Algebraic properties of the coordinate ring of a convex polyomino

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Submitted: Mar 16, 2018; Accepted: Feb 11, 2021; Published: Feb 26, 2021

Abstract

We classify all convex polyominoes whose coordinate rings are Gorenstein. We also give an upper bound for the Castelnuovo-Mumford regularity of the coordinate ring of any convex polyomino in terms of the smallest interval which contains its vertices. We give a recursive formula for computing the multiplicity of a stack polyomino.

Mathematics Subject Classifications: 05E40, 13H10, 13P10

1 Introduction

A *polyomino* $\mathcal{P}$ is a finite connected set of adjacent cells in the cartesian plane $\mathbb{N}^2$. A *cell* in $\mathbb{N}^2$ is simply a unitary square. A polyomino $\mathcal{P}$ is said to be *column convex* (respectively *row convex*) if every column (respectively row) is connected. According to [2], $\mathcal{P}$ is a *convex polyomino* if for every two cells of $\mathcal{P}$ there is a *monotone path* between them, that is a path having only two directions, contained in $\mathcal{P}$. Convex polyominoes include *one-sided ladders, 2-sided ladders and stack polyominoes.*

Let $\mathbb{K}$ be a field and consider the polynomial ring $S = \mathbb{K}[x_{ij}|(i,j) \text{ vertex of } \mathcal{P}]$. The *polyomino ideal* $I_{\mathcal{P}}$ is the ideal of S generated by all 2-inner minors of $\mathcal{P}$, where a 2-inner minor of $\mathcal{P}$ is a 2-minor of the matrix $X = (x_{ij})_{ij}$ which involves only indeterminates of the vertices of $\mathcal{P}$. The *coordinate ring* of $\mathcal{P}$ is defined as the quotient ring $\mathbb{K}[\mathcal{P}] = S/I_{\mathcal{P}}$. The ideal $I_{\mathcal{P}}$ and the ring $\mathbb{K}[\mathcal{P}]$ were first studied by Qureshi in [10]. There it was shown that if $\mathcal{P}$ is a convex polyomino, then $\mathbb{K}[\mathcal{P}]$ is a normal Cohen-Macaulay domain. This was proved by viewing the ring $\mathbb{K}[\mathcal{P}]$ as the edge ring of a suitable bipartite graph $G_{\mathcal{P}}$ associated with $\mathcal{P}$.

Understanding the graded free resolution of polyomino ideals is a difficult task. A first step in this direction was done in [5], where the convex polyomino ideals which are linearly related or have a linear resolution are classified.

In this paper, we continue the study of the algebraic properties of $\mathbb{K}[\mathcal{P}]$.

In Section 1, we recall the basic terminology related to convex polyominoes and their associated bipartite graphs. The first main result of this paper appears in Section 2, where we classify all convex polyominoes whose coordinate rings are Gorenstein (Theorem 21). For this classification, we use a result due to Ohsugi and Hibi ([8]) who classified all 2-connected bipartite graphs whose edge rings are Gorenstein. In the case of stack polyominoes, we recover the classification of all Gorenstein stack polyominoes given in [10, Corollary 4.12]; see Section 3.

In Section 4, we give an upper bound for the Castelnuovo-Mumford regularity of the coordinate ring of any convex polyomino in terms of the smallest interval which contains its vertices (Proposition 37). The computation of the upper bound of the regularity uses as an important tool the formula of the a-invariant of the edge ring of a bipartite graph given in [11].

Finally, in Section 5 we give a recursive formula for computing the multiplicity of $\mathbb{K}[\mathcal{P}]$ if $\mathcal{P}$ is a stack polyomino and we show some concrete cases when this formula may be applied.

2 Preliminaries

To begin with, we recall some concepts and introduce notation about collections of cells and polyominoes.

We consider on $\mathbb{N}^2$ the natural partial order defined as follows: $(i, j) \leqslant (k, l)$ if and only if $i \leqslant k$ and $j \leqslant l$. If $a, b \in \mathbb{N}^2$ with $a \leqslant b$, then the set

$$[a, b] = \{c \in \mathbb{N}^2 \mid a \leqslant c \leqslant b\}$$

is an *interval* in $\mathbb{N}^2$. If $a = (i, j)$ and $b = (k, l) \in \mathbb{N}^2$ have the property that $j = l$ (respectively $i = k$), then the interval $[a, b]$ is called a *horizontal* (respectively *vertical*) *edge interval*.

The interval

$$C = [a, a + (1, 1)]$$

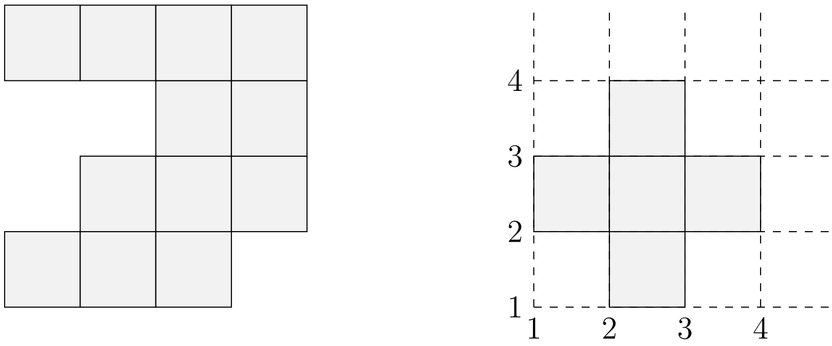
is called a cell in $\mathbb{N}^2$ with lower left corner a. The elements of C are called vertices of C and we denote their set by $V(C)$. The set of edges of C is

$$E(C) = \{\{a, a + (0, 1)\}, \{a, a + (1, 0)\}, \{a + (0, 1), a + (1, 1)\}, \{a + (1, 0), a + (1, 1)\}\}.$$

We consider A and B two cells in $\mathbb{N}^2$ with lower left corners (i, j) and (k, l). Then the set

$$[A, B] = \{E|E \text{ is a cell with lower left corner } (r, s) \text{ such that } i \leqslant r \leqslant k,\ j \leqslant s \leqslant l\}$$

A row convex but not column convex polyomino



A convex polyomino

Figure 1:

is called a *cell interval*. In the case that $j = l$ (respectively $i = k$), the cell interval $[A, B]$ is called a horizontal (respectively vertical) cell interval.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. The vertex set of $\mathcal{P}$ and the edge set of $\mathcal{P}$ are

$$V(\mathcal{P}) = \cup_{C\in\mathcal{P}} V(C) \text{ and } E(\mathcal{P}) = \cup_{C\in\mathcal{P}} E(C),$$

where C are the cells of $\mathcal{P}$. Two cells A and B of $\mathcal{P}$ are connected, if there is a sequence of cells of $\mathcal{P}$ given by $A = A_1, A_2, \dots, A_{n-1}, A_n = B$ such that $A_i \cap A_{i+1}$ is an edge of A_i and A_{i+1} for each $i \in \{1, \dots, n-1\}$. Such a sequence is called a path connecting the cells A and B.

Definition 1. A collection of cells $\mathcal{P}$ is called a polyomino if any two cells of $\mathcal{P}$ are connected.

Definition 2. A polyomino $\mathcal{P}$ is called row (respectively column) convex, if for any two cells A and B of $\mathcal{P}$ with left lower corners $a = (i, j)$ and $b = (k, j)$ (respectively $a = (i, j)$ and $b = (i, l)$), the horizontal (respectively vertical) cell interval $[A, B]$ is contained in $\mathcal{P}$. If $\mathcal{P}$ is row and column convex, then $\mathcal{P}$ is called a convex polyomino.

In Figure 1 we have two examples of polyominoes: the one on the right is a convex polyomino, while the other one is row convex but not column convex, hence it is not convex.

Let $\mathcal{P}$ be a convex polyomino and $[a, b] \subset \mathbb{N}^2$ be the smallest interval which contains $V(\mathcal{P})$. After a shift of coordinates, we may assume that $a = (1, 1)$ and $b = (m, n)$ and thus, we say that $\mathcal{P}$ is a convex polyomino on $[m] \times [n]$, where for a positive integer a, $[a]$ denotes the set $\{1, \dots, a\}$. For example, the right side polyomino in Figure 1 is a convex polyomino on $[4] \times [4]$.

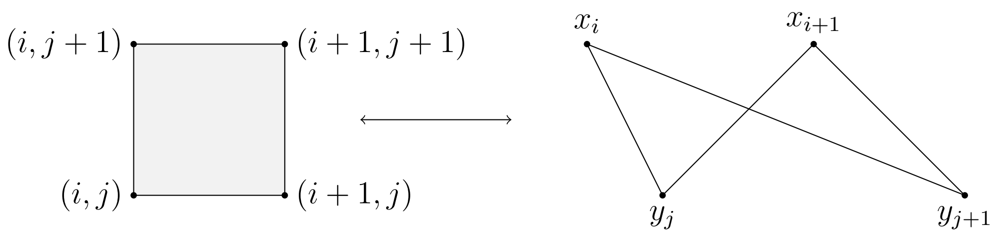


Figure 2: The bipartite graph attached to a cell in $\mathbb{N}^2$

Fix a field $\mathbb{K}$ and a polynomial ring $S = \mathbb{K}[x_{ij} \mid (i, j) \in V(\mathcal{P})]$. We consider the ideal $I_{\mathcal{P}} \subset S$ generated by all binomials $x_{il}x_{kj} - x_{ij}x_{kl}$ for which $[(i, j), (k, l)]$ is an interval in $\mathcal{P}$. The $\mathbb{K}$-algebra $S/I_{\mathcal{P}}$ is denoted $\mathbb{K}[\mathcal{P}]$ and is called the coordinate ring of $\mathcal{P}$. By [10, Theorem 2.2], $\mathbb{K}[\mathcal{P}]$ is a normal Cohen-Macaulay domain.

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. The ring $R = \mathbb{K}[x_iy_j \mid (i, j) \in V(\mathcal{P})] \subset \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ can be viewed as the edge ring of the bipartite graph $G_{\mathcal{P}}$ with vertex set $V(G_{\mathcal{P}}) = X \cup Y$, where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and edge set $E(G_{\mathcal{P}}) = \{\{x_i, y_j\} \mid (i, j) \in V(\mathcal{P})\}$. In Figure 2, we displayed the bipartite graph attached to a cell in $\mathbb{N}^2$. According to [10], $\mathbb{K}[\mathcal{P}]$ can be identified with $\mathbb{K}[G_{\mathcal{P}}]$.

3 Gorenstein convex polyominoes

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. We set $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and, if needed, we identify the point (x_i, y_j) in the plane with the vertex $(i, j) \in V(\mathcal{P})$.

Let A and B be two cells in $\mathcal{P}$. Recall that A and B are connected by a path if there is a sequence of cells in $\mathcal{P}$, $A = A_1, A_2, \ldots, A_{r-1}, A_r = B$, with the property that $A_i \cap A_{i+1}$ is an edge of A_i and A_{i+1}, for each $i \in [r-1]$. We denote by (x_{j_i}, y_{k_i}) the lower left corner of A_i, for all $i \in [r]$. Every path in $\mathcal{P}$ may go in at most four directions which are given below:

1. East if $(x_{j_{i+1}}, y_{k_{i+1}}) - (x_{j_i}, y_{k_i}) = (1, 0)$ for some $i \in [r]$;
2. West if $(x_{j_{i+1}}, y_{k_{i+1}}) - (x_{j_i}, y_{k_i}) = (-1, 0)$ for some $i \in [r]$;
3. South if $(x_{j_{i+1}}, y_{k_{i+1}}) - (x_{j_i}, y_{k_i}) = (0, -1)$ for some $i \in [r]$;
4. North if $(x_{j_{i+1}}, y_{k_{i+1}}) - (x_{j_i}, y_{k_i}) = (0, 1)$ for some $i \in [r]$.

We say that a path connecting two cells is monotone if it goes only in two directions. A characterization of convex polyominoes in terms of paths is given by the following Proposition from [2].

Proposition 3. *[2, Proposition 1] A polyomino $\mathcal{P}$ is convex if and only if for every pair of cells there exists a monotone path connecting them and contained in $\mathcal{P}$.*

In the next proposition we show that the bipartite graph $G_{\mathcal{P}}$ associated with $\mathcal{P}$ is 2-connected. Let us first recall the definition of 2-connectivity.

Definition 4. If G is a finite connected graph on the vertex set V, then given a subset $\emptyset \neq W \subset V$, G_W denotes the induced subgraph of G on W. We say that G is 2-connected if G together with $G_{V\setminus\{v\}}$ are connected for all $v \in V$.

Proposition 5. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then the bipartite graph $G_{\mathcal{P}}$ is 2-connected.*

Proof. Firstly, we prove that the bipartite graph $G_{\mathcal{P}}$ is connected. For that it is sufficient to choose $x, x' \in \{x_1, \dots, x_m\}$ and to find a path between them in $G_{\mathcal{P}}$. Let $x, x', y, y' \in V(G_{\mathcal{P}})$ such that $(x, y), (x', y') \in V(\mathcal{P})$. Without loss the generality, we may suppose that $(x, y-1), (x', y'-1) \notin V(\mathcal{P})$. Since $\mathcal{P}$ is a convex polyomino, there exists a monotone path Γ from a cell containing (x, y) to a cell containing (x', y'), both as outside corners of Γ. Let us consider the sequence γ of vertices of $\mathcal{P}$ belonging to the cells of Γ. Now, let

$$A_\gamma = \{\{x_{i_k}, y_{j_k}\} \mid (x_{i_k}, y_{j_k}) \in \gamma \text{ is a lower outside or inside corner of the path } \Gamma\}.$$

We claim that A_γ is a path in $G_{\mathcal{P}}$ containing x and x'. Clearly, $\{x, y\}$ and $\{x', y'\}$ belong to A_γ by definition of Γ. Since Γ is a monotone path, for every $\{x_{i_k}, y_{j_k}\} \in A_\gamma \setminus \{\{x, y\}, \{x'y'\}\}$, there exist exactly two other edges of the form $\{x_{i_k}, y_r\}$ and $\{x_s, y_{j_k}\}$ in A_γ, with $r \neq j_k$ and $s \neq i_k$.

In order to complete the proof, we show that for any $k \in [m]$, the graph $G_{\mathcal{P}_V}$ is connected, where $V = V(G_{\mathcal{P}}) \setminus \{x_k\}$.

Let $G = G_{\mathcal{P}_V}$ and $x, x', y, y' \in V(G)$ such that $(x, y), (x', y') \in V(\mathcal{P})$. In a similar way as in the first part of the proof, we consider Γ to be a monotone path in $\mathcal{P}$ from a cell containing (x, y) to a cell containing (x', y'). Let

$$A = \{\{x_{i_l}, y_{j_l}\} \mid (x_{i_l}, y_{j_l}) \text{ is a lower outside or inside corner of the path } \Gamma\}.$$

If for any $\{x_{i_l}, y_{j_l}\} \in A$, we have $x_{i_l} \neq x_k$, then A is a path in G containing x and x' by the argument used above.

If there is $\{x_{i_l}, y_{j_l}\} \in A$ such that $i_l = k$, then we have exactly two edges

$$\{x_{i_{l_1}}, y_{j_{l_1}}\}, \{x_{i_{l_2}}, y_{j_{l_2}}\} \in A$$

with $i_{l_1} = i_{l_2} = k$. Since $\mathcal{P}$ is a convex polyomino, $(x_{k-1}, y_{j_{l_1}}), (x_{k-1}, y_{j_{l_2}}) \in V(\mathcal{P})$ or $(x_{k+1}, y_{j_{l_1}}), (x_{k+1}, y_{j_{l_2}}) \in V(\mathcal{P})$. Let

$$A' = A \setminus \{\{x_k, y_{j_{l_1}}\}, \{x_k, y_{j_{l_2}}\}\}.$$

If $(x_{k-1}, y_{j_{l_1}}), (x_{k-1}, y_{j_{l_2}}) \in V(\mathcal{P})$, then $A' \cup \{\{x_{k-1}, y_{j_{l_1}}\}, \{x_{k-1}, y_{j_{l_2}}\}\}$ is a path in G containing x and x' else $A' \cup \{\{x_{k+1}, y_{j_{l_1}}\}, \{x_{k+1}, y_{j_{l_2}}\}\}$ is a path in G containing x and x' by the argument used in the first part of the proof. $\square$

For the characterisation of Gorenstein convex polyominoes we need the following theorem due to Ohsugi and Hibi ([8]).

Let G be a bipartite graph on the vertex set $[m] \cup [n]$ and let

$$S' = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

be the polynomial ring. The edge ring of G is the toric ring

$$\mathbb{K}[G] = \mathbb{K}[x_i y_j \mid \{i, j\} \in E(G)] \subset S'.$$

Let $T \subset V(G)$. We recall that $N(T) = \{y \in V(G) \mid \{x, y\} \in E(G) \text{ for some } x \in T\}$ represents the set of the neighbors of the subset T.

Theorem 6. *[8, Theorem 2.1] Let G be a bipartite graph on $X \cup Y$ and suppose that G is 2-connected. Then the edge ring of G is Gorenstein if and only if $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G]$ and one has $|N(T)| = |T| + 1$ for every subset $T \subset X$ such that $G_{T \cup N(T)}$ is connected and that $G_{(X \cup Y) \setminus (T \cup N(T))}$ is a connected graph with at least one edge.*

Note that $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G]$ if and only if G possesses a perfect matching (i.e. there is a set of edges $E \subset E(G)$ with the property that no two of them have a common vertex and $\cup_{\{x,y\} \in E} \{x, y\} = V(G)$). A characterization of the bipartite graph which possesses a perfect matching is given by Villarreal.

Theorem 7. *[13, Theorem 7.1.9] A bipartite graph G with the vertex set $V = X \cup Y$ possesses a perfect matching if and only if one has $|N(T)| \geqslant |T|$ for every independent subset of vertices $T \subset V$.*

Recall that a subset of vertices of G is called independent if no two of them are adjacent.

From now on, whenever we consider a convex polyomino $\mathcal{P}$, we consider it endowed with its associated bipartite graph $G_{\mathcal{P}}$ on the vertex set $V(G_{\mathcal{P}}) = X \cup Y$.

Corollary 8. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G_{\mathcal{P}}]$ if and only if $|N(T)| \geqslant |T|$ for every $T \subset X$ or $T \subset Y$.*

Proof. If $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G_{\mathcal{P}}]$, then by Theorem 7, we obtain $|N(T)| \geqslant |T|$, for every independent subset of vertices $T \subset X \cup Y$. Notice that all subsets $T \subset X$ and $U \subset Y$ are independent.

Conversely, we suppose $|N(T)| \geqslant |T|$ for every $T \subset X$ or $T \subset Y$. Let

$$T = \{x_{i_1}, \ldots, x_{i_r}, y_{j_1}, \ldots, y_{j_s}\} \subset X \cup Y$$

be an independent set of vertices with $r, s \geqslant 1$. Then, by assumption,

$$|T| = r + s \leqslant |N(\{x_{i_1}, \ldots, x_{i_r}\})| + |N(\{y_{j_1}, \ldots, y_{j_s}\})|.$$

Since $N(\{x_{i_1}, \ldots, x_{i_r}\}) \subset Y$ and $N(\{y_{j_1}, \ldots, y_{j_s}\}) \subset X$, we have

$$\begin{aligned} |N(\{x_{i_1}, \ldots, x_{i_r}\})| + |N(\{y_{j_1}, \ldots, y_{j_s}\})| &= |N(\{x_{i_1}, \ldots, x_{i_r}\}) \cup N(\{y_{j_1}, \ldots, y_{j_s}\})| \\ &= |N(T)|. \end{aligned}$$

Thus, $|T| \leqslant |N(T)|$ and according to Theorem 7, $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G_{\mathcal{P}}]$. □

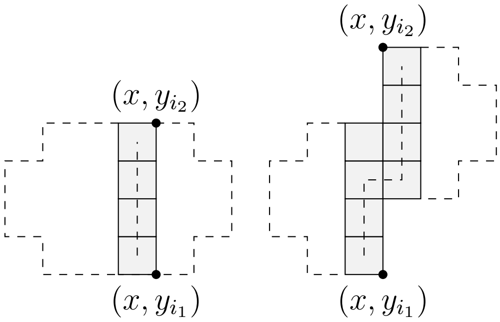


Figure 3: Possible monotone paths

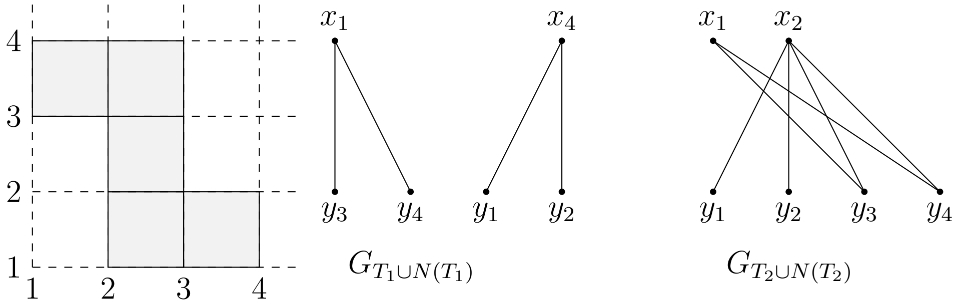


Figure 4:

Remark 9. If a connected bipartite graph G on $X \cup Y$ has a perfect matching, then $m = n$. Indeed, $|X| \leqslant |N(X)| = |Y|$ and $|Y| \leqslant |N(Y)| = |X|$.

Definition 10. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$, $V(G_{\mathcal{P}}) = X \cup Y$ and $T \subset X$. The set $N_Y(T) = \{y \in Y \mid (x, y) \in V(\mathcal{P})$ for some $x \in T\}$ is called a neighbor vertical interval if $N_Y(T) = \{y_a, y_{a+1}, \ldots, y_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $x \in T$ such that $[(x, y_i), (x, y_{i+1})]$ is an edge in $\mathcal{P}$.*

Remark 11. If the subset $T = \{x\} \subset X$ has only one element, then $N_Y(T)$ is a neighbor vertical interval. Indeed, let $y_{i_1}, y_{i_2} \in N_Y(x)$ with $i_1 < i_2$. By Proposition 3, there exists a monotone path between the cells containing (x, y_{i_1}) and (x, y_{i_2}) as corners. We display some of the monotone paths between two cells in Figure 3. Then we have $[(x, y_{i_1}), (x, y_{i_2})] \subset \mathcal{P}$.

Example 12. In the polyomino of Figure 4, let $T_1 = \{x_1, x_4\}$ and $T_2 = \{x_1, x_2\}$. Then $N_Y(T_1) = \{y_1, y_2, y_3, y_4\}$ and $N_Y(T_2) = \{y_1, y_2, y_3, y_4\}$.

We observe that $G_{T_1 \cup N(T_1)}$ is not connected, while $G_{T_2 \cup N(T_2)}$ is connected. Moreover, we notice that $N_Y(T_1)$ and $N_Y(T_2)$ coincide as sets, but $N_Y(T_2)$ is a neighbor vertical interval, while $N_Y(T_1)$ is not.

Lemma 13. *Let $\mathcal{P}$ be a convex polyomino. Then for each $\emptyset \neq T \subsetneq X$, the following conditions are equivalent:*

1. *$N_Y(T)$ is a neighbor vertical interval.*

2. *$G_{T\cup N(T)} := G_{\mathcal{P}_{T\cup N(T)}}$ is a connected graph.*

Proof. For *(1)⇒(2)*, it is sufficient to choose $x, z \in T$ and to find a path d between them in $G_{T\cup N(T)}$. Without loss of generality, we may choose $y_s \in N_Y(x)$ and $y_t \in N_Y(z)$ with $s < t$. Then by hypothesis, $\{y_s, y_{s+1}, \dots, y_t\} \subset N_Y(T)$ and there exist $x_{i_s}, x_{i_{s+1}}, \dots, x_{i_{t-1}} \in T$ such that $[(x_{i_j}, y_j), (x_{i_j}, y_{j+1})]$ is an edge in $\mathcal{P}$, for $j \in \{s, s+1, \dots, t-1\}$. Thus, we have

$$(x, y_s), (x_{i_s}, y_s), (x_{i_s}, y_{s+1}), (x_{i_{s+1}}, y_{s+1}), \dots, (x_{i_{t-1}}, y_{t-1}), (x_{i_{t-1}}, y_t)(z, y_t) \in V(\mathcal{P}).$$

So the path between x and z in $G_{\mathcal{P}}$ is

$$\gamma = \{\{x, y_s\}, \{y_s, x_{i_s}\}, \{x_{i_s}, y_{s+1}\}, \{y_{s+1}, x_{i_{s+1}}\}, \dots, \{x_{i_{t-1}}, y_t\}, \{y_t, z\}\}.$$

For *(2)⇒(1)*, we consider $N_Y(T) = \{y_{i_1}, \dots, y_{i_s} \mid i_1 < i_2 < \dots < i_s\}$ and we prove that for every $j \in [s-1]$, there exists $x_k \in T$ such that

$$[(x_k, y_{i_j}), (x_k, y_{i_{j+1}})]$$

is an edge in $\mathcal{P}$. In particular, it also follows that $i_{j+1} = i_j + 1$ for each $j \in [s-1]$, which will end the proof.

Let $j \in [s-1]$. Since $G_{T\cup N(T)}$ is a connected graph, there is a path between y_{i_j} and $y_{i_{j+1}}$ in $G_{T\cup N(T)}$. In other words, there are $x_{k_1}, \dots, x_{k_{r-1}} \in T$ and $y_{l_1}, \dots, y_{l_{r-2}} \in N_Y(T)$ such that

$$(x_{k_1}, y_{i_j}), (x_{k_1}, y_{l_1}), (x_{k_2}, y_{l_1}), (x_{k_2}, y_{l_2}), \dots, (x_{k_{r-1}}, y_{l_{r-2}}), (x_{k_{r-1}}, y_{i_{j+1}}) \in V(\mathcal{P}).$$

Since there is no $y_l \in N_Y(T)$ between y_{i_j} and $y_{i_{j+1}}$, the only cases that can occur are:

1. there exists $a \in [r-2]$ such that $l_a < i_j < i_{j+1} < l_{a+1}$;

2. for every $a \in [r-2]$, $l_a < i_j < i_{j+1}$;

3. for every $a \in [r-2]$, $i_j < i_{j+1} < l_a$.

If we have $a \in [r-2]$ such that $l_a < i_j < i_{j+1} < l_{a+1}$, then

$$[(x_{k_{a+1}}, y_{i_j}), (x_{k_{a+1}}, y_{i_{j+1}})]$$

is an edge interval in $\mathcal{P}$ because $(x_{k_{a+1}}, y_{l_a}), (x_{k_{a+1}}, y_{l_{a+1}}) \in V(\mathcal{P})$ and $N_Y(x_{k_{a+1}})$ is a neighbor vertical interval by Remark 11. Moreover, $y_{i_j}, y_{i_j+1}, \dots, y_{i_{j+1}} \in N_Y(x_{k_{a+1}}) \subset N_Y(T)$. Thus, $i_{j+1} = i_j + 1$ and $[(x_{k_{a+1}}, y_{i_j}), (x_{k_{a+1}}, y_{i_{j+1}})]$ is an edge in $\mathcal{P}$.

If for all $a \in [r-2]$, $l_a < i_j < i_{j+1}$, then

$$[(x_{k_{r-1}}, y_{i_j}), (x_{k_{r-1}}, y_{i_{j+1}})]$$

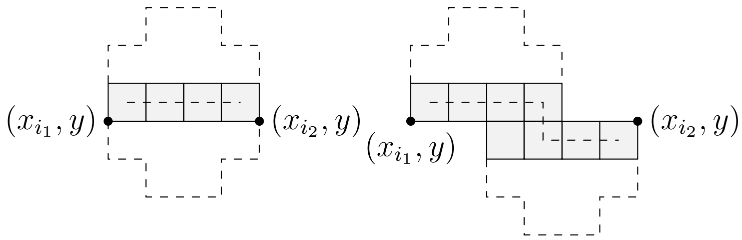


Figure 5: Possible monotone paths

is an edge interval in $\mathcal{P}$, since $(x_{k_{r-1}}, y_{l_{r-2}}), (x_{k_{r-1}}, y_{i_{j+1}}) \in V(\mathcal{P})$ and $N_Y(x_{k_{r-1}})$ is a neighbor vertical interval. So $y_{i_j}, y_{i_j+1}, \ldots, y_{i_{j+1}} \in N_Y(x_{k_{r-1}}) \subset N_Y(T)$ and

$$[(x_{k_{r-1}}, y_{i_j}), (x_{k_{r-1}}, y_{i_{j+1}})]$$

is an edge in $\mathcal{P}$. We proceed in a similar way in the case that for all $a \in [r-2]$ we have $i_j < i_{j+1} < l_a$. $\square$

Definition 14. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$, $V(G_{\mathcal{P}}) = X \cup Y$ and $U \subset Y$. The set $N_X(U) = \{x \in X \mid (x, y) \in V(\mathcal{P})$ for some $y \in U\}$ is called a neighbor horizontal interval if $N_X(U) = \{x_a, x_{a+1}, \ldots, x_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $y \in U$ such that $[(x_i, y), (x_{i+1}, y)]$ is an edge in $\mathcal{P}$.*

Remark 15. If the subset $U = \{y\} \subset Y$, has only one element, then $N_X(U)$ is a neighbor horizontal interval. Indeed, we consider $x_{i_1}, x_{i_2} \in N_X(U)$ with $i_1 < i_2$. Since $\mathcal{P}$ is a convex polyomino, by Proposition 3, there is a monotone path between the cells containing (x_{i_1}, y) and (x_{i_2}, y) as corners. We display some of the monotone paths between two cells in Figure 5. Then we have $[(x_{i_1}, y), (x_{i_2}, y)] \subset \mathcal{P}$.

Example 16. In the polyomino of Figure 6, let $U_1 = \{y_2, y_3\}$ and $U_2 = \{y_1, y_5\}$. We observe that $N_X(U_1) = \{x_1, x_2, x_3, x_4, x_5\}$ is a neighbor horizontal interval, while $N_X(U_2) = \{x_1, x_2, x_3, x_4\}$ is not.

Lemma 17. *If $\mathcal{P}$ is a convex polyomino, then for each $\emptyset \neq T \subset X$,*

$$N_Y(x) \not\subseteq N_Y(T) \text{ for every } x \in X \setminus T$$

if and only if

$$N_X(Y \setminus N_Y(T)) = X \setminus T.$$

Proof. First assume that for every $\emptyset \neq T \subset X$, $N_Y(x) \not\subseteq N_Y(T)$ for every $x \in X \setminus T$. Let $x \in N_X(Y \setminus N_Y(T))$. Then there exists $y \in Y \setminus N_Y(T)$ such that $(x, y) \in V(\mathcal{P})$. If $x \in T$, then $y \in N_Y(x) \subset N_Y(T)$. Thus, $x \in X \setminus T$ and $N_X(Y \setminus N_Y(T)) \subset X \setminus T$.

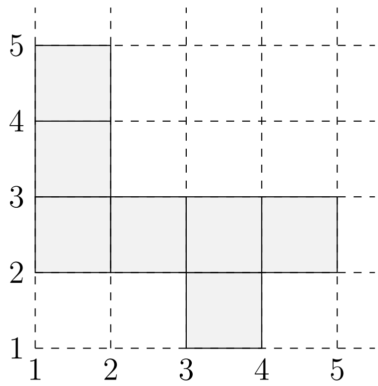


Figure 6:

If $x \in X \setminus T$, then by hypothesis, we obtain $N_Y(x) \nsubseteq N_Y(T)$. So there exists $y \in N_Y(x) \setminus N_Y(T)$. Hence, $y \in Y \setminus N_Y(T)$ and $x \in N_X(y) \subset N_X(Y \setminus N_Y(T))$. In other words, $X \setminus T \subset N_X(Y \setminus N_Y(T))$.

Conversely, let $x \in X \setminus T$. Since $N_X(Y \setminus N_Y(T)) = X \setminus T$, we have that $x \in N_X(Y \setminus N_Y(T))$ and there exists $y \in Y \setminus N_Y(T)$ such that $(x, y) \in V(\mathcal{P})$. This is equivalent to say that $y \in N_Y(x) \setminus N_Y(T)$ and $N_Y(x) \nsubseteq N_Y(T)$. $\square$

Example 18. In Figure 6, let $T = \{x_4, x_5\}$. We observe that $N_X(Y \setminus N_Y(T)) = N_X(\{y_4, y_5\}) = \{x_1, x_2\} \neq \{x_1, x_2, x_3\} = X \setminus T$. On the other hand, $x_3 \notin T$ and $N_Y(x_3) = N_Y(T)$.

Lemma 19. *Let $\mathcal{P}$ be a convex polyomino. Then for each $\emptyset \neq T \subsetneq X$, the following conditions are equivalent:*

1. *$N_X(Y \setminus N_Y(T)) = X \setminus T$ is a neighbor horizontal interval.*

2. *$G_{(X\cup Y)\setminus(T\cup N_Y(T))} := G_{\mathcal{P}_{(X\cup Y)\setminus(T\cup N_Y(T))}}$ is a connected graph with at least one edge.*

Proof. Let T be a subset in X which satisfies the conditions given in *(1)*. By Lemma 17 and the fact that $T \subsetneq X$, there is $x \in X \setminus T$ with $N_Y(x) \nsubseteq N_Y(T)$. In other words, there are $x \in X \setminus T$ and $y \in Y \setminus N_Y(T)$ such that $(x, y) \in V(\mathcal{P})$. This is equivalent to saying that $\{x, y\}$ is an edge in $G_{(X\cup Y)\setminus(T\cup N_Y(T))}$.

For the connectivity of the graph $G_{(X\cup Y)\setminus(T\cup N_Y(T))}$, it is sufficient to choose $y, z \in Y \setminus N_Y(T)$ and to find a path between them in $G_{(X\cup Y)\setminus(T\cup N_Y(T))}$. Without loss of generality, we consider $x_s \in N_X(y)$ and $x_t \in N_X(z)$ with $s < t$. Since $N_X(Y \setminus N_Y(T))$ is a neighbor horizontal interval, $\{x_s, x_{s+1}, \ldots, x_t\} \subset N_X(Y \setminus N_Y(T))$ and there exist $y_{i_s}, y_{i_{s+1}}, \ldots, y_{i_{t-1}} \in Y \setminus N_Y(T)$ such that $[(x_j, y_{i_j}), (x_{j+1}, y_{i_j})]$ is an edge in $\mathcal{P}$, for each $j \in \{s, s+1, \ldots, t-1\}$. It follows that

$$(x_s, y), (x_s, y_{i_s}), (x_{s+1}, y_{i_s}), (x_{s+1}, y_{i_{s+1}}), \ldots, (x_{t-1}, y_{i_{t-1}}), (x_t, y_{i_{t-1}}), (x_t, z) \in V(\mathcal{P}).$$

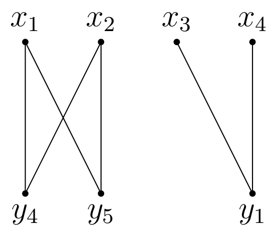


Figure 7: $G_{(X \cup Y)\setminus(T_1 \cup N_Y(T_1))}$

In other words, the path between y and z is

$$\gamma = \{\{y, x_s\}, \{x_s, y_{i_s}\}, \{y_{i_s}, x_{s+1}\}, \{x_{s+1}, y_{i_{s+1}}\}, \dots, \{y_{i_{t-1}}, x_t\}, \{x_t, z\}\}.$$

Conversely, we suppose that $X \setminus T \neq N_X(Y \setminus N_Y(T))$. By Lemma 17, there is $x \in X \setminus T$ with the property that $N_Y(x) \subset N_Y(T)$. So x represents an isolated vertex in $G_{(X \cup Y)\setminus(T \cup N_Y(T))}$.

Now, we consider $N_X(Y \setminus N_Y(T)) = \{x_{i_1}, \dots, x_{i_s} \mid i_1 < \cdots < i_s\}$ and we prove that for every $j \in [s-1]$ there exists $y_k \in Y \setminus N_Y(T)$ such that $[(x_{i_j}, y_k), (x_{i_{j+1}}, y_k)]$ is an edge in $\mathcal{P}$.

Let $j \in [s-1]$. Since $G_{(X \cup Y)\setminus(T \cup N_Y(T))}$ is a connected graph, there is a path between x_{i_j} and $x_{i_{j+1}}$ in $G_{(X \cup Y)\setminus(T \cup N_Y(T))}$. Thus, there exist $x_{l_1}, \dots, x_{l_{r-2}} \in N_X(Y \setminus N_Y(T))$ and $y_{k_1}, \dots, y_{k_{r-1}} \in Y \setminus N_Y(T)$ such that

$$(x_{i_j}, y_{k_1}), (x_{l_1}, y_{k_1}), (x_{l_1}, y_{k_2}), \dots, (x_{l_{r-2}}, y_{k_{r-1}}), (x_{i_{j+1}}, y_{k_{r-1}}) \in V(\mathcal{P}).$$

The claim follows using the same argument of the proof of Lemma 13 (swapping the x_i's with the y_i's and replacing T with $Y \setminus N_Y(T)$). $\square$

Example 20. In Figure 6, let $T_1 = \{x_5\}$ and $T_2 = \{x_1, x_2, x_3\}$. We observe that $G_{(X \cup Y)\setminus(T_1 \cup N_Y(T_1))}$ is not connected because $N_X(Y \setminus N_Y(T_1)) = X \setminus T_1$ is not a neighbor horizontal interval (Figure 7). The graph $G_{(X \cup Y)\setminus(T_2 \cup N_Y(T_2))}$ is represented by the two isolated vertices x_4 and x_5.

Let $\mathcal{P}$ be a convex polyomino. Since the coordinate ring of $\mathcal{P}$ can be viewed as an edge ring of a bipartite graph, by applying Theorem 6, Corollary 8, Lemma 13 and Lemma 19, we get the following result.

Theorem 21. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$.*

Then $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if the following conditions are fulfilled:

1. $|U| \leqslant |N_X(U)|$ *for every* $U \subset Y$ *and* $|T| \leqslant |N_Y(T)|$ *for every* $T \subset X$*;*
2. *For every* $\emptyset \neq T \subsetneq X$ *with the properties*
   (a) $N_Y(T)$ *is a neighbor vertical interval,*
   (b) $N_X(Y \setminus N_Y(T)) = X \setminus T$ *is a neighbor horizontal interval,*

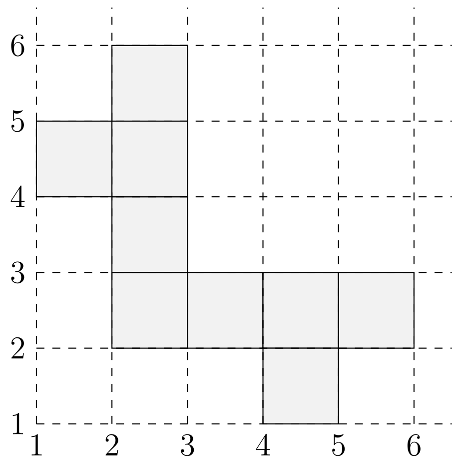


Figure 8:

one has $|N_Y(T)| = |T| + 1$.

Examples 22. Let $\mathcal{P}_1$ be the polyomino of Figure 8.

1. Let $T = \{x_4, x_5, x_6\}$. T satisfies properties $(a), (b)$. Since $|N_Y(T)| = 3 \neq 4 = |T|+1$, $\mathcal{P}_1$ is not a Gorenstein polyomino.

2. For $T = \{x_1, x_4, x_5, x_6\}$, only the property (b) is fulfiled.

3. For $T = \{x_4\}$, we have property (a) and $N_X(Y \setminus N_Y(T))$ is a neighbor horizontal interval, but $X \setminus T \neq N_X(Y \setminus N_Y(T))$.

4. For $T = \{x_6\}$, we have property (a), but $N_X(Y \setminus N_Y(T)) = X \setminus T$ is not a neighbor horizontal interval.

The polyomino $\mathcal{P}_2$ of Figure 9 is Gorenstein, because $x_1y_1 \cdot x_2y_2 \cdot x_3y_4 \cdot x_4y_3 \in \mathbb{K}[\mathcal{P}_2]$ and for each T which satisfies the properties $(a), (b)$, one has $|N_Y(T)| = |T| + 1$. In this case, we need to check the conditions of the Theorem 21 only for two sets:

1. $T = \{x_4\}$ with $N_Y(T) = \{y_2, y_3\}$;

2. $T = \{x_1, x_4\}$ with $N_Y(T) = \{y_1, y_2, y_3\}$.

Definition 23. Let $\mathcal{P}$ be a convex polyomino. A vertex $a \in V(\mathcal{P})$ is called an *interior vertex* of $\mathcal{P}$, if a is a vertex of four distinct cells of $\mathcal{P}$. We denote by $\operatorname{int}(\mathcal{P})$ the set of all interior vertices of $\mathcal{P}$. The set $\partial\mathcal{P} = V(\mathcal{P}) \setminus \operatorname{int}(\mathcal{P})$ is called the *boundary* of $\mathcal{P}$. We say that the vertex $a \in \partial\mathcal{P}$ is an *inside (outside) corner* of $\mathcal{P}$ if it belongs to exactly three (one) different cells of $\mathcal{P}$. (Figure 10)

Let $\mathcal{P}$ be a convex polyomino on $[m]\times[n]$. Then $\mathcal{P}$ is called *two-sided ladder* (Figure 11) if for every $(i,j),(k,l) \in V(\mathcal{P})$ with $i \leqslant k, j \leqslant l$, we have $(i,l),(k,j) \in V(\mathcal{P})$.

As a consequence of Theorem 21, we may recover the characterisation of Gorenstein two-sided ladder polyominoes obtained by Conca in [4, Theorem 5.2].

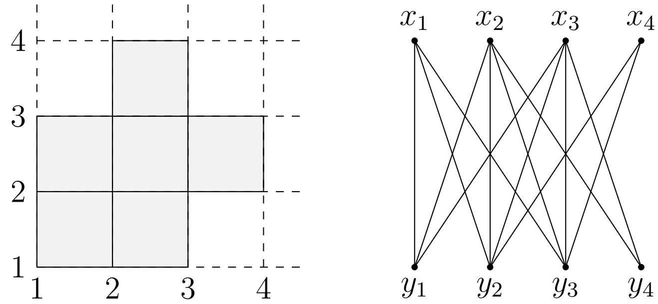


Figure 9:

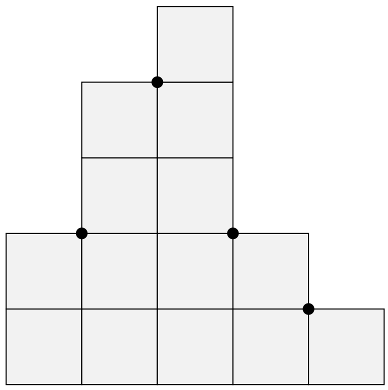


Figure 10: Inside corners

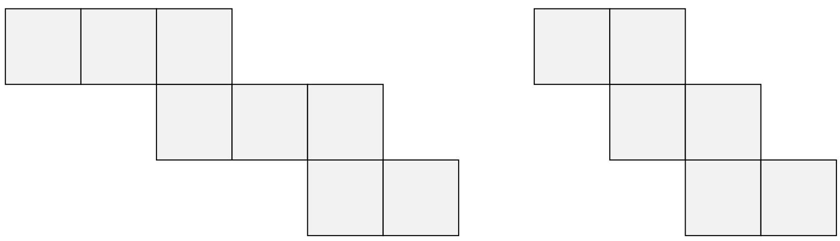


Figure 11: Two-sided ladder polyominoes

Corollary 24. *Let $\mathcal{P}$ be a two-sided ladder polyomino on $[m] \times [n]$. Then $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if $m = n$ and the inside corners of $\mathcal{P}$ lie on the diagonal $\{(x_i, y_j) \in V(\mathcal{P}) | i + j = n + 1\}$.*

Proof. Let $\mathcal{P}$ be a two-sided ladder polyomino on $[m] \times [n]$ such that $\mathbb{K}[\mathcal{P}]$ is Gorenstein. By the first condition of Theorem 21 and Remark 9, we obtain $m = n$.

Let $(x_r, y_t) \in V(\mathcal{P})$ be an inside corner of $\mathcal{P}$. If (x_r, y_t) is a lower inside corner, then we consider $T = \{x_{r+1}, x_{r+2}, \dots, x_n\}$. Since $\mathcal{P}$ is a two-sided ladder polyomino, $N_Y(T) = \{y_1, y_2, \dots, y_t\}$ and $1 < r, t < n$, T satisfies the second condition of Theorem 21. Thus, we obtain $r + t = n - |T| + |N_Y(T)| = n + 1$ and $(x_r, y_t) \in \{(x_i, y_j) \in V(\mathcal{P}) | i + j = n + 1\}$. In the case that $(x_r, y_t) \in V(\mathcal{P})$ is an upper inside corner of $\mathcal{P}$, we proceed in a similar way.

Conversely, we suppose that $m = n$ and the inside corners of $\mathcal{P}$ belong to the set $\{(x_i, y_j) \in V(\mathcal{P}) | i + j = n + 1\}$. According to Corollary 8, for the proof of the first condition of Theorem 21, it is sufficient to show that $(x_i, y_{n+1-i}) \in V(\mathcal{P})$ for every $i \in [n]$. Indeed, if $(x_i, y_{n+1-i}) \in V(\mathcal{P})$ for every $i \in [n]$, we obtain $x_1 \cdots x_n y_1 \cdots y_n \in \mathbb{K}[\mathcal{P}]$.

Let $i \in [n]$ and set $r = \max\{j \in [n] | y_j \in N_Y(x_i)\}$ and $s = \min\{j \in [n] | y_j \in N_Y(x_i)\}$. If $1 < s < r < n$, then (x_j, y_r) and (x_l, y_s) are inside corners of $\mathcal{P}$ for some $j \in \{1, 2, \dots, i-1, i\}$ and some $l \in \{i, i+1, i+2, \dots, n\}$. By hypothesis, $r = n + 1 - j \geqslant n + 1 - i$ and $s = n+1-l \leqslant n+1-i$. In other words, $1 < s \leqslant n+1-i \leqslant r < n$ and $(x_i, y_{n+1-i}) \in V(\mathcal{P})$. If $s = 1$, then (x_k, y_r) is either an inside corner of $\mathcal{P}$ or a top left corner of $\mathcal{P}$ (i.e., (x_1, y_n)) for some $k \in \{1, 2, \dots, i-1, i\}$. Thus, $r = n+1-k \geqslant n+1-i \geqslant 1 = s$ and $(x_i, y_{n+1-i}) \in V(\mathcal{P})$. In the case that $r = n$, (x_k, y_s) is an inside corner of $\mathcal{P}$ or a bottom right corner of $\mathcal{P}$ (i.e., (x_n, y_1)) for some $k \in \{i, i+1, \dots, n\}$. Hence, $s = n + 1 - k \leqslant n + 1 - i \leqslant n = r$ and $(x_i, y_{n+1-i}) \in V(\mathcal{P})$.

Let $\emptyset \neq T \subsetneq X$ such that $N_Y(T)$ is a neighbor vertical interval and $N_X(Y \setminus N_Y(T)) = X \setminus T$ is a neighbor horizontal interval. Notice that $N_Y(T) = \{y_l, \dots, y_n\}$, where $l = \min\{j \in [n] \mid y_j \in N_Y(T)\}$: in fact, if $N_Y(Y) = \{y_l, \dots, y_k\}$ for some $k < n$, then $N_X(Y \setminus N_Y(T))$ is not a neighbor horizontal interval. Moreover, $l = \min\{j \in [n] \mid y_j \in N_Y(T)\} > 1$ because if $l = 1$, then $N_Y(T) = Y$ and $N_X(Y \setminus N_Y(T)) = \emptyset \neq X \setminus T$. Since $\mathcal{P}$ is a two-sided ladder polyomino, $T = \{x_1, x_2, \dots, x_p\}$ for some $p < n$ or $T = \{x_t, x_{t+1}, \dots, x_n\}$ for some $t > 1$. Let $p \in [n-1]$ and $T = \{x_1, x_2, \dots, x_p\}$. Then (x_{p+1}, x_l) is an inside corner of $\mathcal{P}$, where $l = \min\{j \in [n] | y_j \in N_Y(x_p)\} > 1$. Indeed, if (x_{p+1}, x_l) is not an inside corner of $\mathcal{P}$, then $l = \min\{j \in [n] | y_j \in N_Y(x_{p+1})\}$ and $N_Y(x_{p+1}) \subseteq N_Y(T)$. By Lemma 17, we obtain $N_X(Y \setminus N_Y(T)) \neq X \setminus T$. Thus, (x_{p+1}, x_l) is an inside corner of $\mathcal{P}$ and $|N_Y(T)| = |\{y_l, y_{l+1}, \dots, y_n\}| = n + 1 - l = p + 1 = |T| + 1$. Similarly, we obtain $|N_Y(T)| = |T| + 1$ in the case that $T = \{x_t, x_{t+1}, \dots, x_n\}$ for some $t > 1$. Hence, the second condition of Theorem 21 is fulfilled and $\mathbb{K}[\mathcal{P}]$ is Gorenstein. $\square$

4 Gorenstein stack polyominoes

In this section we simplify the characterization of Theorem 21 for the subclass of stack polyominoes, recovering a result of Qureshi [10, Corollary 4.12]. Stack polyominoes have the nice property that $N_Y(T)$ is a neighbor vertical interval for all $\emptyset \neq T \subset X$.

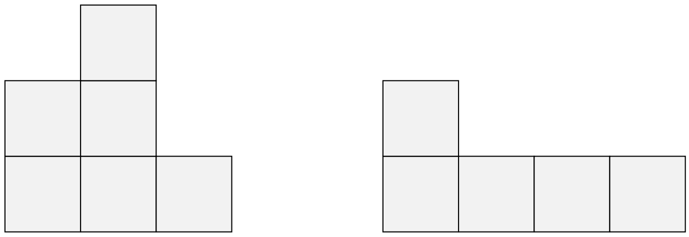


Figure 12: Stack polyominoes

We consider $\mathcal{P}$ to be a polyomino and we may assume that $[(1,1),(m,n)]$ is the smallest interval containing $V(\mathcal{P})$. Then $\mathcal{P}$ is called a stack polyomino (Figure 12), if it is a convex polyomino and for $i \in [m-1]$, the cell $[(i,1),(i+1,2)]$ belongs to $\mathcal{P}$.

Remark 25. If $\mathcal{P}$ is a stack polyomino, then for every $x \in X$ we have $\{y_1, y_2\} \subset N_Y(x)$. Moreover, there exists $x \in X$ such that $N_Y(x) = Y$.

Let $T \neq \emptyset$ be a subset of X and $y_j \in N_Y(T) \setminus \{y_1, y_2\}$. Hence, there exists $x_k \in T$ such that $y_j \in N_Y(x_k)$. Since $N_Y(x_k)$ is a neighbor vertical interval,

$$\{y_1, y_2, \ldots, y_{j-1}, y_j\} \subset N_Y(x_k) \subset N_Y(T).$$

In other words, $N_Y(T) = \{y_1, y_2, \ldots, y_s\}$ is a neighbor vertical interval for all $\emptyset \neq T \subsetneq X$, where $s = \max\{i \in [n] \mid (x_k, y_i) \in V(\mathcal{P}) \text{ for some } x_k \in T\}$.

Lemma 26. *Let $\mathcal{P}$ be a stack polyomino on $[n] \times [n]$. If $x_1 \cdots x_n y_1 \cdots y_n \notin \mathbb{K}[\mathcal{P}]$, then there is a subset $T \subset X$, with $|T| > |N_Y(T)|$, for which the following conditions hold:*

1. *$Y \setminus N_Y(T) \neq \emptyset$ and*
2. *for every $x \in X \setminus T$, $\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}$.*

Proof. We suppose that $x_1 \cdots x_n y_1 \cdots y_n \notin \mathbb{K}[\mathcal{P}]$. By Corollary 8, we find $I \subset X$ with $|I| > |N_Y(I)|$ or $J \subset Y$ with $|J| > |N_X(J)|$.

In the case that $I \subset X$ and $|I| > |N_Y(I)|$, we consider

$$T = I \cup \{x \in X \mid N_Y(x) \subset N_Y(I)\}.$$

We check conditions *(1)* and *(2)* for the set T. Since $\mathcal{P}$ is a stack polyomino, $N_Y(T) = \{y_1, y_2, \ldots, y_s\}$ for some $s \leqslant n$. If $N_Y(T) = Y$, then $|N_Y(T)| = |Y| = n \geqslant |I|$. Hence, $Y \setminus N_Y(T) \neq \emptyset$. Let $x \in X \setminus T$. It follows that $N_Y(x) \nsubseteq N_Y(I) = N_Y(T)$ and $|T| > N_Y(T)$. Thus, there is $l > s$ such that $y_l \in N_Y(x) \setminus N_Y(T)$ and condition *(2)* holds.

If there exists $J \subset Y$ with $|J| > |N_X(J)|$, then we set

$$T = X \setminus N_X(J).$$

We check conditions *(1)* and *(2)* for the set T. For the proof of the first condition, it is sufficient to show that $J \subset Y \setminus N_Y(T)$. Let $y \in J$. If $y \in N_Y(T)$, then there is $x \in T \cap N_X(y)$. Since $y \in J$, we get $x \in N_X(y) \subset N_X(J) = X \setminus T$. Thus, $\emptyset \neq J \subset Y \setminus N_Y(T)$. It follows that $|T| = |X| - |N_X(J)| > |X| - |J| = |Y| - |J| > |N_Y(T)|$, where $|X| = |Y| = n$. Moreover, $X \setminus T = N_X(J) \subset N_X(Y \setminus N_Y(T))$. For each $y \in Y \setminus N_Y(T)$, we have $N_X(y) \cap T = \emptyset$. Consequently, $N_X(y) \subset X \setminus T$ and $N_X(Y \setminus N_Y(T)) = \cup_{y \in Y \setminus N_Y(T)} N_X(y) \subset X \setminus T$. Hence, we proved $X \setminus T = N_X(Y \setminus N_Y(T))$. By Lemma 17 and the previous remark for any $x \in X \setminus T$, $N_Y(x) \nsubseteq N(T)$ and we have the second condition. □

As a consequence of Theorem 21, we may recover the characterisation of Gorenstein stack polyominoes obtained by Qureshi in [10, Corollary 4.12].

Corollary 27. *Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$. The following conditions are equivalent:*

1. *$\mathbb{K}[\mathcal{P}]$ is Gorenstein;*
2. *$m = n$ and for every $T \subset X$ with the properties that $Y \setminus N_Y(T) \neq \emptyset$ and for every $x \in X \setminus T$, $\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}$, one has $|N_Y(T)| = |T| + 1$.*

Proof. For *(1)⇒(2)*, let $T \neq \emptyset$ be a subset of X such that $Y \setminus N_Y(T) \neq \emptyset$ and $\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}$, for every $x \in X \setminus T$. By Remark 25, $N_Y(T)$ is a neighbor vertical interval.

By Lemma 17, we have $X \setminus T = N_X(Y \setminus N_Y(T))$, since $N_Y(T) = \{y_1, y_2, \ldots, y_s\}$ and $N_Y(x) = \{y_1, y_2, \ldots, y_t\}$ with $t > s$, for every $x \in X \setminus T$. Moreover $Y \setminus N_Y(T) = \{y_{s+1}, y_{s+2}, \ldots, y_n\} \neq \emptyset$ and $y_{s+1} \in N_Y(x)$, $\forall\, x \in X \setminus T$. Hence, $N_X(Y \setminus N_Y(T)) = X \setminus T = N_X(y_{s+1})$ and this is a neighbor horizontal interval by Remark 15. By using Theorem 21 and Corollary 8, $|N_Y(T)| = |T| + 1$ and $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[\mathcal{P}]$. Thus, we also obtain $m = n$ by Remark 9.

For *(2)⇒(1)*, we suppose that $m = n$ and $x_1 \cdots x_m y_1 \cdots y_n \notin \mathbb{K}[\mathcal{P}]$.

By Lemma 26, there exists $\emptyset \neq T \subsetneq X$ such that $|T| > |N_Y(T)|$, $Y \setminus N_Y(T) \neq \emptyset$ and $\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}$, for every $x \in X \setminus T$, which contradicts the assumption that $|N_Y(T)| = |T| + 1$. Thus, $x_1 \cdots x_n y_1 \cdots y_n \in \mathbb{K}[\mathcal{P}]$ and we obtain the first condition of Theorem 21 by applying Corollary 8.

Let $\emptyset \neq T \subsetneq X$ such that $N_Y(T)$ is a neighbor vertical interval and $N_X(Y \setminus N_Y(T)) = X \setminus T$ is a neighbor horizontal interval. Since $T \subsetneq X$, there exists $x \in X \setminus T$ with $N_Y(x) \nsubseteq N_Y(T)$, by Lemma 17. It follows that we find $y \in N_Y(x) \setminus N_Y(T) \subset Y \setminus N_Y(T)$. In other words, $Y \setminus N_Y(T) \neq \emptyset$.

If $x \in X \setminus T$, then $\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}$ by Lemma 17 and Remark 25. It implies that $|N_Y(T)| = |T| + 1$ and the second condition of Theorem 21 is fulfilled. Hence, $\mathbb{K}[\mathcal{P}]$ is Gorenstein. □

We may reformulate Corollary 27 as follows.

Corollary 28. *Let $\mathcal{P}$ be a convex stack polyomino and $[(1,1),(m,n)]$ the smallest interval which contains $V(\mathcal{P})$. Then $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if $m = n$ and for each inside corner of $\mathcal{P}$, cutting all the cells of $\mathcal{P}$ which lie below the horizontal edge interval containing the corner, the smallest interval which contains the remaining polyomino is a square.*

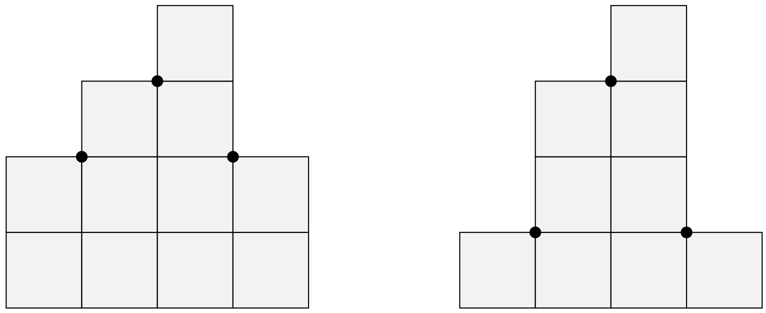


Figure 13:

Proof. Let $\mathbb{K}[\mathcal{P}]$ be Gorenstein and (x_r, y_t) be an inside corner of $\mathcal{P}$. Set $T = \{x \in X \mid \max\{j \in [n] \mid y_j \in N_Y(x)\} \leqslant t\}$. Then

$$\max\{j \in [n] \mid y_j \in N_Y(T)\} = t < n.$$

For $x \in X \setminus T$ we have that $\max\{j \in [n] \mid y_j \in N_Y(x)\} > t$. By Corollary 27, it follows that $|N_Y(T)| = |T| + 1$. Thus, $|T| = t - 1$. In other words, $n - t + 1 = n - |T|$ and the minimal rectangle we are interested in is a square.

Conversely, we suppose that $T \subset X$ is a set with the properties that $Y \setminus N_Y(T) \neq \emptyset$ and for every $x \in X \setminus T$,

$$\max\{j \in [n] \mid y_j \in N_Y(x)\} > \max\{j \in [n] \mid y_j \in N_Y(T)\}.$$

Let $r = \max\{j \in [n] \mid y_j \in N_Y(T)\} < n$.

Since $\mathcal{P}$ is a column convex polyomino, y_r is the y-coordinate of an inside corner. Then by assumption, $|X \setminus T| = n - r + 1$. Hence, $n - |T| = n - r + 1$ and $|T| + 1 = r = |N_Y(T)|$. By Corollary 27, $\mathbb{K}[\mathcal{P}]$ is Gorenstein. □

Notice that Corollaries 27 and 28 extend the classification of Gorenstein one-sided ladder polyominoes given in [3, Theorem 4.9(c)].

Examples 29. By Corollary 28, the first polyomino of Figure 13 is Gorenstein, while the second is not.

5 The regularity of $\mathbb{K}[\mathcal{P}]$

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Recall that the coordinate ring of $\mathcal{P}$ is a finitely generated module over the polynomial ring $S = \mathbb{K}[x_{ij} \mid (i,j) \in V(\mathcal{P})]$. The Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$, denoted $\text{reg}(\mathbb{K}[\mathcal{P}])$, is defined to be the largest integer r such that, for every i, the i^{th} syzygy of $\mathbb{K}[\mathcal{P}]$ is generated in degree at most $r + i$.

We consider $H_{\mathbb{K}[\mathcal{P}]}(t)$ to be the Hilbert series of $\mathbb{K}[\mathcal{P}]$. Then

$$H_{\mathbb{K}[\mathcal{P}]}(t) = \frac{Q(t)}{(1-t)^d}$$

where $Q(t) \in \mathbb{Z}[t]$ and where d is the Krull dimension of $\mathbb{K}[\mathcal{P}]$. According to [10, Theorem 2.2], $d = \dim(\mathbb{K}[\mathcal{P}]) = m + n - 1$.

Since $\mathbb{K}[\mathcal{P}]$ is a Cohen-Macaulay ring, we have

$$\operatorname{reg}(\mathbb{K}[\mathcal{P}]) = \deg(Q(t)) = \dim(\mathbb{K}[\mathcal{P}]) + a(\mathbb{K}[\mathcal{P}]), \tag{1}$$

where the a-invariant $a(\mathbb{K}[\mathcal{P}])$ of $\mathbb{K}[\mathcal{P}]$ is defined as the degree of the Hilbert series of $\mathbb{K}[\mathcal{P}]$, that is $a(\mathbb{K}[\mathcal{P}]) = \deg(Q(t)) - d$. For the proof, we refer, for example, to [12, Corollary B.4.1].

Let $G_{\mathcal{P}}$ be the bipartite graph attached to $\mathcal{P}$ on the vertex set $X \cup Y$. In this section, we consider $G_{\mathcal{P}}$ as a digraph with all its arrows leaving the vertex set Y. Hence, we denote the directed edges by (z, w), where $z \in Y$ and $w \in X$. Following [11], we introduce the following notion.

Definition 30. If $T \subset X \cup Y$, then

$$\delta^+(T) = \{e = (z, w) \in E(G_{\mathcal{P}}) \mid z \in T \text{ and } w \notin T\}$$

is the set of edges leaving the vertex set T and

$$\delta^-(T) = \{e = (z, w) \in E(G_{\mathcal{P}}) \mid z \notin T \text{ and } w \in T\}$$

is the set of edges entering the vertex set T.

The set $\delta^+(T)$ is called a directed cut of the digraph $G_{\mathcal{P}}$ if $\emptyset \neq T \subsetneq X \cup Y$ and $\delta^-(T) = \emptyset$.

Example 31. In the digraph of Figure 14, let $T_1 = \{x_3, y_2, y_3\}$ and $T_2 = \{x_3, y_1, y_2\}$. Then we notice that

$$\emptyset \neq \delta^+(T_1) = \{(y_2, x_1), (y_2, x_2), (y_3, x_1), (y_3, x_2)\} \text{ and } \delta^-(T_1) = \{(y_1, x_3)\} \neq \emptyset,$$

while

$$\emptyset \neq \delta^+(T_2) = \{(y_1, x_1), (y_1, x_2), (y_2, x_1), (y_2, x_2)\} \text{ and } \delta^-(T_2) = \emptyset.$$

Thus, $\delta^+(T_2)$ is a directed cut, while $\delta^+(T_1)$ is not.

Remark 32. Since $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[G_{\mathcal{P}}]$, we consider

$$\delta^+(T) = \{(x, y) \in V(\mathcal{P}) \mid x \notin T \text{ and } y \in T\}$$

and

$$\delta^-(T) = \{(x, y) \in V(\mathcal{P}) \mid x \in T \text{ and } y \notin T\}$$

for all $T \subset X \cup Y$. If $T \subseteq X$, then $\delta^+(T) = \emptyset$. If $T \subseteq Y$, then $\delta^-(T) = \emptyset$ and $\delta^+(T)$ is a directed cut of $G_{\mathcal{P}}$.

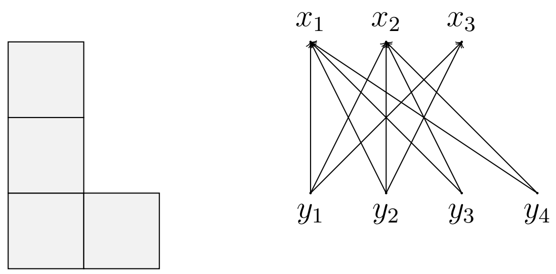


Figure 14: A convex polyomino and its associated digraph

Lemma 33. *Let $\emptyset \neq T \subsetneq X \cup Y$. Then $\delta^+(T)$ is a directed cut of the digraph $G_\mathcal{P}$ if and only if $T = A \cup B$ with $A \subset X$, $B \subset Y$ and $N_Y(A) \subset B$.*

Proof. Let $T \neq \emptyset$ be a proper subset in $X \cup Y$. Then $T = A \cup B$ with $A \subset X$ and $B \subset Y$. By Definition 30 and Remark 32,

$$\delta^+(T) = \{(x, y) \in V(\mathcal{P}) \mid x \notin A \text{ and } y \in B\}$$

is a directed cut of $G_\mathcal{P}$ if and only if

$$\delta^-(T) = \{(x, y) \mid x \in A \text{ and } y \notin B\} = \emptyset.$$

Suppose that $N_Y(A) \not\subset B$. Then there exist $x \in A$ and $y \in Y \setminus B$ such that $(x, y) \in V(\mathcal{P})$. In other words, $(x, y) \in \delta^-(T) \neq \emptyset$.

Conversely, suppose that $\delta^-(T) \neq \emptyset$. Then we find $x \in A$ and $y \in Y \setminus B$ such that $(x, y) \in V(\mathcal{P})$. This is equivalent to saying that $y \in N_Y(x) \setminus B \subset N_Y(A) \setminus B$ and hence, $N_Y(A) \not\subset B$. □

In [11], Valencia and Villarreal show that for any connected bipartite graph G, the a-invariant, $a(\mathbb{K}[G])$ can be interpreted in combinatorial terms as follows.

Proposition 34. *[11, Proposition 4.2] Let G be a connected bipartite graph with $V(G) = X \cup Y$. If G is a digraph with all its arrows leaving the vertex set Y, then*

$$-a(\mathbb{K}[G]) = \textit{the maximum number of disjoint directed cuts of } G.$$

Example 35. In the digraph of Figure 14, $-a(\mathbb{K}[G_\mathcal{P}]) = 4$ and a set of disjoint directed cuts is $\{\delta^+(\{y_1\}), \delta^+(\{y_2\}), \delta^+(\{y_3\}), \delta^+(\{y_4\})\}$.

Remark 36. Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then

$$\delta^+(\{y_i\}) = \{(x, y_i) \in V(\mathcal{P}) \mid x \in N_X(y_i)\} = N_X(y_i) \times \{y_i\} \text{ for } i = 1, \ldots, n$$

are disjoint directed cuts and also,

$$\begin{aligned} \delta^+(\{x_1, x_2, \ldots, \hat{x_i}, \ldots, x_{m-1}, x_m, y_1, y_2, \ldots, y_n\}) &= \{(x_i, y) \in V(\mathcal{P}) \mid y \in N_Y(x_i)\} \\ &= \{x_i\} \times N_Y(x_i) \text{ for } i = 1, \ldots, m \end{aligned}$$

are disjoint directed cuts, where $\hat{x_i}$ means that we skip x_i.

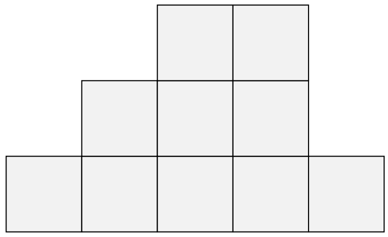


Figure 15:

Proposition 37. *Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then*

$$-a(\mathbb{K}[\mathcal{P}]) \geqslant \max\{m, n\}.$$

In particular,

$$\operatorname{reg}(\mathbb{K}[\mathcal{P}]) \leqslant \min\{m, n\} - 1.$$

Proof. Since $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[G_{\mathcal{P}}]$, we have

$$-a(\mathbb{K}[\mathcal{P}]) = \text{the maximum number of disjoint directed cuts of } G_{\mathcal{P}}.$$

By Proposition 34 and Remark 36, it follows that $-a(\mathbb{K}[\mathcal{P}]) \geqslant \max\{m, n\}$. The inequality for the regularity follows by (1). □

Example 38. Let $\mathcal{P}$ be the stack polyomino of Figure 15. Then $\operatorname{reg}(\mathbb{K}[\mathcal{P}]) = \min\{6, 4\} - 1 = 3$.

In general it is difficult to compute the regularity of $\mathbb{K}[\mathcal{P}]$. Even in the case of stack polyominoes, we have not found a lower bound for the regularity of $\mathbb{K}[\mathcal{P}]$.

Example 39. Let $\mathcal{P}$ be the stack polyomino of Figure 16. Then $\operatorname{reg}(\mathbb{K}[\mathcal{P}]) = 2 < \min\{m, n\} - 1$.

6 The multiplicity of $\mathbb{K}[\mathcal{P}]$

Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$. The multiplicity of $\mathbb{K}[\mathcal{P}]$, denoted $e(\mathbb{K}[\mathcal{P}])$, is given by $Q(1)$, where $Q(t)$ is the numerator of the Hilbert series of $\mathbb{K}[\mathcal{P}]$.

For every $i \in [m]$, we define the height of i as

$$\operatorname{height}(i) = \max\{j \in [n] \mid (i, j) \in V(\mathcal{P})\}.$$

Following the proof of [9, Theorem], we give a total order on the variables x_{ij}, with $(i, j) \in V(\mathcal{P})$, as follows:

$$x_{ij} > x_{kl} \text{ if and only if} \tag{2}$$

$$(\operatorname{height}(i) > \operatorname{height}(k)) \text{ or } (\operatorname{height}(i) = \operatorname{height}(k) \text{ and } i > k) \text{ or } (i = k \text{ and } j > l).$$

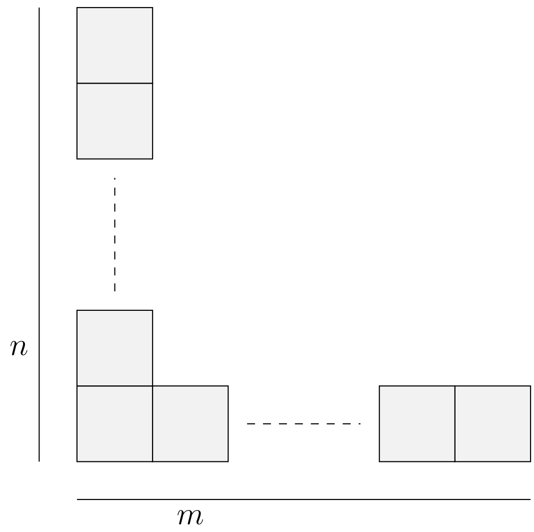


Figure 16:

Let $<$ be the reverse lexicographical order induced by this order of variables. As we have already seen in the previous sections, the ideal $I_{\mathcal{P}}$ can be viewed as the toric ideal of the edge ring $\mathbb{K}[G_{\mathcal{P}}]$, where $G_{\mathcal{P}}$ is the bipartite graph associated to $\mathcal{P}$. As it follows from the proof of [9, Theorem], the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<$ consists of all 2-inner minors of $\mathcal{P}$. In what follows, whenever we consider the Gröbner basis of $I_{\mathcal{P}}$, we assume that the variables x_{ij}, with $(i,j) \in V(\mathcal{P})$ are totally ordered as in (2).

We notice that $\mathrm{in}_<(I_{\mathcal{P}})$ is a squarefree monomial ideal. Thus, we may view $\mathrm{in}_<(I_{\mathcal{P}})$ as the Stanley-Reisner ideal of a simplicial complex $\Delta_{\mathcal{P}}$ on the vertex set $V(\mathcal{P})$. It is known that $\Delta_{\mathcal{P}}$ is a pure shellable simplicial complex by [13, Theorem 9.6.1] and [7, Theorem 9.5.10].

Let $f = (f_0, f_1, \cdots, f_{d-1})$ be the f-vector of $\Delta_{\mathcal{P}}$, where $d = \dim(\mathbb{K}[\mathcal{P}]) = m + n - 1$. We have

$$H_{\mathbb{K}[\mathcal{P}]}(t) = H_{S/\,\mathrm{in}_<(I_{\mathcal{P}})}(t) = H_{\mathbb{K}[\Delta_{\mathcal{P}}]}(t).$$

By [1, Corollary 5.1.9],

$$e(\mathbb{K}[\mathcal{P}]) = f_{d-1} = |\mathcal{F}(\Delta_{\mathcal{P}})|,$$

that is, $e(\mathbb{K}[\mathcal{P}])$ is equal to the number of facets of $\Delta_{\mathcal{P}}$.

Example 40. Let $\mathcal{P}$ be the polyomino of Figure 17. We order the variables as follows $x_{23} > x_{22} > x_{21} > x_{13} > x_{12} > x_{11} > x_{32} > x_{31}$. Then with respect to the reverse lexicographical order induced by this order of variables, we have

$$\mathrm{in}_<(I_{\mathcal{P}}) = (x_{11}x_{32}, x_{21}x_{32}, x_{21}x_{12}, x_{21}x_{13}, x_{22}x_{13}) \text{ and}$$

$$\Delta_{\mathcal{P}} = \langle F_1 = \{(1,1), (2,1), (2,2), (2,3), (3,1)\};$$

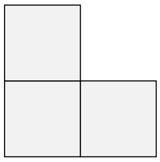


Figure 17:

$$F_2 = \{(1,1),(1,2),(2,2),(2,3),(3,1)\}; F_3 = \{(1,1),(1,2),(1,3),(2,3),(3,1)\};$$

$$F_4 = \{(1,2),(2,2),(2,3),(3,1),(3,2)\}; F_5 = \{(1,2),(1,3),(2,3),(3,1),(3,2)\}\rangle.$$

Let Δ be a simplicial complex on the vertex set V and $v \in V$. Recall that the link of v in Δ is the simplicial complex

$$\mathrm{lk}(v) = \{F \in \Delta \mid v \notin F \text{ and } F \cup \{v\} \in \Delta\}$$

and the deletion of v is the simplicial complex

$$\mathrm{del}(v) = \{F \in \Delta \mid v \notin F\}.$$

Let x_{ij} be the smallest variable in S with respect to $<$ and fix $v = (i, \mathrm{height}(i)) \in V(\mathcal{P})$. If $i = 1$, then we denote by $\mathcal{P}_1$ the polyomino obtained from $\mathcal{P}$ by deleting the only cell which contains the vertex v. Otherwise, $\mathcal{P}_1$ is given by deleting the only cell which contains the vertex $(m, \mathrm{height}(m))$; see Figure 18. Notice that in both cases $\dim(\Delta_{\mathcal{P}_1}) = d - 1 = m + n - 2$.

Remark 41. Since x_{i1} is the smallest variable with respect to $<$, we have $(i, 1) \in F$ for every $F \in \Delta_{\mathcal{P}}$. Indeed, x_{i1} is regular on $S/\operatorname{in}_<(I_{\mathcal{P}})$, thus it does not belong to any of the minimal primes of $\operatorname{in}_<(I_{\mathcal{P}})$ which implies that x_{i1} belongs to all the facets of $\Delta_{\mathcal{P}}$.

In what follows we will sometimes confuse the point (i, j) of $\mathcal{P}$ with the vertex x_{ij} of $\Delta_{\mathcal{P}}$.

Lemma 42. *With respect to the above notation,* $|\mathcal{F}(\Delta_{\mathcal{P}_1})| = |\mathcal{F}(\mathrm{del}(v))|$.

Proof. Let x_{ij} be the smallest variable in S with respect to $<$ and set

$$v = (i, \mathrm{height}(i)) \in V(\mathcal{P}).$$

First, let us consider $\mathrm{height}(i) \geqslant 3$. If $F \in \mathcal{F}(\mathrm{del}(v))$, then applying Algorithm 1, we obtain a facet $F' \in \mathcal{F}(\Delta_{\mathcal{P}_1})$.

Indeed, if $F \in \mathcal{F}(\mathrm{del}(v))$ and $i \neq 1$, then $v \notin F$ and $|F| = m + n - 1$. Notice that, by applying the first "For" loop in Algorithm 1, we obtain F' with $(m, \mathrm{height}(m)) \notin F'$, and we never add this vertex again; hence $F' \subset V(\mathcal{P}_1)$. Since F' is obtained from F by

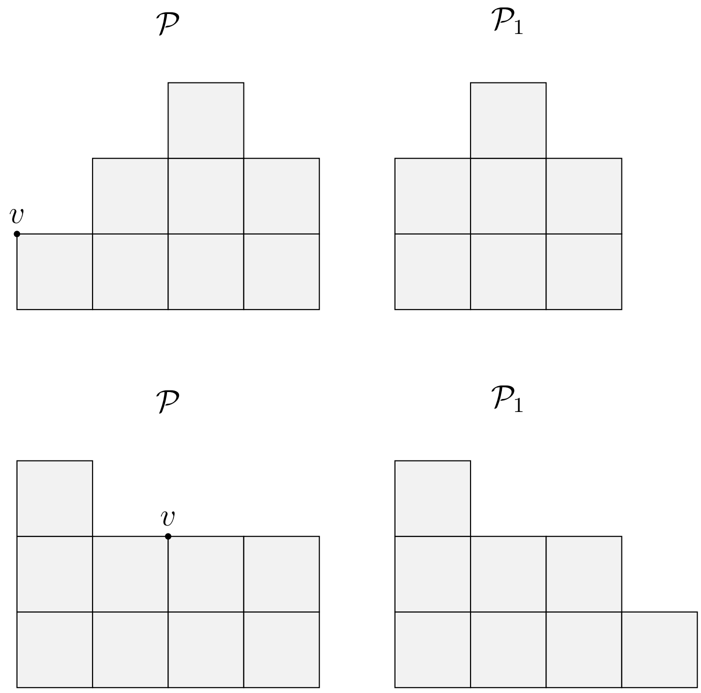


Figure 18:

a circular permutation of the vertices of F which have the x-coordinate greater than or equal to i, we get $|F| = |F'| = m + n - 1 = \dim \Delta_{\mathcal{P}_1} + 1$.

For example, in the polyomino $\mathcal{P}$ of Figure 19, $v = (3, 3)$. In this case, $i = 3$ and $m = 6$. We illustrate all the "For" loops of Algortihm 1 in Figure 19 for the facet

$$F = \{(1, 3), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (5, 3), (6, 3)\} \in \mathrm{del}(v).$$

Following Algorithm 1,

$$F' = \{(1, 3), (1, 4), (2, 4), (3, 2), (3, 3), (4, 3), (5, 3), (6, 1), (6, 2)\}.$$

We depict the points that are in F' by black dots, the points removed from F' by crosses and the points added to F' by empty dots.

Now, we observe that even if the order of the variables for $\mathcal{P}_1$ is not induced by the order of the variables of $\mathcal{P}$, the generators of $\mathrm{in}_<(I_{\mathcal{P}_1})$ are also generators of $\mathrm{in}_<(I_{\mathcal{P}})$, since the 2-inner minors of $\mathcal{P}_1$ are also 2-inner minors of $\mathcal{P}$. Therefore, we may conclude that $F' \in \Delta_{\mathcal{P}_1}$ and so $F' \in \mathcal{F}(\Delta_{\mathcal{P}_1})$.

In the case that $i = 1$, we notice that $F = F'$ and $\mathcal{F}(\Delta_{\mathcal{P}_1}) = \mathcal{F}(\mathrm{del}(v))$. In fact, if $F \in \mathcal{F}(\Delta_{\mathcal{P}_1})$, then $v \notin F$ and $|F| = m+n-1$. Since $F \in \mathrm{del}(v)$ and $\dim \Delta_{\mathcal{P}} = m+n-2$, it follows that $F \in \mathcal{F}(\mathrm{del}(v))$. If $F \in \mathcal{F}(\mathrm{del}(v))$, then $F \in \Delta_{\mathcal{P}_1}$. Since $\dim \mathbb{K}[\Delta_{\mathcal{P}_1}] = m + n - 1$, it follows that $F \in \mathcal{F}(\Delta_{\mathcal{P}_1})$.

Therefore, we have shown that every facet F of $\mathrm{del}(v)$ determines uniquely a facet F' of $\Delta_{\mathcal{P}_1}$, if $\mathrm{height}(i) \geqslant 3$.

Algorithm 1

```
F' := F;
h := height(i);
if i ≠ 1 then
    for k = 1 to h do
        if (m, k) ∈ F then
            F' := F' \ {(m, k)};
        end if
        if (i, k) ∈ F then
            F' := (F' \ {(i, k)}) ∪ {(m, k)};
        end if
    end for
    for j = i + 1 to m − 1 do
        for k = 1 to h do
            if (j, k) ∈ F then
                F' := (F' \ {(j, k)}) ∪ {(j − 1, k)};
            end if
        end for
    end for
    for k = 1 to h do
        if (m, k) ∈ F then
            F' := F' ∪ {(m − 1, k)};
        end if
    end for
end if
return F'
```

Conversely, let F' be a facet of $\Delta_{\mathcal{P}_1}$. Following the steps of Algorithm 1 in reverse order, we obtain a facet F of $\mathrm{del}(v)$. Algorithm 2 gives explicitly all the steps to get F from F'.

We thus get $|\mathcal{F}(\Delta_{\mathcal{P}_1})| = |\mathcal{F}(\mathrm{del}(v))|$ if $\mathrm{height}(i) \geqslant 3$. Moreover, we have equality between the sets $\mathcal{F}(\Delta_{\mathcal{P}_1})$ and $\mathcal{F}(\mathrm{del}(v))$ if and only if $i = 1$ and $\mathrm{height}(i) \geqslant 3$.

In order to complete the proof, let us point out that the same two algorithms work for $\mathrm{height}(i) = 2$. In fact, for $i > 1$ (respectively $i = 1$), F is a facet of $\mathrm{del}(v)$ if and only if F' is a facet of the cone $(m, 1) * \Delta_{\mathcal{P}_1}$ (respectively $(1, 1) * \Delta_{\mathcal{P}_1}$).

For example, if we consider the polyomino $\mathcal{P}$ of Figure 20, $v = (3, 2) \in V(\mathcal{P})$ and if we choose

$$F = \{(1,2),(1,3),(1,4),(2,4),(3,1),(4,1),(4,2),(5,2)\} \in \mathcal{F}(\mathrm{del}(v)),$$

by applying Algorithm 1 with $i = 3$, $h = 2$ and $m = 5$, we obtain

$$F' = \{(1,2),(1,3),(1,4),(2,4),(3,1),(3,2),(4,2),(5,1)\}$$

a facet of $(5, 1) * \Delta_{\mathcal{P}_1}$. □

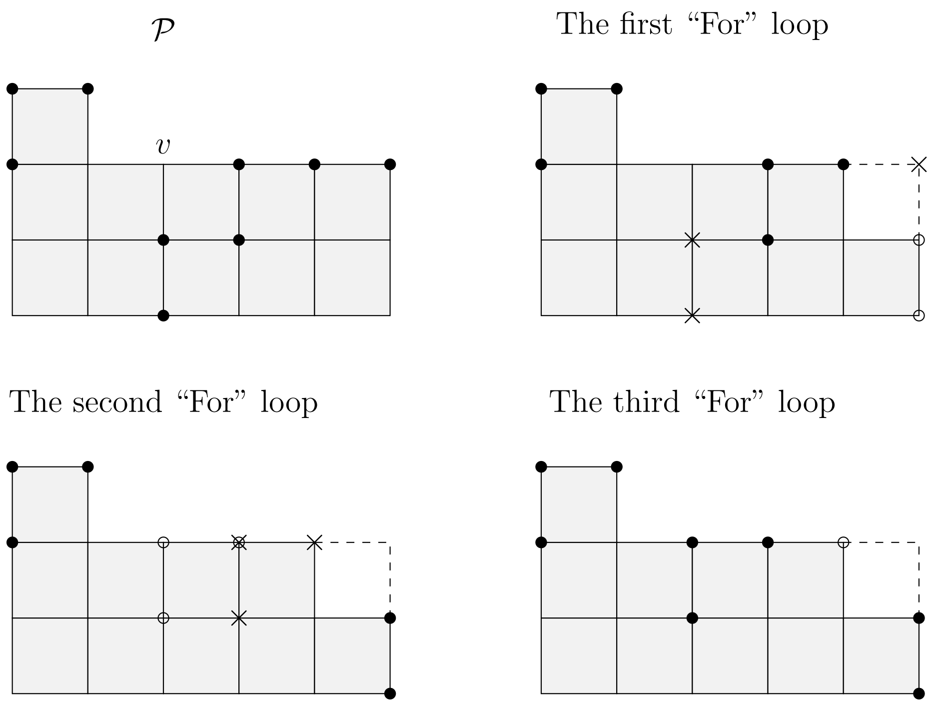


Figure 19:

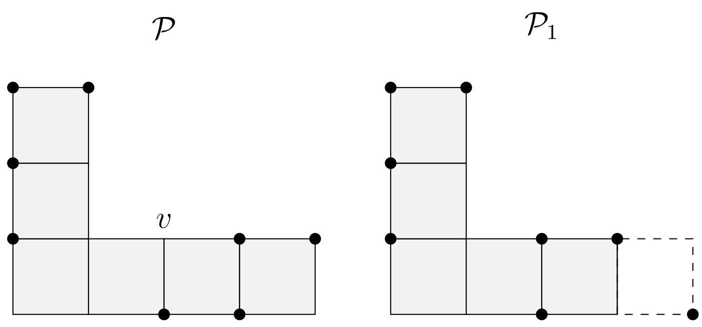


Figure 20:

Algorithm 2

```
$F := F'$;
$h :=$ height$(i)$;
if $i \neq 1$ then
    for $k = 1$ to $h$ do
        if $(m-1, k) \in F'$ then
            $F := F \setminus \{(m-1, k)\}$;
        end if
    end for
    if $i \leqslant m-2$ then
        for $j = m-2$ to $i$ do
            for $k = 1$ to $h$ do
                if $(j, k) \in F'$ then
                    $F := (F \setminus \{(j, k)\}) \cup \{(j+1, k)\}$;
                end if
            end for
        end for
    end if
    for $k = 1$ to $h$ do
        if $(m, k) \in F'$ then
            $F := (F \setminus \{(m, k)\}) \cup \{(i, k)\}$;
        end if
        if $(m-1, k) \in F'$ then
            $F := F \cup \{(m, k)\}$;
        end if
    end for
end if
return $F$
```

Let $\mathcal{P}_2$ be the polyomino obtained from $\mathcal{P}$ by deleting all the cells of $\mathcal{P}$ which lie below the horizontal edge interval containing the vertex v.

Lemma 43. *With respect to the above notation,* $|\mathcal{F}(\Delta_{\mathcal{P}_2})| = |\mathcal{F}(\mathrm{lk}(v))|$.

Proof. Let F be a facet of $\mathrm{lk}(v)$. Then $F \cup \{v\} \in \mathcal{F}(\Delta_{\mathcal{P}})$. We set $j = \mathrm{height}(i)$.

Suppose that $F \cup \{v\} = G_1 \cup G_2$ where $G_1 \in \Delta_{P_2}$ and $G_2 = \{(a,j) \mid (a,j) \in V(\mathcal{P}) \setminus V(\mathcal{P}_2)\} \cup \{(i,j-1),\dots,(i,1)\}$. In fact, since $v \in F \cup \{v\}$, all the vertices of G_2 must belong to $F \cup \{v\}$ and $x_{ij}x_{kl} \in \mathrm{in}_<(I_{\mathcal{P}})$, for every $(k,l) \in V(\mathcal{P}) \setminus G_2$ with $l < j$.

In order to prove that $G_1 \in \mathcal{F}(\Delta_{\mathcal{P}_2})$, it is enough to show that $|G_1| = \dim \Delta_{\mathcal{P}_2} + 1$. We consider the polyomino $\mathcal{P}_2$ to be on $[m-t] \times [n-j+1]$, for some $t \geqslant 1$. It follows that $m+n-1 = |F \cup \{v\}| = |G_1 \cup G_2| = |G_1| + |G_2| = |G_1| + (t+j-1)$, which implies that $|G_1| = (m-t) + (n-j+1) - 1 = \dim \Delta_{\mathcal{P}_2} + 1$. Therefore, $G_1 \in \mathcal{F}(\Delta_{\mathcal{P}_2})$ and $|\mathcal{F}(\mathrm{lk}(v))| \leqslant |\mathcal{F}(\Delta_{\mathcal{P}_2})|$.

Vice versa, let G be a facet of $\Delta_{\mathcal{P}_2}$. By definition of $\mathcal{P}_2$, $v \notin G$ and $G \cup \{v\} \in \Delta_{\mathcal{P}}$. In other words, $G \in \mathrm{lk}(v)$ and there exists $F \in \mathcal{F}(\mathrm{lk}(v))$ such that $G \subset F$. Thus, $F \cup \{v\} \in \mathcal{F}(\Delta_{\mathcal{P}})$. Moreover, $F \cup \{v\} = G \cup G_2$ and $|\mathcal{F}(\Delta_{\mathcal{P}_2})| \leqslant |\mathcal{F}(\mathrm{lk}(v))|$. ☐

We now prove the main result of this section.

Theorem 44. *Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$ and $v = (i,j) \in V(\mathcal{P})$ such that x_{i1} is the smallest variable in S and $j = \mathrm{height}(i)$. Then*

$$e(\mathbb{K}[\mathcal{P}]) = e(\mathbb{K}[\mathcal{P}_1]) + e(\mathbb{K}[\mathcal{P}_2]),$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the polyominoes defined before.

Proof. In order to prove the equality, it is sufficient to show that

$$|\mathcal{F}(\Delta_{\mathcal{P}})| = |\mathcal{F}(\Delta_{\mathcal{P}_1})| + |\mathcal{F}(\Delta_{\mathcal{P}_2})|.$$

We consider F to be a facet in $\Delta_{\mathcal{P}}$. If $v \in F$, then $F \setminus \{v\} \in \mathcal{F}(\mathrm{lk}(v))$. Otherwise, $v \notin F$, thus $F \in \mathcal{F}(\mathrm{del}(v))$. Therefore, we obtain $|\mathcal{F}(\Delta_{\mathcal{P}})| = |\mathcal{F}(\mathrm{lk}(v))| + |\mathcal{F}(\mathrm{del}(v)|$. The claim follows by applying Lemma 42 and Lemma 43. ☐

Example 45. Let $\mathcal{P}$ be the stack polyomino of Figure 21. Then the multiplicity of $\mathbb{K}[\mathcal{P}]$ is equal to 14. The first step in the recursive formula, namely $e(\mathbb{K}[\mathcal{P}]) = e(\mathbb{K}[\mathcal{P}_1]) + e(\mathbb{K}[\mathcal{P}_2])$, is shown in the figure. Next we apply the recursive procedure for each of the polyominoes $\mathcal{P}_1$ and $\mathcal{P}_2$.

Example 46. Let $\mathcal{P}_{m,n}$ be the stack polyomino on $[m] \times [n]$ with $V(\mathcal{P}_{m,n}) = [m] \times [n]$. The multiplicity of $\mathbb{K}[\mathcal{P}_{m,n}]$ was computed in [6, Section 3, Example] and

$$e(\mathbb{K}[\mathcal{P}_{m,n}]) = \binom{m+n-2}{m-1}.$$

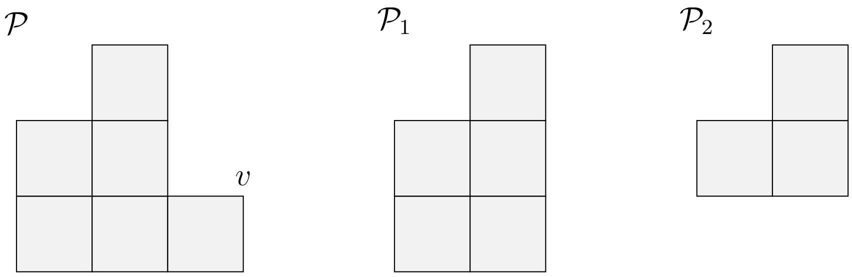


Figure 21: $e(\mathbb{K}[\mathcal{P}]) = 14$

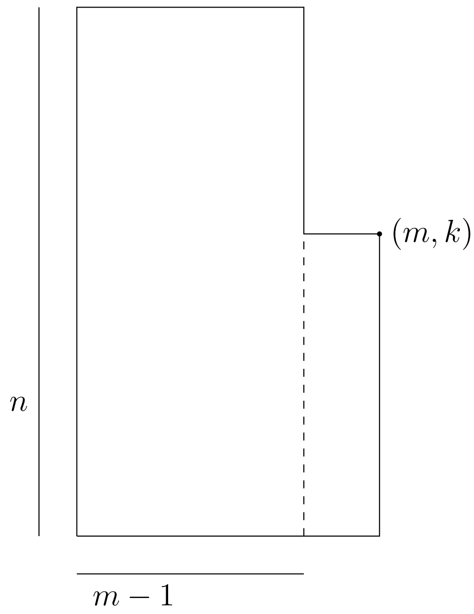


Figure 22:

Now, we consider $k < n$ to be a positive integer and $\mathcal{P}_k$ to be the polyomino of Figure 22. It consists of a rectangle of size $[m-1] \times [n]$ together with a column of cells of height equal to k. By Theorem 44,

$$e(\mathbb{K}[\mathcal{P}_k]) = e(\mathbb{K}[\mathcal{P}_{k-1}]) + e(\mathbb{K}[\mathcal{P}_{m-1,n-k+1}]) = e(\mathbb{K}[\mathcal{P}_{k-1}]) + \binom{m+n-k-2}{m-2}.$$

Applying recursively this formula, we obtain

$$\begin{aligned} e(\mathbb{K}[\mathcal{P}_k]) &= \binom{m+n-3}{m-2} + \binom{m+n-4}{m-2} + \cdots + \binom{m+n-k-2}{m-2} \\ &= \binom{m+n-2}{m-1} - \binom{m+n-k-2}{m-1}. \end{aligned}$$

Example 47. Let $\mathcal{P}(m,n,k_1,k_2,\dots,k_l)$ be the polyomino of Figure 23. This is an example of one-sided ladder with the last l columns of heights $k_1,\dots,k_l$. Hilbert series of one-sided ladders have been considered in [14]. By Theorem 44,

$$
\begin{aligned}
&e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_l)])\\
&\qquad=e(\mathbb{K}[\mathcal{P}(m-1,n-k_l+1,k_1-k_l+1,k_2-k_l+1,\dots,k_{l-1}-k_l+1)])\\
&\qquad\quad+e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_{l-1},k_l-1)])\\
&\qquad=e(\mathbb{K}[\mathcal{P}(m-1,n-k_l+1,k_1-k_l+1,k_2-k_l+1,\dots,k_{l-1}-k_l+1)])\\
&\qquad\quad+e(\mathbb{K}[\mathcal{P}(m-1,n-k_l+2,k_1-k_l+2,k_2-k_l+2,\dots,k_{l-1}-k_l+2)])\\
&\qquad\quad+e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_{l-1},k_l-2)])\\
&\qquad=\\
&\qquad\quad\vdots\\
&\qquad=e(\mathbb{K}[\mathcal{P}(m-1,n-k_l+1,k_1-k_l+1,k_2-k_l+1,\dots,k_{l-1}-k_l+1)])\\
&\qquad\quad+e(\mathbb{K}[\mathcal{P}(m-1,n-k_l+2,k_1-k_l+2,k_2-k_l+2,\dots,k_{l-1}-k_l+2)])\\
&\qquad\quad+\cdots+e(\mathbb{K}[\mathcal{P}(m-1,n-1,k_1-1,k_2-1,\dots,k_{l-1}-1)])\\
&\qquad\quad+e(\mathbb{K}[\mathcal{P}(m-1,n,k_1,k_2,\dots,k_{l-1})]).
\end{aligned}
$$

In other words, we have

$$e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_l)])=\sum_{j_1=0}^{k_l-1}e(\mathbb{K}[\mathcal{P}(m-1,n-j_1,k_1-j_1,k_2-j_1,\dots,k_{l-1}-j_1)]).$$

By iterating the formula, we obtain

$$
\begin{aligned}
&e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_l)])\\
&=\sum_{j_1=0}^{k_l-1}\sum_{j_2=0}^{k_{l-1}-j_1-1}e(\mathbb{K}[\mathcal{P}(m-2,n-j_1-j_2,k_1-j_1-j_2,k_2-j_1-j_2,\dots,k_{l-2}-j_1-j_2)])\\
&=\\
&\ \vdots\\
&=\sum_{j_1=0}^{k_l-1}\sum_{j_2=0}^{k_l-j_1-1}\cdots\sum_{j_{l-1}=0}^{k_2-j_1-\cdots-j_{l-2}-1}e(\mathbb{K}[\mathcal{P}(m-l+1,n-j_1-\cdots-j_{l-1},k_1-j_1-\cdots-j_{l-1})])\\
&=\sum_{j_1=0}^{k_l-1}\sum_{j_2=0}^{k_l-j_1-1}\cdots\sum_{j_{l-1}=0}^{k_2-j_1-\cdots-j_{l-2}-1}\left(\binom{(m-l+1)+(n-j_1-\cdots-j_{l-1})-2}{(m-l+1)-1}-\right.\\
&\qquad\qquad\left.\binom{(m-l+1)+(n-j_1-\cdots-j_{l-1})-(k_1-j_1-\cdots-j_{l-1})-2}{(m-l+1)-1}\right).
\end{aligned}
$$

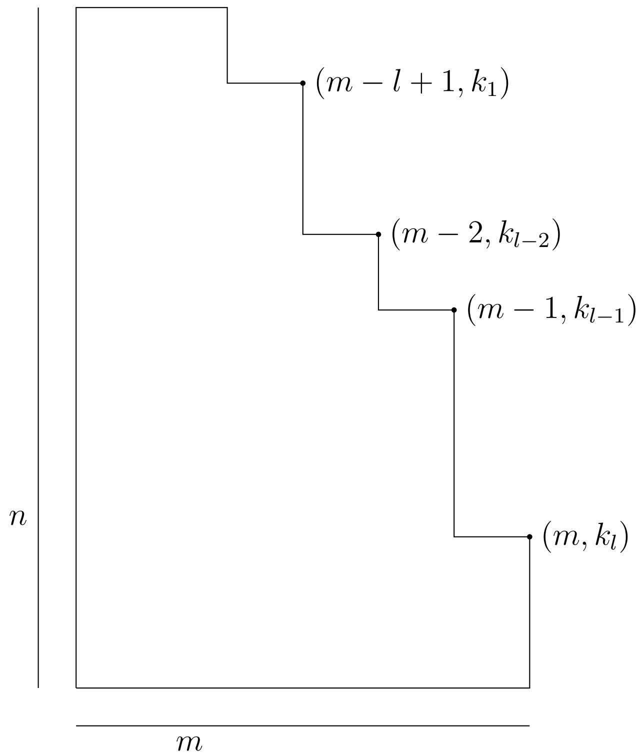


Figure 23:

Thus,

$$\begin{aligned}&e(\mathbb{K}[\mathcal{P}(m,n,k_1,k_2,\dots,k_l)])\\ =&\sum_{j_1=0}^{k_l-1}\sum_{j_2=0}^{k_l-j_1-1}\cdots\sum_{j_{l-1}=0}^{k_2-j_1-\dots-j_{l-2}-1}\left(\binom{m+n-l-j_1-\dots-j_{l-1}-1}{m-l}-\binom{m+n-l-k_1-1}{m-l}\right).\end{aligned}$$

One may, of course, approach the computation of the multiplicity in a recursive way for arbitrary convex polyominoes. Finding the appropriate order of the variables in concordance to the one described in [9] is not difficult as we will see in the example below. What is difficult in the general case is to identify the link of a suitable chosen vertex as a simplicial complex of another polyomino related to the original one. We illustrate part of these difficulties in the following example.

Example 48. Let $\mathcal{P}$ be the convex polyomino of Figure 24. According to the proof of [9], the generators of $I_\mathcal{P}$ form the reduced Gröbner basis of $I_\mathcal{P}$ with respect to the reverse lexicographical order induced by the following order of variables: $x_{32}>x_{33}>x_{34}>x_{31}>x_{22}>x_{23}>x_{24}>x_{12}>x_{13}>x_{14}>x_{42}>x_{43}>x_{41}>x_{52}>x_{53}$. We consider the vertex $v=(5,3)$. The link of v in $\Delta_\mathcal{P}$ is the cone of the vertex $(5,2)$ with the simplicial complex which we may associate to the collection of cells Q displayed in Figure 24 in a similar way to the one we used for stack polyominoes. The problem is that the collection Q is no longer a convex polyomino.

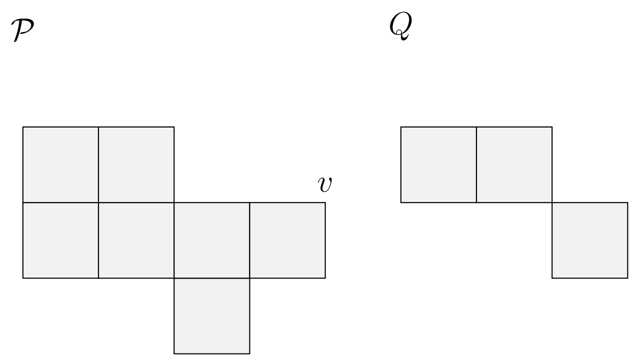


Figure 24:

Acknowledgements

The author wants to express her gratitude to the referee for the valuable suggestions and to Professor Viviana Ene for her support and guidance.

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