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Algebraic Representation of Correlation Functions in Integrable Spin Chains

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Dedicated to the memory of Daniel Arnaudon

Abstract. Taking the XXZ chain as the main example, we give a review of an algebraic representation of correlation functions in integrable spin chains obtained recently. We rewrite the previous formulas in a form which works equally well for the physically interesting homogeneous chains. We discuss also the case of quantum group invariant operators and generalization to the XYZ chain.

1. Introduction

The investigation of integrable spin chains has a long history since Bethe's work [1], in which the Bethe Ansatz method was invented. It was only a start, and later was followed by a line of new ideas and concepts such as commuting transfer matrices, the Yang-Baxter equation, the quantum inverse scattering method, quantum groups and the quantum KZ equation. In a series of papers [2, 3, 4, 5], we studied an algebraic formula for the correlation functions in the infinite XXX, XXZ and XYZ spin chains. Our method is a synthesis of those mentioned above.

The study of correlation functions has been a highlight in the researches of these spin chains. In the early days the only knowledge was the nearest neighbor correlator, written in terms of $\log 2$ for the XXX model. It was a big surprise when Takahashi [29] found $\zeta(3)$ in the next-nearest correlator, where $\zeta(s)$ is the Riemann zeta function. In [11, 12, 13], the quantum vertex operators in the representation theory of the quantum affine $\mathfrak{sl}_2$ algebra were used to obtain multiple integral formulas for the general correlation functions of the XXZ model. Kitanine, Maillet, Slavnov and Terras rederived and further generalized these integral formulas to include magnetic field and time [14, 15] (see [16] for a review). Study of the finite temperature case has also been launched recently by Göhmann, Klümper and Seel [10], and progress has been made in the calculation of long distance asymptotics of some correlators by the Lyon group [17] and Korepin, Lukyanov, Nishiyama and Shiroishi [19]. However it was not immediately understood why $\zeta(3)$ appears in the next-nearest correlators.

In [6, 7] Boos and Korepin explicitly performed the multiple integrals for the next-nearest case and beyond, and again found odd integer values of $\zeta(s)$. Further exact results including the XXZ chain have been obtained by Kato, Nishiyama, Sakai, Sato, Shiroishi and Takahashi [24, 25, 26, 27]. In [8], Boos, Korepin and Smirnov studied the inhomogeneous correlation functions for the XXX model, and arrived at a conjecture on the algebraic structure for the general correlation functions: in brief, one transcendental function is enough to describe all of them. In the limit of the homogeneous chain, the Taylor series expansion of this function produces the special values of $\zeta(s)$ as well as $\log 2$.

In [2, 3, 4], we proved the conjecture by giving an algebraic formula, and obtained similar results in the XXZ and XYZ models. The number of transcendental functions increases to two and three, respectively, as the number of parameters in the models increases. The main idea in the proof was the use of the reduced quantum KZ equation, and the main ingredient in the algebraic formula was the transfer matrix defined via an auxiliary space of non-integer dimensions.

The algebraic formulas presented in these papers had some deficiencies: the beauty of the formula was marred by a chip on the edge of a comb. The relevant transfer matrices are 'incomplete', in that they act on the tensor product where two spaces are omitted. Also the formula for the inhomogeneous model consists of a sum of terms, which have poles when one tries to take the homogeneous limit. They cancel each other only after the summation. In [5], these spots were cleaned up in the XXX model.

In the present paper, we give the algebraic formula for the density matrix in a transparent form, not only in the XXX model but also in the XXZ and the XYZ models. We use the infinite XXZ chain as the main object:

$$H_{\mathrm{XXZ}} = \frac{1}{2}\sum_j (\sigma_j^1\sigma_{j+1}^1 + \sigma_j^2\sigma_{j+1}^2 + \Delta\sigma_j^3\sigma_{j+1}^3). \tag{1.1}$$

The density matrix ρ_n belongs to the space $\mathrm{End}((\mathbb{C}^2)^{\otimes n})^*$ dual to the space of local operators $\mathrm{End}((\mathbb{C}^2)^{\otimes n})$. We consider the space $(\mathbb{C}^2)^{\otimes n}$ as the subchain of the entire infinite chain on which the XXZ Hamiltonian acts. It has the defining property

$$\rho_n(\mathcal{O}) = \langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle.$$

Here the right-hand side is the ground state average of the operator $\mathcal{O} \in \mathrm{End}((\mathbb{C}^2)^{\otimes n})$. We will give the formula for ρ_n in the form

$$\rho_n(\mathcal{O}) = \frac{1}{2^n}\,\mathrm{tr}_{(\mathbb{C}^2)^{\otimes n}}(e^{\Omega_n^*}\mathcal{O}), \tag{1.2}$$

where Ω_n^* is a nilpotent linear operator acting on $\mathrm{End}\big((\mathbb{C}^2)^{\otimes n}\big)$. The formula for Ω_n^* is given by a twofold integral:

$$\Omega_n^* = \frac{1}{2\kappa^2}\iint \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\,\mathrm{tr}_{\mathbb{C}^2\otimes\mathbb{C}^2}\Big(B(\mu_{1,2})(1\otimes\pi^{(1)}(\mathcal{T}_n^*(\mu_2)))(\pi^{(1)}(\mathcal{T}_n^*(\mu_1))\otimes 1)\Big) \\ \times(\omega_1(\mu_{1,2})\mathcal{X}_{1,n}^*(\mu_1,\mu_2)+\omega_2(\mu_{1,2})\mathcal{X}_{2,n}^*(\mu_1,\mu_2)). \quad (1.3)$$

Here, κ is a constant, $B(\mu_{1,2})$ is a 4×4 matrix depending on $\mu_{1,2}=\mu_1-\mu_2$, and ω_i $(i=1,2)$ are certain transcendental functions. The operator $\mathcal{T}_n^*(\mu)$ is given in terms of the L operator $L(\mu)\in U_q(\mathfrak{sl}_2)\otimes\mathrm{End}(\mathbb{C}^2)$ as the monodromy matrix in the adjoint action:

$$\mathcal{T}_n^*(\mu)(\mathcal{O}) = L_1(\mu)^{-1}\cdots L_n(\mu)^{-1}\mathcal{O}L_n(\mu)\cdots L_1(\mu)\in U_q(\mathfrak{sl}_2)\otimes\mathrm{End}((\mathbb{C}^2)^{\otimes n}).$$

The deformation parameter $q=e^{\pi i\nu}$ and the anisotropy parameter Δ in (1.1) are related as $\Delta=\frac{q+q^{-1}}{2}$. We denote the irreducible two-dimensional representation of $U_q(\mathfrak{sl}_2)$ by $\pi^{(1)}$.

The operators $\mathcal{X}_{i,n}^*(\mu_1,\mu_2)$ $(i=1,2)$ is obtained from the monodromy matrix

$$\mathrm{Tr}_{\mu_{1,2}}\,\mathcal{T}_n^*\left(\frac{\mu_1+\mu_2}{2}\right)=\mathcal{X}_{1,n}^*(\mu_1,\mu_2)-\mu_{1,2}\mathcal{X}_{2,n}^*(\mu_1,\mu_2),$$

where Tr_d denotes the trace functional to be defined in the text (see (3.10)).

Formula (1.3) is in the homogeneous case, and the integrand has poles at $\mu_i=0$ $(i=1,2)$. The integral means taking residues at these poles. A similar formula is also given in the inhomogeneous case where the Hamiltonian is replaced with the transfer matrix for the inhomogeneous six vertex model with the spectral parameters $\lambda_1,\dots,\lambda_n$ associated with the tensor components of $(\mathbb{C}^2)^{\otimes n}$. For the details, see Theorem 3.1 and (4.2)–(4.3). In this case, the integrand has poles at $\mu_i=\lambda_j$ $(i=1,2;\ j=1,\dots,n)$. Taking residues at these poles we get the formula obtained in the previous paper [3].

For a general local operator $\mathcal{O}$, we need two functions ω_i $(i=1,2)$ to express its expected value. However, if $\mathcal{O}$ is invariant under the action of $U_q(\mathfrak{sl}_2)$, the formula simplifies, and we need only ω_1. This case is related to the spin chain with an open boundary condition given by the Pasquier-Saleur Hamiltonian [22]. The XXZ Hamiltonian with periodic boundary condition corresponds to the CFT with the central charge $c=1$. In contrast, the Pasquier-Saleur Hamiltonian corresponds to the CFT with $c=1-6\nu^2/(1-\nu)$. The above property of the invariant operators was conjectured in [9]. We give a proof to this conjecture.

We also give a formula similar to (1.2), (1.3) for the XYZ model.

The paper is organized as follows. In Section 2, the density matrix is defined. In Section 3, an algebraic formula of the operator Ω_n, which is dual to Ω_n^*, is given. In Section 4, the algebraic formula is written in an alternative form. In Section 5, the formula for the invariant operators are given. In Section 6, the formula for the XYZ model is given.

The text is followed by three appendices. In Appendix A, we give the derivation of the new formula for Ω_n. In Appendix B, we make a comparison between different conventions used in this paper and in the book [12]. In Appendix C, formulas for the normalization factors are gathered for the XXZ and the XYZ models.

2. Density matrix for the XXZ chain

Consider the XXZ Hamiltonian

$$H_{\mathrm{XXZ}} = \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\sigma_k^1\sigma_{k+1}^1 + \sigma_k^2\sigma_{k+1}^2 + \Delta\sigma_k^3\sigma_{k+1}^3\right), \tag{2.1}$$

where σ^α $(\alpha = 1,2,3)$ are the Pauli matrices and

$$\Delta = \cos\pi\nu$$

is a real parameter. We consider the two regimes, the massive regime $\Delta > 1$, $\nu \in i\mathbb{R}_{>0}$, and the massless regime $|\Delta| < 1$, $0 < \nu < 1$.

Take a sub-interval of the lattice consisting of sites $1,\dots,n$, where n is a positive integer. Let $(E_{\epsilon,\bar\epsilon})_j$ denote the matrix unit $(\delta_{a\epsilon}\delta_{b\bar\epsilon})_{a,b=\pm}$ acting on the site j. By a density matrix, we mean the one whose entries are the ground state averages of products of the $(E_{\epsilon,\bar\epsilon})_j$'s,

$$\begin{aligned}&\rho_n(\lambda_1,\dots,\lambda_n)\\ &\quad= \sum_{\substack{\epsilon,\dots,\epsilon_n\\ \bar\epsilon_1,\dots,\bar\epsilon_n}} {}_{\lambda_1,\dots,\lambda_n}\langle\mathrm{vac}|(E_{\bar\epsilon_1,\epsilon_1})_1\cdots(E_{\bar\epsilon_n,\epsilon_n})_n|\mathrm{vac}\rangle_{\lambda_1,\dots,\lambda_n}(E_{\epsilon_1,\bar\epsilon_1})_1\cdots(E_{\epsilon_n,\bar\epsilon_n})_n.\end{aligned} \tag{2.2}$$

Here we consider the model with inhomogeneities $\lambda_1,\dots,\lambda_n$ attached to each site. More precisely, we mean the following.

Let $V = \mathbb{C}^2$ be the two-dimensional vector space with basis v_+, v_-. Throughout this paper, we set

$$q = e^{\pi i\nu}. \tag{2.3}$$

Denote the standard trigonometric R matrix by

$$R(\lambda) = \frac{\rho(\lambda)}{[\lambda+1]}\,r(\lambda), \tag{2.4}$$

$$r(\lambda) = \begin{pmatrix}[\lambda+1] & 0 & 0 & 0\\ 0 & [\lambda] & 1 & 0\\ 0 & 1 & [\lambda] & 0\\ 0 & 0 & 0 & [\lambda+1]\end{pmatrix} \in \mathrm{End}(V\otimes V).$$

Here the entries are arranged in the order $(++),(+-),(-+),(--)$, and

$$[\lambda] = \frac{q^\lambda - q^{-\lambda}}{q - q^{-1}}.$$

The factor $\rho(\lambda) = \rho(\lambda,2)$ will be given later (see (3.7) and (C.1), (C.2)). Introduce an auxiliary space $V_a \simeq V$ with spectral parameter λ, and denote by $R_{a,j}$ the

R matrix acting on $V_a \otimes V_j$. Using (2.4), we consider the transfer matrix of the inhomogeneous six vertex model

$$\mathrm{tr}_{V_a}\{R_{a,L}(\lambda-\lambda_L)\cdots R_{a,n}(\lambda-\lambda_n)\cdots R_{a,1}(\lambda-\lambda_1)\cdots R_{a,-L}(\lambda-\lambda_{-L})\} \tag{2.5}$$

which acts on the tensor product

$$V_{-L}\otimes\cdots\otimes V_1\otimes\cdots\otimes V_n\otimes\cdots\otimes V_L.$$

With each V_j we associate a spectral parameter λ_j, assuming for definiteness that $\lambda_j = 0$ for $j \le 0$ or $j \ge n+1$. Let $|\mathrm{vac}\rangle^{(L)}_{\lambda_1,\dots,\lambda_n}$ denote the eigenvector of (2.5) corresponding to the lowest eigenvalue. We denote the dual eigenvector by ${}_{\lambda_1,\dots,\lambda_n}^{\quad(L)}\langle\mathrm{vac}|$, normalized so that ${}_{\lambda_1,\dots,\lambda_n}^{\quad(L)}\langle\mathrm{vac}|\mathrm{vac}\rangle^{(L)}_{\lambda_1,\dots,\lambda_n} = 1$. The vacuum expectation value in (2.2) is defined to be the thermodynamic limit

$$\begin{aligned}&{}_{\lambda_1,\dots,\lambda_n}\langle\mathrm{vac}|(E_{\bar\epsilon_1,\epsilon_1})_1\cdots(E_{\bar\epsilon_n,\epsilon_n})_n|\mathrm{vac}\rangle_{\lambda_1,\dots,\lambda_n}\\&\qquad=\lim_{L\to\infty}{}_{\lambda_1,\dots,\lambda_n}^{\quad(L)}\langle\mathrm{vac}|(E_{\bar\epsilon_1,\epsilon_1})_1\cdots(E_{\bar\epsilon_n,\epsilon_n})_n|\mathrm{vac}\rangle^{(L)}_{\lambda_1,\dots,\lambda_n}.\end{aligned}$$

For an arbitrary local operator $\mathcal{O}\in\mathrm{End}(V^{\otimes n})$, we have

$${}_{\lambda_1,\dots,\lambda_n}\langle\mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle_{\lambda_1,\dots,\lambda_n} = \mathrm{tr}_{V^{\otimes n}}(\mathcal{O}\rho_n). \tag{2.6}$$

Our aim is to give an algebraic representation for the density matrix ρ_n.

3. Algebraic formula

The density matrix ρ_n is an operator on $V^{\otimes n}$. To present the result, let us pass from operators to vectors in $V^{\otimes 2n}$. We number the spaces as

$$V_1\otimes\cdots\otimes V_n\otimes V_{\bar n}\otimes\cdots\otimes V_{\bar 1}. \tag{3.1}$$

We use the following convention for the indices: for example, if $u=\sum u'\otimes u''$, $v=\sum v'\otimes v''$ are vectors in $V\otimes V$, then we write

$$u_{1,\bar 1}v_{\bar 2,2}=\sum u'\otimes v''\otimes v'\otimes u'' \quad \in V_1\otimes V_2\otimes V_{\bar 2}\otimes V_{\bar 1}.$$

Similarly, we indicate by suffix the tensor components on which operators act non-trivially.

Introduce a function h_n with values in (3.1) [1]:

$$\begin{aligned}h_n(\lambda_1,\dots,\lambda_n)=\sum\prod_{j=1}^{n}(-\bar\epsilon_j)\langle\mathrm{vac}|(E_{-\bar\epsilon_1,\epsilon_1})_1\cdots(E_{-\bar\epsilon_n,\epsilon_n})_n|\mathrm{vac}\rangle&\\ \times v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n}\otimes v_{\bar\epsilon_n}\otimes\cdots\otimes v_{\bar\epsilon_1}.&\end{aligned} \tag{3.2}$$

In Section 4, we discuss more about the transition from ρ_n to h_n, and vice versa. We mention here only that spectral parameters λ_j, λ_j+1 are attached to the spaces V_j and $V_{\bar j}$, respectively.

[1]There is an erratum in [3]; the right-hand side of the formula seven lines below (13.1) should read $\prod_{j=1}^{n}(-\bar\epsilon_j)\langle\mathrm{vac}|(E_{-\bar\epsilon_1,\epsilon_1})_1\cdots(E_{-\bar\epsilon_n,\epsilon_n})_n|\mathrm{vac}\rangle$.

The function h_n is known to satisfy the following system of equations:

$$h_n(\dots,\lambda_{k+1},\lambda_k,\dots) = \check{R}_{k,k+1}(\lambda_{k,k+1})\check{R}_{\overline{k+1},\overline{k}}(\lambda_{k+1,k})h_n(\dots,\lambda_k,\lambda_{k+1},\dots), \quad (3.3)$$

$$h_n(\lambda_1-1,\lambda_2,\dots,\lambda_n) = A_n(\lambda_1,\dots,\lambda_n)h_n(\lambda_1,\lambda_2,\dots,\lambda_n)\,, \quad (3.4)$$

$$\mathcal{P}^-_{1,\overline{1}}h_n(\lambda_1,\dots,\lambda_n) = \frac{1}{2}s_{1,\overline{1}}h_{n-1}(\lambda_2,\dots,\lambda_n)_{2,\dots,n,\overline{n},\dots,\overline{2}}\,, \quad (3.5)$$

$$\mathcal{P}^-_{n,\overline{n}}h_n(\lambda_1,\dots,\lambda_n) = \frac{1}{2}s_{n,\overline{n}}h_{n-1}(\lambda_1,\dots,\lambda_{n-1})_{1,\dots,n-1,\overline{n-1},\dots,\overline{1}}\,.$$

Here the notation is as follows. We set $\lambda_{i,j} = \lambda_i - \lambda_j$, $\check{R} = P\,R$ with P being the transposition,

$$\mathcal{P}^- = \frac{1}{2}(I-P)$$

is the projection onto $\mathbb{C}s$ where s denotes the vector

$$s = v_+ \otimes v_- - v_- \otimes v_+ \in V \otimes V, \quad (3.6)$$

and

$$A_n(\lambda_1,\dots,\lambda_n) = \\ (-1)^n P_{1,\overline{1}} R_{1,\bar{2}}(\lambda_{1,2}-1)\cdots R_{1,\bar{n}}(\lambda_{1,n}-1)R_{1,n}(\lambda_{1,n})\cdots R_{1,2}(\lambda_{1,2}).$$

We call (3.3)–(3.5) reduced qKZ (rqKZ) equations. We are going to construct a solution of these equations in a certain specific form. For that purpose, we will need three ingredients: monodromy matrix, trace functional Tr_λ, and transcendental functions $\omega_1(\lambda,\nu),\omega_2(\lambda,\nu)$. Let us explain them.

Let E,F,H be the standard generators of $U_q(\mathfrak{sl}_2)$. We consider the L operator

$$L(\lambda) = \frac{\rho(\lambda,d)}{[\lambda+\frac{d}{2}]}\ell(\lambda), \quad (3.7)$$

where

$$\ell(\lambda) = \begin{pmatrix} [\lambda+\frac{1+H}{2}] & Fq^{\frac{H-1}{2}} \\ q^{\frac{1-H}{2}}E & [\lambda+\frac{1-H}{2}] \end{pmatrix} \in U_q(\mathfrak{sl}_2)\otimes \mathrm{End}(V).$$

Here d is related to the central element of $U_q(\mathfrak{sl}_2)$

$$C = \frac{q^{-1+H}+q^{1-H}}{(q-q^{-1})^2} + EF$$

by $C = (q^d+q^{-d})/(q-q^{-1})^2$. In the formula for $\ell(\lambda)$, the first tensor component $U_q(\mathfrak{sl}_2)$ will be represented in the 'auxiliary space' of arbitrary dimension, while the two-dimensional space V in the second component will play the role of the 'quantum space'. The normalization factor $\rho(\lambda,d)$ is chosen to satisfy

$$L(\lambda)L(-\lambda) = 1\otimes I_V \qquad \text{(unitarity relation)},$$

$$\sigma^2 L(\lambda)^t\sigma^2 = -L(-1-\lambda) \qquad \text{(crossing symmetry)}.$$

Here $L(\lambda)^t$ is the transposed matrix with respect to $\mathrm{End}(V)$. For the explicit formula of $\rho(\lambda, d)$, see (C.1), (C.2). We have, in particular,

$$\frac{\rho(\lambda,d)}{[\lambda+\frac{d}{2}]}\frac{\rho(\lambda-1,d)}{[\lambda+\frac{d}{2}-1]}=-\frac{1}{[\lambda-\frac{d}{2}][\lambda+\frac{d}{2}]}. \tag{3.8}$$

We define the monodromy matrix $T_n(\lambda)=T_n(\lambda|\lambda_1,\dots,\lambda_n)$ by

$$T_n(\lambda)=L_{\bar{1}}(\lambda-\lambda_1-1)\cdots L_{\bar{n}}(\lambda-\lambda_n-1)L_n(\lambda-\lambda_n)\cdots L_1(\lambda-\lambda_1). \tag{3.9}$$

The trace functional Tr_λ is the composition map

$$\mathrm{Tr}_\lambda\colon U_q(\mathfrak{sl}_2)\to U_q(\mathfrak{sl}_2)/[U_q(\mathfrak{sl}_2),U_q(\mathfrak{sl}_2)]\to\lambda\mathbb{C}[\zeta,\zeta^{-1}]\oplus\mathbb{C}[\zeta,\zeta^{-1}], \tag{3.10}$$

where $\zeta=q^\lambda$. The first map is the canonical map, and the second is defined by setting for any $m\in\mathbb{Z}$

$$\mathrm{Tr}_\lambda(q^{mH})=\begin{cases}[m\lambda]/[m] & \text{if } m\neq 0;\\ \lambda & \text{if } m=0,\end{cases}$$

and for any $x\in U_q(\mathfrak{sl}_2)$

$$\mathrm{Tr}_\lambda(Cx)=\frac{q^\lambda+q^{-\lambda}}{(q-q^{-1})^2}\,\mathrm{Tr}_\lambda(x).$$

An equivalent way of defining $\mathrm{Tr}_\lambda\, x$ for $x\in U_q(\mathfrak{sl}_2)$ is as follows. It is the unique element of $\lambda\mathbb{C}[\zeta,\zeta^{-1}]\oplus\mathbb{C}[\zeta,\zeta^{-1}]$ such that, for all $(k+1)$-dimensional irreducible representation $\pi^{(k)}\colon U_q(\mathfrak{sl}_2)\to\mathrm{End}(\mathbb{C}^{k+1})$ we have

$$(\mathrm{Tr}_\lambda\, x)\,|_{\lambda=k+1}=\mathrm{tr}_{V^{(k)}}\pi^{(k)}(x)\qquad(k\in\mathbb{Z}_{\geq 0}).$$

With this definition of Tr_λ, the 'trace' of the monodromy matrix has a unique decomposition

$$\mathrm{Tr}_{\mu_{12}}\,T_n\left(\frac{\mu_1+\mu_2}{2}\right)=X_{1,n}(\mu_1,\mu_2|\lambda_1,\dots,\lambda_n)-\mu_{1,2}X_{2,n}(\mu_1,\mu_2|\lambda_1,\dots,\lambda_n), \tag{3.11}$$

where $X_{i,n}(\mu_1,\mu_2)=X_{i,n}(\mu_1,\mu_2|\lambda_1,\dots,\lambda_n)$ $(i=1,2)$ are matrices whose entries are rational functions in the variables $q^{\mu_1},q^{\mu_2},q^{\lambda_1},\dots,q^{\lambda_n}$. Note that, with the substitution $\lambda=\frac{\mu_1+\mu_2}{2}-\lambda_j, d=\mu_{1,2}$, the right-hand side of (3.8) becomes

$$-\frac{1}{[\mu_1-\lambda_j][\mu_2-\lambda_j]}\,. \tag{3.12}$$

Finally, define the functions $\omega_i(\lambda,\nu)$ $(i=1,2)$ [2] by

$$\nu\kappa\, d\log\varphi(\lambda,\nu)=\omega_1(\lambda,\nu)d(\lambda\nu)+\omega_2(\lambda,\nu)d\nu, \tag{3.13}$$

where

$$\varphi(\lambda,\nu)=\rho(\lambda)\left(\frac{[\lambda-1]}{[\lambda+1]}\right)^{1/4},\quad \kappa=\frac{\sin\pi\nu}{\pi\nu},$$

[2] Our ω_1, ω_2 here are denoted ω and $\tilde{\omega}$ in [3].

and ν is given in (2.3). The function $\rho(\lambda) = \rho(\lambda, 2)$ depends also on ν and is defined in (C.1), (C.2) in each regime.

We are now in a position to state the algebraic formula. Set

$$\mathbf{s}_n = \prod_{j=1}^{n} s_{j,\bar{j}}. \tag{3.14}$$

Theorem 3.1. *The following formula gives a solution of the rqKZ equations* (3.3)–(3.5)*:*

$$h_n(\lambda_1, \dots, \lambda_n) = \frac{1}{2^n} e^{\Omega_n(\lambda_1,\dots,\lambda_n)}\, \mathbf{s}_n, \tag{3.15}$$

where

$$\begin{aligned}\Omega_n(\lambda_1, \dots, \lambda_n) = \frac{(-1)^n}{2\kappa^2} \int\!\!\int \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i}\, &(\omega_1(\mu_{1,2}) X_{1,n}(\mu_1, \mu_2) \\ &+ \omega_2(\mu_{1,2}) X_{2,n}(\mu_1, \mu_2))\, \mathrm{Tr}_{2,2}\left(T_n(\mu_1) \otimes T_n(\mu_2) \cdot B(\mu_{1,2})\right)\end{aligned} \tag{3.16}$$

and

$$B(\mu) = \frac{1}{2}\frac{[\mu]}{[\mu-1][\mu+1]} \begin{pmatrix} 0 & & & \\ & q^{\mu}+q^{-\mu} & -q-q^{-1} & \\ & -q-q^{-1} & q^{\mu}+q^{-\mu} & \\ & & & 0 \end{pmatrix}. \tag{3.17}$$

Here $\omega_i(\lambda, \nu)$ *are given in* (3.13)*,* $X_{a,n}$ *are defined by* (3.11)*,* $\mathrm{Tr}_{2,2}(x \otimes y) = \mathrm{Tr}_2(x)\,\mathrm{Tr}_2(y)$*, and the matrix* $B(\mu)$ *acts on the auxiliary space* $\mathbb{C}^2 \otimes \mathbb{C}^2$*. The integral in* μ_i $(i = 1, 2)$ *means taking residues at the poles* $\mu_i = \lambda_1, \dots, \lambda_n$ *which result from* (3.12)*.*

In the massive regime, h_n *coincides with the vector form of the density matrix.*

Conjecturally the same formula gives also the density matrix in the massless regime as well.

In the above formula, we abbreviated all the variables ν, λ_1, etc., other than μ_1, μ_2. This formula was given earlier in [3] in a different form. Since the integrand has no pole at $\lambda_i = \lambda_j$, the present formula is equally valid in the homogeneous chain where $\lambda_1 = \cdots = \lambda_n = 0$. We show the equivalence of the two formulas in Appendix A.

There is another representation for the operator Ω_n using the 'invariant' trace

$$\mathrm{Tr}^q_{\lambda}(A) = \mathrm{Tr}_{\lambda}(q^{-H}A). \tag{3.18}$$

Define $X^q_{1,n}, X^q_{2,n}$ and $B^q(\mu)$ by

$$\mathrm{Tr}^q_{\mu_{12}}\, T_n\left(\frac{\mu_1+\mu_2}{2}\right) = X^q_{1,n}(\mu_1, \mu_2|\lambda_1, \dots, \lambda_n) - \mu_{1,2} X^q_{2,n}(\mu_1, \mu_2|\lambda_1, \dots, \lambda_n), \tag{3.19}$$

$$B^q(\mu) = \frac{[\mu]}{[\mu-1][\mu+1]} \begin{pmatrix} 0 & & & \\ & q & -q^{-\mu} & \\ & -q^{\mu} & q^{-1} & \\ & & & 0 \end{pmatrix}. \tag{3.20}$$

Then Ω_n can also be written as (3.16), with $X_{a,n}(\mu_1,\mu_2)$ and $B(\mu)$ replaced by $X^q_{a,n}(\mu_1,\mu_2)$ and $B^q(\mu)$, respectively.

The existence of this second representation is a peculiar feature of the XXZ model which has analogs neither in the XXX nor in the XYZ models.

Remark 3.2. *The choice of the operator $B(\mu)$ in* (3.17) *is not unique. For example, there is a freedom of adding identity to $B(\mu)$ (see Lemma* A.5*).*

4. An alternative representation

In this section we return from vectors in $V^{\otimes 2n}$ to operators on $V^{\otimes n}$. Let us recall some generalities concerning the action of quantum groups on these spaces.

Denote by $\pi_\lambda\colon U'_q(\widehat{\mathfrak{sl}}_2)\to U_q(\mathfrak{sl}_2)$ the evaluation homomorphism

$$\pi_\lambda(e_0)=q^\lambda F,\quad \pi_\lambda(f_0)=q^{-\lambda}E,\quad \pi_\lambda(q^{\pm h_0/2})=q^{\mp H/2},$$
$$\pi_\lambda(e_1)=q^\lambda E,\quad \pi_\lambda(f_1)=q^{-\lambda}F,\quad \pi_\lambda(q^{\pm h_1/2})=q^{\pm H/2}.$$

For the representation $\pi^{(k)}$ of $U_q(\mathfrak{sl}_2)$, we set $\pi^{(k)}_\lambda=\pi^{(k)}\circ\pi_\lambda$. We use the coproduct

$$\Delta(e_i)=e_i\otimes 1+q^{h_i}\otimes e_i,$$
$$\Delta(f_i)=f_i\otimes q^{-h_i}+1\otimes f_i$$

to define the action of $U'_q(\widehat{\mathfrak{sl}}_2)$ on a tensor product of representations.

Quite generally, for a finite dimensional $U'_q(\widehat{\mathfrak{sl}}_2)$ module W, its dual vector space W^* has two module structures defined via the antipode S as

$$\langle xu,v\rangle=\langle u,S^{\pm1}(x)v\rangle\qquad (x\in U_q(\widehat{\mathfrak{sl}}_2),u\in W^*,v\in W).$$

Denote these structures by $W^{*S^{\pm1}}$. We have canonical isomorphisms

$$(W^{*\phi})^{*\phi^{-1}}\simeq W,\quad (W_1\otimes W_2)^{*\phi}\simeq W_2^{*\phi}\otimes W_1^{*\phi}$$

for $\phi=S^{\pm1}$. The canonical pairing $W^{*S}\otimes W\to\mathbb{C}$ is $U'_q(\widehat{\mathfrak{sl}}_2)$-linear. We regard $\mathrm{End}(W)$ as a $U'_q(\widehat{\mathfrak{sl}}_2)$-module via

$$\mathrm{End}(W)\simeq W\otimes W^{*S}.$$

Using the trace $\mathrm{tr}_W(AB)$, $\mathrm{End}(W)$ may be identified with its dual space. The induced dual module structure becomes

$$\mathrm{End}(W)^{*S^{-1}}\simeq (W\otimes W^{*S})^{*S^{-1}}\simeq W\otimes W^{*S^{-1}}.$$

We are mainly concerned with the 2-dimensional module V where the generators E,F,H act in the basis v_+,v_- as

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad F=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Use the letter $V(\lambda)$ to indicate the evaluation module structure $\pi_\lambda^{(1)}$ on V. We have then an isomorphism of $U_q'(\widehat{\mathfrak{sl}}_2)$-modules

$$V(\lambda)^{*S^{\pm 1}} \simeq V(\lambda \mp 1), \quad v_\epsilon^* \mapsto \epsilon\, v_{-\epsilon},$$

where $\langle v_\epsilon^*, v_{\epsilon'} \rangle = \delta_{\epsilon,\epsilon'}$. In particular, the identity operator $I_V \in \mathrm{End}(V)$ corresponds to $s \in V \otimes V$ given in (3.6).

We started from the tensor product

$$\mathcal{S}_n = V_1 \otimes \cdots \otimes V_n, \qquad V_j = V(\lambda_j),$$

corresponding to the finite interval $1, \ldots, n$ on the lattice. Our space of local operators is the $U_q'(\widehat{\mathfrak{sl}}_2)$-module

$$\begin{aligned} \mathcal{L}_n &= \mathrm{End}(\mathcal{S}_n) \\ &\simeq V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_n - 1) \otimes \cdots \otimes V(\lambda_1 - 1) \end{aligned} \tag{4.1}$$

on which $x \in U_q'(\widehat{\mathfrak{sl}}_2)$ operates by the adjoint action

$$\mathrm{ad}\, x\big(\mathcal{O}\big) = \sum x_i' \mathcal{O} S(x_i'') \qquad (\mathcal{O} \in \mathcal{L}_n),$$

where $\Delta(x) = \sum x_i' \otimes x_i''$. In contrast, density matrix belongs to the dual module

$$\begin{aligned} \mathcal{L}_n^* &= \mathrm{End}(\mathcal{S}_n)^{*S^{-1}} \\ &\simeq V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_n + 1) \otimes \cdots \otimes V(\lambda_1 + 1). \end{aligned}$$

The action of $x \in U_q'(\widehat{\mathfrak{sl}}_2)$ is, in the same notation as above,

$$\mathrm{ad}'\, x\big(\mathcal{O}^*\big) = \sum x_i'' \mathcal{O}^* S^{-1}(x_i') \qquad (\mathcal{O}^* \in \mathcal{L}_n^*).$$

The vector h_n (3.2) is nothing but the image of the density matrix ρ_n under the latter identification.

In passing we note that, for a Hopf subalgebra U of $U_q'(\widehat{\mathfrak{sl}}_2)$, $A \in \mathrm{End}(W)$ belongs to the trivial representation (i.e., $\mathrm{ad}\, x\big(A\big) = \epsilon(x)A$ for all $x \in U$ where ϵ is the co-unit) if and only if $x \cdot A = A \cdot x$ for all $x \in U$. In this case we say A is invariant under U.

With this preparation, let us rewrite our main formula in the matrix formulation. Suppose $\mathcal{O}^* \in \mathcal{L}_n^*$ and $v \in V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_n + 1) \otimes \cdots \otimes V(\lambda_1 + 1)$ are identified. Then the action $v \mapsto L_i(\mu) v$ is translated to the left multiplication

$$\mathcal{O}^* \mapsto L_i(\mu) \mathcal{O}^*,$$

while $v \mapsto L_{\bar{i}}(\mu - 1) v$ is translated to

$$\mathcal{O}^* \mapsto -\mathcal{O}^* L_i(\mu)^{-1}.$$

In view of the cyclicity of the trace, the action of the 'transfer matrix' (3.11) on v turns into

$$(-1)^n \, \mathrm{Tr}_{\mu_{12}} \, \mathcal{T}_n(\mu)(\mathcal{O}^*),$$

where $\mu = (\mu_1 + \mu_2)/2$ and

$$\mathcal{T}_n(\mu)(\mathcal{O}^*) = L_n(\mu - \lambda_n) \cdots L_1(\mu - \lambda_1) \cdot \mathcal{O}^* \cdot L_1(\mu - \lambda_1)^{-1} \cdots L_n(\mu - \lambda_n)^{-1}.$$

Notice that in this formula the normalization factor of the L operator cancels out. Regard the operator Ω_n as acting on $\mathcal{L}_n^*$ via the above formula. Then the density matrix can be written as

$$\rho_n = \frac{1}{2^n} e^{\Omega_n}(I),$$

where I is the identity operator.

Similarly, denote by Ω_n^* the operator corresponding to Ω_n, acting on the space of local operators $\mathcal{L}_n$. Then the main formula (3.15) can be rewritten as

$$\langle \text{vac}|\mathcal{O}|\text{vac}\rangle = \frac{1}{2^n} \operatorname{tr}_{V^{\otimes n}} \left(e^{\Omega_n^*}(\mathcal{O}) \right), \tag{4.2}$$

where

$$\begin{aligned} \Omega_n^*(\lambda_1, \ldots, \lambda_n) &= \frac{1}{2\kappa^2} \iint \frac{d\mu_1}{2\pi i} \frac{d\mu_2}{2\pi i} \operatorname{Tr}_{2,2} \left(B(\mu_{1,2})(I \otimes \mathcal{T}_n^*(\mu_2))(\mathcal{T}_n^*(\mu_1) \otimes I) \right) \\ &\times \left(\omega_1(\mu_{12}) \mathcal{X}_{1,n}^*(\mu_1, \mu_2|\lambda_1, \ldots, \lambda_n) + \omega_2(\mu_{12}) \mathcal{X}_{2,n}^*(\mu_1, \mu_2|\lambda_1, \ldots, \lambda_n) \right), \end{aligned} \tag{4.3}$$

with

$$\mathcal{T}_n^*(\mu)(\mathcal{O}) = L_1(\mu - \lambda_1)^{-1} \cdots L_n(\mu - \lambda_n)^{-1} \ \mathcal{O} \ L_n(\mu - \lambda_n) \cdots L_1(\mu - \lambda_1),$$

and $\mathcal{X}_{1,n}^*$, $\mathcal{X}_{2,n}^*$ are constructed from $\mathcal{T}_n^*(\mu)$ as before.

Let us discuss the last formula briefly. Consider the operators $\mathcal{O}$ which act as identity either on the last or on the first site (i.e., $\mathcal{O} = \mathcal{O}' \otimes I$ or $\mathcal{O} = I \otimes \mathcal{O}'$). For such operators, we have

$$\Omega_n^*(I \otimes \mathcal{O}') = I \otimes \Omega_{n-1}^*(\mathcal{O}'), \tag{4.4}$$

$$\Omega_n^*(\mathcal{O}' \otimes I) = \Omega_{n-1}^*(\mathcal{O}') \otimes I. \tag{4.5}$$

Eq. (4.5) is obvious from the definition, while (4.4) is non-trivial and follows from (A.10). This motivates us to consider a 'universal' operator

$$\begin{aligned} &\mathcal{T}^*(\mu)(\mathcal{O}) \\ &= \lim_{N\to\infty} L_{-N}(\mu - \lambda_{-N})^{-1} \cdots L_N(\mu - \lambda_N)^{-1} \ \mathcal{O} \ L_N(\mu - \lambda_N) \cdots L_{-N}(\mu - \lambda_{-N}). \end{aligned}$$

Introducing further the normalized trace

$$\mathbf{tr} = \lim_{N\to\infty} \left(\frac{1}{2} \operatorname{tr}_{V_{-N}} \cdots \frac{1}{2} \operatorname{tr}_{V_N} \right),$$

and defining Ω^* using $\mathcal{T}^*(\mu)$, we can write down a universal formula

$$\langle \text{vac}|\mathcal{O}|\text{vac}\rangle = \mathbf{tr} \left(e^{\Omega^*}(\mathcal{O}) \right), \tag{4.6}$$

which does not refer to the size n of the subsystem. For any operator acting on a finite sublattice, the right-hand side reduces automatically to this sublattice due to (4.4)–(4.5). Consider the action of the universal $\mathcal{T}^*(\mu)$ on a given operator $\mathcal{O}$. If $\mathcal{O}$ acts on sites $1, \ldots, n$, then the infinite right tail of L-operators L_j with $j > n$ cancels. However, the left tail with $j < 1$ remains. In integrable quantum field theory, this situation is typical for the action of non-local charges which transform local operators into non-local ones with an infinite tail in one direction [20].

Nevertheless, a beautiful feature of our construction is that, when we substitute $\mathcal{T}^*(\mu)$ into the trace and integrate, we obtain the operator Ω^* which sends a local operator to a local one because of (4.5).

In our opinion, this is the most important property of our construction which deserves further understanding.

5. Invariant operators

As we have seen in the previous section, $U'_q(\widehat{\mathfrak{sl}}_2)$ operates on our space of local operators (4.1). The algebra $U'_q(\widehat{\mathfrak{sl}}_2)$ contains two subalgebras isomorphic to $U_q(\mathfrak{sl}_2)$, one generated by $e_0, f_0, q^{\pm h_0/2}$ and the other by $e_1, f_1, q^{\pm h_1/2}$. In this section, we consider the subspace of local operators which are invariant under one of these subalgebras, and show that their correlation functions do not contain the transcendental function $\omega_2(\lambda)$. To fix the idea, let us choose the subalgebra generated by $e_0, f_0, q^{\pm h_0/2}$ and set

$$\mathcal{L}_n^{\mathrm{inv}} = \{\mathcal{O} \in \mathcal{L}_n \mid x \cdot \mathcal{O} = \mathcal{O} \cdot x \quad (x = e_0, f_0, q^{h_0/2})\}.$$

In the present context, it is more convenient to use the formula for Ω_n^* using the invariant trace (see the end of Section 3),

$$\begin{aligned}\Omega_n^*(\lambda_1, \ldots, \lambda_n) &= \frac{1}{2\kappa^2} \iint \frac{d\mu_1}{2\pi i} \frac{d\mu_2}{2\pi i} \operatorname{Tr}_{2,2} \left(B^q(\mu_{1,2})(I \otimes \mathcal{T}_n^*(\mu_2))(\mathcal{T}_n^*(\mu_1) \otimes I)\right) \\ &\quad \times \left(\omega_1(\mu_{12})\mathcal{X}_{1,n}^{q*}(\mu_1, \mu_2|\lambda_1, \ldots, \lambda_n) + \omega_2(\mu_{12})\mathcal{X}_{2,n}^{q*}(\mu_1, \mu_2|\lambda_1, \ldots, \lambda_n)\right). \quad (5.1)\end{aligned}$$

As before, $\mathcal{X}_{1,n}^{q*}$, $\mathcal{X}_{2,n}^{q*}$ are constructed from $\mathcal{T}_n^*(\mu)$ using Tr_μ^q.

Lemma 5.1. *The space $\mathcal{L}_n^{\mathrm{inv}}$ is invariant under the operator $\Omega_n^*(\lambda_1, \ldots, \lambda_n)$.*

Proof. First we show that $\operatorname{Tr}_\mu^q \mathcal{T}_n^*(\mu)$ preserves the space $\mathcal{L}_n^{\mathrm{inv}}$. Abbreviating arguments, we write

$$\mathcal{T}_n^*(\mathcal{O}) = T^{-1}(1 \otimes \mathcal{O})T,$$

where $T = L_n \cdots L_1$, $L_j = L_j(\mu - \lambda_j)$ and $\mu = (\mu_1 + \mu_2)/2$.

Until the end of the proof, we let x stand for $e_0, f_0, q^{h_0/2}$. The operator T belongs to $U_q(sl_2) \otimes \operatorname{End}(W)$ where $W = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$. It satisfies the intertwining property

$$\sum T(x'_i \otimes x''_i) = \sum (x''_i \otimes x'_i)T\,,$$

where $\Delta(x) = \sum x'_i \otimes x''_i$. This equation can be rewritten as

$$T(1 \otimes x) = \sum (x'''_i \otimes x''_i)T(S^{-1}(x'_i) \otimes 1),$$

where we have set $(\Delta \otimes I) \circ \Delta(x) = \sum x'_i \otimes x''_i \otimes x'''_i$. Using the invariance of $\mathcal{O}$ and the intertwining property again, we obtain, in the notation above,

$$T^{-1}(1 \otimes \mathcal{O})T(1 \otimes x) = \sum (x''_i \otimes x'''_i)T^{-1}(1 \otimes \mathcal{O})T(S^{-1}(x'_i) \otimes 1). \quad (5.2)$$

We have $q^{-h_0}xq^{h_0} = S^2(x)$ and $\pi_\lambda(q^{-h_0}) = q^H$, from which follows the invariance property

$$\mathrm{Tr}^q_\lambda(\mathrm{ad}'\,x\,A) = \epsilon(x)\,\mathrm{Tr}^q_\lambda(A)\,.$$

Taking Tr^q_μ of both sides of (5.2) and using the above invariance, we find that

$$x \cdot \mathrm{Tr}^q_\mu\,\mathcal{T}^*(\mathcal{O}) = \mathrm{Tr}^q_\mu\,\mathcal{T}^*(\mathcal{O}) \cdot x.$$

Let us show that the operator

$$\mathrm{Tr}_{2,2}\left(B^q(\mu_{1,2})(I \otimes \mathcal{T}^*_n(\mu_2))(\mathcal{T}^*_n(\mu_1) \otimes I)\right) \tag{5.3}$$

also preserves $\mathcal{L}^{\mathrm{inv}}_n$. The relevant operators act on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes W$. To unburden the notation, let us write $T_{12} = (T(\mu_1)\otimes I)(I\otimes T(\mu_2))$, $B^q_{12} = B^q(\mu_{12})$ and $\mathcal{O}_3 = \mathcal{O}$, indicating the tensor components by the suffix. Thus the action of (5.3) on $\mathcal{O}$ is $\mathrm{Tr}_{2,2}\left(B^q_{12}T^{-1}_{12}\mathcal{O}_3T_{12}\right)$.

Writing again $\Delta(x) = \sum x'_i \otimes x''_i$, we have

$$\sum (x'_i)_1(x''_i)_2B^q_{12} = \sum B^q_{12}(x'_i)_1(x''_i)_2 = \epsilon(x)B^q_{1,2}. \tag{5.4}$$

Using (5.4) together with the intertwining property of T and the invariance of $\mathcal{O}$, we find

$$B^q_{1,2}T^{-1}_{12}\mathcal{O}_3T_{12}x_3 = B^q_{1,2}(x''_i)_3T^{-1}_{12}\mathcal{O}_3T_{12}\Delta(S^{-1}(x'_i))_{12}.$$

Taking trace, moving the last factor by cyclicity and using (5.4) again, we obtain

$$\begin{aligned}
&\mathrm{Tr}_{2,2}\left(B^q(\mu_{1,2})(I \otimes \mathcal{T}^*_n(\mu_2))(\mathcal{T}^*_n(\mu_1) \otimes I)\right)(\mathcal{O}) \cdot x \\
&\qquad = \sum \mathrm{Tr}_{2,2}\big(\Delta(S^{-1}(x'_i))_{12}(x''_i)_3B^q_{12}T^{-1}_{12}\mathcal{O}_3T_{12}\big) \\
&\qquad = x \cdot \mathrm{Tr}_{2,2}\left(B^q(\mu_{1,2})(I \otimes \mathcal{T}^*_n(\mu_2))(\mathcal{T}^*_n(\mu_1) \otimes I)\right)(\mathcal{O}),
\end{aligned}$$

which was to be shown. □

Lemma 5.2. *For an invariant operator $\mathcal{O} \in \mathcal{L}^{\mathrm{inv}}_n$, we have $\mathcal{X}^{q*}_{2,n}(\mu_1,\mu_2)(\mathcal{O}) = 0$.*

Proof. Lemma means that the trace

$$\begin{aligned}
&\mathrm{Tr}^q_{\mu_{12}}(L_1(\mu-\lambda_1)^{-1}\cdots L_n(\mu-\lambda_n)^{-1}\mathcal{O}L_n(\mu-\lambda_n)\cdots L_1(\mu-\lambda_1)) \\
&\qquad = \mathrm{Tr}^q_{\mu_{12}}(L_1(\lambda_1-\mu)\cdots L_n(\lambda_n-\mu)\mathcal{O}L_n(\lambda_n-\mu)^{-1}\cdots L_1(\lambda_1-\mu)^{-1})
\end{aligned}$$

does not produce terms proportional to μ_{12}. We prove this assertion by passing to the vector language and performing a gauge transformation.

Under the isomorphism (4.1), an invariant operator $\mathcal{O}$ is sent to a vector $v \in V(\lambda_1) \otimes \cdots \otimes V(\lambda_n) \otimes V(\lambda_n - 1) \otimes \cdots \otimes V(\lambda_1 - 1)$ invariant under the action of $e_0, f_0, q^{h_0/2}$. Set $g = q^{(1/2)\sum_{j=1}^n(\lambda_j\sigma^3_j + (\lambda_j-1)\sigma^3_{\bar{j}})}$, and introduce the gauge transformation

$$\begin{aligned}
&e = gf_0g^{-1}, \quad f = ge_0g^{-1}, \quad q^{h/2} = q^{-h_0/2}, \\
&\ell'(\lambda) = q^{(\lambda/2)\sigma^3}\ell(\lambda)q^{-(\lambda/2)\sigma^3}.
\end{aligned}$$

Then $v' = gv$ belongs to the subspace $\left(V^{\otimes 2n}\right)^{\mathrm{inv}}$ of vectors invariant under $U_q(\mathfrak{sl}_2)$ generated by $e, f, q^{h/2}$.

For the proof, we show the following slightly more general statement: for any $v' \in \left(V^{\otimes 2n}\right)^{\mathrm{inv}}$ and $\lambda_1, \dots, \lambda_{2n}$,

$$\mathrm{Tr}^q_\mu \left(\ell'_1(\lambda_1) \cdots \ell'_{2n}(\lambda_{2n})\right) v'$$

belongs to $\mathbb{C}[q^{\pm\mu}, q^{\pm\lambda_1}, \dots, q^{\pm\lambda_{2n}}]$.

First consider the case $n = 1$, choosing v' to be $s^q = qv_+ \otimes v_- - v_- \otimes v_+ \in (V^{\otimes 2})^{\mathrm{inv}}$. Note that

$$\mathrm{Tr}^q_\mu(FEA) = \mathrm{Tr}^q_\mu \left(\left[\frac{\mu+1+H}{2}\right] \left[\frac{\mu-1-H}{2}\right] A \right),$$
$$\mathrm{Tr}^q_\mu(EFA) = \mathrm{Tr}^q_\mu \left(\left[\frac{\mu+1-H}{2}\right] \left[\frac{\mu-1+H}{2}\right] A \right).$$

Direct computation using these relations shows that the entries of $\ell'_1(\lambda_1)\ell'_2(\lambda_2)s^q_{12}$ can be reduced to elements of the subalgebra generated by $q^{-H}, Eq^{-H/2}, Fq^{-H/2}$. Since q^H does not appear, Tr^q_μ does not produce terms proportional to μ.

In the general case, the same argument shows that the assertion holds for $v' = \mathbf{s}^q = s^q_{12} \cdots s^q_{2n-1\,2n}$. From the Yang-Baxter relation, the same is true for vectors obtained from $\mathbf{s}^q$ by acting with an arbitrary number of matrices $\check{r}'_{i\,i+1}(\lambda_{i\,i+1})$, where

$$\check{r}'_{i\,i+1}(\lambda) = P_{i\,i+1} q^{\lambda\sigma_i/2} r_{i\,i+1}(\lambda) q^{-\lambda\sigma_i/2}.$$

The operators $\check{r}'_{i\,i+1}(\lambda)$ are linear combinations of 1 and the generators $e_i = \check{r}'_{i\,i+1}(-1)$ $(i = 1, \dots, 2n-1)$ of the Temperley-Lieb algebra, and vice versa. It is well known that the space $\left(V^{\otimes 2n}\right)^{\mathrm{inv}}$ is generated from $\mathbf{s}^q$ by the action of the Temperley-Lieb algebra. Hence the assertion is true for all $v' \in \left(V^{\otimes 2n}\right)^{\mathrm{inv}}$. $\square$

In summary, let us present the final result for invariant operators. In the notation of Section 4, we have

Theorem 5.3. *Consider a local operator $\mathcal{O}$ invariant under the action of $U_q(\mathfrak{sl}_2)$ generated by $e_0, f_0, q^{h_0/2}$. For such an operator, we have*

$$\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle = \mathbf{tr}\left(e^{\Omega^*_{\mathrm{inv}}}(\mathcal{O})\right) \tag{5.5}$$

where

$$\Omega^*_{\mathrm{inv}} = \frac{1}{2\kappa^2} \iint \frac{d\mu_1}{2\pi i}\frac{d\mu_2}{2\pi i} \times \omega_1(\mu_{12})\, \mathrm{Tr}_{2,2}\left(B^q(\mu_{1,2}) \mathcal{T}^*_n(\mu_2) \otimes \mathcal{T}^*_n(\mu_1)\right) \mathrm{Tr}^q_{\mu_{1,2}}\, \mathcal{T}^*_n\left(\frac{\mu_1+\mu_2}{2}\right). \tag{5.6}$$

Formula (5.6) is quite similar to the one in the XXX case [5].

Various methods are known for constructing bases of invariant operators (see, e.g., [21]). For example, a basis for $n = 3$ is I, U_{12}, U_{23}, $U_{12}U_{23}$, $U_{23}U_{12}$. Here the operators $U_{i\,i+1}$, after the gauge transformation $U'_{i\,i+1} = g' U_{i\,i+1} g'^{-1}$ by

$g' = q^{(1/2)\sum_{j=1}^{n}\lambda_j\sigma_j^3}$, are the (negative of the) generators of the Temperley-Lieb algebra,

$$U'_{i\,i+1} = \frac{1}{2}\left(\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 + \frac{q+q^{-1}}{2}\left(\sigma_i^3\sigma_{i+1}^3 - 1\right) + \frac{q-q^{-1}}{2}\left(-\sigma_i^3 + \sigma_{i+1}^3\right)\right).$$

(It is nothing but the local density of the Pasquier-Saleur Hamiltonian, see below.) In the homogeneous case, their expected values are

$$\langle\mathrm{vac}|U_{i,i+1}|\mathrm{vac}\rangle = -\kappa\, a_0,$$

$$\langle\mathrm{vac}|U_{12}U_{23}|\mathrm{vac}\rangle = \langle\mathrm{vac}|U_{23}U_{12}|\mathrm{vac}\rangle = -\frac{1}{\cos\pi\nu}(\kappa a_0 - 3\kappa^3 a_2),$$

where $\kappa = \sin\pi\nu/(\pi\nu)$ and

$$a_m = \int_{-\infty}^{\infty} t^m \frac{\sinh(1/\nu - 1)t}{\sinh(t/\nu)\cosh t}\,dt.$$

Let us explain the physical meaning of correlation functions of invariant operators. We shall consider the case of massless regime $q = e^{\pi i\nu}$ $(0 < \nu < 1)$, because in the present context it is more interesting from the point of view of physics. It is well known that in the continuous limit the massless XXZ model is described by CFT with the central charge $c = 1$. In the continuous field theory, starting with CFT with $c = 1$ one can obtain CFT with $c = 1 - \frac{6\nu^2}{1-\nu}$ by modifying the energy-momentum tensor and introducing screening operators. There is a construction which gives a lattice version of this procedure, and which is closely related to the invariance under the quantum group.

Consider first the model in the finite volume. The XXZ Hamiltonian with the usual periodic boundary condition

$$H_L = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^1\sigma_{j+1}^1 + \sigma_j^2\sigma_{j+1}^2 + \cos(\pi\nu)\sigma_j^3\sigma_{j+1}^3\right), \quad \sigma_{L+1}^a = \sigma_1^a,$$

is not invariant under the quantum group. However, following Pasquier and Saleur [22] one can introduce an invariant Hamiltonian with specific boundary conditions:

$$H_L^{\mathrm{inv}} = \frac{1}{2}\sum_{j=1}^{L-1}\left(\sigma_j^1\sigma_{j+1}^1 + \sigma_j^2\sigma_{j+1}^2 + \cos(\pi\nu)\sigma_j^3\sigma_{j+1}^3\right) + \frac{1}{2}i\sin(\pi\nu)(\sigma_L^3 - \sigma_1^3).$$

In the infinite volume limit, this Hamiltonian has CFT with $c = 1 - \frac{6\nu^2}{1-\nu}$ as the continuous limit. In the finite volume there are significant differences between H_L and H_L^{inv}; for example, Bethe Ansatz equations are very different. However, following the general logic (see, for example, [23]) we believe that in the infinite volume the ground state of the invariant model is obtained by projection of the original ground state onto the invariant subspace:

$$|\mathrm{vac}\rangle_{\mathrm{inv}} = \mathcal{P}_{\mathrm{inv}}|\mathrm{vac}\rangle,$$

where $\mathcal{P}_{\text{inv}}$ denotes the projection operator. Then the correlation function of any operator in the invariant model coincides with that of an invariant operator in the original model:

$$_{\text{inv}}\langle \text{vac}|\mathcal{O}|\text{vac}\rangle_{\text{inv}} = \langle \text{vac}|\left(\mathcal{P}_{\text{inv}}\mathcal{O}\mathcal{P}_{\text{inv}}\right)|\text{vac}\rangle.$$

So, the correlation functions considered in this section describe the lattice version of CFT with $c = 1 - \frac{6\nu^2}{1-\nu}$. The fact that they can be expressed in terms of a single transcendental function was predicted in [8]. Certainly, the most interesting question is that of rational ν when the space of local operators is restricted. We shall consider this situation in the future.

6. XYZ model

A considerable part of the previous sections can be generalized to the case of the XYZ chain

$$H_{\text{XYZ}} = \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(I_1\sigma_k^1\sigma_{k+1}^1 + I_2\sigma_k^2\sigma_{k+1}^2 + I_3\sigma_k^3\sigma_{k+1}^3\right). \tag{6.1}$$

In [4] we put forward a conjectural formula for the density matrix in the elliptic setting. In comparison with the XXX and XXZ chains, however, the results obtained are incomplete: we have so far been unable to verify that the expression written down in [4] satisfies the reduced qKZ equation. In this section, we rewrite the formula conjectured in [4] into a form close to (3.15)–(3.16).

In the following, we denote by $\theta_a(t)$ $(a = 1, 2, 3, 4)$ the Jacobi elliptic theta function with modulus τ. We fix a generic complex number η and use the scaled spectral parameter $t = \lambda\eta$. We deal with functions which have period 1 in the variable t. The parameters in the Hamiltonian (6.1) are given by $(1/2)I^a = \theta_{a+1}(2\eta)/\theta_{a+1}(0)$.

In the XYZ chain, the role of $U_q(\mathfrak{sl}_2)$ in the XXZ chain is played by the Sklyanin algebra [28]. It is an associative algebra $\mathcal{A}$ generated by four symbols S_α $(\alpha = 0, 1, 2, 3)$ and the defining quadratic relations

$$[S_0, S_a] = iJ_{bc}(S_bS_c + S_cS_b), \tag{6.2}$$

$$[S_b, S_c] = i(S_0S_a + S_aS_0). \tag{6.3}$$

Here (a, b, c) runs over cyclic permutations of $(1, 2, 3)$. The structure constants $J_{bc} = -(J_b - J_c)/J_a$ are parametrized as

$$J_a = \frac{\theta_{a+1}(2\eta)\theta_{a+1}(0)}{\theta_{a+1}(\eta)^2}.$$

The algebra $\mathcal{A}$ has two basic central elements, $K_0 = \sum_{\alpha=0}^{3} S_\alpha^2$ and $K_2 = \sum_{a=1}^{3} J_aS_a^2$. We will consider only representations of $\mathcal{A}$ on which these central elements act as scalars,

$$K_0 = 4\left[\frac{d}{2}\right]^2, \quad K_2 = 4\left[\frac{d+1}{2}\right]\left[\frac{d-1}{2}\right]. \tag{6.4}$$

Here and after we set

$$[t]=\frac{\theta_1(2t)}{\theta_1(2\eta)}.$$

The parameter d plays the role of the 'dimension'. For each non-negative integer k, the Sklyanin algebra possesses an analog $\pi^{(k)}$ of the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$, on which the above relations are valid with $d=k+1$.

The L operator associated with the XYZ chain has the following form.

$$L(t):=\frac{\rho(t,d)}{[t+(d/2)\eta]}\,\ell(t)\,,$$
$$\ell(t)=\frac{1}{2}\sum_{\alpha=0}^{3}\frac{\theta_{\alpha+1}(2t+\eta)}{\theta_{\alpha+1}(\eta)}S_\alpha\otimes\sigma^\alpha\,.$$

Here $\rho(t,d)$ is a normalization factor (see (C.3)) which satisfies

$$\rho(t,d)\rho(-t,d)=1,$$
$$\frac{\rho(t,d)}{[t+(d/2)\eta]}\frac{\rho(t-\eta,d)}{[t-\eta+(d/2)\eta]}=\frac{1}{[(d/2)\eta-t][(d/2)\eta+t]}.$$

We define the monodromy matrix by the same formula (3.9).

The R matrix $R(t)$ for the XYZ chain is defined by the above formula with $d=2$ and S_α being represented by σ^α. We need the following three functions[3] $\omega_i=\omega_i(t,\eta,\tau)$ which enter the formula for h_n:

$$\kappa d\log\varphi=\omega_1dt+\omega_2d\eta+\omega_3d\tau, \tag{6.5}$$

where

$$\kappa=\frac{\theta_1(2\eta)}{2\theta_1'(0)},$$
$$\varphi(t):=\rho(t,2)\cdot\left(\frac{[\eta-t]}{[\eta+t]}\right)^{1/4}.$$

In the XYZ case, the trace functional is defined as follows. For each element A of the Sklyanin algebra, there exists a unique entire function $\mathrm{Tr}_\lambda A$ with the properties [4]

(i) $\mathrm{Tr}_\lambda A\big|_{\lambda=d}=\mathrm{tr}\,\pi^{(d)}(A)$ holds for all positive integers d,

(ii) If A is a monomial in the S_α of homogeneous degree n, $\mathrm{Tr}_\lambda A$ has the form

$$\mathrm{Tr}_{\frac{t}{\eta}}A=\theta_1(t)^n\times\begin{cases}g_{A,0}(t) & (n\text{: odd}),\\ g_{A,1}(t)-\frac{t}{\eta}g_{A,2}(t) & (n\text{: even}),\end{cases} \tag{6.6}$$

where $g_{A,0}(t)$, $g_{A,2}(t)$ and $g_{A,3}(t):=g_{A,1}(t+\tau)-g_{A,1}(t)$ are elliptic functions with periods $1,\tau$. In addition, $g_{A,1}(t+1)=g_{A,1}(t)$.

[3] Note that ω_i's here are those in [4] multiplied by $\kappa/4$.

Let us introduce $X_{a,n}$ $(a = 1, 2, 3)$ by

$$\mathrm{Tr}_{s_{12}/\eta}\left(T\Big(\frac{s_1+s_2}{2}\Big)\right) = X_{1,n}(s_1, s_2|t_1, \dots, t_n) - \frac{s_{12}}{\eta} X_{2,n}(s_1, s_2|t_1, \dots, t_n),$$
$$X_{3,n}(s_1, s_2|t_1, \dots, t_n) = X_{1,n}(s_1 + \tau, s_2|t_1, \dots, t_n) - X_{1,n}(s_1, s_2|t_1, \dots, t_n)\,,$$

where $X_{a,n}$ $(a = 1, 2, 3)$ have the following periodicity.

$$X_{a,n}(s_1 + 1, s_2|t_1, \dots, t_n) = X_{a,n}(s_1, s_2|t_1, \dots, t_n) \quad (a = 1, 2, 3),$$
$$X_{a,n}(s_1 + \tau, s_2|t_1, \dots, t_n) = X_{a,n}(s_1, s_2|t_1, \dots, t_n) \quad (a = 2, 3).$$

Proposition 6.1. *The conjectural formula in* [4] *for* h_n *can be written as*

$$\Omega_n(t_1, \dots, t_n) = \frac{(-1)^n}{2\kappa^2} \iint \frac{ds_1}{2\pi i}\frac{ds_2}{2\pi i} \left(\sum_{a=1}^{3} \omega_a(s_{12}) X_{a,n}(s_1, s_2|t_1, \dots, t_n)\right) \times \mathrm{Tr}_{2,2}\left(T_n(s_1) \otimes T_n(s_2) \cdot B(s_{1,2})\right), \quad (6.7)$$

where

$$B(t) = -\frac{1}{4}\frac{[t][2\eta]}{[t+\eta][t-\eta]} \sum_{a=1}^{3} \frac{\theta_{a+1}(2t)}{\theta_{a+1}(2\eta)} \sigma^a \otimes \sigma^a\,.$$

The rewriting procedure is sketched at the end of Appendix A.3.

Appendix A. Connection to previous results

Here we give the details about the derivation of the formula (3.16) and the second one using (3.19) and (3.20). First, in Subsection A.1, we recall the previous result in [3], where an algebraic formula for a solution of the reduced qKZ equation is constructed. Next we rewrite the formula into the exponential form in Subsection A.2 (see Theorem A.3). Finally we obtain the integral formula (3.16) and the second one using (3.19) and (3.20) in Subsection A.3. The above rewriting procedure is applicable also to the elliptic case. We discuss it briefly at the end of Subsection A.3.

A.1. Algebraic construction of a solution to the reduced qKZ equation

In [3] a solution $\{h_n\}_{n=0}^{\infty}$ of the equations (3.3)–(3.5) is constructed in an algebraic way. Let us recall it here.

First we define the operator

$${}_nX^{(i,j)}_{n-2}(\lambda_1, \dots, \lambda_n) \in \mathrm{End}(V^{\otimes 2(n-2)}, V^{\otimes 2n})$$

for $1 \le i < j \le n$ as follows. Set

$${}_nX_{n-2}(\lambda_1, \dots, \lambda_n)(u) := \frac{1}{[\lambda_{1,2}]\prod_{p=3}^{n}[\lambda_{1,p}][\lambda_{2,p}]} \times \mathrm{Tr}_{\lambda_{1,2}}\left(t^{[1]}_n\Big(\frac{\lambda_1+\lambda_2}{2}; \lambda_1, \dots, \lambda_n\Big)\right)(s_{1,\bar{2}}s_{\bar{1},2}u_{3,\dots,n,\bar{n},\dots,\bar{3}}), \quad (A.1)$$

where

$$t_n^{[i]}(\lambda;\lambda_1,\dots,\lambda_n) := \ell_{\bar 1}(\lambda-\lambda_1-1)\cdots\widehat{\ell_{\bar i}(\lambda-\lambda_i-1)}\cdots\ell_{\bar n}(\lambda-\lambda_n-1) \times \ell_n(\lambda-\lambda_n)\cdots\widehat{\ell_i(\lambda-\lambda_i)}\cdots\ell_1(\lambda-\lambda_1).$$

Then ${}_nX_{n-2}^{(i,j)}$ is defined by

$$\begin{aligned}&{}_nX_{n-2}^{(i,j)}(\lambda_1,\dots,\lambda_n)\\ &:= \overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1,\dots,\lambda_n)\,\overleftarrow{P}_n^{(i,j)}\cdot{}_nX_{n-2}(\lambda_i,\lambda_j,\lambda_1,\dots,\widehat{\lambda_i},\dots,\widehat{\lambda_j},\dots,\lambda_n),\end{aligned}$$

where

$$\begin{aligned}\overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1,\dots,\lambda_n) := {} & R_{i,i-1}(\lambda_{i,i-1})\cdots R_{i,1}(\lambda_{i,1})\\ &\times R_{j,j-1}(\lambda_{j,j-1})\cdots\widehat{R_{j,i}(\lambda_{j,i})}\cdots R_{j,1}(\lambda_{j,1})\\ &\times R_{\overline{i-1},\bar i}(\lambda_{i-1,i})\cdots R_{\bar 1,\bar i}(\lambda_{1,i})\\ &\times R_{\overline{j-1},\bar j}(\lambda_{j-1,j})\cdots\widehat{R_{\bar i,\bar j}(\lambda_{i,j})}\cdots R_{\bar 1,\bar j}(\lambda_{1,j})\end{aligned}$$

and

$$\overleftarrow{P}_n^{(i,j)} := P_{i,i-1}\cdots P_{2,1}\cdot P_{j,j-1}\cdots P_{3,2}\cdot P_{\overline{i-1},\bar i}\cdots P_{\bar 1,\bar 2}\cdot P_{\overline{j-1},\bar j}\cdots P_{\bar 2,\bar 3}.$$

Note that the operator $\overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1,\dots,\lambda_n)$ is a rational function in $\zeta_j = e^{\pi i\nu\lambda_j}$ ($j = 1,\dots,n$) because $\rho(\lambda)\rho(-\lambda)=1$.

From the definition (3.10) of Tr_λ, the operator ${}_nX_{n-2}^{(i,j)}$ can be written uniquely in the following form:

$${}_nX_{n-2}^{(i,j)}(\lambda_1,\dots,\lambda_n) = {}_nG_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n) - \lambda_{i,j}\cdot{}_n\tilde G_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n), \tag{A.2}$$

where ${}_nG_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n)$ and ${}_n\tilde G_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n)$ are rational functions in $\zeta_1,\dots,\zeta_n$. Take some meromorphic functions $\omega_j(\lambda)$ ($j=1,2$) and consider the operator

$${}_n\Omega_{n-2}^{(i,j)}(\lambda_1,\dots,\lambda_n) := \omega_1(\lambda_{i,j})\cdot{}_nG_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n) + \omega_2(\lambda_{i,j})\cdot{}_n\tilde G_{n-2}^{(i,j)}(\zeta_1,\dots,\zeta_n).$$

For an ordered set of indices $K=\{k_1,\dots,k_m\}$ ($1\le k_1<\cdots<k_m\le n$), we use the abbreviation

$$\Omega_{K,(k_i,k_j)} = {}_m\Omega_{m-2}^{(i,j)}(\lambda_{k_1},\dots,\lambda_{k_m}).$$

Define

$$\begin{aligned}&h_n(\lambda_1,\dots,\lambda_n)\\ &\quad:= \sum_{m=0}^{[n/2]}\frac{(-1)^m}{2^{n-2m}}\sum\Omega_{K_1,(i_1,j_1)}\cdot\Omega_{K_2,(i_2,j_2)}\cdot\dots\cdot\Omega_{K_m,(i_m,j_m)}(\mathbf{s}_{n-2m}).\end{aligned} \tag{A.3}$$

Here

$$\mathbf{s}_m := \prod_{p=1}^m s_{p,\bar p}\in V^{\otimes 2m}$$

and the second sum in (A.3) is taken over all sequences $i_1, \dots, i_m, j_1, \dots, j_m$ of distinct elements of $K_1 = \{1, \dots, n\}$ such that

$$i_1 < \cdots < i_m, \quad i_1 < j_1, \quad \cdots, \quad i_m < j_m.$$

The sets $K_2, \dots, K_m$ are defined by $K_l = K_{l-1} \setminus \{i_l, j_l\}$ inductively.

Now we state the main results of [3]:

Theorem A.1. *The functions* $\{h_n\}_{n=0}^{\infty}$ *defined by* (A.3) *satisfy the equations* (3.3)–(3.5) *if* $\omega_1(\lambda)$ *and* $\omega_2(\lambda)$ *are solutions of the following system of difference equations:*

$$\omega_1(\lambda - 1) + \omega_1(\lambda) + p_1(\lambda) = 0, \tag{A.4}$$

$$\omega_2(\lambda - 1) + \omega_2(\lambda) + \omega_1(\lambda) + p_1(\lambda) + p_2(\lambda) = 0, \tag{A.5}$$

$$\omega_1(-\lambda) = \omega_1(\lambda), \quad \omega_2(-\lambda) = -\omega_2(\lambda), \tag{A.6}$$

where

$$p_1(\lambda) := \frac{3}{4}\frac{1}{[\lambda][\lambda-1]} - \frac{1}{4}\frac{[3]}{[\lambda-2][\lambda+1]},$$

$$p_2(\lambda) := \frac{1}{2}\frac{1}{[\lambda-1][\lambda-2]} - \frac{1}{4}\frac{[2]}{[\lambda-1][\lambda+1]}.$$

Moreover, h_n *gives the correlation function* (3.2) *in the massive regime when* $\omega_j(\lambda)$ $(j = 1, 2)$ *are given by* (3.13).

A.2. Another formula

In this subsection we construct an operator

$$\widetilde{\Omega}_n(\lambda_1, \dots, \lambda_n) \in \mathrm{End}(V^{\otimes 2n})$$

and rewrite the formula (A.3) by using $\widetilde{\Omega}_n$ (see (A.11)). The procedure of rewriting here is similar to that in Section 11 of [3], but slightly different.

First note that the vector $s_{1,\bar{2}} s_{\bar{1},2}$ in the right-hand side of (A.1) can be replaced by $s_{1,2} s_{\bar{1},\bar{2}}$ because of the following reason. We have the equality

$$s_{1,\bar{2}} s_{\bar{1},2} = s_{1,2} s_{\bar{1},\bar{2}} - s_{1,\bar{1}} s_{2,\bar{2}}.$$

From the crossing symmetry

$$\ell_{\bar{2}}\Big(\frac{\lambda_{1,2}}{2} - 1\Big)\, s_{2,\bar{2}} = -\ell_2\Big(-\frac{\lambda_{1,2}}{2}\Big)\, s_{2,\bar{2}}$$

and the cyclicity of the trace function, we see that

$$\begin{aligned}
&\mathrm{Tr}_{\lambda_{1,2}}\Big(t_n^{[1]}\Big(\frac{\lambda_1+\lambda_2}{2}\Big)\Big)\, s_{2,\bar{2}} \\
&= -\,\mathrm{Tr}_{\lambda_{1,2}}\Big(\ell_{\bar{3}}\Big(\frac{\lambda_1+\lambda_2}{2} - \lambda_3 - 1\Big) \cdots \ell_3\Big(\frac{\lambda_1+\lambda_2}{2} - \lambda_3\Big)\, \ell_2\Big(\frac{\lambda_{1,2}}{2}\Big) \ell_2\Big(-\frac{\lambda_{1,2}}{2}\Big)\Big)\, s_{2,\bar{2}}.
\end{aligned}$$

Use the quantum determinant formula

$$\mathrm{Tr}_d(x\, \ell(\lambda)\ell(-\lambda)) = \left[\frac{d}{2} + \lambda\right]\left[\frac{d}{2} - \lambda\right] \mathrm{Tr}_d(x) \qquad (\forall x \in U_q(sl_2)). \tag{A.7}$$

Then we find

$$\mathrm{Tr}_{\lambda_{1,2}}(t_n^{[1]}(\frac{\lambda_1+\lambda_2}{2}))\, s_{2,\bar{2}} = 0.$$

As a result we obtain

$$\mathrm{Tr}_{\lambda_{1,2}}(t_n^{[1]}(\frac{\lambda_1+\lambda_2}{2}))\, s_{1,\bar{2}} s_{\bar{1},2} = \mathrm{Tr}_{\lambda_{1,2}}(t_n^{[1]}(\frac{\lambda_1+\lambda_2}{2}))\, s_{1,2} s_{\bar{1},\bar{2}}.$$

In the following we use the formula for ${}_nX^{(i,j)}_{n-2}$ where $s_{1,\bar{2}}s_{\bar{1},2}$ is replaced by $s_{1,2}s_{\bar{1},\bar{2}}$.

Introduce the operator ${}_{n-1}\tilde{\Pi}_n \in \mathrm{End}(V^{\otimes 2n}, V^{\otimes 2(n-1)})$ defined by

$$\mathcal{P}^-_{n,\bar{n}}\, u = ({}_{n-1}\tilde{\Pi}_n u)_{1,\dots,n-1,\overline{n-1},\dots,\bar{1}}\, s_{n,\bar{n}},$$

and set ${}_{n-2}\tilde{\Pi}_n := {}_{n-2}\tilde{\Pi}_{n-1}\cdot {}_{n-1}\tilde{\Pi}_n$.

Now we define the operator $\widetilde{X}^{(i,j)}_n \in \mathrm{End}(V^{\otimes 2n})$ by

$$\widetilde{X}^{(i,j)}_n(\lambda_1,\dots,\lambda_n) := -4\, {}_nX^{(i,j)}_{n-2}(\lambda_1,\dots,\lambda_n)\cdot {}_{n-2}\tilde{\Pi}_n \cdot \overrightarrow{P}^{(i,j)}_n\, \overrightarrow{\mathbb{R}}^{(i,j)}_n(\lambda_1,\dots,\lambda_n), \tag{A.8}$$

where

$$\overrightarrow{P}^{(i,j)}_n := P_{n-1,n-2}\cdots P_{i+1,i}\cdot P_{n,n-1}\cdots P_{j+1,j}\cdot P_{\overline{n-2},\overline{n-1}}\cdots P_{\bar{i},\overline{i+1}}\cdot P_{\overline{n-1},\overline{n}}\cdots P_{\bar{j},\overline{j+1}}.$$

and

$$\begin{aligned}\overrightarrow{\mathbb{R}}^{(i,j)}_n(\lambda_1,\dots,\lambda_n) := {} & R_{i,n}(\lambda_{i,n})\cdots \widehat{R_{i,j}(\lambda_{i,j})}\cdots R_{i,i+1}(\lambda_{i,i+1})\\ &\times\ R_{j,n}(\lambda_{j,n})\cdots R_{j,j+1}(\lambda_{j,j+1})\\ &\times\ R_{\overline{n},\bar{i}}(\lambda_{n,i})\cdots \widehat{R_{\bar{j},\bar{i}}(\lambda_{j,i})}\cdots R_{\overline{i+1},\bar{i}}(\lambda_{i+1,i})\\ &\times\ R_{\overline{n},\bar{j}}(\lambda_{n,j})\cdots R_{\overline{j+1},\bar{j}}(\lambda_{j+1,j}).\end{aligned}$$

More explicitly, we have

$$\begin{aligned}\widetilde{X}^{(i,j)}_n(\lambda_1,\dots,\lambda_n) := {} & \frac{-4}{[\lambda_{i,j}]\prod_{p\neq i,j}[\lambda_{i,p}][\lambda_{j,p}]}\overleftarrow{\mathbb{R}}^{(i,j)}_n(\lambda_1,\dots,\lambda_n) \\ &\times \mathrm{Tr}_{\lambda_{i,j}}\left(\ell_{\bar{j}}(\frac{\lambda_{i,j}}{2}-1)\, t^{[i,j]}_n(\frac{\lambda_i+\lambda_j}{2};\lambda_1,\dots,\lambda_n)\,\ell_j(\frac{\lambda_{i,j}}{2})\right)\\ &\times P_{\bar{i},j}\mathcal{P}^-_{i,\bar{i}}\mathcal{P}^-_{j,\bar{j}}\overrightarrow{\mathbb{R}}^{(i,j)}_n(\lambda_1,\dots,\lambda_n),\end{aligned} \tag{A.9}$$

where

$$\begin{aligned} & t^{[i,j]}_n(\lambda;\lambda_1,\dots,\lambda_n)\\ &:= \ell_{\bar{1}}(\lambda-\lambda_1-1)\cdots \ell_{\bar{i}}(\widehat{\lambda-\lambda_i}-1)\cdots \ell_{\bar{j}}(\widehat{\lambda-\lambda_j}-1)\cdots \ell_{\bar{n}}(\lambda-\lambda_n-1)\\ &\times \ell_n(\lambda-\lambda_n)\cdots \ell_{\bar{j}}\widehat{(\lambda-\lambda_j)}\cdots \ell_{\bar{i}}\widehat{(\lambda-\lambda_i)}\cdots \ell_1(\lambda-\lambda_1).\end{aligned}$$

To obtain (A.9) from (A.8) we used

$$\overleftarrow{P}^{(i,j)}_n s_{1,2}s_{\bar{1},\bar{2}}\left({}_{n-2}\tilde{\Pi}_n\cdot \overrightarrow{P}^{(i,j)}_n u\right)_{3,\dots,n,\bar{n},\dots,\bar{3}} = P_{\bar{i},j}\mathcal{P}^-_{i,\bar{i}}\mathcal{P}^-_{j,\bar{j}}u$$

for $u\in V^{\otimes 2n}$.

In the same way as (A.2) the operator $\widetilde{X}_n^{(i,j)}$ is decomposed into two parts:

$$\widetilde{X}_n^{(i,j)}(\lambda_1,\ldots,\lambda_n)=\widetilde{X}_{1,n}^{(i,j)}(\zeta_1,\ldots,\zeta_n)-\lambda_{i,j}\cdot\widetilde{X}_{2,n}^{(i,j)}(\zeta_1,\ldots,\zeta_n),$$

where $\widetilde{X}_{a,n}^{(i,j)}$ $(a=1,2)$ are rational functions in $\zeta_1,\ldots,\zeta_n$ which take values in $\mathrm{End}(V^{\otimes 2n})$. Then take solutions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ of the equations (A.5)–(A.6), and define

$$\widetilde{\Omega}_n^{(i,j)}(\lambda_1,\ldots,\lambda_n):=\sum_{a=1,2}\omega_a(\lambda_{i,j})\widetilde{X}_{a,n}^{(i,j)}(\zeta_1,\ldots,\zeta_n).$$

By definition we set $\widetilde{\Omega}_n^{(j,i)}=\widetilde{\Omega}_n^{(i,j)}$ for $1\le i<j\le n$. Then the operator $\widetilde{\Omega}_n^{(i,j)}$ has the following properties:

Proposition A.2.

(1) *Suppose that $i<j$ and $k<l$. If $\{i,j\}\cap\{k,l\}\neq\emptyset$, we have*

$$\widetilde{\Omega}_n^{(i,j)}(\lambda_1,\ldots,\lambda_n)\cdot{}_n\Omega_{n-2}^{(k,l)}(\lambda_1,\ldots,\lambda_n)=0.$$

If $\{i,j\}\cap\{k,l\}=\emptyset$, we have

$$\begin{aligned}&\widetilde{\Omega}_n^{(i,j)}(\lambda_1,\ldots,\lambda_n)\cdot{}_n\Omega_{n-2}^{(k,l)}(\lambda_1,\ldots,\lambda_n)\\&=-4{}_n\Omega_{n-2}^{(i,j)}(\lambda_1,\ldots,\lambda_n)\cdot{}_{n-2}\Omega_{n-4}^{(k',l')}(\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\widehat{\lambda_j},\ldots,\lambda_n)\\&\quad\times{}_{n-4}\tilde{\Pi}_{n-2}\cdot\overrightarrow{P}_{n-2}^{(i',j')}\overrightarrow{\mathbb{R}}_{n-2}^{(i',j')}(\lambda_1,\ldots,\widehat{\lambda_k},\ldots,\widehat{\lambda_l},\ldots,\lambda_n),\end{aligned}$$

where k' and l' are the positions of λ_k and λ_l in $(\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\widehat{\lambda_j},\ldots,\lambda_n)$, and i' and j' are the positions of λ_i and λ_j in $(\lambda_1,\ldots,\widehat{\lambda_k},\ldots,\widehat{\lambda_l},\ldots,\lambda_n)$.

(2) *The operators $\widetilde{\Omega}_n^{(i,j)}$ satisfy the exchange relations:*

$$\begin{aligned}&\check{R}_{k,k+1}(\lambda_{k,k+1})\check{R}_{\overline{k+1},\overline{k}}(\lambda_{k+1,k})\widetilde{\Omega}_n^{(i,j)}(\ldots,\lambda_k,\lambda_{k+1},\ldots)\\&\quad=\widetilde{\Omega}_n^{(\pi_k(i),\pi_k(j))}(\ldots,\lambda_{k+1},\lambda_k,\ldots)\check{R}_{k,k+1}(\lambda_{k,k+1})\check{R}_{\overline{k+1},\overline{k}}(\lambda_{k+1,k}).\end{aligned}$$

Here π_k is the transposition $(k,k+1)$.

(3) *The operators $\widetilde{\Omega}_n^{(i,j)}$ are commutative:*

$$\widetilde{\Omega}_n^{(i,j)}\widetilde{\Omega}_n^{(k,l)}=\widetilde{\Omega}_n^{(k,l)}\widetilde{\Omega}_n^{(i,j)}\quad\textit{for all}\quad i<j,\ k<l.$$

The proof of Proposition A.2 is quite similar to that of Lemma 12.1, Lemma 12.2 and Lemma 12.3 in [3]. In the proof of (1), use the recurrence relation

$$\begin{aligned}&{}_{n-1}\tilde{\Pi}_n\cdot{}_nX_{n-2}^{(i,j)}(\lambda_1,\ldots,\lambda_n)\\&=\begin{cases}0&\text{if }\ j=n,\\{}_{n-1}X_{n-3}^{(i,j)}(\lambda_1,\ldots,\lambda_{n-1})\cdot{}_{n-3}\tilde{\Pi}_{n-2}&\text{if }\ j<n.\end{cases}\end{aligned}\tag{A.10}$$

Now we define the operator $\widetilde{\Omega}_n$ by

$$\widetilde{\Omega}_n(\lambda_1,\ldots,\lambda_n):=\sum_{1\le i<j\le n}\widetilde{\Omega}_n^{(i,j)}(\lambda_1,\ldots,\lambda_n).$$

Then from Proposition A.2 and the equality

$$P_{1,2}P_{\bar{2},\bar{1}}R_{1,2}(\lambda)R_{\bar{2},\bar{1}}(-\lambda)s_{1,\bar{1}}s_{2,\bar{2}} = s_{1,\bar{1}}s_{2,\bar{2}},$$

we get

Theorem A.3. *The formula* (A.3) *for* h_n *can be rewritten as follows:*

$$h_n(\lambda_1,\dots,\lambda_n) = 2^{-n}e^{\widetilde{\Omega}_n(\lambda_1,\dots,\lambda_n)}\mathbf{s}_n. \tag{A.11}$$

A.3. Derivation of the new formula

Finally we prove that the operator Ω_n defined in (3.16) is equal to $\widetilde{\Omega}_n$.

From the definition (3.11) of $X_{a,n}$ $(a = 1, 2)$, we can easily see that

$$\mathrm{res}_{\mu_1=\lambda_j}\,\mathrm{res}_{\mu_2=\lambda_j}\,X_{a,n}(\mu_1,\mu_2;\lambda_1,\dots,\lambda_n) = 0$$

and

$$\begin{aligned}&\mathrm{res}_{\mu_1=\lambda_i}\,\mathrm{res}_{\mu_2=\lambda_j}\,X_{a,n}(\mu_1,\mu_2;\lambda_1,\dots,\lambda_n)\\ &\qquad = (-1)^a\,\mathrm{res}_{\mu_1=\lambda_j}\,\mathrm{res}_{\mu_2=\lambda_i}\,X_{a,n}(\mu_1,\mu_2;\lambda_1,\dots,\lambda_n).\end{aligned}$$

The operator

$$\mathrm{Tr}_{2,2}\left(T_n(\mu_1)\otimes T_n(\mu_2)\cdot B(\mu_{1,2})\right)$$

is skew-symmetric with respect to μ_1 and μ_2 because of the commutation relation

$$R_{12}(\mu)B_{12}(\mu) = -B_{21}(-\mu)R_{12}(\mu).$$

The functions ω_a $(a = 1, 2)$ satisfy the parity conditions

$$\omega_a(-\lambda) = (-1)^{a-1}\omega_a(\lambda).$$

By using the above properties we have

$$\Omega_n(\lambda_1,\dots,\lambda_n) = \sum_{1\le i<j\le n}\Omega_n^{(i,j)}(\lambda_1,\dots,\lambda_n),$$

where

$$\begin{aligned}\Omega_n^{(i,j)}(\lambda_1,\dots,\lambda_n) := \frac{(-1)^n}{\kappa^2}\sum_{a=1,2}\omega_a(\lambda_{i,j})\,\mathrm{res}_{\mu_1=\lambda_i}\,\mathrm{res}_{\mu_2=\lambda_j}&\\ \times X_{a,n}(\mu_1,\mu_2;\lambda_1,\dots,\lambda_n)\,\mathrm{Tr}_{2,2}(T_n(\lambda_i)\otimes T_n(\lambda_j)\cdot B(\lambda_{i,j})).&\end{aligned}$$

In the following we show the equality $\Omega_n^{(i,j)} = \widetilde{\Omega}_n^{(i,j)}$. To this end we prove two lemmas.

Let B be a linear operator acting on $\mathbb{C}^2\otimes\mathbb{C}^2$. Then

$$\mathrm{Tr}_{2,2}(T_n(\lambda_i)\otimes T_n(\lambda_j)\cdot B)$$

is the operator acting on $V_1\otimes\cdots\otimes V_n\otimes V_{\bar{n}}\otimes\cdots\otimes V_{\bar{1}}$.

Lemma A.4. *We have*

$$\begin{aligned}&\mathrm{Tr}_{2,2}(T_n(\lambda_i)\otimes T_n(\lambda_j)\cdot B)\\ &\qquad = 4\overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1,\dots,\lambda_n)B_{i,j}\mathcal{P}^-_{i,\bar{i}}\mathcal{P}^-_{j,\bar{j}}\overrightarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1,\dots,\lambda_n).\end{aligned} \tag{A.12}$$

Proof. We prepare some notation. Denote by $V_a \otimes V_b$ ($V_a \simeq V_b \simeq \mathbb{C}^2$) the tensor product of the two-dimensional spaces on which the trace $\mathrm{Tr}_{2,2}$ is taken. Set

$$V^Q = V_1 \otimes \cdots \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \otimes V_{\bar{1}}.$$

Let $R_{a,i}(\lambda)$ be the R matrix acting on the a-th and the i-th component of $V_a \otimes V_b \otimes V^Q$. We set

$$T_a(\lambda) = R_{a,\bar{1}}(\lambda - \lambda_1 - 1) \cdots R_{a,\bar{n}}(\lambda - \lambda_n - 1) R_{a,n}(\lambda - \lambda_n) \cdots R_{a,1}(\lambda - \lambda_1).$$

We define $T_b(\lambda)$ similarly. We denote by $B_{a,b}$ the operator B acting on $V_a \otimes V_b$. Similarly, $B_{i,b}$, etc., are defined.

Denote by X the left-hand side of (A.12). Then we have

$$X = \mathrm{tr}_{a,b}(B_{a,b} T_a(\lambda_i) T_b(\lambda_j)).$$

Here $\mathrm{tr}_{a,b}$ means taking trace on $V_a \otimes V_b$. We use the following properties of the R matrix.

$$R_{1,2}(0) = P_{1,2}, \tag{A.13}$$

$$R_{1,2}(-1) = -2\mathcal{P}^-_{1,2}, \tag{A.14}$$

$$R_{1,2}(\lambda_{1,2}) R_{1,3}(\lambda_{1,3}) R_{2,3}(\lambda_{2,3}) = R_{2,3}(\lambda_{2,3}) R_{1,3}(\lambda_{1,3}) R_{1,2}(\lambda_{1,2}), \tag{A.15}$$

$$R_{1,2}(\lambda) R_{2,1}(-\lambda) = 1, \tag{A.16}$$

$$R_{1,3}(\lambda) R_{1,2}(-1) = -R_{3,2}(-1-\lambda) R_{1,2}(-1), \tag{A.17}$$

$$R_{1,2}(-1) R_{1,3}(\lambda) = -R_{1,2}(-1) R_{3,2}(-1-\lambda). \tag{A.18}$$

We abbreviate the arguments of the R-matrices. They can be understood from the space indices $a, b, 1, \dots, n, \bar{1}, \dots, \bar{n}$ through the correspondences $a \leftrightarrow \lambda_i$, $b \leftrightarrow \lambda_j$, $1 \leftrightarrow \lambda_1, \dots, n \leftrightarrow \lambda_n$, $\bar{1} \leftrightarrow \lambda_1 + 1, \dots, \bar{n} \leftrightarrow \lambda_n + 1$. For example, by $R_{i,\bar{j}}$ we mean $R_{i,\bar{j}}(\lambda_{i,j} - 1)$. We extend this convention to T_a and T_b. We also use the abbreviation

$$R_{i,[i-1,1]} = R_{i,i-1} R_{i,i-2} \cdots R_{i,1},$$

etc..

With the above convention, we have

$$\begin{aligned} X = \mathrm{tr}_{a,b}\{&B_{a,b} \\ &\times R_{a,[\bar{1},\overline{i-1}]} R_{a,\bar{i}}(-1) R_{a,[\overline{i+1},\overline{j-1}]} R_{a,\bar{j}} R_{a,[\overline{j+1},\bar{n}]} R_{a,[n,j+1]} R_{a,j} R_{a,[j-1,i+1]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times P_{a,i} R_{a,[i-1,1]} \\ &\times R_{b,[\bar{1},\overline{i-1}]} R_{b,\bar{i}}\, R_{b,[\overline{i+1},\overline{j-1}]} R_{b,\bar{j}}(-1)\, R_{b,[\overline{j+1},\bar{n}]} R_{b,[n,j+1]} P_{b,j}\, R_{b,[j-1,i+1]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times R_{b,i} R_{b,[i-1,1]}\}. \end{aligned}$$

We write the argument -1 in two places in order to emphasize where we can use the crossing symmetries (A.17), (A.18). Using the cyclicity of the trace, we bring $R_{a,[i-1,1]} R_{b,[i-1,1]}$ to the left of $B_{a,b}$. Then, we move $P_{a,i}$ to the left while we change all a to i. Finally, we can eliminate $\mathrm{tr}_a\, P_{a,i}$ because this is equal to the identity operator. Thus, we get

$$\begin{aligned}
X &= R_{i,[i-1,1]}\,\mathrm{tr}_b\{R_{b,[i-1,1]}B_{i,b}\\
&\times R_{i,[\bar{1},\overline{i-1}]}R_{i,\bar{i}}(-1)R_{i,[\overline{i+1},\overline{j-1}]}R_{i,\bar{j}}R_{i,[\overline{j+1},\bar{n}]}R_{i,[n,j+1]}\underline{R_{i,j}R_{i,[j-1,i+1]}}\\
&\times R_{b,[\bar{1},\overline{i-1}]}R_{b,\bar{i}}\,R_{b,[\overline{i+1},\overline{j-1}]}R_{b,\bar{j}}(-1)\,R_{b,[\overline{j+1},\bar{n}]}R_{b,[n,j+1]}\underline{P_{b,j}}\,R_{b,[j-1,i+1]}R_{b,i}\}.
\end{aligned}$$

Using the crossing symmetries (A.17), (A.18), and pushing the underlined terms to the right, we obtain

$$\begin{aligned}
X &= R_{i,[i-1,1]}R_{[\overline{i-1},\bar{1}],\bar{i}}\,\mathrm{tr}_b\{R_{b,[i-1,1]}B_{i,b}\\
&\times R_{i,\bar{i}}(-1)R_{[\bar{n},\overline{j+1}],\bar{i}}R_{\bar{j},\bar{i}}R_{[\overline{j-1},\overline{i+1}],\bar{i}}R_{i,[n,j+1]}\\
&\times R_{[\overline{j-1},\overline{i+1}],\bar{j}}R_{\bar{i},\bar{j}}\,R_{[\overline{i-1},\bar{1}],\bar{j}}R_{b,\bar{j}}(-1)\,R_{[\bar{n},\overline{j+1}],\bar{j}}R_{b,[n,j+1]}\\
&\qquad\qquad\times \underline{R_{i,j}R_{i,[j-1,i+1]}}R_{j,[j-1,i+1]}R_{j,i}\underline{P_{b,j}}\,\}.
\end{aligned}$$

Using the Yang-Baxter equation (A.15) and the unitarity relation (A.16), we reduce $R_{ij}R_{ji}$, and $R_{\bar{j},\bar{i}}R_{\bar{i},\bar{j}}$.

$$\begin{aligned}
X &= R_{i,[i-1,1]}R_{[\overline{i-1},\bar{1}],\bar{i}}\,\mathrm{tr}_b\{R_{b,[i-1,1]}B_{i,b}\\
&\times R_{i,\bar{i}}(-1)R_{[\bar{n},\overline{j+1}],\bar{i}}R_{i,[n,j+1]}\underline{R_{[\overline{j-1},\overline{i+1}],\bar{j}}}R_{[\overline{j-1},\overline{i+1}],\bar{i}}\\
&\times R_{[\overline{i-1},\bar{1}],\bar{j}}R_{b,\bar{j}}(-1)\,R_{[\bar{n},\overline{j+1}],\bar{j}}R_{b,[n,j+1]}\underline{R_{j,[j-1,i+1]}}R_{i,[j-1,i+1]}P_{b,j}\}.
\end{aligned}$$

The underlined terms can be brought outside the trace. Therefore, handling $P_{b,j}$ in the same way as $P_{a,i}$, we obtain

$$\begin{aligned}
X &= R_{i,[i-1,1]}R_{[\overline{i-1},\bar{1}],i}\underline{R_{[\overline{j-1},\overline{i+1}],\bar{j}}R_{j,[j-1,i+1]}}R_{j,[i-1,1]}B_{i,j}\\
&\times R_{i,\bar{i}}(-1)R_{[\bar{n},\overline{j+1}],\bar{i}}R_{i,[n,j+1]}R_{[\overline{j-1},\overline{i+1}],\bar{i}}\\
&\times R_{[\overline{i-1},\bar{1}],\bar{j}}R_{j,\bar{j}}(-1)\,R_{[\bar{n},\overline{j+1}],\bar{j}}R_{j,[n,j+1]}R_{i,[j-1,i+1]}\\
&= \overleftarrow{\mathbb{R}}_n^{(i,j)}B_{i,j}R_{i,\bar{i}}(-1)R_{j,\bar{j}}(-1)\overrightarrow{\mathbb{R}}_n^{(i,j)}.
\end{aligned}$$

This completes the proof. □

To state the second lemma we introduce some notation. The ℓ-operator is written as

$$\ell(\lambda) = \sum_{\alpha=0}^{3} \ell^{\alpha}(\lambda)\otimes\sigma^{\alpha}$$

where

$$\begin{aligned}
\ell^0(\lambda) &= \frac{\left(q^{\lambda+\frac{1}{2}} - q^{-\lambda-\frac{1}{2}}\right)\left(q^{\frac{H}{2}} + q^{-\frac{H}{2}}\right)}{2(q-q^{-1})},\\
\ell^1(\lambda) &= \frac{q^{\frac{1-H}{2}}E + Fq^{\frac{(H-1)}{2}}}{2},
\end{aligned}$$

$$i\ell^2(\lambda) = \frac{q^{\frac{1-H}{2}}E - Fq^{\frac{(H-1)}{2}}}{2},$$
$$\ell^3(\lambda) = \frac{\left(q^{\lambda+\frac{1}{2}} + q^{-\lambda-\frac{1}{2}}\right)\left(q^{\frac{H}{2}} - q^{-\frac{H}{2}}\right)}{2(q-q^{-1})}.$$

For $\alpha, \beta = 0, 1, 2, 3$ we set

$$\epsilon_{\alpha\beta} = \begin{cases} i & \text{if } (\alpha,\beta) = (1,2),(2,3),(3,1), \\ -i & \text{if } (\alpha,\beta) = (2,1),(3,2),(1,3), \\ 1 & \text{otherwise.} \end{cases}$$

Lemma A.5. *Let $B \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be an operator*

$$B = \sum_{\alpha,\beta=0}^{3} \epsilon_{\alpha\beta} B^{\alpha\beta} (\sigma^\alpha \otimes \sigma^\beta) \qquad (B^{\alpha\beta} \in \mathbb{C}).$$

Then we have

$$\mathrm{Tr}_{\lambda_{i,j}} \left(t_n\left(\frac{\lambda_i + \lambda_j}{2}\right) \right) \overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1, \ldots, \lambda_n) B_{i,j} \mathcal{P}^-_{i,\bar{i}} \mathcal{P}^-_{j,\bar{j}} = \overleftarrow{\mathbb{R}}_n^{(i,j)}(\lambda_1, \ldots, \lambda_n)\, \mathrm{Tr}_{\lambda_{i,j}}$$
$$\times \left(\ell_{\bar{j}}\left(\frac{\lambda_{i,j}}{2} - 1\right) t_n^{[i,j]}\left(\frac{\lambda_i + \lambda_j}{2}\right) \ell_j\left(\frac{\lambda_{i,j}}{2}\right) \cdot Y(\lambda_{i,j}) \right) P_{\bar{i},j} \mathcal{P}^-_{i,\bar{i}} \mathcal{P}^-_{j,\bar{j}}, \quad \text{(A.19)}$$

where $Y(\lambda)$ is given by

$$Y(\lambda) := -4 \sum_{\alpha,\beta=0}^{3} \epsilon_{\alpha\beta} C^{\alpha\beta}\, \ell^\beta\left(\frac{\lambda}{2} - 1\right) \ell^\alpha\left(-\frac{\lambda}{2} - 1\right). \quad \text{(A.20)}$$

Here the coefficients $C^{\alpha\beta}$ are determined from the equality

$$\begin{pmatrix} C^{00} \\ C^{11} \\ C^{22} \\ C^{33} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} B^{00} \\ B^{11} \\ B^{22} \\ B^{33} \end{pmatrix} \quad \text{(A.21)}$$

and ones obtained by changing the indices $(00, 11, 22, 33)$ *in the both hand sides of* (A.21) *to* $(01, 10, 23, 32)$, $(02, 13, 20, 31)$ *and* $(03, 12, 21, 30)$*. In particular, if B is the identity, the left-hand side of* (A.19) *becomes zero.*

Proof. We start from the left-hand side of (A.19). The R-matrices $\overleftarrow{\mathbb{R}}_n^{(i,j)}$ go through the ℓ-operators t_n:

$$\overleftarrow{\mathbb{R}}_n^{(i,j)}\, \mathrm{Tr}_{\lambda_{i,j}} \Big(\ell_{\bar{i}}\left(\tfrac{\lambda_{j,i}}{2} - 1\right) \ell_{\bar{j}}\left(\tfrac{\lambda_{i,j}}{2} - 1\right) t_n^{[i,j]}\left(\tfrac{\lambda_i + \lambda_j}{2}\right) \ell_j\left(\tfrac{\lambda_{i,j}}{2}\right) \ell_i\left(\tfrac{\lambda_{j,i}}{2}\right) B_{i,j} \Big) \mathcal{P}^-_{i,\bar{i}} \mathcal{P}^-_{j,\bar{j}}\,. \quad \text{(A.22)}$$

Now it is easy to see that the above operator becomes zero if B is the identity because of the cyclicity of the trace, the crossing symmetry

$$\ell_1(\lambda)\mathcal{P}^-_{\bar{1},2} = -\ell_2(-\lambda-1)\mathcal{P}^-_{\bar{1},2},$$

and the quantum determinant formula (A.7).

Let us proceed the calculation of (A.22). Without loss of generality, we consider the case $i=1$, $j=2$:

$$Z := \mathrm{Tr}_{\lambda_{1,2}}\Big(\ell_{\bar{1}}\left(\tfrac{\lambda_{2,1}}{2}-1\right)\ell_{\bar{2}}\left(\tfrac{\lambda_{1,2}}{2}-1\right)t_x^{[1,2]}(\lambda)\,\ell_2\left(\tfrac{\lambda_{1,2}}{2}\right)\ell_1\left(\tfrac{\lambda_{2,1}}{2}\right)\Big)B_{1,2}\mathcal{P}^-_{1,\bar{1}}\mathcal{P}^-_{2,\bar{2}}\,,$$

where

$$\lambda = \frac{\lambda_1+\lambda_2}{2}.$$

We rewrite the last part of Z by using

$$B_{1,2}\mathcal{P}^-_{1,\bar{1}}\mathcal{P}^-_{2,\bar{2}} = \sum_{\alpha,\beta=0}^{3}\epsilon_{\alpha\beta}C^{\alpha\beta}\sigma^{\alpha}_{\bar{1}}\sigma^{\alpha}_{\bar{2}}P_{\bar{1},2}\mathcal{P}^-_{1,\bar{1}}\mathcal{P}^-_{2,\bar{2}}.$$

Note that

$$P_{\bar{1},2}\mathcal{P}^-_{1,\bar{1}}\mathcal{P}^-_{2,\bar{2}} = \mathcal{P}^-_{1,2}\mathcal{P}^-_{\bar{1},\bar{2}}P_{\bar{1},2}.$$

Using the crossing symmetry, we rewrite Z as

$$\begin{aligned}Z = &\sum_{\alpha,\beta=0}^{3}\epsilon_{\alpha\beta}C^{\alpha\beta}\\ \times\ &\mathrm{Tr}_{\lambda_{1,2}}\Big(\ell_{\bar{1}}\left(\tfrac{\lambda_{2,1}}{2}-1\right)\sigma^{\alpha}_{\bar{1}}\ell_{\bar{1}}\left(\tfrac{\lambda_{2,1}}{2}\right)\mathcal{P}^-_{\bar{1},\bar{2}}\cdot t_n^{[1,2]}(\lambda)\cdot\ell_2\left(\tfrac{\lambda_{1,2}}{2}\right)\sigma^{\beta}_{2}\ell_2\left(\tfrac{\lambda_{1,2}}{2}-1\right)\mathcal{P}^-_{1,2}\Big)P_{\bar{1},2}.\end{aligned}$$

Then (A.19) follows from the quantum determinant formula (A.7), the following Proposition A.6, and the equality

$$\sigma^2\ell(\lambda)^t\cdot\sigma^2 = -\ell(-\lambda-1). \qquad \square$$

Proposition A.6. *Suppose that $m=\sum_{\alpha=0}^{3}m_\alpha\sigma^\alpha$ where m_α is a scalar with respect to the 2-dimensional space on which the Pauli matrices act. We have*

$$\begin{aligned}\sigma^\alpha m &= -\varepsilon_\alpha(\sigma^2 m^t\cdot\sigma^2)\sigma^\alpha + 2m_\alpha,\\ m\sigma^\alpha &= -\varepsilon_\alpha\sigma^\alpha(\sigma^2 m^t\cdot\sigma^2) + 2m_\alpha,\end{aligned}$$

where $m^t=\sum_{\alpha=0}^{3}m_\alpha(\sigma^\alpha)^t$ and

$$\varepsilon_\alpha = \begin{cases}1 & \text{if } \alpha=0;\\ -1 & \text{otherwise.}\end{cases}$$

The proof of Proposition A.6 is straightforward.

Now let us prove that $\Omega_n^{(i,j)} = \widetilde{\Omega}_n^{(i,j)}$. If B is the operator $B(\mu)$ defined in (3.17), the corresponding operator $Y(\lambda)$ determined by (A.20) becomes the identity. Therefore, from the formula (A.9), we obtain

$$\frac{(-1)^n}{\kappa^2} \operatorname{res}_{\mu_1=\lambda_i} \operatorname{res}_{\mu_2=\lambda_j} \operatorname{Tr}_{\mu_{1,2}}(T_n(\frac{\mu_1+\mu_2}{2})) \cdot \operatorname{Tr}_{2,2}(T_n(\lambda_i) \otimes T_n(\lambda_j) \cdot B(\lambda_{i,j})) \\ = \tilde{X}_n^{(i,j)}(\lambda_1, \dots, \lambda_n).$$

This implies the equality $\Omega_n^{(i,j)} = \widetilde{\Omega}_n^{(i,j)}$.

The proof of (3.19), (3.20) is similar. It is easy to see that for the operator $B^q(\mu)$ we have $Y(\lambda) = q^H$. Therefore, the right-hand side of (3.16) where $X_{a,n}(\mu_1, \mu_2)$ and $B(\mu)$ are replaced by $X^q_{a,n}(\mu_1, \mu_2)$ and $B^q(\mu)$, respectively, is equal to Ω_n.

Finally we give a sketch of the calculation in the elliptic case. The ℓ-operator is given by

$$\ell(\lambda) = \sum_{\alpha=0}^{3} w_a(\lambda) S_\alpha \otimes \sigma^\alpha.$$

Here S_α $(\alpha = 0, 1, 2, 3)$ are the generators of Sklyanin algebra and

$$w_\alpha(\lambda) = \frac{\theta_{\alpha+1}(2t+\eta)}{2\theta_{\alpha+1}(\eta)},$$

where $t = \lambda\eta$. If we set

$$B(\lambda) = -\frac{\theta_1(2t)\theta_1(4\eta)}{4\theta_1(2t+2\eta)\theta_1(2t-2\eta)} \sum_{\alpha=1}^{3} \frac{\theta_{\alpha+1}(2t)}{\theta_{\alpha+1}(2\eta)} \sigma^\alpha \otimes \sigma^\alpha,$$

the corresponding operator $Y(\lambda)$ becomes

$$Y(\lambda) = \frac{1}{4}\Big\{ \frac{\theta_1(2t)}{\theta_1(2t-2\eta)} \left(\frac{\theta_1(t-3\eta)\theta_1(t-\eta)}{\theta_1^2(\eta)} K_0 - \frac{\theta_1^2(t-2\eta)}{\theta_1^2(\eta)} K_2 \right) \\ - \frac{\theta_1(4\eta)}{2\theta_1(2t-2\eta)\theta_1(2t-2\eta)\theta_1(2\eta)} \left(\frac{\theta_1(t-\eta)\theta_1(t+\eta)}{\theta_1^2(\eta)} K_0 - \frac{\theta_1^2(t)}{\theta_1^2(\eta)} K_2 \right) \Big\},$$

where K_0 and K_2 are the Casimir elements (6.4). Therefore we obtain

$$\operatorname{Tr}_\lambda(x\, Y(\lambda)) = \frac{\theta_1(2t)}{\theta_1(2\eta)} \operatorname{Tr}_\lambda(x)$$

for any element x of the Sklyanin algebra. This gives the formula (6.7).

Appendix B. Relation with the vertex operator approach

Correlation functions of the XXZ model in the massive regime have been studied in the framework of representation theory [11, 12]. The description in the present paper differs from the above literature by a few minor points. In this appendix, we compare the two in some detail.

Consider an inhomogeneous six-vertex model where an inhomogeneity parameter ζ_j is attached to each column j. By correlation functions we mean those of local operators on a single row. There are two equivalent formulations depending on whether one uses row-to-row or column-to-column transfer matrices. In this paper, correlation functions are expressed as expectation values with respect to the ground state of row-to-row transfer matrices. The latter, and hence the ground state vector, depend on the ζ_j while local operators do not. In [12], on the other hand, column-to-column transfer matrices are employed. Their ground state vectors are independent of ζ_j. The inhomogeneity is encoded rather in local operators, expressed in the form of an insertion of half-column transfer matrices.

Another minor difference between [12] and the present paper is that the R matrices and Hamiltonians are not identical. The parameter $q = e^{\pi i \nu}$ of the present paper and the corresponding parameter q_{JM} in [12] are related by

$$q_{\mathrm{JM}} = -q.$$

With this identification, the R matrix $R_{\mathrm{JM}}(\zeta)$ and the Hamiltonian H_{JM} in [12] are related to $R(\lambda)$ (2.4) and H_{XXZ} (2.1) by the gauge transformation

$$R(\lambda) = (\sigma^3 \otimes 1) R_{\mathrm{JM}}(\zeta)(1 \otimes \sigma^3), \quad H_{\mathrm{XXZ}} = K H_{\mathrm{JM}} K^{-1}, \tag{B.1}$$

where $K = \prod_{j:\,\mathrm{even}} \sigma_j^3$.

Correlation functions in the present paper are related to the mean value of the two expectation values with respect to the two vectors $|\mathrm{vac}\rangle_{(i)}$ considered in [12]. Taking into account the gauge transformation (B.1), we have

$$\begin{aligned}
&\prod_{j=1}^{n}(-\bar{\epsilon}_j)\langle \mathrm{vac}|(E_{-\bar{\epsilon}_1,\epsilon_1})_1 \cdots (E_{-\bar{\epsilon}_n,\epsilon_n})_n|\mathrm{vac}\rangle \qquad\qquad (\mathrm{B.2})\\
&= \frac{1}{2}\sum_{i=0,1}\prod_{j=1}^{n}(-\bar{\epsilon}_j)\cdot {}_{(i)}\langle \mathrm{vac}|K\cdot(E_{-\bar{\epsilon}_1,\epsilon_1})_1 \cdots (E_{-\bar{\epsilon}_n,\epsilon_n})_n \cdot K^{-1}|\mathrm{vac}\rangle_{(i)}\\
&= \frac{1}{2}\sum_{i=0,1}\prod_{j:\ \mathrm{even}}\epsilon_j \prod_{j:\ \mathrm{odd}}(-\bar{\epsilon}_j)_{(i)}\langle \mathrm{vac}|(E_{-\bar{\epsilon}_1,\epsilon_1})_1 \cdots (E_{-\bar{\epsilon}_n,\epsilon_n})_n|\mathrm{vac}\rangle_{(i)}.
\end{aligned}$$

The correlation functions

$${}_{(i)}\langle \mathrm{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\mathrm{vac}\rangle_{(i)}$$

can be constructed in terms of the vertex operators arising from representation theory of the quantum affine algebra $U_{-q} = U_{-q}(\widehat{\mathfrak{sl}}_2)$ as follows (recall that $q_{\mathrm{JM}} = -q$).

Denote by $\Lambda_i\,(i = 0, 1)$ the fundamental weights of U_{-q}. Let $V(\Lambda_i)$ be the irreducible highest weight module with highest weight Λ_i, and $V_\zeta = V \otimes \mathbb{C}[\zeta, \zeta^{-1}]$ the evaluation module in the principal picture. The vertex operator (of type I) is an intertwiner

$$\Phi(\zeta)\colon V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_\zeta.$$

Define the components $\Phi_\epsilon(\zeta)\,(\epsilon=\pm)$ by

$$\Phi_\epsilon(\zeta)\colon V(\Lambda_i)\longrightarrow V(\Lambda_{1-i}),\quad \Phi(\zeta)\,u=\sum_{\epsilon=\pm}(\Phi_\epsilon(\zeta)\,u)\otimes v_\epsilon.$$

Then

$$\begin{aligned}&{}_{(i)}\langle\mathrm{vac}|(E_{-\epsilon_1,\bar\epsilon_1})_1\cdots(E_{-\epsilon_n,\bar\epsilon_n})_n|\mathrm{vac}\rangle_{(i)}\\ &=\chi^{-1}g^n\,\mathrm{tr}_{V(\Lambda_i)}\Big(q^{2D^{(i)}}\Phi_{\epsilon_n}(q^{-1}\zeta_n)\cdots\Phi_{\epsilon_1}(q^{-1}\zeta_1)\Phi_{\bar\epsilon_1}(\zeta_1)\cdots\Phi_{\bar\epsilon_n}(\zeta_n)\Big),\end{aligned}\tag{B.3}$$

where $\zeta_j=q^{\lambda_j}$, $D^{(i)}=-(\Lambda_0+\Lambda_1)+\frac{i}{2}$, and

$$\chi=\frac{1}{(q^2;q^4)_\infty},\quad g=\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}.$$

The vertex operators satisfy the following relations:

$$\Phi_{\epsilon_2}(\zeta_2)\Phi_{\epsilon_1}(\zeta_1)=\sum_{\epsilon_1',\epsilon_2'=\pm}R_{\mathrm{JM}}(\zeta_1/\zeta_2)_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}\Phi_{\epsilon_1'}(\zeta_1)\Phi_{\epsilon_2'}(\zeta_2),\tag{B.4}$$

$$\xi^{D^{(1-i)}}\cdot\Phi_\epsilon(\zeta)\cdot\xi^{-D^{(i)}}=\Phi_\epsilon(\xi\zeta),\tag{B.5}$$

$$g\sum_{\epsilon=\pm}\Phi_{-\epsilon}(\zeta)\Phi_\epsilon(q\zeta)=\mathrm{id}.\tag{B.6}$$

The basic relations (3.3)–(3.4) are simple consequences of (B.3) and (B.4)–(B.6).

Appendix C. The scalar factors

We collect here formulas for the scalar factors which enter the definition of the L-operators.

XXZ case

- Massless regime $(0<\nu<1)$

$$\rho(\lambda,d)=-\frac{S_2\big(1-\frac{d}{2}-\lambda\big)}{S_2\big(1-\frac{d}{2}+\lambda\big)}\frac{S_2\big(2-\frac{d}{2}+\lambda\big)}{S_2\big(2-\frac{d}{2}-\lambda\big)},\tag{C.1}$$

where $S_2(\lambda)=S_2(\lambda|2,1/\nu)$ stands for the double sine function.

- Massive regime $(\nu\in i\mathbb{R}_{>0})$

$$\rho(\lambda,d)=-\zeta\frac{(q^{2-d}\zeta^{-2})_\infty}{(q^{2-d}\zeta^2)_\infty}\frac{(q^{4-d}\zeta^2)_\infty}{(q^{4-d}\zeta^{-2})_\infty}\tag{C.2}$$

where $(z)_\infty=\prod_{j=0}^\infty(1-zq^{4j})$.

<u>XYZ **case**</u>

- Disordered regime ($\eta, t \in i\mathbb{R}$, $-i\eta > 0$)

$$\rho(t,d) = -e^{2\pi i t}\frac{\gamma((4-d)\eta - 2t)}{\gamma((4-d)\eta + 2t)}\frac{\gamma((2-d)\eta + 2t)}{\gamma((2-d)\eta - 2t)}, \tag{C.3}$$

where $\gamma(u) = \Gamma(u, 4\eta, \tau)$ and

$$\Gamma(u,\sigma,\tau) := \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - u)}}{1 - e^{2\pi i(j\tau + k\sigma + u)}}$$

denotes the elliptic Gamma function.

- Ordered regime ($\eta, t \in \mathbb{R}$, $\eta < 0$)

$$\rho(t,d) = e^{-\frac{4\pi i}{\tau}(d-1)\eta t} \times \rho'(t', d),$$

where $\rho'(t', d)$ is obtained from (C.3) by replacing t, η, τ with

$$t' = \frac{t}{\tau}, \quad \eta' = \frac{\eta}{\tau}, \quad \tau' = -\frac{1}{\tau},$$

respectively.

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