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Algebraic-Numerical Method for the Slightly Perturbed Harmonic Oscillator

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Abstract. The solution of slightly perturbed harmonic oscillators can easily be obtained in the form of a series given by Poisson's method. However, this perturbation method leads to secular terms unbounded for large time (the time unit being the fundamental period of the harmonic oscillator), which prevent the use of finite series. The analytical elimination of such terms was first solved by Poincaré and, more recently, generalized by Krylov and Bogoliubov. Unfortunately, these methods are very difficult to handle and are not easily carried out for high orders.

A numerical reinitialization method is combined here with the Poisson perturbation treatment to avoid the growth of secular terms and therefore to get the solution at any time. The advantages of such a method is that the analytical work can be carried to high orders keeping the step of numerical integration to a relatively large value (compared to a purely numerical method). This algorithm has been tested on the Mathieu equation. A method for the computation of the eigenvalues of this equation is given. By properly selecting the order of the perturbation and the time step of reinitialization, we can recover, at any order, all the effects of the slight perturbation (including *all* the unstable zones).

Consequently, such a method is a useful intermediate between purely analytical and purely numerical algorithms.

I. Introduction. The evolution of slightly perturbed physical systems is often described by an equation of the following form:

$$d^2x/dt^2 + x = \epsilon f(x, \dot{x}, t). \tag{1}$$

In Eq. (1), the angular frequency is taken equal to 1 and ϵ is a small parameter. This is the equation of an oscillating system subjected to a small and possibly nonlinear force.

We are interested in finding the solution of Eq. (1) when the deviation from the solution of $\ddot{x} + x = 0$ is large. Such a large deviation will occur after a time T of order $1/\epsilon$.

Using classical numerical methods (a Runge-Kutta algorithm, for example), the resolution of Eq. (1), for large time, presents some difficulties. To get the solution

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correctly, the step of integration Δt must be very small. Consequently, computation time is prohibitively long, and unavoidable numerical difficulties (round-off errors, instability of the method) can, at the end, completely mask the variation itself.

While the solution of the differential equation $\ddot{x} + x = 0$ may be numerically difficult, its analytic form is well known. Therefore, this solution can be used, together with a perturbation method, to solve Eq. (1). In this new process, the step of integration Δt will only have to be much smaller than ϵ^{-1} (instead of $\Delta t \ll 1$ for classical methods). The advantage of this method is that for any value of ϵ, the number of steps to reach the time $1/\epsilon$ is approximately constant and so is the computation time.

Using a perturbation method, we thus seek a solution of (1) in the form:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots + \epsilon^n x_n + \cdots. \tag{2}$$

We identify terms of the same power in ϵ and first solve $\ddot{x}_0 + x_0 = 0$. Then, $\ddot{x}_1 + x_1 = f(x_0, \dot{x}_0, t)$, etc.

The main point is that the initial conditions (at $t = 0$, say) are completely absorbed by the zero order solution x_0, i.e.,

$$\begin{aligned} &x_0(0) = x(0) \quad \text{and} \quad x_n(0) = 0, \qquad n \neq 0, \\ &\dot{x}_0(0) = \dot{x}(0) \quad \text{and} \quad \dot{x}_n(0) = 0, \qquad n \neq 0. \end{aligned} \tag{3}$$

If, as is often the case, $f(x, \dot{x}, t)$ is either a polynomial in $x, \dot{x}$ or a trigonometric function of t (or a combination of both), the analytical expressions of x_1, x_2, etc. $\cdots$ are easily obtained. It can be shown that, in this case, we simply have to solve equations of the type

$$\ddot{y} + y = \sum_{m,p} A_{mp}\, t^m \exp(ipt), \tag{4}$$

which is easily accomplished. The only practical difficulty is that the algebra quickly becomes cumbersome, but, at that point, a formula manipulation language such as FORMAC could be used.

In the general case, the series given by (2) will contain secular terms, i.e., terms which are unbounded as $t \to \infty$. These terms come from the resonance between the solution of $\ddot{y} + y = 0$ and the forcing term $A_{mp}\, t^m \exp(ipt)$ for $p = \pm 1$. Consequently, in the worst case, the first order term x_1 will vary like t, x_2 like t^2, and x_n like t^n, and the series in (2) will be convergent (or at least asymptotically convergent) as soon as $\epsilon t \ll 1$.

If we want to get the solution for a time t such that $\epsilon t > 1$, we have the choice between two methods:

A purely analytical method such as the Krylov-Bogoliubov method [1] which eliminates the secular term and gives the solution for any time. This method proceeds from the same philosophy as the adiabatic invariant method for the harmonic time varying oscillator given by Chandrasekhar [2] or Lewis [3]. However, it must be

pointed out that some effects are lost in the adiabatic method [4] and the elimination of the secular term in the Krylov-Bogoliubov method is analytically difficult and long, practically preventing the calculation of high order effects.

A numerical method which consists in taking a step T such that $\epsilon T \ll 1$, computing $x(T)$ and $\dot{x}(T)$ by (2), (3) and reinitializing the perturbation series by writing

$$\tilde{x}_0(T) = x(T) \quad \text{and} \quad \dot{\tilde{x}}_0(T) = \dot{x}(T).$$

The sign $\sim$ indicates that we build a new set of functions

$$\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n, \cdots.$$

The difficult elimination of secular terms is now entirely handled in a numerical way, and there is no need to solve $\ddot{x} + x = 0$ numerically with time step $\Delta t \ll 1$.

We must point out that this is really a numerical method based upon the fact that $x(T)$ and $\dot{x}(T)$ can be computed as accurately as we like. The order of perturbation to which the series (2) must be pushed enters only through the size of the time step which we use. We may, if we like, stop at the first order $x_0 + \epsilon x_1$ and still recover all the properties of Eq. (1). But the size of the time step T is connected to the order we want to use. This time step can be increased with a compromise to be found between the complexity of the algebraic perturbation calculations and its size.

We will call such a method a "giant step method" or alternatively an "algebraic-numerical method".

II. Application to Mathieu's Equation. Mathieu's equation

$$d^2x/dt^2 + (A - 2Q\cos 2t)x = 0, \quad A > 0 \tag{5}$$

is an interesting and easy to use example for our method.

Let us assume that the relative change of frequency is small. The ratio $\epsilon = 2Q/A$ can then be taken as the perturbation parameter.

In Eq. (5) we introduce a new variable $\theta = t\sqrt{A}$, and we slightly generalize (5) to incorporate the reintialization process, letting $\Theta = T\sqrt{A}$ denote the reduced "giant step". We obtain

$$\frac{d^2x}{d\theta^2} + x = \epsilon x \cos\left(\frac{2}{\sqrt{A}}(\theta + \Theta)\right) \tag{6}$$

which is a special case of Eq. (1).

Using (2) and (3), we first look for the zero order approximation

$$x_0 = R_0 \cos\theta + S_0 \sin\theta$$

with

$$R_0 = x(0), \qquad S_0 = (dx/d\theta)(0).$$

Assuming that x_{n-1} is known and setting $\alpha = 2/\sqrt{A}$, the quantity x_n will be calculated by the recurrence formula

$$d^2x_n/d\theta^2 + x_n = x_{n-1}\cos\alpha(\theta + \Theta). \tag{7}$$

As x_0 is a linear combination of $\cos\theta$ and $\sin\theta$, a particular solution of Eq. (7) will be a linear combination of $\cos[(\alpha \pm 1)\theta + \alpha\Theta]$ and $\sin[(\alpha \pm 1)\theta + \alpha\Theta]$.

Therefore, the first order correction is of the form:

$$x_1 = A_1^1 \cos[(\alpha + 1)\theta + \alpha\Theta] + B_1^1 \sin[(\alpha + 1)\theta + \alpha\Theta]$$
$$+ A_{-1}^1 \cos[(\alpha - 1)\theta + \alpha\Theta] + B_{-1}^1 \sin[(\alpha - 1)\theta + \alpha\Theta]$$
$$+ R_1 \cos\theta + S_1 \sin\theta.$$

The quantities A_1^1, B_1^1, A_{-1}^1, B_{-1}^1 are found by identifying the right and left members of Eq. (7).

After some algebra, we get

$$(8) \qquad A_1^1 = -\frac{R_0}{2\alpha(\alpha + 2)}, \quad A_{-1}^1 = -\frac{R_0}{2\alpha(\alpha - 2)},$$
$$B_1^1 = -\frac{S_0}{2\alpha(\alpha + 2)}, \quad B_{-1}^1 = +\frac{S_0}{2\alpha(\alpha - 2)}.$$

As $x_1(0) = 0$ and $dx_1(0)/d\theta = 0$, R_1 and S_1 must satisfy

$$R_1 = -(A_1^1 + A_{-1}^1)\cos\alpha\Theta - (B_1^1 + B_{-1}^1)\sin\alpha\Theta,$$
$$S_1 = -[B_1^1(\alpha + 1) + B_{-1}^1(\alpha - 1)]\cos\alpha\Theta$$
$$+ [A_1^1(\alpha + 1) + A_{-1}^1(\alpha - 1)]\sin\alpha\Theta.$$

Relations (8) show that the various coefficients can be computed only if $\alpha \neq 2$, that is, $A \neq 1$.

We notice that there is no secular term appearing to first order in ϵ.

It is obvious that the x_2 solution will contain terms such as:

$\cos[(2\alpha \pm 1)\theta + 2\alpha\Theta]$	$\sin[(2\alpha \pm 1)\theta + 2\alpha\Theta]$
$\cos[(\alpha \pm 1)\theta + \alpha\Theta]$	$\sin[(\alpha \pm 1)\theta + \alpha\Theta]$
$\cos\theta$	$\sin\theta$

Moreover, in the product $x_1 \cos[\alpha(\theta + \Theta)]$ terms of the form $\beta\cos\theta$ and $\gamma\sin\theta$ will occur so that the x_2 solution will also contains terms like $-\gamma\theta\cos\theta/2$ and $-\beta\theta\sin\theta/2$. These secular terms are unbounded when $\theta \to \infty$. Therefore, the expansion that is obtained for x will be correct only for the time θ such that $\epsilon^2\theta < \epsilon$, i.e., $\epsilon\theta < 1$.

The second order correction is:

$$x_2 = A_2^2 \cos[(2\alpha + 1)\theta + 2\alpha\Theta] + A_1^2 \cos[(\alpha + 1)\theta + \alpha\Theta]$$

$$+ B_2^2 \sin[(2\alpha + 1)\theta + 2\alpha\Theta] + B_1^2 \sin[(\alpha + 1)\theta + \alpha\Theta]$$

$$+ A_{-2}^2 \cos[(2\alpha - 1)\theta + 2\alpha\Theta] + A_{-1}^2 \cos[(\alpha - 1)\theta + \alpha\Theta]$$

$$+ B_{-2}^2 \sin[(2\alpha - 1)\theta + 2\alpha\Theta] + B_{-1}^2 \sin[(\alpha - 1)\theta + \alpha\Theta]$$

$$+ E_1^2 \theta \cos\theta + F_1^2 \theta \sin\theta + R_2 \cos\theta + S_2 \sin\theta.$$

The secular terms in x_2 will lead to secular terms in x_3 of the form $\theta \cos[(\alpha \pm 1)\theta + \alpha\Theta]$ and $\theta \sin[(\alpha \pm 1)\theta + \alpha\Theta]$.

As a consequence, when computing x_4, the product $x_3 \cos(\theta + \Theta)$ will contain terms like $\theta \cos\theta$ and $\theta \sin\theta$, and, therefore, secular terms such as $\theta^2 \cos\theta$ and $\theta^2 \sin\theta$ will appear in x_4. They are the highest order terms of x_4 (of course there will be also terms in $\theta \cos\theta$ and $\theta \sin\theta$).

By the preceding discussion it can easily be seen that the highest power of θ in x_n will be the entire part of $n/2$. The general form of the solution can thus be written in the following way,

$$x_n = \sum_{j=0}^{\eta} \sum_{k=-n+2j}^{n-2j} A_{k,j}^n \cos[(k\alpha + 1)\theta + k\Theta] + B_{k,j}^n \sin[(k\alpha + 1)\theta + K\Theta] \tag{9}$$

with η the entire part of $n/2$.

The coefficients $A_{k,j}^n$, and $B_{k,j}^n$, can be obtained by a recurrence formula, $A_{0,0}^n$ and $B_{0,0}^n$ being determined by the initial conditions $x_n(0) = \dot{x}_n(0) = 0$. It can be seen, that the $A_{k,j}^n$, and $B_{k,j}^n$ contain expressions of the form $\delta/2k\alpha(k\alpha - 2)$, $(k = 1, 2, \cdots, n)$ so that we must assume $k\alpha \neq 2$, or A is not one of the integer squares included in the range $1, n$.

Once the analytical expression of x_{n-1} is known, it is more convenient to compute the x_n contribution through the recurrence integral relation

$$x_n(\theta) = \int_0^\theta x_{n-1}(\theta') \sin(\theta - \theta') \cos\alpha\theta' d\theta'. \tag{10}$$

The coefficients $A_{k,j}^n$ and $B_{k,j}^n$ can be obtained through an algorithm directly derived from Eq. (10). This way of computing x_n has been worked out to any order n. The convergence of the series given by (9) can easily be checked. We found that the ϵ expansion of the solution of the Mathieu equation (which is known to have an infinite radius of convergence) is identical to our expansion. Numerical results support this statement.

However, in the general case, only few terms are known and the reinitialization becomes essential. That is why we present results only for $n = 1$ to 4 with a

variable number of steps and a small $\epsilon\Theta$ (for example $\epsilon\Theta = .1$ if we stop at the second order, $\epsilon\Theta = .25$ if we take the third order term, etc.).

It is in the reinitialization process that our method basically differs from the analytical schemes (high order adiabatic invariant Krylov-Bogoliubov· · ·) which try to approximate the solution for any time. These schemes are obtained by time-averaging solutions, which, in the Krylov-Bogoliubov method, accqmplishes the elimination of the secular terms and, in the adiabatic method, allows the substitution of Ω (the slowly varying frequency of a time-dependent harmonic oscillator) and all its derivatives at a given time to the exact past history of Ω.

But the exact consequences of these approximations are generally not completely understood (we know, for example, that the high order invariants' method introduces difficult asymptotic convergence problems and wipes out nonadiabatic effects). Our method is more modest and resorts to a numerical scheme to get rid of these difficulties by putting a time limit on the validity of the series and then reinitializing the solution as in a numerical algorithm. The advantage is that, by varying the giant step Θ, we can check the convergence of the results.

III. Computation of the Eigenvalues of Mathieu's Equation. The numerical test of the method will be the determination of the eigenvalues of Mathieu's equation.

The solution of Eq. (5) is given by Floquet's theorem:

$$x = Me^{i\mu t}F(t) + Ne^{-i\mu t}F(-t),$$

$F(t)$ being a periodic function with period $T = \pi$, M and N being arbitrary constants and μ the desired eigenvalue.

To compute these eigenvalues, let us explicitly write down the matrix that gives $x(T)$ and $dx(T)/dt$, starting from $x(0)$ and $dx(0)/dt$.

After a little algebra, we get

$$(11)\qquad \begin{bmatrix} x(T) \\ \dot{x}(T) \end{bmatrix} = \begin{bmatrix} \cos \mu T & i\dfrac{F(0)}{\dot{F}(0) + i\mu F(0)} \cdot \sin \mu T \\ i\dfrac{\dot{F}(0) + i\mu F(0)}{F(0)} \cdot \sin \mu T & \cos \mu T \end{bmatrix} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$$

The dot indicates differentiation with respect to t.

The eigenvalues λ of the matrix given by Eq. (11) are solutions of

$$(12)\qquad \lambda^2 - 2\lambda \cos \mu T + 1 = 0.$$

Denoting the solutions satisfying the special initial conditions by $x^0(t), x^1(t)$:

$$x^0(0) = 0,\ \dot{x}^0(0) = 1; \qquad x^1(0) = 1,\ \dot{x}^1(0) = 0,$$

We have $x(t) = x(0)x^1(t) + \dot{x}(0)x^0(t)$, giving for the matrix in (11) the alternative form

$$\begin{bmatrix} x^1(T) & x^0(T) \\ \dot{x}^1(T) & \dot{x}^0(T) \end{bmatrix}.$$

Comparing the characteristic equation of this matrix with (12) and taking θ as new variable, we find

$$\mu = \frac{1}{\pi} \operatorname{Arc\,cos} \frac{x^1(\pi\sqrt{A}) + \dot{x}^0(\pi\sqrt{A})}{2} \tag{13}$$

$x^1(\pi\sqrt{A})$ and $\dot{x}^0(\pi\sqrt{A})$ were computed with the series given by Eq. (9) and for different number of steps in the period $\pi\sqrt{A}$. The results are compared with those obtained by solving Hill's determinant which is very simple for Mathieu's equation where the frequency contains only one harmonic.

The results obtained with the "giant steps" method are shown in Fig. 1 and Fig. 2 where ϵ has been taken equal to .1.

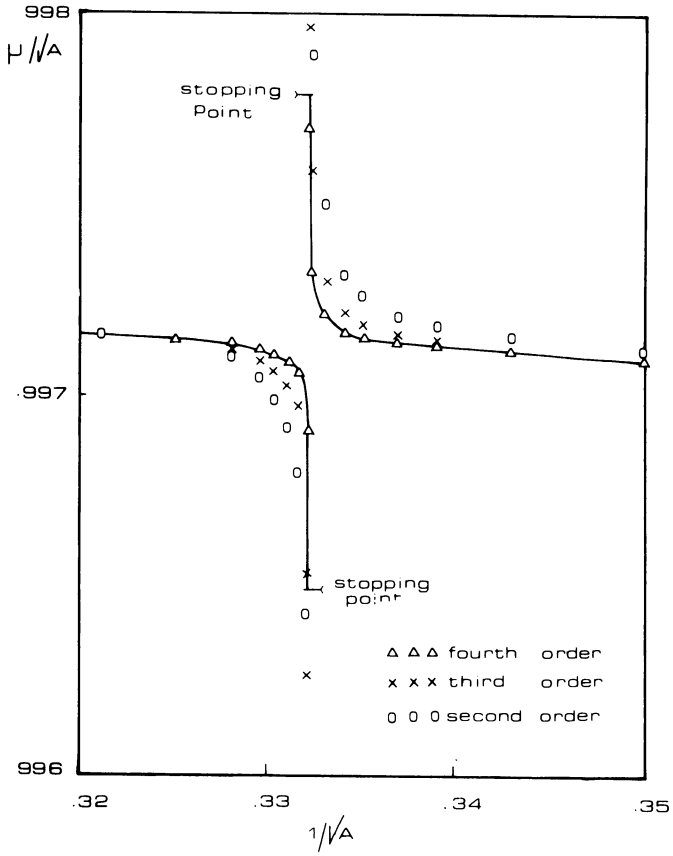

FIGURE 1. $\mu/\sqrt{A}$ versus $1/\sqrt{A}$ for the second, third and fourth order approximation in the neighborhood of the discontinuity $A = 9$; the number of steps per period has been taken equal to 4. The solid line represents the exact value of $\mu/\sqrt{A}$, obtained by solving the Hill determinant. The stopping points of the two branches are obtained with great accuracy in the fourth order calculation.

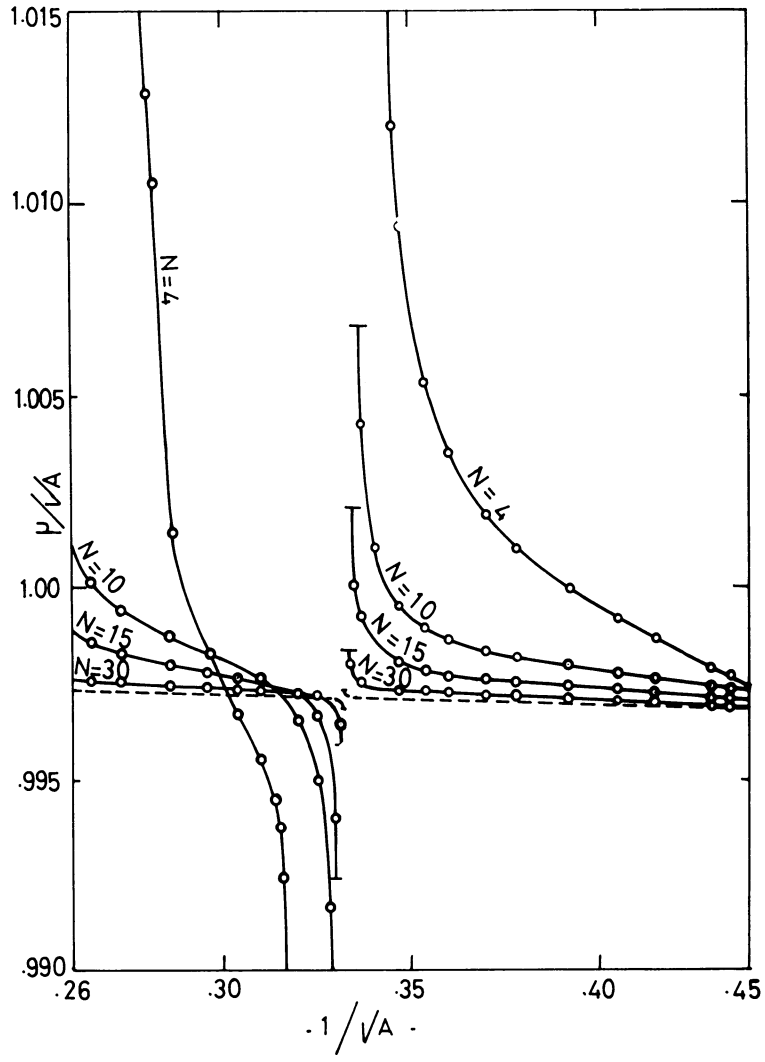

FIGURE 2. $\mu/\sqrt{A}$ versus $1/\sqrt{A}$ for the first order approximation. The dashed line is the exact curve. N is the number of steps in one period $\pi\sqrt{A}$.

Fig. 1 shows the different order (2nd, 3rd, 4th) approximation of the normalized eigenvalues $\mu/\sqrt{A}$ of Mathieu's equation versus $1/\sqrt{A}$ (A being roughly in the range 8 to 10), and for 4 steps in one period $\pi\sqrt{A}$. The first order, being completely out of scale, has not been represented.

The 4th order is strictly on Hill's curve. The stopping points of the two branches are obtained with great accuracy. When, in the adiabatic case, the solution crosses over the discontinuity (around $A = 9$) [4], we must notice that in our method the unstable zone is entirely recovered.

Fig. 2 shows the convergence of the first order approximation to the exact curve for A going from 5 to 15. The parameter N is the number of steps in one period $\pi\sqrt{A}$.

In that case, we notice that no secular term is present. Although the unstable zone around $A = 9$ has a growth rate of order ϵ^3, we numerically recover this growth rate if we take N sufficiently large although our method uses only first order term expansion at each step. In the same way, we can use a first, second or higher order algorithm to numerically solve a differential equation and recover the exact solutions for all these algorithms provided we choose the time steps correctly (which for the first order algorithm will probably have to be very small).

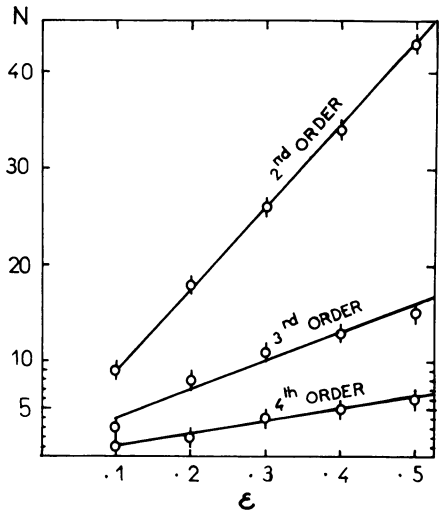

FIGURE 3. The number of steps N is clearly a linear function (at a given order) of the perturbation parameter ϵ. The small dashes on both sides of the obtained value represent the ΔN error (obviously equal to ± 1).

We can see, in Fig. 3, that the number of steps which permits for the value $A = 7$ to obtain the eigenvalues with a given precision (here 10^{-6}) is a linear function of ϵ. As $\Theta = \pi\sqrt{A}/N$, this result shows that $\epsilon\Theta$ is constant for a given order and that it is the only quantity which has to be considered for reinitialization.

For $\epsilon = .1$ and using a fourth order formula, we can take a step as large as the period $\pi\sqrt{A}$ and, consequently, we do not have to reinitialize to compute $x^1(\pi\sqrt{A})$, and $\dot{x}^0(\pi\sqrt{A})$ and the eigenvalue.

Conclusion. We tested our method in the case of Mathieu's equation. Such an equation with its narrow unstable bands provides a severe numerical test. We found that:

Although no secular term arises in the first perturbation expansion, we nevertheless recover the unstable zones of Mathieu's equations (and not only around $A = 1$ but also $A = 4$ and $A = 9$ etc.) as shown in Fig. 2. But, to obtain good agreement, we must use a rather small ($N \sim 30$ per period) step; at least for this problem, a first order perturbation has no practical advantage over a purely numerical method.

On the other hand, an expansion up to the 4th power in ϵ allows for $\epsilon = .1$ to proceed with a step as big as one period while giving good accuracy. For larger ϵ (up to $\epsilon = .5$), we need less than 6 steps per period. Since such a formula is of the 4th power in ϵ, it plays a similar role for the "giant step" method as a 4th order in Δt in Runge-Kutta development–an algorithm widely used in numerical analysis.

The key of the problem is the determination of a sufficiently high order formula for the ϵ expansion. For Mathieu's equation, the 4th order can be obtained without great difficulty. For more complex equations, we probably have to resort to formula manipulation languages. It is, of course, fundamental that the chain of equations can be solved and this implies that $f(x, \dot{x}, t)$ either has a simple expression or can be expanded into a simple series.

A problem for which this method seems especially promising is the motion of a particle in a time-independent, slowly varying (in space) magnetic field given by a Taylor expansion around the position of the center guide. The algebra corresponding to third order effects is easily obtainable. Such a problem is currently under investigation.

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