mathcal{G}} & \mathbf{CAT}/\mathcal{M}^{\mathbf{2}} \\ (\mathbb{C},\mathbb{F}) & \longmapsto & \mathbb{C}\textbf{-coalg} \end{array}$$

The unit $\lambda\colon \mathcal{J} \to \mathbb{C}\textbf{-coalg}$ is universal among morphisms of nwfs

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\lambda"] \arrow[dr] & \mathbb{C}\textbf{-coalg} \arrow[d, dashed, "\xi_*"] \\ & \mathbb{C}'\textbf{-coalg} \end{tikzcd}$$

i.e., it is initial in the slice category $\mathcal{J}/\mathcal{G}$.

Change of base

Garner's small object argument satisfies a stronger universal property.

Two categories cofibered over $\mathbf{CAT}_{\mathsf{ladj}}$

- Let $\mathbf{NWFS}_{\mathsf{ladj}}$ be the category of nwfs over any base whose morphisms are adjunctions of nwfs.
- Let $\mathbf{CAT}/(-)^{\mathbf{2}}_{\mathsf{ladj}}$ be the category of categories sliced over arrow categories, with morphisms the left adjoints of specified adjunctions between the base categories with specified lifts.

Theorem (R.)

Garner's construction produces a reflection along

$$\mathbf{NWFS}_{\mathsf{ladj}} \xrightarrow{\mathcal{G}^{\mathsf{ladj}}} \mathbf{CAT}/(-)^{\mathbf{2}}_{\mathsf{ladj}}$$

i.e., the units $\lambda \colon \mathcal{J} \to \mathbb{C}\textbf{-coalg}$ are universal among adjunctions of nwfs.

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Further details

Further details can be found in the preprint "Algebraic model structures" arXiv:0910.2733v2 available at `www.math.uchicago.edu/~eriehl`.