

# ALGEBRAIC PERIODIC POINTS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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ABSTRACT. We prove the existence of transcendental entire functions  $f$  having a property studied by Mahler, namely that  $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$  and  $f^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ , and in addition having a prescribed number of  $k$ -periodic algebraic orbits, for all  $k \geq 1$ . Under a suitable topology, such functions are shown to be dense in the set of all entire transcendental functions.

## 1. INTRODUCTION

Let  $D$  be a subset of  $\mathbb{C}$ . Recall that a function  $f : D \rightarrow \mathbb{C}$  is called *transcendental* if the only two-variable polynomial  $P \in \mathbb{C}[x, y]$  such that  $P(z, f(z)) = 0$  for all  $z \in D$  is the zero polynomial. For the purposes of this article, let us define a *Mahler function* to be a transcendental entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  with the property that

$$(1) \quad f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}} \quad \text{and} \quad f^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}},$$

where  $\overline{\mathbb{Q}}$  denotes the field of the algebraic numbers, i.e., the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . If  $f$  is a Mahler function and  $K$  is a dense subset of  $\mathbb{C}$  such that  $\overline{K \cap \overline{\mathbb{Q}}} = \mathbb{C}$  (for instance  $K = \mathbb{Q}[i]$ ), we say that  $f$  is *defined over  $K$*  if every coefficient of the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

belongs to  $K$ . This terminology is motivated by a question posed by Mahler [3, p. 53] in 1976: *Does there exist a transcendental entire function  $f$  for which (1) holds, and such that every coefficient of its Taylor series is rational?* Mahler mentions that the answer seems to be unknown even if the coefficients are allowed to be arbitrary *complex* numbers.

The question of existence of Mahler functions was settled in recent work of the second and third authors [4], who proved that there exist uncountably many Mahler functions defined over  $\mathbb{Q}$ , thus answering Mahler's question. In this article we study some of the dynamical properties of Mahler functions.

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By virtue of the property (1), every Mahler function  $f$  can be regarded as a discrete dynamical system on  $\overline{\mathbb{Q}}$ . One can then ask how many algebraic periodic orbits  $f$  has of every period. Our main result is Theorem 1, which states, loosely speaking, that given any transcendental entire function  $g$ , one can obtain – via a small perturbation of  $g$  – a Mahler function having prescribed dynamical behavior on  $\overline{\mathbb{Q}}$ .

To state our results precisely, we introduce some notation and terminology. For every dynamical system  $f : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$ , we denote by  $\text{Per}(k, f)$  the set of all  $k$ -periodic points of  $f$ , i.e., the algebraic numbers  $\alpha$  for which  $k$  is the smallest positive integer with  $f^k(\alpha) = \alpha$  (where  $f^1 = f$  and  $f^k = f \circ f^{k-1}$  for  $k \geq 2$ ). The *orbit* of an algebraic number  $\alpha$  is the set  $\{\alpha, f(\alpha), f^2(\alpha), \dots\}$ . The orbit of a  $k$ -periodic point is a *k-cycle*. Finally, we denote by  $\text{Orb}(k, f)$  the set of all  $k$ -cycles of  $f$ . Note that  $\#\text{Orb}(k, f) = \#\text{Per}(k, f)/k$ .

Let  $\mathcal{E}$  denote the set of all entire functions. We place a topology on  $\mathcal{E}$  with a sub-base consisting of sets  $V_{g, \vartheta}$  for every function  $g \in \mathcal{E}$  and every sequence  $\vartheta$  of positive real numbers: If  $g(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $\vartheta = (\theta_k)_{k \geq 0}$ , the open set  $V_{g, \vartheta}$  consists of all functions  $\sum_{n=0}^{\infty} b_n z^n \in \mathcal{E}$  satisfying  $|a_n - b_n| < \theta_n$  for all  $n$ . Notice that the set  $\mathcal{T}$  of all transcendental entire functions is open and dense in  $\mathcal{E}$  in this topology.

**Theorem 1.** *Let  $K$  be a subset of  $\mathbb{C}$  such that  $K \cap \overline{\mathbb{Q}}$  is dense in  $\mathbb{C}$  (in the Euclidean topology), and let  $\sigma = (s_k)_{k \geq 0}$  be a sequence in  $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ . Let  $\mathcal{M}(\sigma, K)$  denote the set of all Mahler functions  $f$  that are defined over  $K$  and satisfy  $\#\text{Orb}(k, f) = s_k$  for all  $k \geq 0$ . Then  $\mathcal{M}(\sigma, K)$  is dense in  $\mathcal{T}$  (and thus in  $\mathcal{E}$ ).*

Taking  $K = \mathbb{C}$  in the theorem, and choosing an arbitrary sequence  $\sigma$ , we obtain the following new result illustrating the ubiquity of Mahler functions.

**Corollary 2.** *The set of Mahler functions is dense in  $\mathcal{E}$ .*

The proof of our main theorem relies primarily on a theorem of Bergweiler [1, 2] concerning periodic orbits of transcendental entire functions, and builds on techniques developed by the second and third authors in [4, 5].

## 2. PROOF OF THEOREM 1

**2.1. Auxiliary results.** Throughout this section we use the familiar notation  $[a, b] = \{a, a + 1, \dots, b\}$  for integers  $a < b$ .

We begin our discussion of the proof of Theorem 1 by stating several preliminary results that will be used in the proof. The first result concerns holomorphic dynamics, and is due to Bergweiler [1].

Recall that a point  $\alpha \in \text{Per}(k, f)$  is called *repelling* if  $|(f^k)'(\alpha)| > 1$ .

**Lemma 3** (Bergweiler). *Every transcendental entire function has infinitely many repelling  $k$ -periodic points, for all  $k \geq 2$ .*

In order to construct prescribed 1-periodic points (which are not covered by the previous result – for example,  $e^z + z \in \mathcal{T}$  does not have fixed points), we shall need the classical Picard theorems.

**Lemma 4** (Great Picard Theorem). *Suppose that a holomorphic function  $f$  (whose domain contains a punctured disk centered at  $w$ ) has an essential singularity at  $w$ . Then, on any punctured neighborhood of  $w$ ,  $f$  assumes all possible complex values, with at most a single exception, infinitely often.*

**Remark 5.** Note that every function  $g \in \mathcal{T}$  has an essential singularity at infinity. Moreover, if  $P(z)$  is any nonzero polynomial, then  $g(z)/P(z)$  has an essential singularity at infinity (which is equivalent to say that  $g(1/z)/P(1/z)$  has an essential singularity at 0). The previous result implies that  $g(z)/P(z)$  assumes all possible complex values, with at most a single exception, infinitely often.

**Lemma 6.** *Let  $g(z)$  be an entire function and set  $f(z) = g(z) + \epsilon P(z)$ , where  $\epsilon > 0$  is a real number and  $P(z) \in \mathbb{C}[z]$  is a nonzero polynomial. Then, for all  $k \geq 1$ , we have that  $f^k(z) = g^k(z) + \epsilon \phi_k(\epsilon, z)$ , where  $\phi_k(\epsilon, z)$  is a nonzero analytic function in both variables  $\epsilon \in \mathbb{C} \setminus \{0\}$  and  $z \in \mathbb{C}$ .*

*Proof.* The proof is by induction on  $k$ . The base step  $k = 1$  follows immediately by choosing  $\phi_1(\epsilon, z) = P(z)$ . Suppose that the result is valid for  $k$ ; then

$$\begin{aligned} f^{k+1}(z) &= f(f^k(z)) = f(g^k(z) + \epsilon \phi_k(\epsilon, z)) \\ &= g(g^k(z) + \epsilon \phi_k(\epsilon, z)) + \epsilon P(g^k(z) + \epsilon \phi_k(\epsilon, z)). \end{aligned}$$

Now use the identity

$$g(y+h) - g(y) = h \int_0^1 g'(y+th) dt,$$

which follows from the fact that the derivative of the function  $u(t) = g(y+th)$  is  $h \cdot g'(y+th)$ , to write

$$g(g^k(z) + \epsilon \phi_k(\epsilon, z)) = g^{k+1}(z) + \epsilon \phi_k(\epsilon, z) \int_0^1 g'(g^k(z) + t\epsilon \phi_k(\epsilon, z)) dt,$$

and the result follows with

$$\phi_{k+1}(\epsilon, z) = \phi_k(\epsilon, z) \int_0^1 g'(g^k(z) + t\epsilon \phi_k(\epsilon, z)) dt + P(g^k(z) + \epsilon \phi_k(\epsilon, z)).$$

□

**Lemma 7.** *Let  $g(z)$  be a transcendental entire function,  $P(z) \in \mathbb{C}[z]$  and  $(k, M) \in \mathbb{Z}_{\geq 1}^2$ . There, for every  $\delta > 0$ , there are  $\delta_1, \delta_2$  with  $0 < \delta_1 < \delta_2 < \delta$  and  $R \in \mathbb{R}_{>M}$  such that, for all  $\epsilon \in (\delta_1, \delta_2)$ , the function  $f(z) := g(z) + \epsilon P(z)$  has at least  $M$  fixed points  $w_j, j \in [1, M]$  with  $f'(w_j) \notin \{0, 1\}, \forall j \leq M$  and  $M$  expansive  $t$ -cycles entirely contained in  $B(0, R)$  for all  $t \in [1, k]$ .*

*Proof.* First, by Remark 5, the meromorphic function  $h(z) := (z - g(z))/P(z)$  has an essential singularity at infinity. We deduce from Lemma 4 that there exists  $\hat{\delta} \in (0, \delta)$  such that  $h^{-1}(\epsilon)$  is an infinite set for all  $\epsilon \in (0, \hat{\delta})$ . Observe that any element of  $h^{-1}(\epsilon)$  is a fixed point of  $f(z) = g(z) + \epsilon P(z)$ . Now, given

a constant  $c$ , the set of  $\epsilon \in \mathbb{C}$  such that, for some  $z$  with  $P(z) \neq 0$  we have  $g(z) + \epsilon P(z) = z$  and  $g'(z) + \epsilon P'(z) = c$  is countable. Indeed, these equalities imply  $g(z)P'(z) + \epsilon P(z)P'(z) = zP'(z)$  and  $g'(z)P(z) + \epsilon P(z)P'(z) = cP(z)$ , so  $g'(z)P(z) - g(z)P'(z) = cP(z) - zP'(z)$ . Notice that  $cP(z) - zP'(z)$  is a polynomial and  $g'(z)P(z) - g(z)P'(z)$  is a transcendental entire function - indeed, the latter is equal to  $P(z)^2 \cdot (g(z)/P(z))'$ , so, if it is a polynomial, then  $u(z) := (g(z)/P(z))'$  is a rational function; however, in that case, since  $g(z)/P(z)$  is holomorphic in a neighbourhood of  $\{z \in \mathbb{C} : |z| \geq T\}$  for some  $T > 0$ ,  $|g(z)/P(z)|$  is bounded in  $\{|z| = T\}$ , so there is  $m \geq 1$  such that  $|u(z)/z^m|$  is bounded for  $|z| \geq T$  and, for  $|z| > T$ ,

$$g(z)/P(z) = g(Tz/|z|)/P(Tz/|z|) + \int_T^{|z|} u(sz/|z|) \cdot \frac{z}{|z|} ds.$$

It follows that  $|\frac{g(z)}{P(z)z^{m+1}}|$  is bounded for  $|z| \geq T$ , but, since  $P(z)$  is a polynomial, this implies that  $g(z)$  is a polynomial, which is a contradiction. Hence the set of solutions of the equality  $g'(z)P(z) - g(z)P'(z) = cP(z) - zP'(z)$  has no accumulation points, and thus is countable, and since in this case  $\epsilon = (z - g(z))/P(z)$ , the set of corresponding values of  $\epsilon$  is also countable. Given that the set of zeros of  $P(z)$  is finite, this implies that there is  $\tilde{\delta} \in (0, \hat{\delta})$  such that if  $\tilde{f}(z) = g(z) + \tilde{\delta}P(z)$ , then  $\tilde{f}(z)$  has infinitely many fixed points  $w_j, j \geq 1$  with  $\tilde{f}'(w_j) \notin \{0, 1\}, \forall j \geq 1$ .

Bergweiler's theorem 3 implies that the transcendental entire function  $\tilde{f}(z) = g(z) + \tilde{\delta}P(z)$  has infinitely many expansive  $t$ -cycles for every  $t$  with  $t \in [2, k]$ . We can choose  $R > 0$  such that  $\tilde{f}$  has at least  $M$  fixed points as before, which are not zeros of  $P(z)$ , and  $M$  expansive  $t$ -cycles contained in  $B(0, R)$  for every  $t \in [2, k]$ . Since those fixed points and  $t$ -cycles persist for small perturbations of  $\tilde{f}$ , there is  $\eta > 0$  such that  $(\tilde{\delta} - \eta, \tilde{\delta} + \eta) \subset (0, \delta)$  and, for all  $\epsilon \in (\delta_1, \delta_2) = (\tilde{\delta} - \eta, \tilde{\delta} + \eta)$ , the function  $f(z) := g(z) + \epsilon P(z)$  has at least  $M$  fixed points  $w_j, j \in [1, M]$  with  $f'(w_j) \notin \{0, 1\}, \forall j \leq M$  and  $M$  expansive  $t$ -cycles entirely contained in  $B(0, R)$  for all  $t \in [1, k]$ .  $\square$

**Lemma 8.** *Let  $R \in (0, +\infty)$ ,  $g(z)$  be an entire function and  $P(z) \in \mathbb{C}[z]$  a nonzero polynomial. Suppose that  $\alpha$  is a complex number such that  $g(z) \neq \alpha$  for all  $z \in \partial B(0, R)$ . Then there exists a positive real number  $\delta$  such that, for all  $\epsilon \in (0, \delta)$ , the function  $f(z) := g(z) + \epsilon P(z)$  satisfies*

$$\#(f^{-1}(\alpha) \cap B(0, R)) = \#(g^{-1}(\alpha) \cap B(0, R)).$$

*Proof.* The hypothesis implies that  $\min_{z \in \partial B(0, R)} |g(z) - \alpha| > 0$ . Since also  $\max_{z \in \partial B} |P(z)| > 0$ , we may define  $\delta > 0$  by

$$\delta := \frac{\min_{z \in \partial B} |g(z) - \alpha|}{\max_{z \in \partial B} |P(z)|}.$$

Clearly, for any  $\epsilon \in (0, \delta)$ , we have  $|\epsilon P(z)| < |g(z) - \alpha|$  for all  $z \in \partial B(0, R)$ . By Rouché's theorem, the functions  $g(z) - \alpha + \epsilon P(z)$  and  $g(z) - \alpha$  have the

same number of zeros in  $B(0, R)$ . Setting  $f(z) = g(z) + \epsilon P(z)$ , we conclude that  $\#(f^{-1}(\alpha) \cap B(0, R)) = \#(g^{-1}(\alpha) \cap B(0, R))$ , as desired.  $\square$

Having proved all the necessary auxiliary results, we now proceed to the proof of Theorem 1.

**2.2. The proof.** Let  $g(z) = \sum_{k \geq 0} b_k z^k \in \mathcal{T}$  and let  $(\theta_k)_{k \geq 0}$  be a sequence of positive real numbers. We shall prove the existence of an entire function  $\phi$  such that  $f := g + \phi$  is a Mahler function defined over  $K$ , and that moreover,  $\#\text{Orb}(k, f) = s_k$  and  $|a_k - b_k| < \theta_k$  for all  $k \geq 0$ , where  $f(z) = \sum_{k \geq 0} a_k z^k$ . We may assume  $\theta_k < 1/k!, \forall k \geq 0$ , so  $f(z)$  will automatically be an entire function.

Several functions appearing in the process of constructing  $f$  will not be Mahler functions, so we make the following general definition. For an arbitrary function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , we say that a  $k$ -periodic orbit is *algebraic* if every term in the cycle is an algebraic number.

Let  $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$  be an enumeration of  $\overline{\mathbb{Q}}$  with  $\alpha_1 = 0$ . In what follows, we denote by  $L(P)$  the *length* of a polynomial  $P$ , i.e., the sum of the absolute values of the coefficients of  $P$ .

We construct our desired function inductively. Choose  $\epsilon_0 \in B(0, \theta_0)$  such that  $b_0 + \epsilon_0 \in (K \setminus \{0\}) \cap \overline{\mathbb{Q}}$ , and set  $f_1(z) := g(z) + \epsilon_0$ . We wish to recursively construct a sequence of entire functions  $f_2(z), f_3(z), \dots$  of the form

$$f_{m+1}(z) = f_m(z) + z^{m+2} h_m(z) P_m(z),$$

with the following properties being satisfied:

- (i)  $h_m, P_m \in \mathbb{C}[z]$  and  $f_m(z) = g(z) + \sum_{i=0}^{t_m} a_i z^i$  with  $t_m \geq m$ ;
- (ii) Letting  $X_m = \{\alpha_2, \alpha_3, \dots, \alpha_m\}$ ,  
 $\tilde{X}_m = \{\tau \in f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) \mid \tau \neq 0, \tau \in \overline{\mathbb{Q}}, f'_m(\tau) \neq 0\}$ ,

and

$$Y_m = \bigcup_{k=1}^m Y_m^{(k)}$$

- where, for  $k \in [1, m]$ ,  $Y_m^{(k)}$  is a union of  $\min\{m, s_k\}$  algebraic periodic orbits of period  $k$  of  $f_m$  contained in  $B(0, r_m)$ , for some suitable choice of  $r_m$  with  $r_m > \max\{m, r_{m-1}\}$ , we have  $P_m(z) = \prod_{\tau \in X_m \cup \tilde{X}_m \cup Y_m} (x - \tau)^2$ . Moreover,  $P_m(z) \mid P_{m+1}(z)$  for all  $m \geq 1$ ;
- (iii)  $f_m(\tau) \in \overline{\mathbb{Q}}, f'_m(\tau) \neq 0 \forall \tau \in X_m$  and, for each integer  $j \in [1, m]$ , there is  $\tau \in \tilde{X}_m$  such that  $f_m(\tau) = \alpha_j$ ;
- (iv)  $0 < L(h_m P_m) < \nu_m := \frac{1}{m^{m+2} + \deg(h_m P_m)}$ ;
- (v)  $a_k + b_k \in K$  and  $|a_k| < \theta_k$ , for  $k \in [1, m]$ ;
- (vi)  $f_{m+1}^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap \overline{B(0, r_m)} = \tilde{X}_m \subseteq \overline{\mathbb{Q}}$ ;
- (vii) If  $P_{m-1}(\alpha_m) \neq 0$  then  $P_m(f_m(\alpha_m)) \neq 0$ ; if  $\tau \in \tilde{X}_m$  and  $P_{m-1}(\tau) \neq 0$  then  $\tau \notin \{f_m(w); P_m(w) = 0\}$ .

The polynomials  $h_n$  have the form  $\sum_{j=0}^{s_n+\ell_n} \epsilon_{n,j} z^{1-\hat{\delta}_{j,0}} \tilde{P}_{n,j}(z)$ , where

$$s_n \leq \hat{s}_n := 1 + n \max\{\deg f_n, n + 1 + \deg P_n\}$$

and

$$\ell_n \leq \hat{\ell}_n := \#\{\alpha \in \text{Orb}(k, f_n) : \text{Orb}(k, f_n) \subseteq B(0, r_n), k \in [1, n]\}$$

are natural numbers to be chosen later. Here,  $\hat{\delta}_{j,0}$  is 1 if  $j = 0$  and 0 otherwise, and  $\tilde{P}_{n,j}$  are monic polynomials with complex coefficients.

The required function will have the form  $f(z) = \lim_{m \rightarrow \infty} f_m(z)$ , where, for each  $m \geq 1$ ,  $f_{m+1}(z) = g(z) + \sum_{1 \leq n \leq m} \sum_{j=0}^{s_n+\ell_n} \epsilon_{n,j} z^{n+2-\hat{\delta}_{j,0}} P_{n,j}(z)$ , with  $P_{n,j}(z) = P_n(z) \tilde{P}_{n,j}(z)$ . Thus

$$f(z) = g(z) + \epsilon_0 + \sum_{n \geq 2} \sum_{j=0}^{s_n+\ell_n} \epsilon_{n,j} z^{n+2-\hat{\delta}_{j,0}} P_{n,j}(z).$$

We will have, for  $0 \leq j < s_n + \ell_n$ , that  $P_n(z) \mid P_{n,j}(z) \mid P_{n,j+1}(z) \mid P_{n+1}(z)$ .

At each step, we shall choose  $\epsilon_{n,j}$  with

$$(2) \quad 0 < |\epsilon_{n,j}| < \nu_{n,j},$$

where

$$\nu_{n,j} := \frac{1}{L(P_{n,j})(\hat{s}_n + \hat{\ell}_n)(n + \frac{2}{\Theta_{n+2}})^{n+3+\deg P_{n,j}}} \quad \text{and} \quad \Theta_k := \min_{j \in [1, k]} \{\theta_j\}.$$

Since  $|P_{n,j}(z)| \leq L(P_{n,j}) \max\{1, |z|\}^{\deg P_{n,j}}$ , we have the following for  $z$  belonging to the open ball  $B(0, R)$ :

$$\left| \sum_{j=0}^{s_n+\ell_n} \epsilon_{n,j} P_{n,j}(z) \right| < \frac{R+1}{n} \left( \frac{\max\{1, R\}}{n} \right)^{n+2+\deg P_{n,j}}.$$

Thus the series

$$\sum_{n \geq 2} \sum_{j=0}^{s_n+\ell_n} \epsilon_{n,j} z^{n+2-\hat{\delta}_{j,0}} P_{n,j}(z)$$

converges uniformly in any of these balls (note that  $\deg P_{n,j}$  is a non-decreasing function in  $j$ ).

Suppose that we have a function  $f_n$  satisfying (i)-(viii). We now construct  $f_{n+1}$  with the desired properties.

Before continuing the proof, we shall make some conventions in order to make the text simpler and clearer. For a positive real number  $\epsilon$ , we say that  $h_1(z)$  is an  $\epsilon$ -perturbation of  $h_2(z)$  if  $(h_2(z) - h_1(z))/\epsilon$  is a nonzero polynomial with coefficients bounded by 1 in norm. Note that all the intermediate functions (from  $f_n(z)$  to  $f_{n+1}(z)$ ) will have the form  $f_J(z) = f_{J-1}(z) + \epsilon_J z^{n+2} P_J(z)$ , where  $P_J(z)$  is a polynomial and  $f_{J-1} \in \mathcal{E}$ . We then shall say that a complex number  $\gamma$  is *nailed* in  $f_J(z)$  if  $P_J(\gamma) = 0$ . Moreover, we define two types of periodic orbits: a  $k$ -periodic orbit  $\{\beta, f_J(\beta), \dots, f_J^{k-1}(\beta)\}$  is a *nailed* orbit if  $f_J^j(\beta)$  is nailed, for all  $j \in [0, k-1]$ . On the other hand, the

orbit is said to be of the *free* if  $f_j^j(\beta)$  is not nailed, for all  $j \in [0, k-1]$ . Also, by the definition of  $P_j(z)$ , a nailed orbit must consist of only algebraic numbers.

Let  $B_n = B(0, r_n)$ . We define  $f_{n,0}(z)$  as

$$f_{n,0}(z) = f_n(z) + \epsilon_{n,0} z^{n+1} P_{n,0}(z),$$

for some choice of an admissible  $\epsilon_{n,0}$  (here  $P_{n,0}(z) = P_n(z)$ ). By the inductive hypothesis, one has that  $f_n^{-1}(\{\alpha_2, \dots, \alpha_{n-1}\}) \cap \partial B_n = \emptyset$ , and any  $\tau$  belonging to  $f_n^{-1}(\{\alpha_2, \dots, \alpha_{n-1}\}) \cap B_n$  is an algebraic number for which  $f_n'(\tau) \neq 0$  (it follows that  $\tau$  is a double zero of  $P_n$ ). Since  $\epsilon_{n,0}$  is admissible, then the number of zeros (counted with multiplicity) of  $f_n(z) - \alpha_i$  and  $f_{n,0}(z) - \alpha_i$  belonging to  $B_n$  are equal. However, every zero of  $f_n(z) - \alpha_i$  in  $B_n$  is a zero of  $f_{n,0}(z) - \alpha_i$  and every zero of  $f_{n,0}(z) - \alpha_i$  in  $B_n$  is simple. Therefore  $f_{n,0}^{-1}(\alpha_i) \cap B_n = f_n^{-1}(\alpha_i) \cap B_n$  for all  $i \in [1, n-1]$ . (This argument ensures that, in our construction, no new pre-image under  $f_{n,0}$  of  $\{\alpha_1, \dots, \alpha_n\}$  lying in  $B_{n+1}$  will appear apart from pre-images under  $f_n$ .)

Next we show that, except for a countable set of values of  $\epsilon_{n,0}$ , for any  $w \in \mathbb{C}$  with  $f_{n,0}(w) \in \{\alpha_2, \dots, \alpha_{n+1}\}$  we have  $f_{n,0}'(w) \neq 0$ , and for each  $j \in [2, n+1]$ , there is  $w \in \mathbb{C}$  with  $f_{n,0}(w) = \alpha_j$ . Notice that for each  $j \in [2, n+1]$ , there is at most one value of  $\epsilon_{n,0}$  for which the image of  $f_{n,0}$  does not contain  $\alpha_j$  (since  $\frac{\alpha_j - f_n(z)}{z^{n+1} P_{n,0}(z)}$  has an essential singularity at  $\infty$ ). If  $w$  is a root of  $P_{n,0}(z)$ , then  $f_{n,0}'(w) = f_n'(w) \neq 0$ . Then  $w$  is a simple root of  $f_n(z) - \beta_i$ . Otherwise,  $P_{n,0}(w) \neq 0$ . We have  $f_{n,0}(z) = f_n(z) + \epsilon_{n,0} g(z)$ , where  $g(z) = z^{n+1} P_{n,0}(z)$ . If  $f_{n,0}'(w) = 0$ , we should have  $f_n(w) + \epsilon_{n,0} g(w) = \alpha_i$  and  $f_n'(w) + \epsilon_{n,0} g'(w) = 0$ . Defining  $h_i(z) = f_n(z) - \alpha_i$ , we have  $h_i'(z) = f_n'(z)$ , and thus

$$h_i(w) + \epsilon_{n,0} g(w) = 0 \quad \text{and} \quad h_i'(w) + \epsilon_{n,0} g'(w) = 0.$$

Now let  $\psi_i(z) = -h_i(z)/g(z)$ . Since  $h_i(w) + \epsilon_{n,0} g(w) = 0$ , we have  $\psi_i(w) = -h_i(w)/g(w) = \epsilon_{n,0}$ . Moreover,  $\psi_i'(w) = \frac{h_i(w)g'(w) - h_i'(w)g(w)}{g(w)^2} = 0$  since  $h_i(w)g'(w) - h_i'(w)g(w) = (h_i(w) + \epsilon_{n,0} g(w))g'(w) - (h_i'(w) + \epsilon_{n,0} g'(w))g(w) = 0$ . This implies that  $\epsilon_{n,0}$  is a singular value of  $\psi_i(w)$ , and the set of singular values of a holomorphic function is countable; this concludes the argument.

Since  $f_{n,0}(z)$  is a transcendental function, we can apply lemmas 3 and 7 to ensure the existence of a real number  $r_{n+1} > \max\{n+1, r_n\}$  such that  $f_{n,0}^{-1}(\{\alpha_1, \dots, \alpha_{n+1}\})$  does not intersect  $\partial B(0, r_{n+1})$ , where  $B_{n+1} := B(0, r_{n+1})$  contains at least  $n+1 + D_n$  repelling  $k$ -periodic orbits of  $f_{n,0}(z)$ , for all  $k \in [2, n+1]$  and at least  $n+1 + D_n$  fixed points whose eigenvalues do not belong to  $\{0, 1\}$ , where  $D_n$  is the number of distinct roots of  $P_n(z)$ . It follows that  $B_{n+1}$  contains at least  $n+1$  free repelling  $k$ -periodic orbits of  $f_{n,0}(z)$ , for all  $k \in [2, n+1]$  and at least  $n+1$  free fixed points whose eigenvalues do not belong to  $\{0, 1\}$ .

Let  $f_{n,0}^{-1}(\{\alpha_2, \dots, \alpha_{n+1}\}) \cap B_{n+1} = \{\tau_1, \tau_2, \dots, \tau_{m_n}\}$ . Notice that each element of  $\{\alpha_2, \dots, \alpha_{n+1}\}$  has at least one pre-image in  $\{\tau_1, \tau_2, \dots, \tau_{m_n}\}$ .

For  $j \in [1, m_n]$ , fix a small ball  $B(\tau_j, \eta_j)$  which does not intersect  $\partial B_{n+1}$  in such a way that the balls  $B(\tau_j, \eta_j)$  are disjoint. The number  $s_n$  of further steps in the construction of  $f_{n+1}$  will be the number of elements of  $\{\alpha_{n+1}\} \cup \{\tau_1, \tau_2, \dots, \tau_{m_n}\} \setminus (X_n \cup \tilde{X}_n)$ . The following  $s_n$  perturbations  $f_{n,j}$  (with  $j \in [1, s_n]$ ) of  $f_{n,0}$  will be taken so close to  $f_{n,0}$  that they will be  $\epsilon$ -perturbations (as defined before) with the further requirement that the number of zeros (counted with multiplicity) of  $f_{n,0}(z) - \alpha_i$  and  $f_{n,j}(z) - \alpha_i$  in the ball  $B(\tau_j, \eta_j)$  ( $j \leq m_n$ ) are equal, for all  $i \in [1, n+1]$  and  $\ell \in \{0, 1\}$  (from now on,  $\epsilon$  will be called admissible by including this extra requirement). Notice that in the balls  $B(\tau_j, \eta_j)$ ,  $j \leq m_n$  these numbers are equal to one, since  $\tau_j$  is a simple zero of  $f_{n,0}(z) - \alpha_i$  for some  $i \in [2, n+1]$ . We will take  $f_{n+1,0} := f_{n,s_n}$ , and so the number of zeros (counted with multiplicity) of  $f_{n,0}(z) - \alpha_i$  and  $f_{n+1,0}(z) - \alpha_i$  in the balls  $B_{n+1}$ ,  $B_n$  and  $B(\tau_j, \eta_j)$ ,  $j \leq m_n$  will be equal, for all  $i \in [2, n+1]$ . In particular, as before,  $f_{n+1,0}^{-1}(\alpha_i) \cap B_n = f_n^{-1}(\alpha_i) \cap B_n$ , for all  $i \in [1, n-1]$  and, for  $j \in [1, m_n]$ ,  $f_{n+1,0}$  has only one zero in  $B(\tau_j, \eta_j)$ , which is simple.

Now, we shall choose an admissible  $\epsilon_{n,0}$  for which  $f_{n,0}(z)$  will satisfy the property (v). In fact, for any  $f \in \mathcal{E}$ , let  $c_j(f)$  be the  $j$ th coefficient of its power series. Then

$$c_{n+1}(f_{n,0}) = b_n + c_{n+1}(F_n) + \epsilon_{n,0}P_{n,0}(0),$$

where  $F_n(z) := \sum_{k=1}^{n-1} \sum_{j=0}^{s_k+\ell_k} \epsilon_{k,j} z^{k+1-\hat{\delta}_{j,0}} P_{k,j}(z)$ . Since  $P_{n,0}(0) \neq 0$  and  $K$  is dense, there exists an admissible  $\epsilon_{n,0} \in B_{n,0}$  for which  $c_{n+1}(f_{n,0}) \in K \setminus \{0\}$ . Since in all further perturbations we will add multiples of  $z^{n+2}$ ,  $a_{n+1}$  will be equal to this coefficient, and thus belongs to  $K \setminus \{0\}$ . Now, we must prove that  $|c_{n+1}(F_n) + \epsilon_{n,0}P_{n,0}(0)| < \theta_{n+1}$ . For that, by (2), observe that  $|\epsilon_{n,0}P_{n,0}(0)| \leq |\epsilon_{n,0}|L(P_{n,0}) < \theta_{n+1}/2$  and that

$$\begin{aligned} c_{n+1}(F_n) &\leq L(F_n) \leq \sum_{k=1}^{n-1} \sum_{j=0}^{s_k+\ell_k} |\epsilon_{k,j}|L(P_{k,j}) \\ &\leq \sum_{k=1}^{n-1} \sum_{j=0}^{s_k+\ell_k} \nu_{k,j}L(P_{k,j}) \leq \sum_{k=1}^{n-1} \sum_{j=0}^{s_k+\ell_k} \frac{1}{(\hat{s}_k + \hat{\ell}_k)(k + 2/\Theta_{k+2})^{k+3}} \\ &\leq \sum_{k=1}^{n-1} \frac{1}{(k + 2/\Theta_{k+2})^k} < \sum_{k \geq 1} \frac{1}{(1 + 2/\theta_{n+1})^k} = \frac{\theta_{n+1}}{2}. \end{aligned}$$

In conclusion, we have

$$|c_{n+1}(F_n) + \epsilon_{n,0}P_{n,0}(0)| \leq |c_{n+1}(F_n)| + |\epsilon_{n,0}P_{n,0}(0)| < \theta_{n+1},$$

as desired. Next, we define  $f_{n,1}(z)$  by

$$f_{n,1}(z) = f_{n,0}(z) + \epsilon_{n,1}z^{n+2}P_{n,1}(z)$$

(so  $P_{n,1}(z)$  is also equal to  $P_{n,0}(z)$ ). Note that  $f'_{n,1}(y) \neq 0$ , for all  $y \in \{\alpha_1, \dots, \alpha_n\} \cup f_n^{-1}(\{\alpha_2, \dots, \alpha_{n+1}\})$ , by choosing  $\epsilon_{n,1}$  an admissible number



of small norm. If  $\alpha_{n+1}$  is a root of  $P_{n,1}(z)$ , then  $f_{n,1}(\alpha_{n+1}) = f_{n,0}(\alpha_{n+1}) \in \overline{\mathbb{Q}}$ , by the definition of the roots of  $P_n(z)$  in (ii). Hence, in this case, we are done. Thus, suppose that  $\alpha_{n+1}$  is not a zero of  $P_{n,1}(z)$ . Then, by density of  $\overline{\mathbb{Q}}$ , there exists such an  $\epsilon_{n,1}$  such that

$$f_{n,1}(\alpha_{n+1}) = f_{n,0}(\alpha_{n+1}) + \epsilon_{n,1}\alpha_{n+1}^{n+2}P_{n,1}(\alpha_{n+1}) \in \overline{\mathbb{Q}}.$$

Define

$$f_{n,2}(z) = f_{n,1}(z) + \epsilon_{n,2}z^{n+2}P_{n,2}(z),$$

where  $P_{n,2}(z) = P_{n,1}(z)(z - \alpha_{n+1})^2$  and set  $g_{n,2}(z) = z^{n+2}P_{n,2}(z)$ . Since we are supposing that  $\alpha_{n+1}$  is not a root of  $P_{n,1}(z)$ , then  $g'_{n,2}(\alpha_{n+1}) \neq 0$  with at most one exception; we can then choose  $\epsilon_{n,2}$  admissible in  $I_{n,2}$  such that  $f'_{n,2}(\alpha_{n+1}) \neq 0$ .

Let  $J$  be the set of indices  $j \in [1, m_n]$  such that  $\tau_j$  does not belong to  $\{\alpha_{n+1}\} \cup X_n \cup \tilde{X}_n \cup \{0\}$ . Let  $J = \{j_1, j_2, \dots, j_{s_n-1}\}$ . For each  $i \in [1, s_n - 2]$ , we will do a perturbation as below. Let  $z_i$  be the only element of  $f_{n,1+i}^{-1}(\{\alpha_1, \dots, \alpha_{n+1}\})$  in  $B(\tau_{j_i}, \eta_{j_i})$ . In the  $i$ -th step we will guarantee that the element of  $f_{n,2+i}^{-1}(\{\alpha_1, \dots, \alpha_{n+1}\})$  in  $B(\tau_{j_i}, \eta_{j_i})$  will belong to  $\overline{\mathbb{Q}}$ , and in subsequent steps the images of these elements by the maps  $f_{n,1+j}$  will remain unchanged.

Define  $f_{n,3}(z)$  by

$$f_{n,3}(z) = f_{n,2}(z) + \epsilon_{n,3}z^{n+2}P_{n,3}(z),$$

where  $P_{n,3}(z) = P_{n,2}(z)$ . To simplify, we write  $\epsilon := \epsilon_{n,3}$  and  $F(\epsilon, z) = f_{n,3}(z)$ . Let  $w = f_{n,2}(z_1) \in \{\alpha_1, \dots, \alpha_{n+1}\}$ . For each algebraic number  $\tau$  close to  $z_1$ , let

$$\epsilon(\tau) = \frac{w - f_{n,2}(\tau)}{\tau^{n+2}P_{n,3}(\tau)}$$

be such that  $f_{n,3}(\tau) = w$ . Since  $\frac{\partial F}{\partial z}(0, z_1) = f'_{n,2}(z_1) \neq 0$ , for  $\tau$  close enough to  $z_1$ , we have  $f'_{n,3}(\tau) \neq 0$ . We may also choose  $\tau$  not belonging to  $\{f_{n,2}(z); P_{n,3}(z) = 0\}$ .

Thus, following this construction we will obtain at the end a function  $f_{n,s_n}$  such that  $f_{n,s_n}^{-1}(\{\alpha_1, \dots, \alpha_{n+1}\}) \cap (B_{n+1} \cup (\cup_{j=1}^{m_n} B(\tau_j, \eta_j)))$  is a subset of  $\overline{\mathbb{Q}}$ . We set  $f_{n+1,0}(z) := f_{n,s_n}(z)$ . This function will satisfy the items (i), (iii)-(vi) (with  $n$  replaced by  $n+1$ ).

To finish the construction, we need to deal with the periodic orbits of  $f_{n+1,0}$  lying in  $B_{n+1}$ . Since at each step we have taken  $\epsilon$ -perturbations, there must exist at least  $n+1$   $k$ -periodic orbits of  $f_{n+1,0}$  lying inside of  $B_{n+1}$ , for all  $k \in [1, n+1]$ . Moreover, if  $\alpha \in \text{Per}(k, f_n)$ , then  $P_{n,s_n-1}(f_n^j(\alpha)) = 0$ , for all  $j \in [0, k]$  and so  $f_{n+1,0}^j(\alpha) = f_n^j(\alpha)$  for all  $j \in [1, k]$  (in other words, the whole orbit of  $\alpha$  is nailed).

Suppose that

$$\{\beta, f_{n+1,0}(\beta), \dots, f_{n+1,0}^{k-1}(\beta)\}$$

is a free repelling  $k$ -periodic orbit of  $f_{n+1,0}(z)$ , for some  $k \in [1, n+1]$ . Notice that  $\beta$  is not nailed (as well as all the elements of its orbit). Take algebraic numbers  $\gamma_0$  very close to  $\beta$  and  $\gamma_1$  very close to  $f_{n+1,0}(\beta)$  and let  $P_{n+1,1}(z) := P_{n+1,0}(z)$ . Since  $P_{n+1,1}(\gamma_0) \neq 0$ , there exists a very small and admissible  $\epsilon_{n+1,1}$  for which  $f_{n+1,0}(\gamma_0) + \epsilon_{n+1,1}(\gamma_0)^{n+2}P_{n+1,1}(\gamma_0) = \gamma_1$ . Now, we define  $f_{n+1,2}(z)$  as

$$f_{n+1,2}(z) = f_{n+1,1}(z) + \epsilon_{n+1,2}z^{n+2}P_{n+1,2}(z),$$

where  $P_{n+1,2}(z) := P_{n+1,1}(z)(z - \gamma_0)^2$ . We continue this construction until we arrive at a function  $f_{n+1,k}(z)$  and algebraic numbers  $\gamma_0, \dots, \gamma_{k-1}$  such that  $\gamma_i$  is very close to  $f_{n+1,0}^i(\beta)$  and  $f_{n+1,k}(\gamma_j) = \gamma_{j+1}$ , for all  $(i, j) \in [0, k-1] \times [0, k-2]$ . We also require the property that for any  $j \in [1, k-1]$ , we have Finally, we define  $f_{n+1,k+1}(z)$  as

$$f_{n+1,k+1}(z) = f_{n+1,k}(z) + \epsilon_{n+1,k+1}z^{n+2}P_{n+1,k+1}(z),$$

where  $P_{n+1,k+1}(z) := P_{n+1,1}(z)(z - \gamma_0)^2 \cdots (z - \gamma_{k-2})^2$ . To finish, we take a small and admissible  $\epsilon_{n+1,k+1}$  for which  $f_{n+1,k+1}(\gamma_{k-1}) = \gamma_0$ . This construction works since our original periodic orbit is free, and so we are able to create new algebraic periodic orbits of period  $k$  until reaching the required number (and we do not create undesirable nailed algebraic periodic orbits in the process).

2.2.1. *The final part.* By construction, the function

$$f(z) = \lim_{m \rightarrow \infty} f_m(z) = g(z) + \sum_{n \geq 1} \sum_{j=0}^{s_n + \ell_n} \epsilon_{n,j} z^{n+2-\delta_{j,0}} P_{n,j}(z) = \sum_{n \geq 0} a_n z^n$$

is entire and satisfies the desired properties:  $f(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ ,  $f^{-1}(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ ,  $a_n \in K$  for all  $n \geq 0$ , and  $\#\{\text{Orb}(k, f)\} = s_k$  for all  $k \geq 1$ . Indeed, for each  $m \geq 1$ ,  $f_{k+1}(\alpha_m) = f_k(\alpha_m) \in \overline{\mathbb{Q}}$  for all  $k$  such that  $k \geq m$ . In particular since  $f = \lim_{k \rightarrow \infty} f_k$ , we have  $f(\alpha_m) = f_m(\alpha_m) \in \overline{\mathbb{Q}}$ . Hence  $f(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ . On the other hand, since  $f_{m+1}^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = \tilde{X}_m \cap B(0, r_m) \subseteq \overline{\mathbb{Q}}$  and  $B(0, r_m) \subset B(0, r_{m+1})$  for all  $m \geq 1$ , it follows by induction that, for each  $k \geq m$ ,

$$f_k^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m).$$

This implies that

$$f^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) \subseteq \overline{\mathbb{Q}}.$$

Indeed,  $f = \lim_{k \rightarrow \infty} f_k$ , and so

$$f^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) \supset f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m).$$

On the other hand, if there were another element  $w$  of  $f^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m)$ , it should be at positive distance of the finite set  $f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m)$ , but, since  $f = \lim_{n \rightarrow \infty} f_n$ , arbitrarily close to  $w$  there should be,

for  $k$  large, an element of  $f_k^{-1}(\{\alpha_2, \dots, \alpha_m\})$  (also by Rouché's theorem), which contradicts the equality

$$f_k^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m) = f_m^{-1}(\{\alpha_2, \dots, \alpha_m\}) \cap B(0, r_m).$$

Hence  $f^{-1}(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$  (and  $f^{-1}(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ , since  $f(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ ). Moreover, since, for each integer  $j \in [1, n]$ , there is  $\tau \in \tilde{X}_n$  such that  $f_n(\tau) = \alpha_j$ , and, since  $P_n(z) \mid P_{n+1}(z) \forall n \geq 1$  we shall have, for each  $k \geq m$ , that  $f_k(\tau) = \alpha_j$  and so  $f(\tau) = \alpha_j$ . In conclusion,  $f(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$  and  $f^{-1}(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ .

We also prove, in an analogous way, that  $\#\{\text{Orb}(k, f)\} = s_k, \forall k \geq 1$  (since, for every natural number  $m, Y_m^{(k)}$  is a union of exactly  $\min\{m, s_k\}$  algebraic periodic orbits of period  $k$  of  $f_m$  contained in  $B(0, r_m)$ , and is contained in the set of roots of  $P_m$ ).

Moreover, the function  $f(z) = \sum_{n \geq 0} a_n z^n$  is transcendental, since it is a non-polynomial entire function (recall that  $a_n \neq 0 \forall n \geq 0$ ). This completes the proof.  $\square$

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