ALGEBRAIC MODELS FOR MEASURE PRESERVING TRANSFORMATIONS

BY

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1. **Introduction.** The purpose of this paper is to study measure preserving transformations T on probability measure spaces (X, Σ, μ) by means of algebraic models (Γ, U, φ) (see Definitions 1 and 2).

The results obtained here contain those obtained in [3] concerning algebraic models (Γ, φ) of measure spaces (X, Σ, μ).

Each transformation possesses algebraic models and conversely every algebraic system is a model for a certain transformation (Theorem 2). Algebraic models determine transformations uniquely up to a conjugacy (Theorem 1).

Transformations with discrete models (see Definition 3) are uniquely determined by (Γ, U) (Theorem 3). Such transformations are characterized by the existence of an orthonormal basis $\Gamma' \subset L^2(\mu)$ of functions $|f| \equiv 1$, which is also a multiplicative group, such that $U_T\Gamma' \subset C \cdot \Gamma'$ (direct product), where C is the circle group (Theorem 5). In certain cases, conjugacy does no more involve U either (Theorem 4). Continuous automorphisms and rotations on an abelian compact group—equipped with Haar measure—are examples of transformations with discrete model (Corollary of Theorem 5), and in fact, every *invertible* transformation with discrete model is a superposition of an automorphism and a rotation (Theorem 6).

The class of transformations with discrete models contains the transformations with quasi-discrete spectrum (see Abramov [1]) and the transformations with discrete spectrum (see Halmos [5]). Necessary and sufficient conditions are given for algebraic systems in order to be models for transformations with quasi-discrete spectrum (Theorem 10) or with discrete spectrum (Theorem 11). We mention also Theorem 12 which gives necessary and sufficient conditions in order that $\Gamma_1 = \Gamma_\infty$.

In Theorems 7 and 9, ergodicity of transformations is characterized by means of algebraic models.

2. **Preliminaries.** Let (X, Σ, μ) be a probability measure space and $T: X \to X$ a measure preserving transformation.

(1) We denote by $\Gamma(\mu)$ the multiplicative group of the (equivalence classes of) functions $f \in L^\infty(\mu)$ with $|f| \equiv 1$, by φ_μ the function of positive type on $\Gamma(\mu)$ defined by

$$\varphi_\mu(f) = \int f \, d\mu, \quad \text{for } f \in \Gamma(\mu)$$

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and by U_T the linear isometry defined on $L^2(\mu)$ by $U_T f = f \circ T$. Then [3, Proposition 1],

$$\varphi_\mu(f) = 1 \quad \text{if and only if } f = 1$$

and U_T (or, more precisely, the restriction of U_T to $\Gamma(\mu)$) is an *injective homomorphism* of $\Gamma(\mu)$ into itself, such that

$$U_T c = c, \quad \text{for } c \in C \text{ (the circle group)}$$

and

$$\varphi_\mu(U_T f) = \varphi_\mu(f), \quad \text{for } f \in \Gamma(\mu).$$

If T is invertible, then U_T is an *automorphism* of $\Gamma(\mu)$.

(2) For every function $f \in \Gamma(\mu)$ put

$$w_T(f) = U_T f \cdot \bar{f}.$$

Then w_T is a *homomorphism* of $\Gamma(\mu)$ into itself and we have

$$U_T f = w_T(f) \cdot f.$$

$w_T(f)$ is called the *generalized proper value* corresponding to the *generalized proper function* f of U_T.

A subgroup $\Gamma \subset \Gamma(\mu)$ is invariant under U_T (that is $U_T \Gamma \subset \Gamma$) if and only if Γ is invariant under w_T (that is $w_T \Gamma \subset \Gamma$).

(3) For every integer $n \geqq 0$ put

$$\Gamma_n = \Gamma_n(T) = w_T^{-n}(C) = \{f \in \Gamma(\mu);\ w_T^n(f) \in C\}.$$

In particular, $\Gamma_0 = C$ and Γ_1 is the set of the proper functions of U_T belonging to $\Gamma(\mu)$. Each Γ_n is a group invariant under U_T and $\Gamma_n \subset \Gamma_{n+1}$ for every n. The set

$$\Gamma_\infty = \Gamma_\infty(T) = \bigcup_{n=0}^{\infty} \Gamma_n$$

is also a subgroup of $\Gamma(\mu)$ invariant under U_T. Moreover, if $\Gamma \subset \Gamma(\mu)$ is a group such that

$$C \subset \Gamma \quad \text{and} \quad w_T^{-1}\Gamma = \Gamma$$

then $\Gamma_\infty \subset \Gamma$. (In fact, for every n we have $w_T^{-n}(C) \subset w_T^{-n}\Gamma \subset \Gamma$).

In particular, if $\Gamma_{n+1} = \Gamma_n$ for some n, then $\Gamma_\infty = \Gamma_n$.

(4) For every integer $k \geqq 0$, U_T^k is an injective homomorphism of $\Gamma(\mu)$ into itself and

$$\varphi_\mu(U_T^k f) = \varphi_\mu(f), \quad \text{for } f \in \Gamma(\mu).$$

If $\Gamma \subset \Gamma(\mu)$ is invariant under U_T, then Γ is invariant under U_T^k.

For every n we have

$$\Gamma_n(T) \subset \Gamma_n(T^k)$$

therefore

$$\Gamma_\infty(T) \subset \Gamma_\infty(T^k).$$

(5) Let Γ be an abelian group containing a subgroup C' of the circle group C, and suppose that $\Gamma = C' \cdot \Gamma'$ (direct product), where Γ' is a subgroup of Γ. Let further $U\colon \Gamma \to \Gamma$ be an injective homomorphism such that

$$Uc = c, \quad \text{for } c \in C'.$$

For every $\gamma \in \Gamma'$ we have $U\gamma \in \Gamma$, therefore, there exists a number $\rho(\gamma) \in C'$ and an element $V\gamma \in \Gamma'$ such that

$$U\gamma = \rho(\gamma)V\gamma.$$

Then $\rho\colon \Gamma' \to C'$ is a homomorphism and $V\colon \Gamma' \to \Gamma'$ is an injective homomorphism. Moreover, for every n there exists a homomorphism $\rho_n\colon \Gamma' \to C'$ such that

$$U^n\gamma = \rho_n(\gamma)V^n\gamma, \quad \text{for } \gamma \in \Gamma'.$$

In particular, if $\Gamma \subset \Gamma(\mu)$ and $U = U_T$, then $V\gamma = \gamma$ and $\rho(\gamma) = w_T(\gamma)$, for $\gamma \in \Gamma' \cap \Gamma_1(T)$.

Conversely, if $\rho\colon \Gamma' \to C'$ is a homomorphism and $V\colon \Gamma' \to \Gamma'$ is an injective homomorphism, then the equality

$$U(c\gamma) = c\rho(\gamma)V(\gamma), \quad \text{for } c \in C' \quad \text{and} \quad \gamma \in \Gamma'$$

defines an injective homomorphism $U\colon \Gamma \to \Gamma$ which satisfies

$$Uc = c, \quad \text{for } c \in C'$$

and

$$U\gamma = \rho(\gamma)V\gamma, \quad \text{for } \gamma \in \Gamma'.$$

(6) Let (X', Σ', μ') be a probability measure space and $T'\colon X' \to X'$ a measure preserving transformation.

The transformations T and T' are *conjugate* (see [5, pp. 44–45]) if there exists a *linear isometry*

$$\phi\colon L^2(\mu) \to L^2(\mu')$$

such that

$$\phi L^2(\mu) = L^2(\mu'),$$

$$\phi(fg) = \phi f \cdot \phi g, \quad \text{for } f, g \in L^\infty(\mu)$$

and

$$\phi U_T = U_{T'}\phi.$$

It follows then that $\phi L^\infty(\mu) = L^\infty(\mu')$ and

$$\|\phi f\|_\infty = \|f\|_\infty, \quad \text{for } f \in L^\infty(\mu).$$

REMARK. To say that T and T' are conjugate means that the measures μ and μ' are conjugate (see [3, Definition 1]) by means of a linear isometry $\phi\colon L^2(\mu) \to L^2(\mu')$ which satisfies in addition the equality

$$\phi U_T = U_{T'}\phi.$$

The following proposition gives some conjugacy invariants connected to $\Gamma(\mu)$, U_T and ϕ_μ.

PROPOSITION 1. *If T and T' are conjugate, then there exists an injective homomorphism $\phi\colon \Gamma(\mu) \to \Gamma(\mu')$ having the following properties*:

(i) $\phi\Gamma(\mu) = \Gamma(\mu')$;

(ii) $\phi c = c$, *for* $c \in C$;

(iii) *If* $\Gamma \subset \Gamma(\mu)$ *generates* $L^2(\mu)$, *then* $\phi\Gamma$ *generates* $L^2(\mu')$;

(iv) *If* $\Gamma \subset \Gamma(\mu)$ *is an orthonormal system in* $L^2(\mu)$ *then* $\phi\Gamma$ *is orthonormal in* $L^2(\mu')$;

(v) $\phi\Gamma_n(T) = \Gamma_n(T')$ *and* $\phi\Gamma_\infty(T) = \Gamma_\infty(T')$;

(vi) $\varphi_\mu(f) = \varphi_{\mu'}(\phi f)$, *for* $f \in \Gamma(\mu)$;

(vii) $\phi U_T = U_{T'}\phi$ *and* $\phi w_T = w_{T'}\phi$.

In fact, if ϕ is a linear isometry of $L^2(\mu)$ onto $L^2(\mu')$ realizing the conjugacy between T and T', then the restriction of ϕ to $\Gamma(\mu)$, still denoted by ϕ, is the required isomorphism (see also [3, Proposition 2]).

REMARK. We shall see (corollary of Theorem 1) that, conversely, if ϕ is an isomorphism of $\Gamma(\mu)$ onto $\Gamma(\mu')$ satisfying conditions (vi) and (vii), then T and T' are conjugate.

3. **Algebraic models.** The considerations of the preceding section lead to the following

DEFINITION 1. *A system* (Γ, U, φ) *consisting of an abelian group* Γ *with unit* 1, *an injective homomorphism* $U\colon \Gamma \to \Gamma$ *and a complex function of positive type* φ *on* Γ *such that* $\varphi(\gamma) = 1$ *if and only if* $\gamma = 1$ *and* $\varphi(U\gamma) = \varphi(\gamma)$, *for* $\gamma \in \Gamma$, *is called an algebraic ergodic system* (*a.e. system*).

Two a.e. systems (Γ, U, φ) *and* (Γ', U', φ') *are said to be isomorphic if there exists an isomorphism* ϕ *of* Γ *onto* Γ' *such that*

$$\varphi(\gamma) = \varphi'(\phi\gamma), \quad \textit{for } \gamma \in \Gamma$$

and

$$\phi U = U'\phi.$$

If we define the homomorphisms $w\colon \Gamma \to \Gamma$ by

$$w(\gamma) = U\gamma\cdot\gamma^{-1}, \quad \text{for } \gamma \in \Gamma$$

and the homomorphism $w'\colon \Gamma' \to \Gamma'$ in a similar way, then condition $\phi U = U'\phi$ above is equivalent to condition $\phi w = w'\phi$.

EXAMPLE. If T is a measure preserving transformation on a probability measure space (X, Σ, μ), then (C, U_T, φ_μ) and $(\Gamma(\mu), U_T, \varphi_\mu)$ are a.e. systems. More generally, for every group $\Gamma \subset \Gamma(\mu)$ invariant under U_T, $(\Gamma, U_T, \varphi_\mu)$ is an a.e. system.

We shall see (Theorem 2) that every a.e. system can be obtained in this way.

REMARKS. 1°. To say that (Γ, U, φ) is an a.e. system, means that (Γ, φ) is a measure system (see [3, Definition 2]) and that $U: \Gamma \to \Gamma$ is an injective homomorphism satisfying $\varphi(U\gamma)=\varphi(\gamma)$ for $\gamma \in \Gamma$. Then $(\Gamma, \varphi \circ U)$ is also a measure system. Moreover, if $U\Gamma=\Gamma$, then (Γ, φ) and $(\Gamma, \varphi \circ U)$ are isomorphic measure systems.

Conversely if (Γ, φ) and (Γ, φ') are isomorphic measure systems by means of an isomorphism $U: \Gamma \to \Gamma$, then (Γ, U, φ) is an a.e. system and $U\Gamma=\Gamma$.

2°. To say that two a.e. systems (Γ, U, φ) and (Γ', U', φ') are isomorphic, means that (Γ, φ) and (Γ', φ') are isomorphic measure systems, by means of an isomorphism $\phi: \Gamma \to \Gamma'$ which satisfies $\phi U = U'\phi$.

Conversely, if (Γ, φ) and (Γ', φ') are isomorphic measure systems, then taking $U: \Gamma \to \Gamma$ and $U': \Gamma' \to \Gamma'$ the identity mappings, the a.e. systems (Γ, U, φ) and (Γ', U', φ') are isomorphic.

3°. If (Γ, U, φ) is an a.e. system, then the set $C'=\{\gamma \in \Gamma; |\varphi(\gamma)|=1\}$ is a group, and φ is an injective homomorphism of C' into the circle group C. If we identify an element $\gamma \in C'$ with the number $\varphi(\gamma)=c$, we have (see [3, corollary of Proposition 3])

$$\varphi(c\gamma) = c\varphi(\gamma), \quad \text{for } c \in C' \quad \text{and} \quad \gamma \in \Gamma.$$

Moreover,

$$Uc = c, \quad \text{for } c \in C'.$$

In fact, if $c \in C'$, then $\varphi(Uc)=\varphi(c)=c$, therefore $Uc \in C'$ and $Uc=c$.

If C' is divisible, then there exists a group $\Gamma' \subset \Gamma$ such that

$$\Gamma = C' \cdot \Gamma' \text{ (direct product).}$$

The a.e. system (Γ, U, φ) can be embedded in an a.e. system $(\Gamma_1, U_1, \varphi_1)$ such that

$$\{\gamma \in \Gamma_1; |\varphi_1(\gamma)| = 1\} = C$$

and then

$$\Gamma_1 = C \cdot \Gamma_1' \text{ (direct product).}$$

In case $U\gamma=\gamma$ (or, equivalently, $w(\gamma)=1$) implies $\gamma \in C$, the group Γ' can be precised:

PROPOSITION 2. *Let* (Γ, U, φ) *be an a.e. system, let* $C'=\{\gamma \in \Gamma; \varphi(\gamma) \in C\}$ *and* $w(\gamma)=U\gamma \cdot \gamma^{-1}$, *for* $\gamma \in \Gamma$.

If C' *is divisible (in particular if* $C'=C$*) and if* $w(\gamma)=1$ *implies* $\gamma \in C'$, *then every injective homomorphism* $a \to \gamma_a$ *of a group* $G \subset w\Gamma$ *into* Γ *such that* $w(\gamma_a)=a$ *for* $a \in G$ *(in particular the homomorphism* $1 \to \gamma_1=1$ *of* $G=\{1\}$*) can be extended to an injective homomorphism* $a \to \gamma_a$ *of* $w\Gamma$ *into* Γ, *such that* $w(\gamma_a)=a$, *for* $a \in w\Gamma$.

If we put $\Gamma'=\{\gamma_a; a \in w\Gamma\}$, *then* $\Gamma=C' \cdot \Gamma'$ *(direct product).*

The proof is similar to that given in [5, p. 46], for ergodic transformations with discrete spectrum.

For every $a \in w\Gamma$ choose $\mu_a \in \Gamma$ with $U\mu_a = a\mu_a$, that is $w(\mu_a) = a$. If $a \in G$ we take $\mu_a = \gamma_a$. We have

$$U\mu_{ab} = ab\mu_{ab} \quad \text{and} \quad U\mu_a\mu_b = ab\mu_a\mu_b$$

whence

$$w(\mu_{ab}) = w(\mu_a\mu_b) = ab.$$

By hypothesis, there exists a number $\gamma(a, b) \in C'$ such that

$$\mu_a\mu_b = \gamma(a, b)\mu_{ab}.$$

If $a, b \in G$, then $\gamma(a, b) = 1$. Consider the group $\{c\gamma_a; c \in C', a \in G\}$ and the homomorphism p of this group into C' defined by $p(c\gamma_a) = c$. We have, in particular, $p(c) = c$ for $c \in C'$ and $p(\gamma_a) = 1$ for $a \in G$. Since C' is divisible, p can be extended to a homomorphism, still denoted by p, of $w\Gamma$ into C'.

If we now define

$$\gamma_a = \overline{p(\mu_a)}\mu_a, \quad \text{for } a \in w\Gamma$$

then the requirements of the proposition are fulfilled.

Remark. Condition: $w(\gamma) = 1$ implies $\gamma \in C'$, is satisfied, for example, if $U = U_T$, where T is an *ergodic* transformation.

Definition 2. *Let* (X, Σ, μ) *be a probability measure space and* $T: X \to X$ *a measure preserving transformation. We say that an a.e. system* (Γ, U, φ) *is an algebraic model of the transformation* T *if there exists an injective homomorphism* $J: \Gamma \to \Gamma(\mu)$ *such that*:

(a) $J\Gamma$ *generates* $L^2(\mu)$;

(b) $\varphi(\gamma) = \varphi_\mu(J\gamma)$, *for* $\gamma \in \Gamma$;

(c) $JU = U_TJ$.

It follows that if $\Gamma \subset \Gamma(\mu)$ is a group generating $L^2(\mu)$, and invariant under U_T, then $(\Gamma, U_T, \varphi_\mu)$ is an algebraic model for T.

If (Γ, U, φ) is an algebraic model of T by means of an isomorphism J, then, identifying Γ and $J\Gamma$ we can consider that $\Gamma \subset \Gamma(\mu)$, $U = U_T$ and $\varphi = \varphi_\mu$.

If (Γ, U, φ) is an algebraic model of T, then T is invertible (that is $U_TL^2(\mu) = L^2(\mu)$) if and only if U is an automorphism of Γ (that is $U\Gamma = \Gamma$). In particular, a transformation T having $(\Gamma_1(T), U_T, \varphi_\mu)$ as algebraic model, is always invertible (since $U_T\Gamma_1 = \Gamma_1$).

Remark. To say that (Γ, U, φ) is an algebraic model for T means that (Γ, φ) is an algebraic model for the measure μ (see [3, Definition 3]), by means of an isomorphism $J: \Gamma \to \Gamma(\mu)$ which satisfies, in addition, $JU = U_TJ$.

Conversely, if (Γ, φ) is an algebraic model of the measure μ and if $U: \Gamma \to \Gamma$ is the identity mapping, then (Γ, U, φ) is an algebraic model for the identity transformation $T: X \to X$.

Algebraic models determine the transformations uniquely up to a conjugacy:

THEOREM 1. *Two measure preserving transformations are conjugate if and only if they possess isomorphic algebraic models.*

Let T and T' be two measure preserving transformations on the probability measure spaces (X, Σ, μ) respectively (X', Σ', μ').

If T and T' are conjugate, then from Proposition 1 we deduce that their algebraic models $(\Gamma(\mu), U_T, \varphi_\mu)$ and $(\Gamma(\mu'), U_{T'}, \varphi_{\mu'})$ are isomorphic.

Conversely, suppose that T and T' possess isomorphic models (Γ, U, φ) respectively (Γ', U', φ'). We may consider $\Gamma \subset \Gamma(\mu)$, $U = U_T$, $\varphi = \varphi_\mu$ and $\Gamma' \subset \Gamma(\mu')$, $U' = U'_T$ and $\varphi' = \varphi_{\mu'}$.

If ϕ is an isomorphism of Γ onto Γ' such that

$$\varphi_\mu = \varphi_{\mu'} \circ \phi \quad \text{and} \quad \phi U_T = U_{T'} \phi$$

then (see [3, Theorem 2]), ϕ can be extended to a linear isometry $\phi: L^2(\mu) \to L^2(\mu')$ such that

$$\phi L^2(\mu) = L^2(\mu') \quad \text{and} \quad \phi L^\infty(\mu) = L^\infty(\mu'),$$

and

$$\phi(fg) = \phi f \cdot \phi g, \quad \text{for } f, g \in L^\infty(\mu).$$

The equality

$$\phi U_T f = U_{T'} \phi f, \quad \text{for } f \in \Gamma$$

remains true first for linear combinations of functions of Γ and then for every $f \in L^2(\mu)$, so that T and T' are conjugate.

COROLLARY. *The transformations T and T' are conjugate if and only if the a.e. systems* $(\Gamma(\mu), U_T, \varphi_\mu)$ *and* $(\Gamma(\mu'), U_{T'}, \varphi_{\mu'})$ *are isomorphic.*

The following theorem states that every a.e. system is an algebraic model for some transformation.

THEOREM 2. *Every a.e. system* (Γ, U, φ) *is an algebraic model for a continuous measure preserving homomorphism τ on an abelian compact group G equipped with a suitable regular Borel measure μ.*

Moreover, if $U\Gamma = \Gamma$, then τ is an automorphism of G.

Consider on Γ the discrete topology and take $G = \Gamma^\wedge$. Let μ be the unique regular Borel measure on G such that (Bochner's theorem),

$$\varphi(\gamma) = \int \langle x, \gamma \rangle \, d\mu(x), \quad \text{for } \gamma \in \Gamma.$$

Then the mapping $J: \Gamma \to \Gamma(\mu)$ defined by

$$J\gamma = \langle \cdot, \gamma \rangle, \quad \text{for } \gamma \in \Gamma$$

is an injective homomorphism, $J\Gamma$ generates $L^2(\mu)$ and

$$\varphi(\gamma) = \varphi_\mu(J\gamma), \quad \text{for } \gamma \in \Gamma.$$

We define now the mapping $\tau: G \to G$ by

$$\langle \tau x, \gamma \rangle = \langle x, U\gamma \rangle, \quad \text{for } x \in G \quad \text{and} \quad \gamma \in \Gamma.$$

Then τ is a continuous homomorphism of G into itself, and

$$JU = U_\tau J.$$

If $\Gamma U = \Gamma$, then τ is injective and $\tau G = G$, therefore τ is an automorphism of G.

It remains to prove that τ is measure preserving.

Consider the regular Borel measure ν defined on G by

$$\nu(A) = \mu(\tau^{-1}A), \quad \text{for every Borel set } A \subset G.$$

Then for every $\gamma \in \Gamma$ we have

$$\varphi(\gamma) = \varphi(U\gamma) = \int \langle x, U\gamma \rangle \, d\mu(x) = \int \langle \tau x, \gamma \rangle \, d\mu(x) = \int \langle x, \gamma \rangle \, d\nu(x).$$

By the uniqueness of μ we deduce that $\mu = \nu$, therefore $\mu(\tau^{-1}A) = \mu(A)$, for every Borel set $A \subset G$ consequently τ is measure preserving.

REMARK. The proof of Theorem 2 was used in [4] to prove the following

COROLLARY. *Every measure preserving transformation T on a probability measure space (X, Σ, μ) is conjugate to a continuous homomorphism τ on an abelian compact group G equipped with a suitable regular Borel measure. If T is invertible then τ is an automorphism of G.*

4. Discrete algebraic models.

DEFINITION 3. *An a.e. system (Γ, U, φ) is said to be discrete if $C \subset \Gamma$ and*

$$\varphi(\gamma) = \gamma, \quad \textit{for } \gamma \in C,$$
$$= 0, \quad \textit{for } \gamma \notin C.$$

REMARKS. 1°. An a.e. system (Γ, U, φ) is discrete if and only if (Γ, U^n, φ) is discrete.

2°. We have

$$Uc = c, \quad \text{and} \quad w(c) = 1, \quad \text{for } c \in C$$

where $w(\gamma) = U\gamma \cdot \gamma^{-1}$ for $\gamma \in \Gamma$ (see Remark 3 after Definition 1).

3°. Let (Γ, U, φ) be an a.e. system with $C\subset\Gamma$. Then $\Gamma=C\cdot\Gamma'$ (direct product) where Γ' is a subgroup of Γ. To say that (Γ, U, φ) is discrete, means that

$$\begin{aligned}\varphi(\gamma) &= 1 \quad \text{for } \gamma = 1,\\ &= 0 \quad \text{for } \gamma\in\Gamma', \gamma\neq 1.\end{aligned}$$

4°. Let (Γ, U, φ) be an a.e. system such that

$$|\varphi(\gamma)| < 1 \quad \text{implies } \varphi(\gamma) = 0.$$

Then (Γ, U, φ) is "essentially" a discrete system. In fact we can consider (Γ, U, φ) as a model of a measure preserving transformation T on a probability measure space, and consider $\Gamma\subset\Gamma(\mu)$, $U=U_T$ and $\varphi=\varphi_\mu$. Consider then the group $\Gamma_1=\{c\gamma; c\in C, \gamma\in\Gamma\}$; then $(\Gamma_1, U_T, \varphi_\mu)$ is a discrete model of T and contains the initial model (Γ, U, φ).

For a discrete system (Γ, U, φ), the function φ is completely determined by Γ, so that the system itself is completely determined by (Γ, U).

PROPOSITION 3. *Let Γ be an abelian group containing C and let $U: \Gamma\to\Gamma$ be an injective homomorphism such that*

$$Uc = c, \quad \text{for } c\in C.$$

If we define

$$\begin{aligned}\varphi(\gamma) &= \gamma \quad \text{if } \gamma\in C,\\ &= 0 \quad \text{if } \gamma\notin C,\end{aligned}$$

then (Γ, U, φ) is a discrete system.

In fact, φ is of positive type:

$$\begin{aligned}\sum_{i,j=1}^{n} \alpha_i\bar{\alpha}_j\varphi(\gamma_i\gamma_j^{-1}) &= \sum_{\gamma_i\sim\gamma_j} \alpha_i\bar{\alpha}_j\varphi(\gamma_i\gamma_j^{-1}) = \sum_k \sum_{\gamma_i,\gamma_j\in C\gamma_k} \alpha_i\bar{\alpha}_j\varphi(\gamma_i\gamma_j^{-1})\\ &= \sum_k \sum_{\gamma_i\in C\gamma_k} |\alpha_i\varphi(\gamma_i\gamma_k^{-1})|^2 \geqq 0\end{aligned}$$

where $\gamma_i\sim\gamma_j$ means $\gamma_i\gamma_j^{-1}\in C$ and $C\gamma_k$ the equivalence classes.

If $\gamma\in C$ then $U\gamma=\gamma$, therefore

$$\varphi(U\gamma) = \varphi(\gamma);$$

if $\gamma\notin C$, then $U\gamma\notin C$ (since U is injective), therefore $\varphi(\gamma)=0$ and $\varphi(U\gamma)=0$, consequently

$$\varphi(U\gamma) = \varphi(\gamma).$$

Moreover, $\varphi(\gamma)=1$, if and only if $\gamma=1$, so that (Γ, U, φ) is a discrete system.

For discrete systems, isomorphism does no more involve functions of positive type.

THEOREM 3. *Two discrete systems* (Γ, U, φ) *and* (Γ', U', φ') *are isomorphic if and only if there exists an isomorphism* ϕ *of* Γ *onto* Γ' *such that*

$$\phi c = c, \quad \textit{for } c \in C$$

and

$$\phi U = U'\phi.$$

In fact, if the systems are isomorphic by an isomorphism ϕ, then for every $c \in C$ we have

$$\varphi'(\phi c) = \varphi(c) = c \neq 0$$

therefore $\phi c \in C$, and then

$$\varphi'(\phi c) = \phi c$$

consequently $\phi c = c$.

Conversely, let $\phi: \Gamma \to \Gamma'$ be an isomorphism such that $\phi c = c$ for $c \in C$ and $\phi U = U'\phi$. We have to prove that $\varphi = \varphi' \circ \phi$. For $c \in C$ we have $\phi c = c$, therefore

$$\varphi'(\phi c) = \phi c = c = \varphi(c).$$

If $\gamma \notin C$, then $\phi\gamma \notin C$ (since ϕ is injective), therefore $\varphi'(\phi\gamma) = 0$ and $\varphi(\gamma) = 0$, consequently

$$\varphi'(\phi\gamma) = \varphi(\gamma).$$

REMARK. If (Γ, U, φ) is a discrete system, we shall say also that (Γ, U) is a discrete system. If (Γ, U_T) is a discrete system and $\Gamma \subset \Gamma(\mu)$, for some transformation T on a measure space (X, Σ, μ), we understand that $\varphi = \varphi_\mu$.

From Proposition 3 it follows that (Γ, U) is a discrete system provided that Γ is an abelian group containing C and $U: \Gamma \to \Gamma$ is an injective homomorphism such that $Uc = c$ for $c \in C$.

For certain discrete models (Γ, U) isomorphism does no more involve homomorphisms U either:

THEOREM 4. *Let* (Γ_1, U_1) *and* (Γ_2, U_2) *be two discrete systems and put*

$$w_i(\gamma) = U_i\gamma \cdot \gamma^{-1}, \quad \textit{for } \gamma \in \Gamma_i, \qquad i = 1, 2.$$

Suppose that

$$\gamma \in \Gamma_i \quad \textit{and} \quad w_i(\gamma) = 1 \quad \textit{imply } \gamma \in C, \qquad i = 1, 2.$$

Then (Γ_1, U_1) *and* (Γ_2, U_2) *are isomorphic, if and only if the groups* $w_1\Gamma_1$ *and* $w_2\Gamma_2$ *are isomorphic by an isomorphism* ϕ *such that* $\phi w_1 = w_2\phi$ *and* $\phi c = c$ *for* $c \in C \cap w_1\Gamma_1$.

If (Γ_1, U_1) and (Γ_2, U_2) are isomorphic by means of an isomorphism $\phi: \Gamma_1 \to \Gamma_2$ such that

$$\phi U_1 = U_2\phi \quad \text{and} \quad \phi c = c \quad \text{for } c \in C,$$

then we have also

$$\phi w_1 = w_2\phi.$$

From $\phi\Gamma_1=\Gamma_2$ we deduce then $\phi w_1\Gamma_1=w_2\Gamma_2$. The restriction of ϕ to $w_1\Gamma_1$ is the required isomorphism.

Conversely, suppose that $w_1\Gamma_1$ and $w_2\Gamma_2$ are isomorphic by means of an isomorphism ϕ: $w_1\Gamma_1\to w_2\Gamma_2$ such that $\phi w_1=w_2\phi$ and $\phi c=c$ for $c\in C\cap w_1\Gamma_1$.

By Proposition 2 there exists an injective homomorphism $a\to\gamma_a$ of $w_1\Gamma_1$ into Γ_1 such that $w_1(\gamma_a)=a$ for $a\in w_1\Gamma_1$; then $\Gamma_1=C\cdot\Gamma_1'$ (direct product) where $\Gamma_1'=\{\gamma_a; a\in w_1\Gamma_1\}$.

Consider the groups $G_1=w_1^2\Gamma_1$ and $G_2=w_2^2\Gamma_2$. Since $\phi w_1\Gamma_1=w_2\Gamma_2$ and $\phi w_1=w_2\phi$ we have $G_2=\phi G_1$.

If $a\in G_1$ then $w_1(\gamma_a)=a$ and $a=w_1(b)$ for some $b\in w_1\Gamma_1$, therefore $\gamma_a=cb$ for some $c\in C$; if we have also $\gamma_a=c_1b_1$ with $c_1\in C$ and $b_1\in w_1\Gamma_1$, then $c\bar{c}_1=b_1b^{-1}\in w_1\Gamma_1$, therefore, by hypothesis,

$$c\bar{c}_1 = \phi(c\bar{c}_1) = \phi(b_1)\overline{\phi(b)},$$

whence $c\phi b=c_1\phi b_1$. We define then unambiguously

$$\gamma_{\phi a} = c\phi b, \quad \text{if } \gamma_a = cb \text{ with } c\in C \text{ and } b\in w_1\Gamma_1.$$

It is easy to see that $\phi a\to\gamma_{\phi a}$ is an injective homomorphism of G_2 into Γ_2 such that $w_2(\phi a)=\phi a$. By Proposition 2, this homomorphism can be extended to an injective homomorphism $a\to\gamma_a$ of $w_2\Gamma_2$ into Γ_2 such that $w_2(\gamma_a)=a$ for $a\in w_2\Gamma_2$.

We extend now ϕ from $w_1\Gamma_1$ to Γ_1 by

$$\psi c\gamma_a = c\gamma_{\phi a} \quad \text{for } c\in C \text{ and } a\in w_1\Gamma_1.$$

ψ is an extension of ϕ, since if $b\in w_1\Gamma_1$, then $b=c\gamma_a$ for some $c\in C$ and $a\in w_1\Gamma_1$, whence $a=w_1(\gamma_a)=w_1(b)\in w_1^2\Gamma_1$ and $\gamma_a=\bar{c}b$, therefore $\gamma_{\phi a}=\bar{c}\phi b$; it follows then that $\phi b=c\gamma_{\phi a}=\psi(c\gamma_a)=\psi b$.

Moreover, ψ is an isomorphism of Γ_1 onto Γ_2 and $\psi c=c$ for $c\in C$. Finally, if $c\in C$ and $\gamma_a\in\Gamma_1'$, we have

$$\psi U_1c\gamma_a = \psi ca\gamma_a = \psi a\cdot\psi c\gamma_a = \phi a\cdot c\gamma_{\phi a} = U_2c\gamma_{\phi a} = U_2\psi c\gamma_a$$

therefore $\psi U_1=U\psi_2$. By Theorem 3, (Γ_1, U_1) and (Γ_2, U_2) are isomorphic.

For transformations with discrete models we have the following characterization:

THEOREM 5. *A measure preserving transformation T on a probability measure space (X, Σ, μ) has a discrete model if and only if there exists a set $\Gamma'\subset\Gamma(\mu)$ such that*

(a) Γ' *is a group;*

(b) Γ' *is an orthonormal basis of $L^2(\mu)$;*

(c) $U_T\Gamma'\subset C\Gamma'$.

We remark first that if Γ' is a group and an orthonormal basis in $L^2(\mu)$, then Γ' contains no constant function except 1, so that $C\cdot\Gamma'$ is a direct product.

If conditions a, b and c are satisfied, then $(C\cdot\Gamma', U_T, \varphi_\mu)$ is a discrete algebraic model for T. In fact, $C\subset C\cdot\Gamma'$ and $C\cdot\Gamma'$ generates $L^2(\mu)$; if $c\in C$, then

$$\varphi_\mu(c) = \int c\, d\mu = c$$

while if $\gamma \notin C$, then $\gamma = c \cdot \gamma'$ for some $c \in C$ and $\gamma' \in \Gamma'$ with $\gamma' \neq 1$, therefore

$$\varphi_\mu(\gamma) = c \int \gamma' \, d\mu = c(\gamma'|1) = 0.$$

Conversely, let (Γ, U, φ) be a discrete algebraic model for T; we may suppose $\Gamma \subset \Gamma(\mu)$, $U = U_T$ and $\varphi = \varphi_\mu$. Write Γ as a direct product $\Gamma = C \cdot \Gamma'$, where Γ' is a subgroup of Γ, containing no constant function except 1. Finally, Γ' is an orthonormal system, since for $\gamma' \in \Gamma'$ we have

$$\begin{aligned} \int \gamma' \, d\mu = \varphi(\gamma') &= 1 \quad \text{if } \gamma' = 1, \\ &= 0 \quad \text{if } \gamma' \neq 1. \end{aligned}$$

COROLLARY. *If G is an abelian compact group, equipped with Haar measure μ, then continuous automorphisms τ' and rotations R on G, as well as their superpositions $\tau = R\tau'$, have discrete model.*

We remark first that continuous automorphisms τ' and rotations R, therefore, their superpositions $\tau = R\tau'$, are measure preserving.

The group of characters $\Gamma' = G^\wedge$ is an orthogonal system in $L^2(\mu)$ and $U_{\tau'}\Gamma' \subset \Gamma'$; if R is defined on G by $Rx = cx$, for some $c \in G$, then

$$U_\tau \gamma(x) = \gamma(\tau x) = \gamma(c)\gamma(\tau' x) = \gamma(c) U_{\tau'} \gamma(x)$$

for every $\gamma \in \Gamma'$, therefore $U_\tau \Gamma' \subset C \cdot \Gamma'$. By Theorem 5, τ has discrete model. Conversely:

THEOREM 6. *Every invertible measure preserving transformation T, with discrete model (Γ, U_T), on a probability measure space (X, Σ, μ), is conjugate to the superposition of a continuous automorphism and a rotation on an abelian compact group, equipped with Haar measure.*

Consider $\Gamma = C \cdot \Gamma'$ (direct product) and

$$U_T \gamma = \rho(\gamma) V \gamma, \quad \text{for } \gamma \in \Gamma'$$

where ρ is a character of Γ' and V is an injective homomorphism of Γ'. Since T is invertible, we have $U_T L^2(\mu) = L^2(\mu)$, therefore $V\Gamma' = \Gamma'$. Consider Γ' endowed with the discrete topology and consider the Haar measure ν on the abelian compact group $G = \Gamma'^\wedge$. Then $\rho \in G$. We define the continuous homomorphism τ' on G by

$$\langle \tau' x, \gamma \rangle = \langle x, V\gamma \rangle, \quad \text{for } x \in G \text{ and } \gamma \in \Gamma'.$$

Since $V\Gamma' = \Gamma'$, τ' is an automorphism. Consider finally the mapping $\tau: G \to G$ defined by

$$\tau(x) = \rho\tau'(x), \quad \text{for } x \in G.$$

Then $(C\cdot G^\wedge, U_\tau)$ is a discrete model for τ, and the mapping $\phi\colon C\cdot\Gamma'\to C\cdot G^\wedge$ defined by

$$\phi c\gamma = c\langle\cdot,\gamma\rangle, \quad \text{for } c\in C \text{ and } \gamma\in\Gamma'$$

is an isomorphism such that $\phi c=c$ for $c\in C$. Moreover, for $\gamma\in\Gamma'$ we have

$$\begin{aligned}\phi U_T\gamma &= \phi\rho(\gamma)V\gamma = \rho(\gamma)\langle\cdot, V\gamma\rangle\\ &= \rho(\gamma)\langle\tau'\cdot,\gamma\rangle = \langle\rho\tau'\cdot,\gamma\rangle = \langle\tau\cdot,\gamma\rangle\\ &= U_\tau\langle\cdot,\gamma\rangle = U_\tau\phi\gamma\end{aligned}$$

and this equality remains valid for $\gamma\in\Gamma$, therefore $\phi U_T=U_\tau\phi$. By Theorem 3, T and τ are conjugate.

Corollary 1. *A measure preserving transformation T on a probability measure space (X,Σ,μ) is conjugate to a continuous automorphism on a compact abelian group, equipped with Haar measure, if and only if there exists a set $\Gamma'\subset\Gamma(\mu)$ such that*

(a) *Γ' is a group;*
(b) *Γ' is an orthonormal basis of $L^2(\mu)$;*
(c) *$U_T\Gamma'=\Gamma'$.*

Corollary 2. *A measure preserving transformation T on a probability measure space (X,Σ,μ) is conjugate to a rotation on an abelian compact group, equipped with Haar measure, if and only if T has a discrete model (Γ, U_T) with $\Gamma\subset\Gamma_1$.*

We mention also the following property of discrete models.

Proposition 4. *Let T be a measure preserving transformation on a probability measure space (X,Σ,μ) and let $(\Gamma, U_T, \varphi_\mu)$, $(\Gamma', U_T, \varphi_\mu)$ be two discrete systems. If $(\Gamma, U_T, \varphi_\mu)$ is a discrete model for T and if $\Gamma\subset\Gamma'$, then $\Gamma=\Gamma'$.*

In fact, let $f\in\Gamma'$. If for every $g\in\Gamma$ we had $fg\notin C$, then

$$\int fg\,d\mu = 0$$

therefore $f\equiv 0$, which would contradict $|f|\equiv 1$.

It follows that there exists $g\in\Gamma$ with $fg\in C$.

Then $f\in\bar{g}C\subset\Gamma$, therefore $\Gamma'=\Gamma$.

5. **Ergodic transformations.** In this section we give some characterizations of ergodic transformations by means of their algebraic models.

Let (X,Σ,μ) be a probability measure space and $T\colon X\to X$ a measure preserving transformation. The transformation T is ergodic if $f\in L^2(\mu)$ and $U_Tf=f$ imply $f=$constant.

Proposition 5. *If T is ergodic, then $(\Gamma_1(T), U_T)$ is a discrete system.*

In fact if $\gamma \in \Gamma_1(T) - C$, then $U_T\gamma = c\gamma$ for some $c \neq 1$ (because T is ergodic), therefore

$$\int \gamma \, d\mu = \int U\gamma \, d\mu = c \int \gamma \, d\mu$$

consequently

$$\int \gamma \, d\mu = 0.$$

REMARKS. 1°. If T^n is ergodic for some n, then T is ergodic, therefore $(\Gamma_1(T), U_T)$ is a discrete system. Theorem 7 below states a somewhat converse property.

2°. We shall see (Corollary 1 of Proposition 6) that if T^n is ergodic for every n, then $(\Gamma_\infty(T), U_T)$ is a discrete system.

LEMMA. *If T has a discrete model (Γ, U_T) and if $\Gamma_1(T) \subset \Gamma$, then for every natural n we have*

$$\Gamma_1(T^n) \cap \Gamma = \Gamma_1(T).$$

Consider $\Gamma = C \cdot \Gamma'$, where Γ' is a group and an orthonormal basis of $L^2(\mu)$. Consider the homomorphisms $\rho_n: \Gamma' \to C$ and $V: \Gamma' \to \Gamma'$ such that $U_T^n\gamma = \rho_n(\gamma)V^n(\gamma)$, for $\gamma \in \Gamma'$.

Let $\gamma \in \Gamma_1(T^n) \cap \Gamma'$. Then $U_T^n\gamma = c\gamma$, for some $c \in C$, therefore $\rho_n(\gamma) = c$ and $V^n\gamma = \gamma$. Let $k \leqq n$ be the least natural number such that $V^k\gamma = \gamma$ and consider the k-dimensional space K generated by $\gamma, V\gamma, \ldots, V^{k-1}\gamma$. Then K is invariant under U_T, therefore there exists a basis $f_1, \ldots, f_k$ of K consisting of proper functions of U_T:

$$U_T f_i = c_i f_i, \quad \text{with } c_i \in C.$$

Then $f_i \in \Gamma_1(T) \subset \Gamma$. Moreover, we may take $f_i \in \Gamma'$ (multiplying each f_i by a suitable number of C). The basis $(f_1, \ldots, f_k)$ must then coincide with the basis $(\gamma, V\gamma, \ldots, V^{k-1}\gamma)$; for example $f_1 = \gamma$, therefore $U_T\gamma = c_1\gamma$.

It follows that $\gamma \in \Gamma_1(T)$, therefore $\Gamma_1(T^n) \cap \Gamma' \subset \Gamma_1(T)$, consequently $\Gamma_1(T^n) \cap \Gamma \subset \Gamma_1(T)$.

The converse inclusion follows from $\Gamma_1(T) \subset \Gamma_1(T^n)$.

THEOREM 7. *Suppose that T has a discrete model (Γ, U_T) and let n be a natural number. If:*

(a) *either* $\Gamma \subset \Gamma_1(T)$, *or* $\Gamma_1(T) \subset \Gamma$;

(b) $\gamma \in \Gamma$ *and* $U_T^n\gamma = \gamma$ *imply* $\gamma \in C$;

then T^n is ergodic.

Let $f \in L^2(\mu)$ be a function such that $U_T^n f = f$ and prove that f is constant.

Consider $\Gamma = C \cdot \Gamma'$, where Γ' is a group and an orthonormal basis of $L^2(\mu)$. Then

$$f = \sum_{\gamma \in \Gamma'} \alpha(\gamma)\gamma$$

where

$$\alpha(\gamma) = \int f\bar{\gamma} \, d\mu, \quad \text{for every } \gamma \in \Gamma'.$$

For every natural number $k \in N$ we have

$$U_T^{kn} f = f$$

and

$$U_T^{kn} f = \sum_{\gamma \in \Gamma'} \alpha(\gamma) \rho_{kn}(\gamma) V^{kn} \gamma$$

therefore

$$\alpha(V^{kn}\gamma) = \alpha(\gamma)\rho_{kn}(\gamma), \quad \text{for } k \in N \quad \text{and} \quad \gamma \in \Gamma',$$

whence

$$|\alpha(V^{kn}\gamma)| = |\alpha(\gamma)|, \quad \text{for } k \in N \quad \text{and} \quad \gamma \in \Gamma'.$$

We shall prove that for every element $\gamma \neq 1$ of Γ' we have $\alpha(\gamma) = 0$. Let therefore $\gamma \in \Gamma'$ be such that $\gamma \neq 1$.

If $V^{kn}\gamma = \gamma$ for some k, then $\gamma \in \Gamma_1(T^n)$. In fact, if $\Gamma \subset \Gamma_1(T)$, then $\gamma \in \Gamma_1(T^n)$ without any other assumption, while if $\Gamma_1(T) \subset \Gamma$, then by the preceding lemma

$$\gamma \in \Gamma_1(T^{kn}) \cap \Gamma = \Gamma_1(T) = \Gamma_1(T^n) \cap \Gamma.$$

Writing now the equality $\alpha(V^{kn}\gamma) = \alpha(\gamma)\rho_{kn}(\gamma)$ for $k = 1$ we obtain

$$\alpha(\gamma) = \alpha(\gamma)\rho_n(\gamma)$$

therefore either $\alpha(\gamma) = 0$ or $\rho_n(\gamma) = 1$. But $\rho_n(\gamma) = 1$ means $U_T^n \gamma = \gamma$, which by hypothesis implies $\gamma = 1$ and we get a contradiction. It follows that $\alpha(\gamma) = 0$.

If $V^{kn}\gamma \neq \gamma$ for every k, then the functions γ, $V^n\gamma$, $V^{2n}\gamma$, are different from each other, therefore

$$\sum_{k=0}^{\infty} |\alpha(V^{kn}\gamma)|^2 \leqq \sum_{\gamma' \in \Gamma'} |\alpha(\gamma')|^2 < \infty$$

consequently $|\alpha(V^{kn}\gamma)| \to 0$ as $k \to \infty$, whence $\alpha(\gamma) = 0$.

It follows that $f = \alpha(1)1$, that is f is constant, consequently T^n is ergodic.

REMARKS. 1°. Is it possible to drop condition (a) in the preceding theorem? The answer is positive if condition (b) is satisfied for every n (see Theorem 9 below).

2°. Is it true that if T is ergodic, then $\Gamma_1(T) \subset \Gamma$ for every discrete model (Γ, U_T) of T?

The answer is positive if, in addition, T^n is ergodic for every n. Moreover, in this case we have $\Gamma_\infty(T) \subset \Gamma$ for every discrete model (Γ, U_T) of T (see Corollary 2 of Proposition 6).

For ergodic transformations, we have the following conjugacy criterion:

THEOREM 8. *Two ergodic transformations T and T' with discrete model, are conjugate if and only if there exist discrete models (Γ, U_T) and $(\Gamma', U_{T'})$ of T and T' respectively, such that the groups $w_T\Gamma$ and $w_{T'}\Gamma'$ are isomorphic by an isomorphism ϕ such that $\phi w_T = w_{T'}\phi$ and $\phi c = c$ for $c \in C \cap w_T\Gamma$.*

We use Theorem 4.

6. **Transformations with ergodic iterates.** Let (X, Σ, μ) be a probability measure space and $T: X \to X$ a measure preserving transformation.

PROPOSITION 6. *Suppose that T^n is ergodic for every n. If (Γ, U_T) is a discrete system, then $(\bigcup_{n=0}^{\infty} w_T^{-n}(\Gamma), U_T)$ is again a discrete system.*

We prove first that $(w_T^{-1}(\Gamma), U_T)$ is a discrete system.

It is clear that $w_T^{-1}(\Gamma)$ is a subgroup of $\Gamma(\mu)$ invariant under U_T and containing C. We have to prove that

$$\varphi_\mu(f) = \int f\, d\mu = 0, \quad \text{for } f \in w_T^{-1}(\Gamma) - C.$$

Let $f \in w_T^{-1}(\Gamma) - C$. There are two possibilities:

(a) $\int U_T^n f \cdot \bar{f}\, d\mu = 0$, for every $n \geqq 1$.

Then $f, U_T f, U_T^2 f, \ldots$ is an orthonormal system in $L^2(\mu)$. If g is the projection of 1 on the space generated by this sequence, we have

$$g = \sum_{n=0}^{\infty} a_n U_T^n f \quad \text{with } \sum |a_n|^2 < \infty.$$

Then

$$\int f\, d\mu = \int U_T^n f\, d\mu = (U_T^n f \mid 1) = (U_T^n f \mid g) = a_n$$

and $a_n \to 0$, therefore $\int f\, d\mu = 0$.

(b) There exists n such that

$$\int U_T^n f \cdot \bar{f}\, d\mu \neq 0.$$

Since

$$U_T^n f \cdot \bar{f} = w_T(f \cdot U_T f \cdot \cdots \cdot U_T^{n-1} f) \in \Gamma$$

and since (Γ, U_T) is a discrete system, we have

$$U_T^n f \cdot \bar{f} \in C$$

that is

$$U_T^n f = cf \quad \text{for some } c \in C.$$

Since f is not constant and T^n is ergodic, we have $c \neq 1$.

Suppose that n is the least natural number satisfying $U_T^n f = cf$. The n-dimensional space K generated by $f, U_T f, \ldots, U_T^{n-1} f$ is invariant under U_T, therefore there exists a basis $f_1, \ldots, f_n$ of K consisting of proper functions of U_T:

$$U_T f_i = c_i f_i, \quad \text{with } c_i \in C.$$

Each f_i is of the form

$$f_i = \sum_{k=0}^{n-1} \alpha_{ik} U_T^k f$$

therefore

$$U_T^n f_i = c f_i.$$

On the other hand

$$U_T^n f_i = c_i^n f_i$$

therefore

$$c_i^n = c, \quad \text{for every } i.$$

Then $c_i \neq 1$ for each i and

$$\int f_i \, d\mu = \int U_T f_i \, d\mu = c_i \int f_i \, d\mu$$

therefore $\int f_i \, d\mu = 0$. From

$$f = \sum_{i=1}^{n} \alpha_i f_i$$

we deduce that $\int f \, d\mu = 0$.

By induction we deduce then that for every n, $(w_T^{-n}(\Gamma), U_T)$ is a discrete system, therefore $(\bigcup_{n=0}^{\infty} w_T^{-n}(\Gamma), U_T)$ is also a discrete system.

COROLLARY 1. *If T^n is ergodic for every n, then* $(\Gamma_\infty(T), U_T)$ *is a discrete system.*

In fact (C, U_T) is a discrete system, and $\Gamma_\infty(T) = \bigcup_{n=1}^{\infty} w_T^{-n}(C)$.

COROLLARY 2. *Suppose that T has a discrete model* (Γ, U_T). *If T^n is ergodic for every n, then*

$$w_T^{-1}\Gamma = \Gamma \quad \textit{and} \quad \Gamma_\infty(T) \subset \Gamma.$$

In fact, in this case $(w_T^{-1}\Gamma, U_T)$ is again a discrete system and $\Gamma \subset w^{-1}\Gamma$, therefore, by Proposition 3, $w^{-1}\Gamma = \Gamma$. Then $\Gamma_\infty(T) \subset \Gamma$.

THEOREM 9. *Suppose that T has a discrete model* (Γ, U_T).

If for every natural number n, conditions $\gamma \in \Gamma$ and $U^n\gamma = \gamma$ imply $\gamma \in C$, then T^n is ergodic for every n.

Consider Γ as a direct product $\Gamma = C \cdot \Gamma'$, where Γ' is a group and an orthonormal basis of $L^2(\mu)$.

Consider the homomorphisms $\rho_n: \Gamma' \to C$ and $V: \Gamma' \to \Gamma'$ such that $U_T^n\gamma = \rho_n(\gamma)V^n\gamma$, for $\gamma \in \Gamma'$ and $n \in N$.

We shall prove first that T is ergodic. Let $f \in L^2(\mu)$ be such that $U_T f = f$ and prove that f is constant. We have

$$f = \sum_{\gamma \in \Gamma'} \alpha(\gamma)\gamma$$

where $\alpha(\gamma) = (f \mid \gamma)$. For every n we have $U_T^n f = f$ and

$$U_T^n f = \sum_{\gamma \in \Gamma'} \alpha(\gamma)\rho_n(\gamma)V^n\gamma$$

therefore

$$\alpha(V^n\gamma) = \alpha(\gamma)\rho_n(\gamma), \quad \text{for } \gamma \in \Gamma' \quad \text{and} \quad n \in N$$

whence

$$|\alpha(V^n\gamma)| = |\alpha(\gamma)|, \quad \text{for } \gamma \in \Gamma' \quad \text{and} \quad n \in N.$$

We shall prove that $\alpha(\gamma)=0$ if $\gamma\neq 1$. We remark that the hypothesis implies that if $\gamma\neq 1$, then $\rho_n(\gamma)\neq 1$ for every n.

If $\gamma\neq 1$ and $V^n\gamma=\gamma$ for some n, then the equality $\alpha(V^n\gamma)=\alpha(\gamma)\rho_n(\gamma)$ becomes

$$\alpha(\gamma) = \alpha(\gamma)\rho_n(\gamma)$$

therefore $\alpha(\gamma)=0$.

If $\gamma\neq 1$ and $V^n\gamma\neq\gamma$ for every n, then the functions $\gamma, V\gamma, V^2\gamma,\dots$ are different from each other, therefore

$$\sum_{n=1}^{\infty} |\alpha(V^n\gamma)|^2 \leqq \sum_{\gamma'\in\Gamma'} |\alpha(\gamma')|^2 = \|f\|_2^2 < \infty$$

consequently $|\alpha(V^n\gamma)|^2\to 0$ as $n\to\infty$, whence $\alpha(\gamma)=0$.

We deduce that $f=\alpha(1)1$, that is f is constant, therefore T is ergodic.

We remark now that for every n, (Γ, U_T^n) is a discrete model for T^n, satisfying the conditions of the theorem with respect to U_T^n, therefore T^n is ergodic.

7. **Transformations with quasi-discrete spectrum.** Let (X, Σ, μ) be a probability measure space and $T: X\to X$ a measure preserving transformation.

DEFINITION 4. *We say that T has quasi-discrete spectrum if T has a discrete model (Γ, U_T) with $\Gamma\subset\Gamma_\infty(T)$.*

To say that T has quasi-discrete spectrum means that there exists a group $\Gamma'\subset\Gamma_\infty(T)$ which is an orthonormal basis of $L^2(\mu)$, such that $U_T\Gamma'\subset C\cdot\Gamma'$.

Here are some properties of transformations with quasi-discrete spectrum:

(1) *If T has quasi-discrete spectrum, then $(\Gamma_\infty(T), U_T, \varphi_\mu)$ is an (not necessarily discrete) algebraic model of T.*

(2) *If T has quasi-discrete spectrum, then T^n has quasi-discrete spectrum, for every n (since $\Gamma_\infty(T)\subset\Gamma_\infty(T^n)$).*

(3) *If T is conjugate with a transformation with quasi-discrete spectrum, then T has itself quasi-discrete spectrum* (see Proposition 1).

(4) *Two transformations T and T' with quasi-discrete spectrum are conjugate if and only if the a.e. systems $(\Gamma_\infty(T), U_T, \varphi_\mu)$ and $(\Gamma_\infty(T'), U_{T'}, \varphi_{\mu'})$ are isomorphic.*

We use Proposition 1 and Theorem 1.

For transformations T for which $(\Gamma_\infty(T), U_T)$ is itself a discrete model we have, in addition, the following properties:

(5) *Let T and T' be two measure preserving transformations having $(\Gamma_\infty(T), U_T)$ respectively $(\Gamma_\infty(T'), U_{T'})$ as discrete models.*

Then T and T' are conjugate if and only if there exists an isomorphism J of $\Gamma_\infty(T)$ onto $\Gamma_\infty(T')$ such that

$$Jc = c, \quad \text{for } c\in C$$

and

$$JU_T = U_{T'}J.$$

(6) *If T has quasi-discrete spectrum and if there exists a discrete system* (Γ^*, U_T) *with* $\Gamma_\infty(T) \subset \Gamma^*$, *then* $(\Gamma_\infty(T), U_T)$ *is a discrete model of T and there is no other discrete model of T containing or contained in* $(\Gamma_\infty(T), U_T)$.

We use Proposition 3.

For transformations with all iterates ergodic we have some more properties:

(7) *If T^n is ergodic for every n and if* $\Gamma_\infty(T)$ *generates* $L^2(\mu)$, *then* $(\Gamma_\infty(T), U_T)$ *is a discrete model of T and there is no other discrete model of T.*

In fact, by Corollary 1 of Proposition 6, $(\Gamma_\infty(T), U_T)$ is a discrete system, therefore $(\Gamma_\infty(T), U_T)$ is a discrete model of T. By Corollary 2 of Proposition 6, for any other discrete model (Γ, U_T) of T we have $\Gamma_\infty(T) \subset \Gamma$, therefore $\Gamma_\infty(T) = \Gamma$.

(8) *Let T and T' be two transformations with quasi-discrete spectrum and all iterates T^n and T'^n ergodic.*

Then T and T' are conjugate if and only if $w_T\Gamma_\infty(T)$ *and* $w_{T'}\Gamma_\infty(T')$ *are isomorphic by an isomorphism ϕ such that* $\phi w_T = w_{T'}\phi$ *and* $\phi c = c$ *for* $c \in C \cap w_T\Gamma_\infty(T)$.

We use Theorem 8 and property (5) above.

The following theorem gives a characterization of discrete systems which are models for transformations with quasi-discrete spectrum.

THEOREM 10. *If* (Γ, U) *is a discrete system such that*

$$\Gamma = \bigcup_{n=0}^{\infty} w^{-n}(C), \quad \text{where } w(\gamma) = U\gamma\cdot\gamma^{-1}, \quad \text{for} \quad \gamma \in \Gamma,$$

then the corresponding transformation T has quasi-discrete spectrum.

If, in addition, for every natural number $n \in N$, $\gamma \in \Gamma$ *and* $U^n\gamma = \gamma$ *imply* $\gamma \in C$, *then T^n is ergodic for every n.*

In fact w is the restriction of w_T to Γ, therefore $\Gamma \subset \Gamma_\infty(T)$, consequently T has quasi-discrete spectrum.

For the second part of the theorem we use Theorem 9 to deduce that all the iterates T^n are ergodic. In this case we have $\Gamma = \Gamma_\infty(T)$.

REMARK. Theorem 10 and property (8) were proved by Abramov [1].

Example of transformation with discrete model but without quasi-discrete spectrum. Let $X_n = \{-1, 1\}$ and $\mu_n(\{-1\}) = \mu_n(\{1\}) = \frac{1}{2}$ for $n = 0, \pm 1, \pm 2, \ldots$. Consider the product $X = \prod_{n=-\infty}^{\infty} X_n$, equipped with the product measure μ and the bilateral shift $T(x_n) = (y_n)$, where $y_n = x_{n+1}$ for every n. Then T^n is ergodic for every n and the only proper value of T is 1, so that $w_T^{-1}(C) = C$. It follows that $\Gamma_\infty(T) = C$ so that T has not quasi-discrete spectrum.

On the other hand, consider the function $f_0\colon X \to R$ defined by

$$\begin{aligned} f_0((x_n)) &= -1 \quad \text{if } x_0 = -1, \\ &= 1 \quad\ \ \text{if } x_0 = 1, \end{aligned}$$

and the group Γ generated by $U_T^n f_0$, $n = 0, \pm 1, \pm 2, \ldots$ and by the constants. Then (Γ, U_T) is a discrete model of T.

8. **Transformations with discrete spectrum.** Let (X, Σ, μ) be a probability measure space and $T: X \to X$ a measure preserving transformation.

DEFINITION 5. *We say that T has discrete spectrum if T has a discrete model (Γ, U_T) with $\Gamma \subset \Gamma_1(T)$.*

To say that T has a discrete spectrum means that there exists a group $\Gamma' \subset \Gamma_1(T)$ of proper functions of U_T which is an orthonormal basis of $L^2(\mu)$.

Here are some properties of transformations with discrete spectrum:

(1) *Every transformation with discrete spectrum is invertible (since $U_T\Gamma_1 = \Gamma_1$).*

(2) *A transformation has discrete spectrum if and only if it is conjugate to a rotation on a compact abelian group equipped with Haar measure (see Corollary 2 of Theorem 6).*

(3) *Every transformation with discrete spectrum has quasi-discrete spectrum.*

(4) *If T has discrete spectrum, then $(\Gamma_1(T), U_T, \varphi_\mu)$ is an (not necessarily discrete) algebraic model of T.*

(5) *If T has discrete spectrum, then T^n has discrete spectrum for every n.*

(6) *If T is conjugate with a transformation with discrete spectrum, then T has itself discrete spectrum.*

(7) *Two transformations T and T' with discrete spectrum are conjugate if and only if the a.e. systems $(\Gamma_1(T), U_T, \varphi_\mu)$ and $(\Gamma_1(T'), U_{T'}, \varphi_{\mu'})$ are isomorphic.*

Transformations T for which $(\Gamma_1(T), U_T)$ is itself a discrete model, have additional properties:

(8) *Let T and T' be two measure preserving transformations having $(\Gamma_1(T), U_T)$ respectively $(\Gamma_1(T'), U_{T'})$ as discrete models. Then T and T' are conjugate if and only if there exists an isomorphism J of $\Gamma_1(T)$ onto $\Gamma_1(T')$ such that:*

$$Jc = c, \quad \text{for } c \in C$$

and

$$JU_T = U_{T'}J.$$

(9) *If T has discrete spectrum and if there exists a discrete system (Γ^*, U_T) with $\Gamma_1(T) \subset \Gamma^*$ then $(\Gamma_1(T), U_T)$ is a discrete model of T, and there is no other discrete model of T containing or contained in $(\Gamma_1(T), U_T)$.*

For ergodic transformations we have some more properties:

(10) *If T is ergodic and if $\Gamma_1(T)$ generates $L^2(\mu)$, then $(\Gamma_1(T), U_T)$ is a discrete model of T, and there is no other discrete model containing or contained in $(\Gamma_1(T), U_T)$.*

(11) *Let T and T' be two ergodic transformations with discrete spectrum.*

Then T and T' are conjugate if and only if U_T and $U_{T'}$ have the same spectrum [4, p. 46].

We use Theorem 8 remarking that $w_T\Gamma_1(T)$ is the spectrum of U_T and $w_{T'}\Gamma_1(T')$ is the spectrum of T'.

The characterization of discrete systems which are models for transformations with discrete spectrum, is given by the following:

THEOREM 11. *If* (Γ, U) *is a discrete system such that*

$$w^{-1}(C)=\Gamma, \quad \text{where } w(\gamma) = U\gamma\cdot\gamma^{-1}, \quad \text{for} \quad \gamma\in\Gamma$$

then the corresponding transformation T has discrete spectrum.

If, in addition, there exists $n\in N$ *such that* $\gamma\in\Gamma$ *and* $U^n\gamma=\gamma$ *imply* $\gamma\in C$, *then* T^n *is ergodic.*

In fact, w is the restriction of w_T to Γ, therefore $\Gamma\subset\Gamma_1(T)$, consequently T has discrete spectrum. The second part follows from Theorem 9. In this case we have $\Gamma=\Gamma_1(T)$.

When does Γ_1 coincide with Γ_∞?

THEOREM 12. *Let T be an ergodic transformation with discrete spectrum, on a probability measure space* (X, Σ, μ). *We have* $\Gamma_1(T)=\Gamma_\infty(T)$ *if and only if the point spectrum of* U_T *contains no root of* 1 (*except* 1 *itself*).

Suppose first that $\Gamma_1=\Gamma_\infty$. Let ξ be a proper value of a function $f\in\Gamma_1$:

$$U_Tf = \xi f.$$

We shall prove that if $\xi^n=1$ for some n, then $\xi=1$. In fact, suppose that N is the least natural number with $\xi^N=1$. We have then

$$U_Tf^N = f^N$$

therefore, (since T is ergodic) f^N is constant, and we may suppose that $f^N\equiv 1$, multiplying f by a suitable number, if necessary. Then f takes on the values $1, \xi,\dots, \xi^{N-1}$ on the corresponding sets $A_0, A_1,\dots, A_{N-1}$

$$f = \sum_{k=0}^{N-1} \xi^k\varphi_{A_k}.$$

Since $f(Tx)=\xi f(x)$, we have

$$\sum_{k=0}^{N-1} \xi^k\varphi_{T^{-1}A_k} = \sum_{k=0}^{N-1} \xi^{k+1}A_k$$

therefore $TA_k=A_{k+1}$ for $k=0, 1,\dots, N-1$, where $A_N=A_0$. It follows that $\mu(A_k)=\mu(A_0)>0$ for every k.

If N is odd we take ν such that $\nu^N=1$; if N is even, we take ν such that $\nu^N=-1$.

Define now the function

$$g = \sum_{k=0}^{N-1} \nu^k\xi^{k(k-1)/2}\varphi_{A_k}.$$

Then we have

$$Ug = \nu fg.$$

If fact, if $x\in A_k$ and $k=0, 1,\dots, N-2$, we have $Tx\in A_{k+1}$, therefore

$$Ug(x) = g(Tx) = \nu^{k+1}\xi^{k(k+1)/2} = \nu\xi^k\nu^k\xi^{k(k-1)/2} = \nu f(x)g(x)$$

and for $x\in A_{N-1}$ we have $Tx\in A_N=A_0$, therefore

$$Ug(x) = g(Tx) = 1 = \nu^N\xi^{N(N-1)/2} = \nu\xi^{N-1}\nu^{N-1}\xi^{(N-1)(N-2)/2} = \nu f(x)g(x)$$

since $\xi^{N(N-1)/2}$ is equal to 1 if N is odd and to -1 if N is even.

It follows that $g \in \Gamma_2 = \Gamma_1$, therefore $\nu f = \alpha \in C$, consequently $f \in C$. We deduce then that $Uf = f$, therefore $\xi = 1$.

Conversely, suppose that $\Gamma_1 \neq \Gamma_2$ and prove that U_T has at least a proper value $\alpha \neq 1$ such that $\alpha^N = 1$ for some N.

Let $g_0 \in \Gamma_2 - \Gamma_1$. Since Γ_1 is an orthonormal basis of $L^2(\mu)$ and $g_0 \neq 0$, there exists $h \in \Gamma_1$ such that

$$(g_0, \bar{h}) = \int g_0 h \, d\mu \neq 0.$$

If we put $g = g_0 h$, we have $g \in \Gamma_2 - \Gamma_1$ and $\int g \, d\mu \neq 0$. There exists a function $f \in \Gamma_1$ such that

$$U_T g = fg.$$

There exists also a number $\lambda \in C$ such that

$$U_T f = \lambda f.$$

By induction, we deduce that for every n we have

$$U_T^n g = \lambda^{n(n-1)/2} f^n g.$$

We have $f^n \in C$ for some n. In fact, if we had $f^n \notin C$ for every n, then (since $f^n \in \Gamma_1$),

$$\int U_T^n g \bar{g} \, d\mu = \lambda^{n(n-1)/2} \int f^n \, d\mu = 0$$

therefore, the sequence $g, Ug, U^2 g, \ldots$ would be orthonormal, consequently

$$\int g \, d\mu = \int U_T^n g \, d\mu = (U_T^n g, 1) \to 0$$

and we would get a contradiction.

Let N be the least integer $\geqq 0$ such that

$$f^N = \mu \in C.$$

Then

$$U^N g = \lambda^{N(N-1)/2} \mu g = \xi g.$$

We have

$$\xi \int g \, d\mu = \int U_T^n g \, d\mu = \int g \, d\mu \neq 0$$

therefore $\xi = 1$, consequently

$$U^N g = g.$$

Since g is not constant and since Γ_1 generates $L^2(\mu)$, there exists a proper function $k \neq 1$ of Γ_1 such that $(g, k) \neq 0$. If α is the corresponding proper value:

$$U_T k = \alpha k$$

we have $\alpha \neq 1$ and

$$(g, k) = (U^N g, k) = (g, U^{-N} k) = (g, \alpha^{-N} k) = \alpha^N (g, k)$$

therefore $\alpha^N = 1$.

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