ALGEBRAIC PRECONDITIONING FOR BIOT-BARENBLATT POROELASTIC SYSTEMS

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Abstract. Poroelastic systems describe fluid flow through porous medium coupled with deformation of the porous matrix. In this paper, the deformation is described by linear elasticity, the fluid flow is modelled as Darcy flow. The main focus is on the Biot-Barenblatt model with double porosity/double permeability flow, which distinguishes flow in two regions considered as continua. The main goal is in proposing block diagonal preconditionings to systems arising from the discretization of the Biot-Barenblatt model by a mixed finite element method in space and implicit Euler method in time and estimating the condition number for such preconditioning. The investigation of preconditioning includes its dependence on material coefficients and parameters of discretization.

Keywords: poroelasticity; double permeability; preconditioning; Schur complement

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1. Introduction

The basic model describing fluid flow through porous medium which undergoes mechanical deformations of the porous matrix is the Biot's poroelastic model. This model can be described by the following system of equations, which holds in a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $n = 2, 3$:

$$-\operatorname{div}(C : \varepsilon(u)) + \alpha \operatorname{grad}(p) = f, \tag{1.1}$$

$$K^{-1}v + \operatorname{grad}(p) = g, \tag{1.2}$$

$$\alpha \frac{\partial}{\partial t} \operatorname{div}(u) + \operatorname{div}(v) + c_{pp} \frac{\partial}{\partial t} p = 0. \tag{1.3}$$

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Above, $\varepsilon(u)$ is the small strain tensor given by the symmetric part of the gradient of the displacement u, $2\varepsilon(u) = \nabla u + (\nabla u)^{\mathsf{T}}$. We will assume isotropic and homogeneous linear elastic media with elasticity tensor C defined by the identity

$$C : \varepsilon(u) = 2\mu\varepsilon(u) + \lambda \operatorname{div}(u) I,$$

where I is the identity tensor and λ and μ are two elastic constants (Lamé moduli).

The flow is described by the velocity v and fluid pressure p, K is the hydraulic conductivity of the porous media given by the ratio of permeability and dynamic viscosity of the fluid, and $c_{pp} > 0$ is storativity. The constant α is the Biot-Willis constant.

This model is completed by boundary and initial conditions, e.g.

$$u(x,t) = \hat{u}(x,t), \quad x \in \Gamma_u,\ t \in T, \tag{1.4}$$

$$\sigma(x,t) \cdot n(x) = \hat{\sigma}(x,t), \quad x \in \Gamma_\sigma,\ t \in T, \tag{1.5}$$

$$v(x,t) \cdot n(x) = \hat{v}(x,t), \quad x \in \Gamma_v,\ t \in T, \tag{1.6}$$

$$p(x,t) = \hat{p}(x,t), \quad x \in \Gamma_p,\ t \in T, \tag{1.7}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.8}$$

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.9}$$

$$p(x,0) = p_0(x), \quad x \in \Omega, \tag{1.10}$$

where $\Gamma_u \cap \Gamma_\sigma = \emptyset$, $\Gamma_u \cup \Gamma_\sigma = \partial\Omega$, $\Gamma_p \cap \Gamma_v = \emptyset$, $\Gamma_p \cup \Gamma_v = \partial\Omega$, n denotes the unit outward normal to $\partial\Omega$ and $\hat{u}$, $\hat{\sigma}$ and $\hat{v}$, $\hat{p}$ are given vector and scalar functions, respectively.

The poroelasticity model is successfully applied in many situations when the porous material is homogeneous or not too heterogeneous.

However, the porous material is frequently highly heterogeneous, e.g. due to fractures in the rock mass or systems of bigger and smaller pores in the soil or filter. In such cases the fractures or bigger pores create preferable flow paths, whereas the rock matrix or smaller soil pores, which cover substantially larger part of the volume of pores, provide larger fluid storage capacity. As explicit modelling of fractures or big pores is normally above the available computing capacity, a continuous macroscopic phenomenological double porosity/double permeability model was suggested (first in [6], [21]) and found a lot of important applications.

This model also known as Biot-Barenblatt model considers fluid flow as a superposition of two coupled flows in two overlapping continua with different conductivities. Moreover, both the flows are coupled with deformation of the porous material. According to our field of applications, we denote the continua as rock matrix and system of fractures, but as already mentioned, the field of physical applications also includes other situations.

In this paper, the Biot-Barenblatt model will be described by five variables-displacement u, two fluxes v^1, v^2 and two corresponding pressures p^1, p^2 describing the fluid flow in matrix and fractures, respectively. The equations describing the model are [19]

$$\operatorname{div}(C : \varepsilon(u)) - \alpha_1 \operatorname{grad}(p^1) - \alpha_2 \operatorname{grad}(p^2) = f, \tag{1.11}$$

$$K_1^{-1}(x)v^1 - \operatorname{grad}(p^1) = g_1, \tag{1.12}$$

$$K_2^{-1}(x)v^2 - \operatorname{grad}(p^2) = g_2, \tag{1.13}$$

$$\alpha_1 \frac{\partial}{\partial t} \operatorname{div}(u) + \operatorname{div}(v^1) + c_{pp}^1 \frac{\partial}{\partial t} p^1 + \beta(p^1 - p^2) = 0, \tag{1.14}$$

$$\alpha_2 \frac{\partial}{\partial t} \operatorname{div}(u) + \operatorname{div}(v^2) + c_{pp}^2 \frac{\partial}{\partial t} p^2 - \beta(p^1 - p^2) = 0. \tag{1.15}$$

Above, we distinguish different permeability, different storativity and different Biot-Willis constants for fluid in the matrix and the fracture systems. Moreover, equations (1.14)–(1.15) involve terms enabling fluid transfer between matrix and fractures. The transfer depends only on the difference in pressures and the constant $\beta > 0$ which describes the strength of the coupling.

The Biot-Barenblatt model will be discretized by a mixed finite element method in space and the implicit Euler method in time. Solvability, stability and other properties of the model and properties of the proposed discretization are not analysed in this paper, for this we can refer e.g. [14], [15], [16], [19], [20]. Instead, we consider systems to be solved in each time step of the implicit Euler method and propose and analyse block diagonal preconditioners, which generalize the preconditioner considered in [3], [4].

The rest of the paper is organized as follows. Section 2 describes variational formulation and discretization of the Biot-Barenblatt problem. Section 3 deals with augmented type algebraic preconditioning applicable for both single and double porosity flow systems. The paper then describes numerical experiments in Section 4 and ends with conclusions.

2. variational formulation and discretization

For the variational formulation of the problem (1.11)–(1.15) with proper boundary conditions, we consider the spaces

$$U = \{u \in [H^1(\Omega)]^n \mid u = u_D \text{ on } \Gamma_u\}, \tag{2.1}$$

$$V = \{v \in H(\operatorname{div}, \Omega) \mid v \cdot n = \hat{v} \text{ on } \Gamma_v\}, \tag{2.2}$$

$$P = L^2(\Omega). \tag{2.3}$$

Let us denote $U_0 = U$ for $u_D = 0$ and $V_0 = V$ for $\hat{v} = 0$ respectively. Moreover, we will consider a bilinear form $a(u, v)$ given by

$$a(u, w) = 2\mu \int_\Omega \varepsilon(u) : \varepsilon(w) + \lambda \int_\Omega \operatorname{div} u \operatorname{div} w. \tag{2.4}$$

The three field variational formulation of the single porosity Biot's problem is then as follows [14], [20]. Find $(u, v, p) \in U \times V \times P$ such that

$$a(u, w) - \alpha(p, \operatorname{div} w) = (\hat{f}, w) \quad \forall\, w \in U_0, \tag{2.5}$$

$$(K^{-1}v, \eta) - (p, \operatorname{div} \eta) = (\hat{g}, \eta) \quad \forall\, \eta \in V_0, \tag{2.6}$$

$$\alpha\Big(\frac{\partial}{\partial t} \operatorname{div} u, q\Big) + (\operatorname{div} v, q) + \Big(c_{pp} \frac{\partial}{\partial t} p, q\Big) = 0 \quad \forall\, q \in P, \tag{2.7}$$

where $(\cdot, \cdot)$ denotes the L^2 scalar product. Instead of the L^2 inner product, $(\hat{f}, \cdot)$, $(\hat{g}, \cdot)$ denote bounded linear functionals that include the contribution of the given boundary conditions.

For the double porosity/double permeability problem, boundary conditions are imposed for all flow variables. If $v^1 \cdot n = v^2 \cdot n = \hat{v}$ on Γ_v then a similar five field formulation looks like

$$a(u, w) - \alpha_1(p^1, \operatorname{div} w) - \alpha_2(p^2, \operatorname{div} w) = (\hat{f}, v) \quad \forall\, w \in U, \tag{2.8}$$

$$(K_1^{-1}v^1, \eta) - (p^1, \operatorname{div} \eta) = (\hat{g}_1, \eta) \quad \forall\, \eta \in V, \tag{2.9}$$

$$(K_2^{-1}v^2, \eta) - (p^2, \operatorname{div} \eta) = (\hat{g}_2, \eta) \quad \forall\, \eta \in V, \tag{2.10}$$

$$\alpha_1\Big(\frac{\partial}{\partial t} \operatorname{div} u, q\Big) + (\operatorname{div} v^1, q) + \Big(c_{pp} \frac{\partial}{\partial t} p^1, q\Big) + \beta(p^1 - p^2, q) = 0 \quad \forall\, q \in P, \tag{2.11}$$

$$\alpha_2\Big(\frac{\partial}{\partial t} \operatorname{div} u, q\Big) + (\operatorname{div} v^2, q) + \Big(c_{pp} \frac{\partial}{\partial t} p^2, q\Big) - \beta(p^1 - p^2, q) = 0 \quad \forall\, q \in P. \tag{2.12}$$

The system of partial differential equations (2.8)–(2.12) can be discretized first in space using the finite element method. Piecewise linear $P(1)$ elements are used for representation of displacement u, lowest order $RT(0)$ elements are used for v, v^1, v^2, and $P(0)$ elements for p, p^1, and p^2. The difference $RT(0) - P(0)$ is a stable pair of mixed finite elements, see [8]. Using the same elements pairs for both the velocities and pressures is natural. There are other types of finite elements that can be considered for discretization, but the further analysis exploits the fact that $P(0)$ elements are used for pressures.

In this context, it should be mentioned that recently a significant progress was obtained in the analysis of an appropriate choice of discretization spaces for full parameter robust stability of the classical three-field formulation (displacement, Darcy velocity, pore pressure) of Biot's consolidation model, see [14], [20]. In the present

paper, we consider systems arising from $P(1) - RT(0) - P(0)$ discretization, which are not uniformly stable but work well in many engineering applications.

Discretization in time is done by the implicit Euler scheme with timestep τ. More details about discretization for the single permeability system can be found e.g. in [4], [14], [20]. Discretization of double permeability system is considered e.g. in [15], [16], [19].

The space and time discretization leads to a time-stepping algorithm with the solution of a linear system in each time step. The matrix of this time step system can be symmetrized by a row scaling. The symmetrized matrix has the form

$$\mathcal{A} = \begin{bmatrix} A & & B_u^{\mathsf{T}} \\ & \tau M & \tau B_v^{\mathsf{T}} \\ B_u & \tau B_v & -D \end{bmatrix}, \tag{2.13}$$

where for the single porosity Biot's problem the matrices A, M, D, B_u, B_v represent the bilinear forms in the corresponding finite element spaces. If $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product in the algebraic vector space and $\xi_h \leftrightarrow \xi$ denotes the correspondence between the functions in finite element spaces and their algebraic representation, then

$$\begin{aligned} \langle Au, v \rangle &= a(u_h, v_h) \quad \text{on } U_h \times U_h, \\ \langle Mv, \eta \rangle &= m(v_h, \eta_h) = (K^{-1} v_h, \eta_h) \quad \text{on } V_h \times V_h, \\ \langle Dp, q \rangle &= d(p_h, q_h) = (c_{pp} p_h, q_h) \quad \text{on } P_h \times P_h, \\ \langle B_u u, q \rangle &= b_u(u_h, q_h) = -(q_h, \operatorname{div}(u_h)) \quad \text{on } U_h \times P_h, \\ \langle B_v u, v \rangle &= b_v(v_h, p_h) = -(p_h, \operatorname{div}(v_h)) \quad \text{on } V_h \times P_h. \end{aligned}$$

For the double porosity Biot's problem the matrices M, D, B_u, B_v have doubled block structure and we get

$$\bar{\mathcal{A}} = \begin{bmatrix} A & & \bar{B}_u^{\mathsf{T}} \\ & \tau \bar{M} & \tau \bar{B}_v^{\mathsf{T}} \\ \bar{B}_u & \tau \bar{B}_v & -\bar{D} \end{bmatrix}, \tag{2.14}$$

$$\bar{M} = \begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix}, \quad \bar{B}_u = \begin{bmatrix} \alpha_1 B_u \\ \alpha_2 B_u \end{bmatrix}, \quad \bar{B}_v = \begin{bmatrix} B_v & \\ & B_v \end{bmatrix},$$

and

$$\bar{D} = \begin{bmatrix} D_1 + \beta\tau D_0 & -\beta\tau D_0 \\ -\beta\tau D_0 & D_2 + \beta\tau D_0 \end{bmatrix},$$

where D_0, D_1, D_2 are pressure mass matrices, $\langle D_0 p, q \rangle = (p_h, q_h)$ on $P_h \times P_h$, D_1, D_2 include weights c_{pp}^1 and c_{pp}^2, respectively. For constant c_{pp}^i, $D_i = c_{pp}^i D_0$, $i = 1, 2$. For the piecewise constant $P(0)$ the elements for pressure, D_0, D_1, D_2 are diagonal.

Further, M, M_1 and M_2 are the velocity mass matrices weighted by the inverse of the respective conductivities. The difference between M_1 and M_2 is only in different conductivities. Matrices B_v and B_u are discretizations of the divergence of flux and the displacement. Note that the repetition of the blocks in doubled structure is due to the use of the same finite elements both for pressures and fluxes.

We will assume that the elasticity stiffness matrix A is symmetric and positive definite (SPD). Then for the single porosity, the system matrix $\mathcal{A}$ is symmetric and indefinite with respect to its (saddle point) structure with diagonal blocks $\begin{bmatrix} A & \\ & M \end{bmatrix}$ and $-D$, where the first block is symmetric positive definite and the second is symmetric negative definite.

The same argument can be used for double porosity matrix $\bar{\mathcal{A}}$, just the diagonal blocks are $\begin{bmatrix} A & \\ & \bar{M} \end{bmatrix}$ and $-\bar{D}$,

$$\bar{D} = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} + \begin{bmatrix} \beta\tau D_0 & -\beta\tau D_0 \\ -\beta\tau D_0 & \beta\tau D_0 \end{bmatrix}, \tag{2.15}$$

is SPD as the first matrix in the sum is positive definite and the second is positive semidefinite (see the accompanying quadratic form).

In the case of constant c^1_{pp} and c^2_{pp}, we get

$$\bar{D} = \begin{bmatrix} (c^1_{pp} + \beta\tau)D_0 & -\beta\tau D_0 \\ -\beta\tau D_0 & (c^2_{pp} + \beta\tau)D_0 \end{bmatrix} \tag{2.16}$$

and $\bar{D}$ also admits the tensor product expression, see [12],

$$\bar{D} = \begin{bmatrix} c^1_{pp} + \beta\tau & -\beta\tau \\ -\beta\tau & c^2_{pp} + \beta\tau \end{bmatrix} \otimes [\, D_0 \,] \tag{2.17}$$

which allows a simple computation of the inverse of $\bar{D}$,

$$\bar{D}^{-1} = \frac{1}{\xi} \begin{bmatrix} (c^2_{pp} + \beta\tau){D_0}^{-1} & \beta\tau {D_0}^{-1} \\ \beta\tau {D_0}^{-1} & (c^1_{pp} + \beta\tau){D_0}^{-1} \end{bmatrix}, \tag{2.18}$$

where

$$\xi = (c^1_{pp} + \beta\tau)(c^2_{pp} + \beta\tau) - \beta^2\tau^2. \tag{2.19}$$

The tensor product expression (2.17) and the fact that the eigenvalues of the tensor product are products of the eigenvalues of the factors allows to compute the eigenvalues of $\bar{D}$,

$$\lambda_{i,1,2} = \delta_i \frac{c^1_{pp} + c^2_{pp} + 2\beta\tau \pm \sqrt{(c^1_{pp} - c^2_{pp})^2 + 4\beta^2\tau^2}}{2}, \tag{2.20}$$

where δ_i are eigenvalues of D_0. For uniform mesh we have $D_0 = \delta I$, which means that all δ_i are equal to δ.

3. Preconditioning

For poroelasticity with single porosity, two SPD preconditioners for the system with matrix (2.13) were suggested and analysed in [3], [4], namely

$$\mathcal{P}_F = \begin{bmatrix} S_{11} & S_{21} & \\ S_{12} & S_{22} & \\ & & +D \end{bmatrix}, \quad \mathcal{P}_D = \begin{bmatrix} S_{11} & & \\ & S_{22} & \\ & & +D \end{bmatrix},$$

where

$$S = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} = \begin{bmatrix} A & \\ & M \end{bmatrix} + \begin{bmatrix} B_u^{\mathsf{T}} \\ B_v^{\mathsf{T}} \end{bmatrix} D^{-1} \begin{bmatrix} B_u & B_v \end{bmatrix}.$$

Our aim is to extend these preconditioners and their analysis to Biot-Barenblatt double porosity/double permeability matrix $\bar{\mathcal{A}}$ introduced in (2.14). It means that the inner structure and doubled $\bar{M}, \bar{D}, \bar{B}_u, \bar{B}_v$ have to be taken into account.

$$\bar{\mathcal{P}}_F = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{21} & \\ \bar{S}_{12} & \bar{S}_{22} & \\ & & +\bar{D} \end{bmatrix}, \quad \bar{\mathcal{P}}_D = \begin{bmatrix} \bar{S}_{11} & & \\ & \bar{S}_{22} & \\ & & +\bar{D} \end{bmatrix}, \tag{3.1}$$

$$\bar{S} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{21} \\ \bar{S}_{12} & \bar{S}_{22} \end{bmatrix} = \begin{bmatrix} A & \\ & \bar{M} \end{bmatrix} + \begin{bmatrix} \bar{B}_u^{\mathsf{T}} \\ \bar{B}_v^{\mathsf{T}} \end{bmatrix} \bar{D}^{-1} \begin{bmatrix} \bar{B}_u & \bar{B}_v \end{bmatrix}.$$

Note that $\mathcal{P}_F$ and $\bar{\mathcal{P}}_F$ are SPD block diagonal preconditioners belonging to a broader class of preconditioners for (generalized) saddle point matrices, see e.g. [7], [10]. We choose an augmented type preconditioner, because it utilizes the benefit of special structure and explicit inverse (2.18) of the lower-right block. Moreover, we choose SPD block diagonal preconditioner for possibility of spectral analysis of the properties of the preconditioned system. On the other hand, the related triangular preconditioner can be more efficient.

With respect to the Schur complement structure, we immediately get a theorem on localization of the spectrum $\sigma(\bar{\mathcal{P}}_F^{-1}\bar{\mathcal{A}})$.

Theorem 1. *The following spectral estimate holds for the preconditioned system* $\bar{\mathcal{P}}_F^{-1}\bar{\mathcal{A}}$*:*

$$\sigma(\bar{\mathcal{P}}_F^{-1}\bar{\mathcal{A}}) \subset \left\langle \frac{-1-\sqrt{5}}{2}, -1 \right\rangle \cup \left\langle \frac{-1+\sqrt{5}}{2}, 1 \right\rangle. \tag{3.2}$$

P r o o f. Let us consider the eigenvalue problem $\bar{\mathcal{A}}x = \lambda\bar{\mathcal{P}}_F x$, where the following partitioning for $\bar{\mathcal{A}}$ and $\mathcal{P}_F$ are used:

$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}, \quad \bar{\mathcal{P}_F} = \begin{bmatrix} \bar{S} & \\ & \bar{D} \end{bmatrix}, \quad \mathcal{A}_{22} = -\bar{D}, \quad \bar{S} = \mathcal{A}_{11} + \mathcal{A}_{12}\bar{D}^{-1}\mathcal{A}_{21}.$$

As $\mathcal{A}_{22}$ is SPD, we can use the factorization of $\bar{\mathcal{A}}$, which yields

$$\begin{bmatrix} I_1 & \mathcal{A}_{12}\mathcal{A}_{22}^{-1} \\ & I_2 \end{bmatrix} \begin{bmatrix} \bar{S} & \\ & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} I_1 & \\ \mathcal{A}_{22}^{-1}\mathcal{A}_{21} & I_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} \bar{S} & \\ & \bar{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\begin{bmatrix} \bar{S} & \\ & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} I_1 & \\ \mathcal{A}_{22}^{-1}\mathcal{A}_{21} & I_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} I_1 & -\mathcal{A}_{12}\mathcal{A}_{22}^{-1} \\ & I_2 \end{bmatrix} \begin{bmatrix} \bar{S} & \\ & \bar{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$\begin{bmatrix} \bar{S} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} \bar{S} & \mathcal{A}_{12} \\ 0 & -\mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

i.e.

$$\bar{S}x_1 = \lambda\bar{S}x_1 + \lambda\mathcal{A}_{12}x_2, \tag{3.3}$$

$$\mathcal{A}_{21}x_1 + \mathcal{A}_{22}x_2 = -\lambda\mathcal{A}_{22}x_2. \tag{3.4}$$

For $\lambda \neq -1$ we get

$$(1+\lambda)\mathcal{A}_{22}x_2 = -\mathcal{A}_{21}x_1,$$

$$x_2 = -\frac{1}{1+\lambda}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}x_1,$$

$$(1-\lambda)\bar{S}x_1 = -\frac{\lambda}{1+\lambda}\mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}x_1 = \frac{\lambda}{1+\lambda}\bar{S}_0x_1.$$

As $\bar{S}$ and $\mathcal{A}_{22}$ are nonsingular, $\mathcal{A}$ is also nonsingular and $0 \notin \sigma(\mathcal{P}_F^{-1}\mathcal{A})$. Thus $\lambda \neq 0$ and we get $(1-\lambda^2)/\lambda = \mu \in \sigma(\bar{S}_0, \bar{S})$ and consequently $\lambda = \frac{1}{2}(-\mu \pm \sqrt{\mu^2+4})$. From the positive definiteness of $\bar{S}$ and positive semidefiniteness of $\bar{S}_0$, we get $\mu \geqslant 0$, and due to positive definiteness of $\bar{S} - \bar{S}_0$, we have $\mu < 1$ for $\mu \in \sigma(\bar{S}_0, \bar{S})$. It is also easy to show that both branches $-\mu \pm \sqrt{\mu^2+4}$ are decreasing, so the limits are given by $\mu = 0, 1$. This gives (3.2). □

Note that the proof uses only the algebraical Schur complement structure and does not require any other properties of matrices $\bar{\mathcal{A}}$ and $\bar{\mathcal{P}}_F$.

Further, we will investigate the possibility of simplification of the Schur complement $\bar{S} = \left[\begin{smallmatrix} \bar{S}_{11} & \bar{S}_{21} \\ \bar{S}_{12} & \bar{S}_{22} \end{smallmatrix}\right]$ in the preconditioner $\bar{\mathcal{P}}_F$ by its block diagonal part $\bar{S}_D =$

$\begin{bmatrix} \bar{S}_{11} & \\ & \bar{S}_{22} \end{bmatrix}$ appearing in the preconditioner $\bar{\mathcal{P}}_D$. To this end we analyse the blocks

$$\bar{S}_{11} = A + \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u, \tag{3.5}$$

$$\bar{S}_{22} = \tau \bar{M} + \tau^2 \bar{B}_v^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v, \tag{3.6}$$

$$\bar{S}_{12} = \bar{S}_{21}^{\mathsf{T}} = \tau \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v. \tag{3.7}$$

We will do this investigation under the assumption that $D_1 = c_{pp}^1 D_0$ and $D_2 = c_{pp}^2 D_0$.

The analysis for one porosity Biot problem was performed in [4], the following theorem extends the analysis to the double porosity/permeability Biot-Barenblatt problem.

Theorem 2. *There is a constant $\bar{\gamma} \in \langle 0, 1)$ such that*

$$(1 - \bar{\gamma}) \bar{S}_D \leqslant \bar{S} \leqslant (1 + \bar{\gamma}) \bar{S}_D,$$

where the Loewner ordering between matrices is used, i.e. $X \leqslant Y$ for two symmetric and positive matrices X and Y means that $Y - X$ is positive semidefinite.

Note that in the proof, we derive an estimate for $\bar{\gamma}$, which involves the quantities $\alpha_n = \sqrt{\alpha_1^2 + \alpha_2^2}$, c_d and c_{el}. The constant c_d is the minimum eigenvalue of $\bar{D}_0^{-1} \bar{D}$, $\bar{D}_0 = \begin{bmatrix} D_0 & \\ & D_0 \end{bmatrix}$ is the $L_2(\Omega) \times L_2(\Omega)$ mass matrix for $P(0)$ elements. Then c_d is given by (2.20) for nonsingular meshes:

$$c_d = \frac{c_{pp}^1 + c_{pp}^2 + 2\beta\tau + \sqrt{(c_{pp}^1 - c_{pp}^2)^2 + 4\beta^2\tau^2}}{2}. \tag{3.8}$$

For the special case of $c_{pp}^1 = c_{pp}^2 = c_{pp}$ we have

$$c_d = c_{pp} + 2\beta\tau \tag{3.9}$$

The list of constants is completed by $c_{el} = \sqrt{\lambda + 2\mu/n}$ for isotropic elasticity in $\Omega \subset \mathbb{R}^n$.

P r o o f. The analysis uses a strengthened Cauchy-Schwarz inequality, see [18]. We have

$$\begin{aligned}
|\langle \bar{S}_{12} v, u \rangle| &= |\langle \tau \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v v, u \rangle| = |\langle \tau \bar{D}^{-1/2} \bar{B}_v v, \bar{D}^{-1/2} \bar{B}_u u \rangle| \\
&\leqslant \sqrt{\langle \tau \bar{D}^{-1/2} \bar{B}_v v, \tau \bar{D}^{-1/2} \bar{B}_v v \rangle} \sqrt{\langle \bar{D}^{-1/2} \bar{B}_u u, \bar{D}^{-1/2} \bar{B}_u u \rangle} \\
&\leqslant \sqrt{\langle \tau^2 \bar{B}_v^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v v, v \rangle} \sqrt{\langle \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u u, u \rangle} \\
&\leqslant \bar{\gamma} \sqrt{\langle (\tau \bar{M} + \tau^2 \bar{B}_v^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v) v, v \rangle} \sqrt{\langle (A + \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u) u, u \rangle} \\
&= \bar{\gamma} \sqrt{\langle S_{11} u, u \rangle} \sqrt{\langle S_{22} v, v \rangle}.
\end{aligned}$$

The constant $\bar{\gamma} < 1$ comes from the inequality $\bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u \leqslant (\alpha_n^2/(c_{el}c_d)^2)A$, which can be derived as follows:

$$\begin{aligned}(3.10)\quad \langle \bar{B}_u^{\mathsf{T}} \underbrace{\bar{D}^{-1}\bar{B}_u}_{q} u, u\rangle = \langle \bar{B}_u^{\mathsf{T}} q, u\rangle &= \int_\Omega \alpha_1 \operatorname{div}(u_h)\cdot q_h^1 \,\mathrm{d}x + \int_\Omega \alpha_2 \operatorname{div}(u_h)\cdot q_h^2 \,\mathrm{d}x \\ &\leqslant \alpha_n \| \operatorname{div}(u_h)\|_{L_2(\Omega)} \|q_h\|_{L_2(\Omega)\times L_2(\Omega)} \\ &\leqslant \frac{\alpha_n}{c_{el}} \langle Au, u\rangle^{1/2} \langle \bar{D}_0 q, q\rangle^{1/2} \\ &\leqslant \frac{\alpha_n}{c_{el}c_d} \langle Au, u\rangle^{1/2} \langle \bar{D} q, q\rangle^{1/2} \\ &\leqslant \frac{\alpha_n}{c_{el}c_d} \langle Au, u\rangle^{1/2} \langle \bar{B}_u u, \bar{D}^{-1}\bar{B}_u u\rangle^{1/2}.\end{aligned}$$

Finally,

$$\Big(1 + \frac{(c_{el}c_d)^2}{\alpha_n^2}\Big)\bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u \leqslant A + \bar{B}_u^{\mathsf{T}} \bar{D}^{-1} \bar{B}_u,$$

i.e.

$$(3.11)\qquad \bar{\gamma} \leqslant \frac{1}{1 + (c_{el}c_d)^2 \alpha_n^{-2}}.$$

□

Note that the upper bound of $\bar{\gamma}$ does not depend on permeabilities nor on spatial discretization. Generally, low c_{pp} and $\beta\tau$ decreases the efficiency of the estimate. The analysis could be extended to involve a contribution from $\bar{B}_v^{\mathsf{T}} \bar{D}^{-1} \bar{B}_v \leqslant c\bar{M}$, which can be also substantial. The results from Theorems 1 and 2 can provide bounds for the spectra $\sigma(\bar{\mathcal{P}}_D^{-1}\bar{\mathcal{A}})$ according to the following simple lemma.

Lemma 1. *Let X is a symmetric matrix, P and Q are SPD matrices, $c_0 Q \leqslant P \leqslant c_1 Q$. For $X_P = P^{-1/2} X P^{-1/2}$, let $\sigma(X_p) \subset \langle -a, -b\rangle \cup \langle c, d\rangle$, where a, b, c, d are positive, $\alpha^2 I \leqslant X_P^2 \leqslant \beta^2 I$, where α and β are positive, $\alpha^2 = \min\{b^2, c^2\}$, $\beta^2 = \max\{a^2, d^2\}$.*

Then for $X_Q = Q^{-1/2} X Q^{-1/2}$,

$$(3.12)\qquad c_0^2\alpha^2 I \leqslant X_Q^2 \leqslant c_1^2\beta^2 I \quad \text{and} \quad \sigma(X_Q) \subset \langle -c_1\beta, -c_0\alpha\rangle \cup \langle c_0\alpha, c_1\beta\rangle.$$

P r o o f. The spectral equivalence $c_0 Q \leqslant P \leqslant c_1 Q$ implies that $c_0 I \leqslant Q^{-1/2} P Q^{-1/2} = EE^{\mathsf{T}} \leqslant c_1 I$. For $E^{\mathsf{T}}E = P^{1/2}Q^{-1}P^{1/2}$ we get the same bounds $c_0 I \leqslant P^{1/2}Q^{-1}P^{1/2} = E^{\mathsf{T}}E \leqslant c_1 I$ as $\sigma(EE^{\mathsf{T}}) = \sigma(E^{\mathsf{T}}E)$.

Further,

$$X_Q = Q^{-1/2} X Q^{-1/2} = Q^{-1/2} P^{1/2} X_P P^{1/2} Q^{-1/2}$$

and

$$(X_Q^2 x, x) = (X_Q x, X_Q x) = (Q^{-1/2}P^{1/2}X_P P^{1/2}Q^{-1/2}x, Q^{-1/2}P^{1/2}XP^{1/2}Q^{-1/2}x)$$
$$= ((P^{1/2}Q^{-1}P^{1/2})X_P P^{1/2}Q^{-1/2}x, A_P P^{1/2}Q^{-1/2}x).$$

Thus,

$$\begin{aligned}(X_Q^2 x, x) &\geqslant c_0(X_P P^{1/2}Q^{-1/2}x, X_P P^{1/2}Q^{-1/2}x)\\ &\geqslant c_0\alpha^2(P^{1/2}Q^{-1/2}x, P^{1/2}Q^{-1/2}x)\\ &\geqslant c_0\alpha^2(Q^{-1/2}P^1Q^{-1/2}x, x)\\ &\geqslant c_0^2\alpha^2(x,x).\end{aligned}$$

Similarly, $(X_Q^2 x, x) \leqslant c_1^2\beta^2(x,x)$. Therefore, the estimates (3.12) hold. □

Properties of $\bar{S}_{11}$. We can exploit the expression for inverse of $\bar{D}$, see (2.18). It provides

$$\bar{S}_{11} = A + \frac{1}{\xi}(\alpha_1^2 c_{pp}^1 + \alpha_2^2 c_{pp}^2 + (\alpha_1+\alpha_2)^2\beta\tau)B_u^{\mathsf{T}} {D_0}^{-1} B_u. \tag{3.13}$$

The matrix A represents the bilinear form (2.4), which for the isotropic elastic material takes the form

$$\langle Au, v\rangle = \int_\Omega \lambda \operatorname{div}(u_h)\operatorname{div}(v_h) + \int_\Omega 2\mu\varepsilon(u_h) : \varepsilon(v_h).$$

For $RT(0)-P(0)$ discretization, it holds that D_0 corresponds to elements E of the FE division and the area/volume $|E|$ is the corresponding diagonal element. Moreover, $\operatorname{div} u_h$ is constant on E. Consequently,

$$\begin{aligned}\langle B_u^{\mathsf{T}} {D_0}^{-1} B_u u, v\rangle &= \langle {D_0}^{-1} B_u u, B_u v\rangle \sum_E |E|^{-1} \int_E \operatorname{div} u_h \int_E \operatorname{div} v_h\\ &= \sum_E \int_E \operatorname{div} u_h \operatorname{div} v_h = \int_\Omega \operatorname{div} u_h \operatorname{div} v_h.\end{aligned} \tag{3.14}$$

Thus, for $RT(0)$ elements, $\bar{S}_{11}$ can be viewed as an elasticity stiffness matrix

$$\begin{aligned}\langle \bar{S}_{11}u, v\rangle = &\int_\Omega \Big[\lambda + \frac{1}{\xi}(\alpha_1^2 c_{pp}^1 + \alpha_2^2 c_{pp}^2 + (\alpha_1+\alpha_2)^2\beta\tau)\Big]\operatorname{div} u_h \operatorname{div} v_h\\ &+ \int_\Omega 2\mu\varepsilon(u_h) : \varepsilon(v_h).\end{aligned}$$

In case of higher order finite elements for velocity, S_{11} will be not exactly in the form of an elasticity stiffness matrix, but will be spectrally equivalent to one. Any technique that effectively solves elastic systems can be used to solve systems with $\bar{S}_{11}$, e.g. algebraic multigrid like AMG Boomer [13].

Properties and further simplification of $\bar{S}_{22}$. The block $\bar{S}_{22}$ has the form

$$S_{22} = \tau \begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix} + C,$$

where

$$\mathcal{C} = \begin{bmatrix} \tau^2 \frac{c^2_{pp}+\beta\tau}{\xi} B^\mathsf{T} {D_0}^{-1} B & \tau^2 \frac{\beta\tau}{\xi} B^\mathsf{T} {D_0}^{-1} B \\ \tau^2 \frac{\beta\tau}{\xi} B^\mathsf{T} {D_0}^{-1} B & \tau^2 \frac{c^1_{pp}+\beta\tau}{\xi} B^\mathsf{T} {D_0}^{-1} B \end{bmatrix}$$

$$= \frac{\tau^2}{\xi} \begin{bmatrix} (c^2_{pp}+\beta\tau) & \beta\tau \\ \beta\tau & (c^1_{pp}+\beta\tau) \end{bmatrix} \otimes \left[B^\mathsf{T} {D_0}^{-1} B \right].$$

This form enables us to show that

$$|\langle Cp, q\rangle| \leqslant \gamma_C \langle Cp, p\rangle^{1/2} \langle Cq, q\rangle^{1/2},$$

where

$$\gamma_C = \frac{\beta\tau}{\sqrt{(c^1_{pp}+\beta\tau)(c^2_{pp}+\beta\tau)}} < 1. \tag{3.15}$$

Consequently (see e.g. [18]), for Loewner ordering we get

$$(1-\gamma_C) \begin{bmatrix} \tau M_1 + C_{11} & \\ & \tau M_2 + C_{22} \end{bmatrix} \leqslant \bar{S}_{22} \leqslant (1+\gamma_C) \begin{bmatrix} \tau M_1 + C_{11} & \\ & \tau M_2 + C_{22} \end{bmatrix}.$$

Hence, the system with $\bar{S}_{22}$ can be solved by using some inner iterative method, e.g. preconditioned conjugate gradients, using the block diagonal preconditioning, where individual blocks can be solved e.g. by a special algorithm introduced in [17]. We can also perform a second step of diagonalization of $\mathcal{P}_F$ and use the third type of preconditioner

$$\bar{\mathcal{P}}_{DD} = \begin{bmatrix} S_{11} & & & & \\ & \tau M_1 + C_{11} & & & \\ & & \tau M_2 + C_{22} & & \\ & & & (c^1_{pp}+\beta\tau) D_0 & \beta\tau D_0 \\ & & & \beta\tau D_0 & (c^2_{pp}+\beta\tau) D_0 \end{bmatrix}. \tag{3.16}$$

4. Numerical experiments

Following [16], we will consider a model problem on the unit square Ω, see Fig. 1.

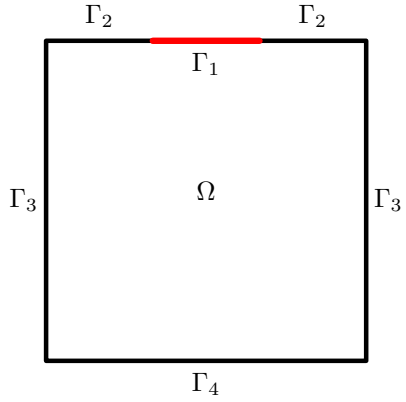

Figure 1. Computational domain.

The boundary conditions are set as follows:

- $\sigma(x) \cdot n = 100(x - 0.4)(0.6 - x)$ on Γ_1, $\sigma \cdot n = 0$ on Γ_2, $u \cdot n = 0$ and $(\sigma \cdot n) \times n = 0$ on Γ_3 and $u = 0$ on Γ_4,
- $p_m = p_f = 0$ on Γ_1 and $\partial p_m / \partial n = \partial p_f / \partial n = 0$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Initial conditions are zero displacement and zero fluid pressures, i.e. $u(x, 0) = 0$ and $p_m(x, 0) = p_f(x, 0) = 0$ for $x \in \Omega$.

The problem is discretized by $P(1) - RT(0) - P(0)$ elements in space and implicit Euler method in time on a uniform grid with 100×100 cells each divided to two triangles and with $\tau = 0.05$. The problem parameters are taken from [16] and given in Table 1.

parameter	unit	value
λ	MPa	4.2
μ	MPa	2.4
k^1	$10^{-15}\,\mathrm{m}^2$	6.18
k^2	$10^{-15}\,\mathrm{m}^2$	27.2
c_{pp}^1	$(\mathrm{GPa})^{-1}$	54
c_{pp}^2	$(\mathrm{GPa})^{-1}$	14
α_1	–	0.95
α_2	–	0.12
β	$10^{-10}\,\mathrm{kg}/(\mathrm{m} \cdot \mathrm{s})$	5

Table 1. Problem parameters.

For solving the systems with matrix $\bar{\mathcal{A}}$ (given by (2.14)) we use MINRES method with preconditioners $\bar{\mathcal{P}}_D$ (defined by (3.1)) or $\bar{\mathcal{P}}_{DD}$ (defined by (3.16)). In Tables 2 and 3, we compare numbers of iterations for solving the system $\bar{\mathcal{A}}$ with $\bar{\mathcal{P}}_D$ and $\bar{\mathcal{P}}_F$. The accuracy is driven by the Euclidean norm of residual and the iterations are stopped when$\|r^i\|/\|r^0\| \leqslant 10^{-6}$, the initial guess is zero. The inner block subsystems are solved by a direct solver. Tables 2 and 3 include results for the basic set of parameters from Table 1 as well as for modified conductivities and in Table 3 also for a changed water transfer coefficient.

	$k_1 \cdot 10^{-2}$	$k_1 \cdot 10^{-1}$	k_1
k_2	21/18	21/18	21/18
$k_2 \cdot 10^1$	29/21	29/21	29/21
$k_2 \cdot 10^2$	47/29	47/29	47/29

Table 2. Numbers of iterations for solving the system with matrix $\bar{\mathcal{A}}$ and preconditioner $\bar{\mathcal{P}}_D/\bar{\mathcal{P}}_F$ up to relative residual accuracy $\varepsilon = 10^{-6}$.

	$k_1 \cdot 10^{-2}$	$k_1 \cdot 10^{-1}$	k_1
k_2	19/16	19/16	19/16
$k_2 \cdot 10^1$	29/19	29/19	29/19
$k_2 \cdot 10^2$	46/25	46/25	46/25

Table 3. Numbers of iterations for solving the system with matrix $\bar{\mathcal{A}}$ and preconditioner $\bar{\mathcal{P}}_D/\bar{\mathcal{P}}_F$ up to relative residual accuracy $\varepsilon = 10^{-6}$. The constant β is changed to $\beta = 10^{-8}$.

Note that the same number of iterations as reported in Tables 2 and 3 will be obtained if $\bar{\mathcal{P}}_D$ is replaced by $\bar{\mathcal{P}}_{DD}$. This fact is not surprising as the strengthened Cauchy-Schwarz constant γ_C is $\gamma_C = 0.0009$ for $\beta = 5 \cdot 10^{-10}$ and $\gamma_C = 0.082$ for $\beta = 5 \cdot 10^{-8}$.

Note also that further increasing β does not deteriorate the behaviour with $\bar{\mathcal{P}}_F$ (estimate from Theorem 1 does not depend on β) nor the behaviour with $\bar{\mathcal{P}}_D$ (estimate from Theorem 2 with estimate (3.11), which decreases with increased β). The experiments carried out with parameter β in range from 10^{-10} to 10^{-2} showed no significant difference in the observed numbers of iterations for preconditioners $\bar{\mathcal{P}}_D$, $\bar{\mathcal{P}}_F$ and even $\bar{\mathcal{P}}_{DD}$. The observed general trend is that a bigger β leads to a smaller number of iterations. The convergence depends on β less than on the permeabilities k^1 and k^2. The interesting fact is that $\bar{\mathcal{P}}_{DD}$ performs almost equally well as P_D even for γ_C close to 1. This means that our analysis of the relation between $\bar{\mathcal{P}}_D$ and $\bar{\mathcal{P}}_{DD}$ gives for larger β a somehow pessimistic estimate of quality of $\bar{\mathcal{P}}_{DD}$ as a preconditioner for $\bar{\mathcal{A}}$.

To compare the numbers of iterations for different preconditioners $\bar{\mathcal{P}}$ we watch the norm of residuals $\|r^i\|$, not the norm $\|r^i\|_{\bar{\mathcal{P}}^{-1}}$, which is minimized by MINRES, see e.g. [10]. It also avoids the problem with large drop of $\|r^i\|_{\bar{\mathcal{P}}^{-1}}$ in the first iterations. An idea about the decrease of different error indicators for a selected case with the original input data $k^1, k^2, c^1_{pp}, c^2_{pp}$ and β can be seen in Fig. 2.

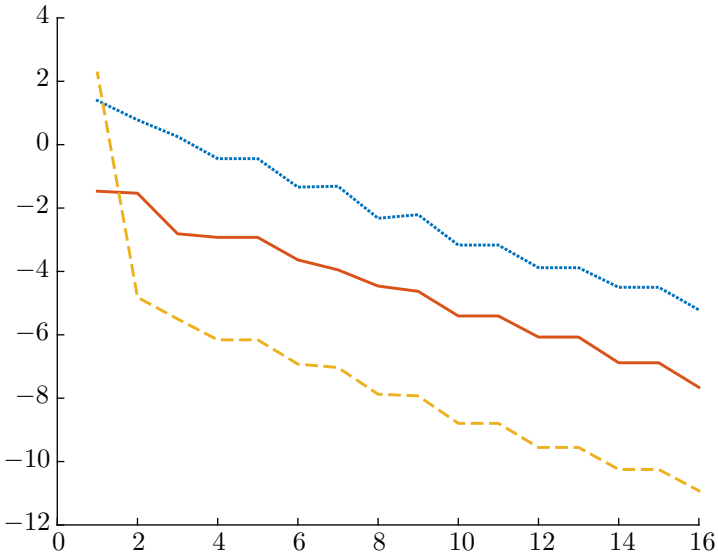

Figure 2. Plot of the decrease of the logarithm of norms of residual $\|r^i\|$ (bold line), $\|r^i\|_{\bar{\mathcal{P}}^{-1}}$ (dashed line) and for the pseudo-error $\|u^i - u^{\star d}\|$ (dotted line), where $u^{\star d}$ is the solution of the system with matrix $\bar{\mathcal{A}}$ by a direct solver. The preconditioner $\bar{\mathcal{P}} = \bar{\mathcal{P}}_D$.

5. Conclusion

Dual permeability model has been used for a long time for simulation of flow in naturally fractured reservoirs (see e.g. [5]). Our research was motivated by a study of flow in rock mass fractured naturally as well as by excavation, especially for flow in excavated damage zone (EDZ). Such analysis is important for the performance assessment of nuclear waste repository [9]. The application can be also important for investigation of enhanced geothermal systems. The model can be further extended to cover variably saturated flow, see e.g. [11].

Inspired by preconditioning introduced for the single porosity Biot's system [3], [4], an algebraic block-type preconditioning is presented for the dual permeability model and analysed. Preconditioners $\bar{\mathcal{P}}_F$ (defined by (3.1)), $\bar{\mathcal{P}}_D$ (defined by (3.1)) and $\bar{\mathcal{P}}_{DD}$ (defined by (3.16)) are proposed to solve systems with matrix $\bar{\mathcal{A}}$ (given

by (2.14)). The block diagonal preconditioner $\bar{\mathcal{P}}_D$ is analysed as a preconditioning for $\bar{\mathcal{P}}_F$.

We have shown that the block diagonal preconditioning is viable for the double permeability model, at least for some range of physical parameters for which $\bar{\gamma}$ and γ_C are far from 1. We can quantify this range by the provided estimates of $\bar{\gamma}$ for the whole preconditioner and γ_C for the $\bar{S}_{22}$ block of the preconditioner. Note that block diagonal preconditioners create space for natural parallelization, but the use of extended, triangular preconditioner is also possible.

Solvers for individual blocks $\bar{S}_{11}$, $\bar{S}_{22}$, and $\bar{D}$ are needed for implementation of $\bar{\mathcal{P}}$. The algebraic multigrid is proposed to be used for $\bar{S}_{11}$, inner preconditioned conjugate gradients using the block diagonal as a preconditioning can be used for $\bar{S}_{22}$, and inversion of $\bar{D}$ is trivial, see (2.18). Solvers for $H(\mathrm{div})$ type problems on the block diagonal of $\bar{S}_{22}$ can be chosen as e.g. in [1], [2] or [17].

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