

ALGEBRAIC MONOIDS WHOSE NONUNITS ARE PRODUCTS OF IDEMPOTENTS

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(Communicated by Bhama Srinivasan)

ABSTRACT. Let M be a connected regular linear algebraic monoid with zero and group of units G . Suppose G is nearly simple, i.e. the center of G is one dimensional and the derived group G' is a simple algebraic group. Then it is shown that $S = M \setminus G$ is an idempotent generated semigroup. If M has a unique nonzero minimal ideal, the converse is also proved. It follows that if G_0 is any simple algebraic group defined over an algebraically closed field K and if $\Phi: G_0 \rightarrow GL(n, K)$ is any representation of G_0 , then the nonunits of the monoid $M(\Phi) = K\Phi(G_0)$ form an idempotent generated semigroup.

It has been shown by J. Erdos [3] (see also [2]) that any nonsingular matrix over a field is a product of idempotent matrices. Let K be an algebraically closed field. Our interest is in connected linear algebraic monoids with zero. This means by definition that the underlying set is an irreducible affine variety and that the product map is a morphism (i.e. a polynomial map). Let G denote the group of units of M , $S = M \setminus G$. We are interested in knowing when S is idempotent generated. We will only consider the situation when G is a *reductive group*. This means [1] that the unipotent radical of G is trivial. Then by [8, 10], M is *unit regular*, i.e. $M = E(M)G$ where $E = E(M) = \{e \in M | e^2 = e\}$. If $X \subseteq M$, let $E(X) = X \cap E(M)$ and $\langle X \rangle$ the semigroup generated by X . Let $\mathcal{J}, \mathcal{R}, \mathcal{L}, \mathcal{H}$ denote the usual Green's relations on M (see [5]). If $a, b \in M$, then $a \mathcal{J} b$ means $MaM = MbM$, $a \mathcal{R} b$ means $aM = bM$, $a \mathcal{L} b$ means $Ma = Mb$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. The following result was proved by the author [9, Theorem 2.7].

THEOREM 1 [9]. *Suppose $S = M \setminus G$ has a maximum \mathcal{J} -class (i.e. S is an irreducible variety). Then S is an idempotent generated semigroup.*

Let R, G' denote the radical, derived group of G , respectively. Then [1, §1.8], R is contained in the center of G , G' is semisimple and $G = RG'$. Suppose $\dim R > 1$. Then there exists $e \in E(\overline{R})$, $e \neq 0, 1$. There exists $g \in G$ such that $eg \neq e$. If $eg \in \langle E(M) \rangle$, then $eg = e_1 \cdots e_k$ for some $e_i \in E(M)$, $i = 1, \dots, k$. Then since e is a central idempotent, $eg = e'_1 \cdots e'_k$, where $e'_i = ee_i \in E(M)$ and $e'_i \leq e$. This is a contradiction since M is a matrix semigroup. Thus $S = M \setminus G$ is not idempotent generated. We will say that G is *nearly simple* if $\dim R = 1$ and G' is a simple algebraic group, i.e. G' has no nontrivial closed connected normal subgroups.

THEOREM 2. *Suppose G is nearly simple. Then $S = M \setminus G$ is an idempotent generated semigroup.*

Received by the editors February 1, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20G99, 20M10.

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0002-9939/88 \$1.00 + \$.25 per page

PROOF. Let R denote the radical of G , G' the derived group of G . Since $\dim R = 1$, we see that $E(\overline{R}) = \{0, 1\}$. Also, in any faithful linear representation of M , every element of G' has determinant 1. It follows that G' is closed in M . Let $e \in E = E(M)$, $e \neq 0, 1$. Let G_e denote the connected component of the group $\{g \in G \mid ge = eg = e\}$. Then by [5], $e \in \overline{G_e}$. Let G_1 denote the subgroup of G generated by $x^{-1}G_ex$ ($x \in G$). Then G_1 is a closed connected normal subgroup of G , $e \in \overline{G_1}$. So $G_1 \not\subseteq R$, $G_1 \not\subseteq G'$. Since G is nearly simple, we see that $G_1 = G$. Let $E_0 = \langle E(J) \rangle$,

$$H = \{g \in G \mid eg \in \langle E_0 \rangle\}.$$

Let $f \in E_0$, $g \in H$. By [6, Lemma 1.12], there exist $e_1, f_1 \in E_0$ such that $e \mathcal{L} e_1 \mathcal{R} f_1 \mathcal{L} f$. Then $e_1g = e_1(eg) \in \langle E_0 \rangle$. So $ge_1 = g(e_1g)g^{-1} \in \langle E_0 \rangle$. Hence $gf_1 = (ge_1)f_1 \in \langle E_0 \rangle$. So $f_1g = g^{-1}(gf_1)g \in \langle E_0 \rangle$. Thus $fg = f(f_1g) \in \langle E_0 \rangle$. It follows that $E_0H \subseteq \langle E_0 \rangle$. In particular $x^{-1}Hx = H$ for all $x \in G$. Now let $a, b \in H$. Then $ea, eb \in \langle E_0 \rangle$. So $be = b(eb)b^{-1} \in \langle E_0 \rangle$. Thus $bea = (be)(ea) \in \langle E_0 \rangle$. Hence $eab = b^{-1}(bea)b \in \langle E_0 \rangle$. Thus $ab \in H$. Since $G_e \subseteq H$, we see that $G = G_1 \subseteq H$. Hence $H = G$. Since M is unit regular, we see that $S \subseteq \langle E \rangle$.

COROLLARY 1. Let G_0 be a simple algebraic group, $\Phi: G_0 \rightarrow GL(n, K)$ any representation. Let $M(\Phi)$ denote the Zariski closure of $K\Phi(G_0)$ in $\mathcal{M}_n(K)$. Then the nonunits of $M(\Phi)$ form an idempotent generated semigroup.

We now consider the situation dual to that in Theorem 1, namely when $M \setminus \{0\}$ has a minimum \mathcal{J} -class. Equivalently, the intersection of all nonzero ideals of M is again nonzero. Renner [11, Corollary 8.3.3] has characterized these monoids as being exactly those having an idempotent separating (equivalently finite in the algebraic geometry sense) irreducible, linear representation. Moreover, in such a case the radical R of G is one dimensional [11, Lemma 8.3.2].

THEOREM 3. Suppose M has a unique nonzero minimal ideal. Then $S = M \setminus G$ is an idempotent generated semigroup if and only if G is nearly simple.

PROOF. Let R denote the one dimensional radical of G and J_0 the nonzero minimum \mathcal{J} -class of M . So $J_0 \cup \{0\}$ is the minimum nonzero ideal of M . If G is nearly simple, then by Theorem 2, S is idempotent generated. Conversely suppose that $S \subseteq \langle E \rangle$ where $E = E(M)$. Suppose G is not nearly simple. Then $G = G_1G_2$ where G_1, G_2 are proper closed connected normal subgroups of G containing R , and such that $g_1g_2 = g_2g_1$ for all $g_i \in G_i$, $i = 1, 2$. Let M_i denote the closure of G_i in M , $i = 1, 2$. Since $R \neq G_1$, there exists $e \in E(M_1)$, $e \neq 0, 1$. By [5], $J = GeG$ is the \mathcal{J} -class of e in M . By [5], any element of $E(J)$ is conjugate to e . Since G_2 centralizes M_1 and $G = G_1G_2$, we see that $E(J) \subseteq M_1$. Let $g \in G$, $a = eg$. Then $a = e_1 \dots e_m$ for some $e_1, \dots, e_m \in E$. Let

$$e'_i = e_i \dots e_m g^{-1} e e_1 \dots e_i \in E(J), \quad i = 1, \dots, m.$$

Then with $e_0 = e$, we have for $i = 1, \dots, m$,

$$e_0 \dots e_{i-1} e'_i = e_0 \dots e_m g^{-1} e e_1 \dots e_i = e g g^{-1} e e_1 \dots e_i = e e_1 \dots e_i.$$

So $ee'_1 \dots e'_m = ee_1 \dots e_m = a$. Thus $a \in \langle E(J) \rangle \subseteq M_1$. Hence $ge = gag^{-1} \in \langle E(J) \rangle \subseteq M_1$. It follows that $J = GeG \subseteq M_1$. Hence $MeM = \overline{J} \subseteq M_1$. Thus $J_0 \subseteq M_1$. So J_0 is centralized by G_2 . Similarly $J_0 \subseteq M_2$ and J_0 is centralized by

G_1 . Thus J_0 is centralized by G . So if $f \in E(J_0)$, then by [7, Theorem 2.3], $f \in \bar{R}$. This contradicts the fact that $\dim R = 1$, proving the theorem.

The difference between Theorems 1 and 3 is illustrated by the following elementary examples.

EXAMPLE 1. Let $M = \{A \oplus B \mid A, B \in \mathcal{M}_2(K), \det A = \det B\}$, G the group of units of M . Then $S = M \setminus G$ has a maximum \mathcal{J} -class and by Theorem 1, S is idempotent generated. Note that G is not nearly simple.

EXAMPLE 2. Let $M = \{A \otimes B \mid A, B \in \mathcal{M}_2(K)\}$, G the group of units of M , $S = M \setminus G$. Then M has a unique nonzero minimal ideal. Clearly G is not nearly simple and hence by Theorem 3, S is not idempotent generated.

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