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Algebraic Modelling and Performance Evaluation of Acyclic Fork-Join Queueing Networks

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Abstract: Simple lower and upper bounds on service cycle times in stochastic acyclic fork-join queueing networks are derived using a (max, +)-algebra based representation of network dynamics. The behaviour of the bounds under various assumptions concerning the service times in the networks is discussed, and related numerical examples are presented.

Keywords and phrases: (max,+)-algebra, dynamic state equation, acyclic fork-join queueing networks, stochastic dynamic systems, service cycle time

1.1 Introduction

Fork-join networks, as introduced in Baccelli and Makowski (1989), Baccelli et al. (1989), present a class of queueing systems which allow customers (jobs, tasks) to be split into several parts, and to be merged into one when they circulate through the system. The fork-join formalism proves to be useful in the description of dynamical processes in a variety of actual complex systems, including production processes in manufacturing, transmission of messages in communication networks, and parallel data processing in multi-processor computer systems. As a natural illustration of the fork and join operations, one can consider respectively splitting a message into packets in a communication network, each intended for transmitting via separate ways, and merging packets at a destination node of the network to restore the message. Further examples can be found in Baccelli and Makowski (1989).

The usual way to represent the dynamics of fork-join queueing networks relies on the implementation of recursive state equations of the Lindley type [Baccelli and Makowski (1989)]. Since the recursive equations associated with

the fork-join networks can be expressed only in terms of the operations of maximum and addition, there is a possibility to represent the dynamics of the networks in terms of the (max,+)-algebra which is actually an algebraic system just supplied with the same two operations [Cuninghame-Green (1979), Baccelli et al. (1992), Maslov and Kolokoltsov (1994)]. In fact, (max,+)-algebra models offer a more compact and unified way of describing network dynamics, and, moreover, lead to equations closely analogous to those in the conventional linear system theory [Baccelli et al. (1992), Krivulin (1994, 1995, 1996a, 1996b)]. In that case, the (max,+)-algebra approach gives one the chance to exploit results and numerical procedures available in the algebraic system theory and computational linear algebra.

One of the problems of interest in the analysis of stochastic queueing networks is to evaluate the service cycle time of a network. Both the cycle time and its inverse which can be regarded as a throughput present performance measures commonly used to describe efficiency of the network operation.

It is often rather difficult to evaluate the cycle time exactly, even though the network under study is quite simple. To get information about the performance measure in this case, one can apply computer simulation to produce reasonable estimates. Another approach is to derive bounds on the cycle time. Specifically, a technique which allows one to establish bounds based on results of the theory of large deviations as well as the Perron-Frobenius spectral theory has been introduced in Baccelli and Konstantopoulos (1991).

In this paper we propose an approach to get bounds on the service cycle time, which exploits the (max,+)-algebra representation of acyclic fork-join network dynamics derived in Krivulin (1996a, 1996b). This approach is essentially based on pure algebraic manipulations combined with application of bounds on extreme values, obtained in Gumbel (1954), Hartly and David (1954).

The rest of the paper is organized as follows. Section 1.2 presents basic (max,+)-algebra definitions and related results which underlie the development of network models and their analysis in the subsequent sections. In Section 1.3, further algebraic results are included which provide a basis for derivation of bounds on the service cycle time.

A (max,+)-algebra representation of the fork-join network dynamics and related examples are given in Section 1.4. Furthermore, Section 1.5 offers some monotonicity property for the networks, which is exploited in Section 1.6 to get algebraic bounds on the service cycle completion time. Stochastic extension of the network model is introduced in Section 1.7. This section concludes with a result which provides simple bounds on the network cycle time. Finally, Section 1.8 presents examples of calculating bounds and related discussion.

1.2   Preliminary Algebraic Definitions and Results

The $(\max, +)$-algebra presents an idempotent commutative semiring (idempotent semifield) which is defined as $\mathbb{R}_{\max} = \langle \underline{\mathbb{R}}, \oplus, \otimes \rangle$ with $\underline{\mathbb{R}} = \mathbb{R} \cup \{\varepsilon\}$, $\varepsilon = -\infty$, and binary operations $\oplus$ and $\otimes$ defined as

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y, \quad \text{for all} \quad x, y \in \underline{\mathbb{R}}.$$

There are the null and identity elements in the algebra, namely ε and 0, to satisfy the conditions $x \oplus \varepsilon = \varepsilon \oplus x = x$, and $x \otimes 0 = 0 \otimes x = x$, for any $x \in \mathbb{R}$. The null element ε and the operation $\otimes$ are related by the usual absorption rule involving $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

As it is easy to see, the operations $\oplus$ and $\otimes$ retain most of the properties of the ordinary addition and multiplication, including associativity, commutativity, and distributivity of multiplication over addition. However, the operation $\oplus$ is idempotent; that is, for any $x \in \mathbb{R}$, one has $x \oplus x = x$.

Non-negative integer power of any $x \in \mathbb{R}$ can be defined as $x^0 = 0$, and $x^q = x \otimes x^{q-1} = x^{q-1} \otimes x$ for $q \geq 1$. Clearly, the $(\max, +)$-algebra power x^q corresponds to qx in ordinary notations. We will use the power notations only in the $(\max, +)$-algebra sense.

The $(\max, +)$-algebra of matrices is readily introduced in the regular way. Specifically, for any $(n \times n)$-matrices $X = (x_{ij})$ and $Y = (y_{ij})$, the entries of $U = X \oplus Y$ and $V = X \otimes Y$ are calculated as

$$u_{ij} = x_{ij} \oplus y_{ij}, \quad \text{and} \quad v_{ij} = \bigoplus_{k=1}^{n} x_{ik} \otimes y_{kj}.$$

As the null and identity elements, the matrices

$$\mathcal{E} = \begin{pmatrix} \varepsilon & \cdots & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & \cdots & \varepsilon \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & & \varepsilon \\ & \ddots & \\ \varepsilon & & 0 \end{pmatrix}$$

are respectively taken in the algebra.

The matrix operations $\oplus$ and $\otimes$ possess monotonicity properties; that is, the matrix inequalities $X \leq U$ and $Y \leq V$ result in

$$X \oplus Y \leq U \oplus V, \qquad X \otimes Y \leq U \otimes V$$

for any matrices of appropriate size.

Let $X \neq \mathcal{E}$ be a square matrix. In the same way as in the conventional algebra, one can define $X^0 = I$, and $X^q = X \otimes X^{q-1} = X^{q-1} \otimes X$ for any integer $q \geq 1$. However, idempotency leads, in particular, to the matrix identity

$$(X \oplus Y)^q = X^q \oplus X^{q-1} \otimes Y \oplus \cdots \oplus Y^q.$$

As direct consequences of the above identity, one has

$$(X \oplus Y)^q \geq X^p \otimes Y^{q-p}, \qquad (I \oplus X)^q \geq (I \oplus X)^p \geq X^p,$$

for all $p = 0, 1, \ldots, q$.

For any matrix X, one can define the matrix function

$$\|X\| = \bigoplus_{i,j} x_{ij} = \max_{i,j} x_{ij}.$$

Note that the function $\|\cdot\|$ possesses properties similar to those of the ordinary matrix norm. Specifically, for any matrix X, it holds $\|X\| \geq \varepsilon$, and $\|X\| = \varepsilon$ if and only if $X = \mathcal{E}$. Furthermore, we have $\|c \otimes X\| = c \otimes \|X\|$ for any $c \in \underline{\mathbb{R}}$, as well as additive and multiplicative properties involving

$$\|X \oplus Y\| = \|X\| \oplus \|Y\|, \qquad \|X \otimes Y\| \leq \|X\| \otimes \|Y\|$$

for any two conforming matrices X and Y. For any $c > 0$, we also have $\|cX\| = c\|X\|$. The matrix function $\|\cdot\|$ will be referred to as $(\max, +)$-algebra norm, or simply as norm.

Consider an $(n \times n)$-matrix X with its entries $x_{ij} \in \mathbb{R}$. It can be treated as an adjacency matrix of an oriented graph with n nodes, provided each entry $x_{ij} \neq \varepsilon$ implies the existence of the arc (i, j) in the graph, while $x_{ij} = \varepsilon$ does the lack of the arc.

It is easy to verify that for any integer $q \geq 1$, the matrix X^q has its entry $x_{ij}^{(q)} \neq \varepsilon$ if and only if there exists a path from node i to node j in the graph, which consists of q arcs. Furthermore, if the graph associated with the matrix X is acyclic, we have $X^q = \mathcal{E}$ for all $q > p$, where p is the length of the longest path in the graph. Otherwise, provided that the graph is not acyclic, one can construct a path of any length, lying along circuits, and then it holds that $X^q \neq \mathcal{E}$ for all $q \geq 0$.

Consider the implicit equation in an unknown vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$,

$$\boldsymbol{x} = U \otimes \boldsymbol{x} \oplus \boldsymbol{v}, \qquad (1.1)$$

where $U = (u_{ij})$ and $\boldsymbol{v} = (v_1, \ldots, v_n)^T$ are respectively given $(n \times n)$-matrix and n-vector. Suppose that the entries of the matrix U and the vector $\boldsymbol{v}$ are either positive or equal to ε. It is easy to verify (see, e.g. Cuninghame-Green (1979), Cohen et al. (1985)) that equation (1.1) has the unique bounded solution if and only if the graph associated with U is acyclic. Provided that the solution exists, it is given by

$$\boldsymbol{x} = (I \oplus U)^p \otimes \boldsymbol{v}, \qquad (1.2)$$

where p is the length of the longest path in the graph.

1.3 Further Algebraic Results

Consider a square matrix X, and denote the adjacency $(\varepsilon\text{–}0)$-matrix of the graph associated with X by G. The matrix G is normally referred to as support of X.

Proposition 1.3.1 *For any matrix X, it holds*

$$X \leq \|X\| \otimes G,$$

where G is the support of X.

Proposition 1.3.2 *Suppose that matrices $X_1, \ldots, X_k$ have a common associated acyclic graph, p is the length of the longest path in the graph, and*

$$X = X_1^{m_1} \otimes \cdots \otimes X_k^{m_k},$$

where $m_1, \ldots, m_k$ are nonnegative integers.
If it holds that $m_1 + \cdots + m_k > p$, then $X = \mathcal{E}$.

Proof: It follows from Proposition 1.3.1 that

$$X = X_1^{m_1} \otimes \cdots \otimes X_k^{m_k} \leq \|X_1\|^{m_1} \otimes \cdots \otimes \|X_k\|^{m_k} \otimes G^{m_1 + \cdots + m_k},$$

where G is the common support of the matrices $X_1, \ldots, X_k$.

Since the graph is acyclic, it holds that $G^q = \mathcal{E}$ for all $q > p$. Therefore, with $q = m_1 + \cdots + m_k > p$, we arrive at the inequality $X \leq \mathcal{E}$ which leads us to the desired result. ∎

Lemma 1.3.1 *Suppose that matrices $X_1, \ldots, X_k$ have a common associated acyclic graph, and p is the length of the longest path in the graph.*
If $\|X_i\| \geq 0$ for all $i = 1, \ldots, k$, then it holds

$$\left\| \bigotimes_{i=1}^{k} (I \oplus X_i)^{m_i} \right\| \leq \left(\bigoplus_{i=1}^{k} \|X_i\| \right)^{p}$$

for any nonnegative integers $m_1, \ldots, m_k$.

Proof: Consider the matrix

$$
\begin{aligned}
X &= \bigotimes_{i=1}^{k} (I \oplus X_i)^{m_i} = \left(\bigoplus_{i_1=0}^{m_1} X_1^{i_1} \right) \otimes \cdots \otimes \left(\bigoplus_{i_k=0}^{m_k} X_k^{i_k} \right) \\
&= \bigoplus_{i_1=0}^{m_1} \cdots \bigoplus_{i_k=0}^{m_k} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k} \leq \bigoplus_{0 \leq i_1 + \cdots + i_k \leq m} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k},
\end{aligned}
$$

where $m = m_1 + \cdots + m_k$. From Proposition 1.3.2 we may replace m with p in the last term to get

$$X \leq \bigoplus_{0 \leq i_1 + \cdots + i_k \leq p} X_1^{i_1} \otimes \cdots \otimes X_k^{i_k}.$$

Proceeding to the norm, with its additive and multiplicative properties, we arrive at the inequality

$$\|X\| \leq \bigoplus_{0 \leq i_1 + \cdots + i_k \leq p} \|X_1\|^{i_1} \otimes \cdots \otimes \|X_k\|^{i_k}.$$

Since for all $i = 1, \ldots, k$, it holds $0 \leq \|X_i\| \leq \|X_1\| \oplus \cdots \oplus \|X_k\|$, we finally have

$$\|X\| \leq \bigoplus_{i=0}^{p} (\|X_1\| \oplus \cdots \oplus \|X_k\|)^p = \left(\bigoplus_{i=0}^{k} \|X_i\| \right)^p .$$

$\blacksquare$

1.4 An Algebraic Model of Queueing Networks

We consider a network with n single-server nodes and customers of a single class. The topology of the network is described by an oriented acyclic graph $\mathcal{G} = (\mathbf{N}, \mathbf{A})$, where the set $\mathbf{N} = \{1, \ldots, n\}$ represents the nodes, and $\mathbf{A} = \{(i, j)\} \subset \mathbf{N} \times \mathbf{N}$ represents the arcs determining the transition routes of customers.

For every node $i \in \mathbf{N}$, we denote the sets of its immediate predecessors and successors respectively as $\mathbf{P}(i) = \{j \,|\, (j, i) \in \mathbf{A}\}$ and $\mathbf{S}(i) = \{j \,|\, (i, j) \in \mathbf{A}\}$. In specific cases, there may be one of the conditions $\mathbf{P}(i) = \emptyset$ and $\mathbf{S}(i) = \emptyset$ encountered. Each node i with $\mathbf{P}(i) = \emptyset$ is assumed to represent an infinite external arrival stream of customers; provided that $\mathbf{S}(i) = \emptyset$, it is considered as an output node intended to release customers from the network.

Each node $i \in \mathbf{N}$ includes a server and its buffer with infinite capacity, which together present a single-server queue operating under the first-come, first-served (FCFS) discipline. At the initial time, the server at each node i is assumed to be free of customers, whereas in its buffer, there may be r_i, $0 \leq r_i \leq \infty$, customers waiting for service. The value $r_i = \infty$ is set for every node i with $\mathbf{P}(i) = \emptyset$, which represents an external arrival stream of customers.

For the queue at node i, we denote the kth arrival and departure epochs respectively as $u_i(k)$ and $x_i(k)$. Furthermore, the service time of the kth customer at server i is indicated by τ_{ik}. We assume that $\tau_{ik} \geq 0$ are given for all $i = 1, \ldots, n$, and $k = 1, 2, \ldots$, while $u_i(k)$ and $x_i(k)$ are considered as unknown state variables. With the condition that the network starts operating at time zero, it is convenient to set $x_i(0) \equiv 0$, and $x_i(k) \equiv \varepsilon$ for all $k < 0$, $i = 1, \ldots, n$.

It is easy to set up an equation which relates the system state variables. In fact, the dynamics of any single-server node i with an infinite buffer, operating on the FCFS basis, is described as

$$x_i(k) = \tau_{ik} \otimes u_i(k) \oplus \tau_{ik} \otimes x_i(k-1). \tag{1.3}$$

With the vector-matrix notations

$$\boldsymbol{u}(k) = \begin{pmatrix} u_1(k) \\ \vdots \\ u_n(k) \end{pmatrix}, \quad \boldsymbol{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix}, \quad \mathcal{T}_k = \begin{pmatrix} \tau_{1k} & & \varepsilon \\ & \ddots & \\ \varepsilon & & \tau_{nk} \end{pmatrix},$$

we may rewrite equation (1.3) in a vector form, as

$$\boldsymbol{x}(k) = \mathcal{T}_k \otimes \boldsymbol{u}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{x}(k-1). \tag{1.4}$$

1.4.1 Fork-Join Queueing Networks

In fork-join networks, in addition to the usual service procedure, special join and fork operations are performed in its nodes, respectively before and after service. The join operation is actually thought to cause each customer which comes into node i, not to enter the buffer at the server but to wait until at least one customer from every node $j \in \mathbf{P}(i)$ arrives. As soon as these customers arrive, they, taken one from each preceding node, are united into one customer which then enters the buffer to become a new member of the queue.

The fork operation at node i is initiated every time the service of a customer is completed; it consists in giving rise to several new customers instead of the original one. As many new customers appear in node i as there are succeeding nodes included in the set $\mathbf{S}(i)$. These customers simultaneously depart the node, each being passed to separate node $j \in \mathbf{S}(i)$. We assume that the execution of fork-join operations when appropriate customers are available, as well as the transition of customers within and between nodes require no time.

As it immediately follows from the above description of the fork-join operations, the kth arrival epoch into the queue at node i is represented as

$$u_i(k) = \begin{cases} \displaystyle\bigoplus_{j \in \mathbf{P}(i)} x_j(k - r_i), & \text{if } \mathbf{P}(i) \neq \emptyset, \\ \varepsilon, & \text{if } \mathbf{P}(i) = \emptyset. \end{cases} \tag{1.5}$$

In order to get this equation in a vector form, we first define the number $M = \max\{r_i \,|\, r_i < \infty,\ i = 1, \ldots, n\}$. Now we may rewrite (1.5) as

$$u_i(k) = \bigoplus_{m=0}^{M} \bigoplus_{j=1}^{n} g_{ji}^m \otimes x_j(k - m),$$

where the numbers g_{ij}^m are determined by the condition

$$g_{ij}^m = \begin{cases} 0, & \text{if } i \in \mathbf{P}(j) \text{ and } m = r_j, \\ \varepsilon, & \text{otherwise.} \end{cases} \tag{1.6}$$

Let us introduce the matrices $G_m = \left(g_{ij}^m\right)$ for each $m = 0, 1, \ldots, M$. In fact, G_m presents an adjacency matrix of the partial graph $\mathcal{G}_m = (\mathbf{N}, \mathbf{A}_m)$ with $\mathbf{A}_m = \{(i, j) \mid (i, j) \in \mathbf{A}; \ r_j = m\}$. Since the graph of the entire network is acyclic, all its partial graphs $\mathcal{G}_m, \ m = 0, 1, \ldots, M$, possess the same property.

With these matrices, equation (1.5) may be written in the vector form

$$\boldsymbol{u}(k) = \bigoplus_{m=0}^{M} G_m^T \otimes \boldsymbol{x}(k - m), \tag{1.7}$$

where G_m^T denotes the transpose of the matrix G_m.

By combining equations (1.4) and (1.7), we arrive at the equation

$$\boldsymbol{x}(k) = \mathcal{T}_k \otimes G_0^T \otimes \boldsymbol{x}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{x}(k - 1)$$
$$\oplus \mathcal{T}_k \otimes \bigoplus_{m=1}^{M} G_m^T \otimes \boldsymbol{x}(k - m). \tag{1.8}$$

Clearly, it is actually an implicit equation in $\boldsymbol{x}(k)$, which has the form of (1.1), with $U = \mathcal{T}_k \otimes G_0^T$. Taking into account that the matrix $\mathcal{T}_k$ is diagonal, one can prove the following statement (see also Krivulin (1996a, 1996b)).

Theorem 1.4.1 *Suppose that in the fork-join network model, the graph $\mathcal{G}_0$ associated with the matrix G_0 is acyclic. Then equation (1.8) can be solved to produce the explicit dynamic state equation*

$$\boldsymbol{x}(k) = \bigoplus_{m=1}^{M} A_m(k) \otimes \boldsymbol{x}(k - m), \tag{1.9}$$

with the state transition matrices

$$A_1(k) = (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (I \oplus G_1^T), \tag{1.10}$$
$$A_m(k) = (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_m^T, \quad m = 2, \ldots, M, \tag{1.11}$$

where p is the length of the longest path in $\mathcal{G}_0$.

As one can see, with $M = 1$ and $A_1(k) = A(k)$, we have the dynamic equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k - 1). \tag{1.12}$$

Note that this equation differs in appearance from those used in the linear system theory, which normally have the form of $\boldsymbol{x}(k) = A(k - 1) \otimes \boldsymbol{x}(k - 1)$.
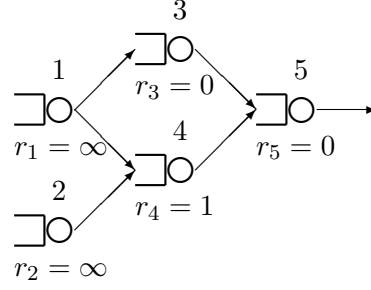
Figure 1.1: An acyclic fork-join network.

1.4.2 Examples of Network Models

An example of an acyclic fork-join network with $n = 5$ is shown in Fig. 1.1.

Since for the network $M = 1$, we have from (1.6)

$$
G_0 = \begin{pmatrix} \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \qquad G_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.
$$

Taking into account that for the graph $\mathcal{G}_0$, the length of its longest path $p = 2$, we arrive at equation (1.12) with the state transition matrix calculated from (1.10) as

$$
\begin{aligned}
A(k) &= (I \oplus \mathcal{T}_k \otimes G_0^T)^2 \otimes \mathcal{T}_k \otimes (I \oplus G_1^T) \\
&= \begin{pmatrix} \tau_{1k} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_{2k} & \varepsilon & \varepsilon & \varepsilon \\ \tau_{1k} \otimes \tau_{3k} & \varepsilon & \tau_{3k} & \varepsilon & \varepsilon \\ \tau_{4k} & \tau_{4k} & \varepsilon & \tau_{4k} & \varepsilon \\ (\tau_{1k} \otimes \tau_{3k} \oplus \tau_{4k}) \otimes \tau_{5k} & \tau_{4k} \otimes \tau_{5k} & \tau_{3k} \otimes \tau_{5k} & \tau_{4k} \otimes \tau_{5k} & \tau_{5k} \end{pmatrix}.
\end{aligned}
$$

Note that open tandem queueing systems (see Fig. 1.2) can be considered as trivial networks in which no fork and join operations are actually performed.
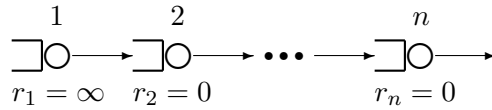

Figure 1.2: Open tandem queues.

For the system in Fig. 1.2, we have $M = 0$, and $p = n - 1$. Its related state transition matrix $A(k)$ has the entries [Krivulin (1994, 1995)]

$$
a_{ij}(k) = \begin{cases} \tau_{jk} \otimes \tau_{j+1k} \otimes \cdots \otimes \tau_{ik}, & \text{if } i \geq j, \\ \varepsilon, & \text{otherwise.} \end{cases}
$$

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1.5 A Monotonicity Property

In this section, a property of monotonicity is established which shows how the system state vector $\boldsymbol{x}(k)$ may vary with the initial numbers of customers r_i. It is actually proven that the entries of $\boldsymbol{x}(k)$ for all $k = 1, 2, \ldots$, do not decrease when the numbers r_i with $0 < r_i < \infty$, $i = 1, \ldots, n$, are reduced to zero.

As it is easy to see, the change in the initial numbers of customers results only in modifications to partial graphs $\mathcal{G}_m$ and so to their adjacency matrices G_m. Specifically, reducing these numbers to zero leads us to new matrices $\widetilde{G}_0 = G_0 \oplus G_1 \cdots \oplus G_M$, and $\widetilde{G}_m = \mathcal{E}$ for all $m = 1, \ldots, M$.

We start with a lemma which shows that replacing the numbers $r_i = 1$ with $r_i = 0$ does not decrease the entries of the matrix $A_1(k)$ defined by (1.10).

Lemma 1.5.1 *For all $k = 1, 2, \ldots$, it holds*

$$A_1(k) \leq \widetilde{A}(k)$$

with $\widetilde{A}(k) = (I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^q \otimes \mathcal{T}_k$, where $\widetilde{G}_0 = G_0 \oplus G_1$, and q is the length of the longest path in the graph associated with the matrix $\widetilde{G}_0$.

Proof: Consider the matrix $A_1(k)$ and represent it in the form

$$A_1(k) = ((I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k) \oplus ((I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T),$$

where p is the length of the longest path in the graph associated with G_0.

As one can see, to prove the lemma, it will suffice to verify both inequalities

$$
\begin{align}
\widetilde{A}(k) &\geq (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k, \tag{1.13} \\
\widetilde{A}(k) &\geq (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T. \tag{1.14}
\end{align}
$$

Let us write the obvious representation

$$(I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^q = \bigoplus_{i=0}^{q} (I \oplus \mathcal{T}_k \otimes G_0^T)^i \otimes (\mathcal{T}_k \otimes G_1^T)^{q-i}.$$

Since $q \geq p$, we get from the representation

$$(I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^q \geq (I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^p = ((I \oplus \mathcal{T}_k \otimes G_0^T) \oplus \mathcal{T}_k \otimes G_1^T)^p \geq (I \oplus \mathcal{T}_k \otimes G_0^T)^p.$$

It remains to multiply both sides of the above inequality by $\mathcal{T}_k$ on the right so as to arrive at (1.13).

To verify (1.14), let us first assume that $q > p$. In this case, we obtain

$$
\begin{align}
(I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^q &\geq (I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^{p+1} \\
&= ((I \oplus \mathcal{T}_k \otimes G_0^T) \oplus \mathcal{T}_k \otimes G_1^T)^{p+1} \geq (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T.
\end{align}
$$

Suppose now that $q = p$. Then it is necessary that $G_1 \otimes G_0^p = \mathcal{E}$. If this were not the case, there would be a path in the graph associated with the matrix $\widetilde{G}_0 = G_0 \oplus G_1$, which has its length greater than p, and we would have $q > p$.

Clearly, the condition $G_1 \otimes G_0^p = \mathcal{E}$ results in $(\mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T = \mathcal{E}$, and thus we get

$$(I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^q = ((I \oplus \mathcal{T}_k \otimes G_0^T) \oplus \mathcal{T}_k \otimes G_1^T)^p$$
$$\geq (I \oplus \mathcal{T}_k \otimes G_0^T)^{p-1} \otimes \mathcal{T}_k \otimes G_1^T = (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes G_1^T.$$

Since it holds $(I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^p \otimes \mathcal{T}_k \geq (I \oplus \mathcal{T}_k \otimes \widetilde{G}_0^T)^p$, one can conclude that inequality (1.14) is also valid. ∎

Theorem 1.5.1 *In the acyclic fork-join queueing network model (1.9–1.11), reducing the initial numbers of customers from any finite values to zero does not decrease the entries of the system state vector $\boldsymbol{x}(k)$ for all $k = 1, 2, \ldots$.*

Proof: Let $\boldsymbol{x}(k)$ be determined by (1.9–1.11). Suppose that the vector $\widetilde{\boldsymbol{x}}(k)$ satisfies the dynamic equation

$$\widetilde{\boldsymbol{x}}(k) = \widetilde{A}(k) \otimes \widetilde{\boldsymbol{x}}(k - 1)$$

with

$$\widetilde{A}(k) = \left(I \oplus \mathcal{T}_k \otimes \bigoplus_{m=0}^{M} G_m^T \right)^q \otimes \mathcal{T}_k = (I \oplus \mathcal{T}_k \otimes G^T)^q \otimes \mathcal{T}_k,$$

where q is the length of the longest path in the graph associated with the matrix $G = G_0 \oplus G_1 \oplus \cdots \oplus G_m$.

Now we have to show that for all $k = 1, 2, \ldots$, it holds

$$\boldsymbol{x}(k) \leq \widetilde{\boldsymbol{x}}(k).$$

Since $\boldsymbol{x}(k_1) \leq \boldsymbol{x}(k_2)$ for any $k_1 < k_2$, we have from (1.9)

$$\boldsymbol{x}(k) = \bigoplus_{m=1}^{M} A_m(k) \otimes \boldsymbol{x}(k - m) \leq \left(\bigoplus_{m=1}^{M} A_m(k) \right) \otimes \boldsymbol{x}(k - 1).$$

Consider the matrix

$$\widetilde{A}_1(k) = \bigoplus_{m=1}^{M} A_m(k) = (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes \left(I \oplus \bigoplus_{m=1}^{M} G_m^T \right).$$

By applying Lemma 1.5.1, we have

$$\widetilde{A}_1(k) \leq \left(I \oplus \mathcal{T}_k \otimes G_0^T \oplus \bigoplus_{m=1}^{M} G_m^T \right)^q \otimes \mathcal{T}_k = \widetilde{A}(k).$$

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Starting with the condition $\boldsymbol{x}(0) = \widetilde{\boldsymbol{x}}(0)$, we successively verify that the relations

$$\boldsymbol{x}(k) \leq \widetilde{A}_1(k) \otimes \boldsymbol{x}(k-1) \leq \widetilde{A}(k) \otimes \boldsymbol{x}(k-1) \leq \widetilde{A}(k) \otimes \widetilde{\boldsymbol{x}}(k-1) = \widetilde{\boldsymbol{x}}(k)$$

are valid for each $k = 1, 2, \ldots$. ∎

1.6 Bounds on the Service Cycle Completion Time

We consider the evolution of the system as a sequence of service cycles: the 1st cycle starts at the initial time, and it is terminated as soon as all the servers in the network complete their 1st service, the 2nd cycle is terminated as soon as the servers complete their 2nd service, and so on. Clearly, the completion time of the kth cycle can be represented as

$$\max_i x_i(k) = \|\boldsymbol{x}(k)\|$$

with $\boldsymbol{x}(0) = \boldsymbol{0}$.

The next lemma provides simple algebraic lower and upper bounds for the kth cycle completion time.

Lemma 1.6.1 *With the condition that $\boldsymbol{x}(0) = \boldsymbol{0}$, for all $k = 1, 2, \ldots$, it holds*

$$\left\| \sum_{i=1}^{k} \mathcal{T}_i \right\| \leq \|\boldsymbol{x}(k)\| \leq \sum_{i=1}^{k} \|\mathcal{T}_i\| + p \left(\bigoplus_{i=1}^{k} \|\mathcal{T}_i\| \right).$$

Proof: To prove the left inequality first note that

$$
\begin{aligned}
A_1(k) &= (I \oplus \mathcal{T}_k \otimes G_0^T)^p \otimes \mathcal{T}_k \otimes (I \oplus G_1^T) \\
&= \left(I \oplus \mathcal{T}_k \otimes G_0^T \oplus \cdots \oplus (\mathcal{T}_k \otimes G_0^T)^p \right) \otimes \mathcal{T}_k \otimes (I \oplus G_1^T) \geq \mathcal{T}_k.
\end{aligned}
$$

With this condition, we have from (1.9)

$$\boldsymbol{x}(k) = \bigoplus_{m=1}^{M} A_m(k) \otimes \boldsymbol{x}(k-m) \geq A_1(k) \otimes \boldsymbol{x}(k-1) \geq \mathcal{T}_k \otimes \boldsymbol{x}(k-1).$$

Now we can write

$$\boldsymbol{x}(k) \geq \mathcal{T}_k \otimes \boldsymbol{x}(k-1) \geq \mathcal{T}_k \otimes \mathcal{T}_{k-1} \otimes \boldsymbol{x}(k-2) \geq \cdots \geq \mathcal{T}_k \otimes \cdots \otimes \mathcal{T}_1 \otimes \boldsymbol{x}(0),$$

where $\boldsymbol{x}(0) = \boldsymbol{0}$. Taking the norm, and considering that $\mathcal{T}_i$, $i = 1, \ldots, k$, present diagonal matrices, we get

$$\|\boldsymbol{x}(k)\| \geq \|\mathcal{T}_k \otimes \cdots \otimes \mathcal{T}_1\| = \|\mathcal{T}_1 + \cdots + \mathcal{T}_k\|.$$

To obtain an upper bound, let us replace the general system (1.9–1.11) with that governed by the equation

$$\widetilde{\boldsymbol{x}}(k) = \widetilde{A}(k) \otimes \widetilde{\boldsymbol{x}}(k-1) \tag{1.15}$$

with $\widetilde{A}(k) = (I \oplus \mathcal{T}_k \otimes \widetilde{G}^T)^q \otimes \mathcal{T}_k$, where $\widetilde{G} = G_0 \oplus G_1 \oplus \cdots \oplus G_m$, and q is the length of the longest path in the graph associated with $\widetilde{G}$. As it follows from Theorem 1.5.1, one has $\boldsymbol{x}(k) \leq \widetilde{\boldsymbol{x}}(k)$ for all $k = 1, 2, \ldots$.

Let us denote $\widetilde{A}_k = \widetilde{A}(k) \otimes \cdots \otimes \widetilde{A}(1)$. With the condition $\widetilde{\boldsymbol{x}}(0) = \boldsymbol{x}(0) = \boldsymbol{0}$, we get from (1.15)

$$\|\widetilde{\boldsymbol{x}}(k)\| = \|\widetilde{A}(k) \otimes \cdots \otimes \widetilde{A}(1)\| = \|\widetilde{A}_k\|.$$

With Proposition 1.3.2 we have

$$\widetilde{A}_k = \bigotimes_{i=1}^{k} (I \oplus \mathcal{T}_{k-i+1} \otimes \widetilde{G}^T)^q \otimes \mathcal{T}_{k-i+1} \leq \bigotimes_{i=1}^{k} \|\mathcal{T}_i\| \otimes \bigotimes_{i=1}^{k} (I \oplus \mathcal{T}_{k-i+1} \otimes \widetilde{G}^T)^q.$$

Proceeding to the norm and using Lemma 1.3.1, we arrive at the inequality

$$\|\widetilde{A}_k\| \leq \bigotimes_{i=1}^{k} \|\mathcal{T}_i\| \otimes \left(\bigoplus_{i=1}^{k} \|\mathcal{T}_i\| \right)^q = \sum_{i=1}^{k} \|\mathcal{T}_i\| + q \left(\bigoplus_{i=1}^{k} \|\mathcal{T}_i\| \right).$$

which provides us with the desired result. ∎

1.7   Stochastic Extension of the Network Model

Suppose that for each node $i = 1, \ldots, n$, the service times $\tau_{i1}, \tau_{i2}, \ldots$, form a sequence of independent and identically distributed (i.i.d.) non-negative random variables with $\mathbb{E}[\tau_{ik}] < \infty$ and $\mathbb{D}[\tau_{ik}] < \infty$ for all $k = 1, 2, \ldots$.

As a performance measure of the stochastic network model, we consider the service cycle time which can be defined as [Baccelli et al. (1992)]

$$\gamma = \lim_{k \to \infty} \frac{1}{k} \|\boldsymbol{x}(k)\| \tag{1.16}$$

provided that the above limit exists. Another performance measure of interest is the throughput defined as $\pi = 1/\gamma$.

Since it is frequently rather difficult to evaluate the cycle time exactly, even though the network under study is quite simple, one can try to derive bounds on γ. In this section, we show how these bounds may be obtained based on (max,+)-algebra representation of the network dynamics.

We start with some preliminary results which include properties of the expectation operator, formulated in terms of (max,+)-algebra operations.

1.7.1 Some Properties of Expectation

Let $\xi_1, \ldots, \xi_k$ be random variables taking their values in $\mathbb{R}$, and such that their expected values $\mathbb{E}[\xi_i]$, $i = 1, \ldots, k$, exist.

First note that ordinary properties of expectation leads us to the obvious relations

$$\mathbb{E}\left[\bigoplus_{i=1}^{k} \xi_i\right] \leq \bigotimes_{i=1}^{k} \mathbb{E}[\xi_i], \quad \text{and} \quad \mathbb{E}\left[\bigotimes_{i=1}^{k} \xi_i\right] = \bigotimes_{i=1}^{k} \mathbb{E}[\xi_i].$$

Furthermore, the next statement is valid.

Lemma 1.7.1 *It holds*

$$\mathbb{E}\left[\bigoplus_{i=1}^{k} \xi_i\right] \geq \bigoplus_{i=1}^{k} \mathbb{E}[\xi_i].$$

Proof: The statement of the lemma for $k = 2$ follows immediately from the identity

$$x \oplus y = \frac{1}{2}(x + y + |x - y|), \quad \text{for all } x, y \in \mathbb{R}$$

and ordinary properties of expectation. It remains to extend the statement to the case of arbitrary k by induction. ∎

The next result [Gumbel (1954), Hartly and David (1954)] provides an upper bound for the expected value of the maximum of i.i.d. random variables.

Lemma 1.7.2 *Let $\xi_1, \ldots, \xi_k$ be i.i.d. random variables with $\mathbb{E}[\xi_1] < \infty$ and $\mathbb{D}[\xi_1] < \infty$. Then it holds*

$$\mathbb{E}\left[\bigoplus_{i=1}^{k} \xi_i\right] \leq \mathbb{E}[\xi_1] + \frac{k-1}{\sqrt{2k-1}}\sqrt{\mathbb{D}[\xi_1]}.$$

Consider a random matrix X with its entries x_{ij} taking values in $\mathbb{R}$. We denote by $\mathbb{E}[X]$ the matrix obtained from X by replacing each entry x_{ij} by its expected value $\mathbb{E}[x_{ij}]$.

Lemma 1.7.3 *It holds*

$$\mathbb{E}\|X\| \geq \|\mathbb{E}[X]\|.$$

Proof: It follows from Lemma 1.7.1 that

$$\mathbb{E}\|X\| = \mathbb{E}\left[\bigoplus_{i,j} x_{ij}\right] \geq \bigoplus_{i,j} \mathbb{E}[x_{ij}] = \|\mathbb{E}[X]\|.$$

∎

1.7.2 Existence of the Cycle Time

In the analysis of the cycle time, one first has to convince himself that the limit at (1.16) exists. As a standard tool to verify the existence of the above limit, the next theorem proposed in Kingman (1973) is normally applied. One can find examples of the implementation of the theorem in the $(\max, +)$-algebra framework in Baccelli and Konstantopoulos (1991), Baccelli et al. (1992).

Theorem 1.7.1 *Let* $\{\xi_{lk}|\; l, k = 0, 1, \ldots;\; l < k\}$ *be a family of random variables which satisfy the following properties:*

Subadditivity: $\xi_{lk} \leq \xi_{lm} + \xi_{mk}$ *for all* $l < m < k$;

Stationarity: both families $\{\xi_{l+1k+1}|\; l < k\}$ *and* $\{\xi_{lk}|\; l < k\}$ *have the same joint distributions;*

Boundedness: for all $k = 1, 2, \ldots$, *there exists* $\mathbb{E}[\xi_{0k}] \geq -ck$ *for some finite number* c.

Then there exists a constant γ, *such that it holds*

1. $\lim_{k \to \infty} \xi_{0k}/k = \gamma$ *with probability 1,*

2. $\lim_{k \to \infty} \mathbb{E}[\xi_{0k}]/k = \gamma$.

For simplicity, we examine the existence of the cycle time for a network with the maximum of the initial numbers of customers in nodes $M \leq 1$. As it follows from representation (1.9–1.11), the dynamics of the system may be described by the equation

$$\boldsymbol{x}(k) = A(k) \otimes \boldsymbol{x}(k-1)$$

with the matrix $A(k) = A_1(k)$ determined by (1.10). Clearly, in the case of $M > 1$, a similar representation can be easily obtained by going to an extended model with a new state vector which combines several consecutive state vectors of the original system.

To prove the existence of the cycle time, first note that τ_{ik} with $k = 1, 2, \ldots$, are i.i.d. random variables for each $i = 1, \ldots, n$, and consequently, $\mathcal{T}_k$ are i.i.d. random matrices, whereas $\|\mathcal{T}_k\|$ present i.i.d. random variables with $\mathbb{E}\|\mathcal{T}_k\| < \infty$ and $\mathbb{D}\|\mathcal{T}_k\| < \infty$ for all k.

Furthermore, since the matrix $A(k)$ depends only on $\mathcal{T}_k$, the matrices $A(1), A(2), \ldots$, also present i.i.d. random matrices. It is easy to verify that $0 \leq \mathbb{E}\|A(k)\| < \infty$ for all $k = 1, 2, \ldots$.

In order to apply Theorem 1.7.1 to stochastic system (1.9) with transition matrix (1.10), one can define the family of random variables $\{\xi_{lk}|\; l < k\}$ with

$$\xi_{lk} = \|A(k) \otimes \cdots \otimes A(l+1)\|.$$

Since $A(i)$, $i = 1, 2, \ldots$, present i.i.d. random matrices, the family $\{\xi_{lk}|\; l < k\}$ satisfies the stationarity condition of Theorem 1.7.1. Furthermore, the multiplicative property of the norm endows the family with subadditivity. The

boundedness condition can be readily verified based on the condition that $0 \leq \mathbb{E}[\tau_{ik}] < \infty$ for all $i = 1, \ldots, n$, and $k = 1, 2, \ldots$.

1.7.3 Calculating Bounds on the Cycle Time

Now we are in a position to present our main result which offers bounds on the cycle time.

Theorem 1.7.2 *In the stochastic dynamical system (1.9) the cycle time γ satisfies the double inequality*

$$\|\mathbb{E}[\mathcal{T}_1]\| \leq \gamma \leq \mathbb{E}\|\mathcal{T}_1\|. \tag{1.17}$$

Proof: Since Theorem 1.7.1 holds true, we may write

$$\gamma = \lim_{k \to \infty} \frac{1}{k} \mathbb{E}\|\boldsymbol{x}(k)\|.$$

Let us first prove the left inequality in (1.17). From Lemmas 1.6.1 and 1.7.3, we have

$$\frac{1}{k}\mathbb{E}\|\boldsymbol{x}(k)\| \geq \frac{1}{k}\mathbb{E}\left\|\sum_{i=1}^{k}\mathcal{T}_i\right\| \geq \left\|\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}[\mathcal{T}_i]\right\| = \|\mathbb{E}[\mathcal{T}_1]\|,$$

independently of k.

With the upper bound offered by Lemma 1.6.1, we get

$$\frac{1}{k}\mathbb{E}\|\boldsymbol{x}(k)\| \leq \mathbb{E}\|\mathcal{T}_1\| + \frac{p}{k}\mathbb{E}\left[\bigoplus_{i=1}^{k}\|\mathcal{T}_i\|\right].$$

From Lemma 1.7.2, the second term on the right-hand side may be replaced by that of the form

$$\frac{p}{k}\left(\mathbb{E}\|\mathcal{T}_1\| + \frac{k-1}{\sqrt{2k-1}}\sqrt{\mathbb{D}\|\mathcal{T}_1\|}\right),$$

which tends to 0 as $k \to \infty$. ∎

1.8   Discussion and Examples

Now we discuss the behaviour of the bounds (1.17) under various assumptions concerning the service times in the network. First note that the derivation of the bounds does not require the kth service times τ_{ik} to be independent for all $i = 1, \ldots, n$. As it is easy to see, if $\tau_{ik} = \tau_k$ for all i, we have $\|\mathbb{E}[\mathcal{T}_1]\| = \mathbb{E}\|\mathcal{T}_1\|$, and so the lower and upper bound coincide.

To show how the bounds vary with strengthening the dependency, we consider the network with $n = 5$ nodes, depicted in Fig. 1.1. Let $\tau_{i1} = \sum_{j=1}^{5} a_{ij}\xi_{j1}$,

where ξ_{j1}, $j = 1, \ldots, 5$, are i.i.d. random variables with the exponential distribution of mean 1, and

$$a_{ij} = \begin{cases} a, & \text{if } i = j, \\ \frac{1}{4}(1 - a), & \text{if } i \neq j, \end{cases}$$

where a is a number such that $1 \leq a \leq 1/5$.

It is evident that for $a = 1$, one has $\tau_{i1} = \xi_{i1}$, and then τ_{i1}, $i = 1, \ldots, 5$, present independent random variables. As a decreases, the service times τ_{i1} become dependent, and with $a = 1/5$, we will have $\tau_{i1} = (\xi_{11} + \cdots + \xi_{51})/5$ for all $i = 1, \ldots, 5$.

Table 1.1 presents estimates of the cycle time $\widehat{\gamma}$ obtained via simulation after performing 100000 service cycles, together with the corresponding lower and upper bounds calculated from (1.17).

a	$\|\mathbb{E}[\mathcal{T}_1]\|$	$\widehat{\gamma}$	$\mathbb{E}\|\mathcal{T}_1\|$
1	1.0	1.005718	2.283333
1/2	1.0	1.002080	1.481250
1/3	1.0	1.000871	1.213889
1/4	1.0	1.000279	1.080208
1/5	1.0	1.000000	1.000000

Table 1.1: Numerical results for a network with dependent service times.

Let us now consider the network in Fig. 1.1 under the assumption that the service times τ_{i1} are independent exponentially distributed random variables. We suppose that $\mathbb{E}[\tau_{i1}] = 1$ for all i except for one, say $i = 4$, with $\mathbb{E}[\tau_{41}]$ essentially greater than 1. One can see that the difference between the upper and lower bounds will decrease as the value of $\mathbb{E}[\tau_{41}]$ increases. Table 1.2 shows how the bounds vary with different values of $\mathbb{E}[\tau_{41}]$.

Let us discuss the effect of decreasing the variance $\mathbb{D}[\tau_{i1}]$ on the bounds on γ. Note that if τ_{i1} were degenerate random variables with zero variance, the lower and upper bounds in (1.17) would coincide. One can therefore expect that with decreasing the variance of τ_{i1}, the accuracy of the bounds increases.

As an illustration, consider a tandem queueing system (see Fig. 1.2) with $n = 5$ nodes. Suppose that $\tau_{i1} = \xi_{i1}/r$, where ξ_{i1}, $i = 1, \ldots, 5$, are i.i.d. random variables which have the Erlang distribution with the probability density function

$$f_r(t) = \begin{cases} t^{r-1}e^{-t}/(r-1)!, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Clearly, $\mathbb{E}[\tau_{i1}] = 1$ and $\mathbb{D}[\tau_{i1}] = 1/r$. Related numerical results including estimates $\widehat{\gamma}$ evaluated by simulating 100000 cycles are shown in Table 1.3.

$\mathbb{E}[\tau_{41}]$	$\|\mathbb{E}[\mathcal{T}_1]\|$	$\widehat{\gamma}$	$\mathbb{E}\|\mathcal{T}_1\|$
1.0	1.0	1.005718	2.283333
2.0	2.0	2.004857	2.896032
3.0	3.0	3.004242	3.685531
4.0	4.0	4.003627	4.554525
5.0	5.0	5.003013	5.465368
6.0	6.0	6.002398	6.400835
7.0	7.0	7.001783	7.351985
8.0	8.0	8.001168	8.313731
9.0	9.0	9.000553	9.282968
10.0	10.0	10.000008	10.257692

Table 1.2: Results for a network with a dominating service time.

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r	$\|\mathbb{E}[\mathcal{T}_1]\|$	$\widehat{\gamma}$	$\mathbb{E}\|\mathcal{T}_1\|$
1	1.0	1.042476	2.928968
2	1.0	1.026260	2.311479
3	1.0	1.019503	2.045538
4	1.0	1.015637	1.890824
5	1.0	1.013110	1.787242
6	1.0	1.010864	1.711943
7	1.0	1.009920	1.654154
8	1.0	1.008409	1.608064
9	1.0	1.007726	1.570232
10	1.0	1.006657	1.538479

Table 1.3: Results for tandem queues at changing variance.

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