Algebraic Shifting and Sequentially Cohen-Macaulay Simplicial Complexes

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Abstract

Björner and Wachs generalized the definition of shellability by dropping the assumption of purity; they also introduced the *h-triangle*, a doubly-indexed generalization of the h-vector which is combinatorially significant for nonpure shellable complexes. Stanley subsequently defined a nonpure simplicial complex to be *sequentially Cohen-Macaulay* if it satisfies algebraic conditions that generalize the Cohen-Macaulay conditions for pure complexes, so that a nonpure shellable complex is sequentially Cohen-Macaulay.

We show that algebraic shifting preserves the h-triangle of a simplicial complex K if and only if K is sequentially Cohen-Macaulay. This generalizes a result of Kalai's for the pure case. Immediate consequences include that nonpure shellable complexes and sequentially Cohen-Macaulay complexes have the same set of possible h-triangles.

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1 Introduction

A simplicial complex is pure if all of its facets (maximal faces, ordered by inclusion) have the same dimension. Cohen-Macaulayness, algebraic shifting, shellability, and the h-vector are significantly interrelated for pure simplicial complexes. We will be concerned with extending some of these relations to nonpure complexes, but first, we briefly review the pure case. More detailed definitions are in later sections.

A simplicial complex is Cohen-Macaulay if its face-ring is a Cohen-Macaulay ring (an algebraic property), or, equivalently, if the complex satisfies certain topological conditions (see, *e.g.*, [St3, St6]). In particular, the complex must be pure. A pure simplicial complex is shellable if it can be constructed one facet at a time, subject to certain conditions (see, *e.g.*, [Bj1, BW1]). A shellable complex is Cohen-Macaulay, and the h-vector of a Cohen-Macaulay or shellable complex has natural combinatorial interpretations.

Algebraic shifting is a procedure that defines, for every simplicial complex K, a new complex $\Delta(K)$ with the same h-vector as K and a nice combinatorial structure ($\Delta(K)$ is shifted). Additionally, algebraic shifting preserves many algebraic and topological properties of the original complex, including Cohen-Macaulayness; a simplicial complex is Cohen-Macaulay if and only if $\Delta(K)$ is Cohen-Macaulay, which, in turn, holds if and only if $\Delta(K)$ is pure. Thus, it is easy to tell whether K is Cohen-Macaulay, if $\Delta(K)$ is known. (See, *e.g.*, [BK1, BK2].)

Now we are ready for the nonpure case.

Björner and Wachs' generalization of shellability to nonpure simplicial complexes, made by simply dropping the assumption of purity [BW2, BW3], generated a great deal of interest, and sparked the generalization of several other related concepts [SWa, SWe, BS, DR]. In particular, Stanley introduced sequential Cohen-Macaulayness [St6, Section III.2], a nonpure generalization of Cohen-Macaulayness, and designed the (algebraic) definition so that a nonpure shellable complex is sequentially Cohen-Macaulay, much as a shellable complex is Cohen-Macaulay. Meanwhile, joint work with L. Rose [DR] shows that algebraic shifting preserves the h-triangle (a nonpure generalization of the h-vector) of nonpure shellable complexes. These developments prompted A. Björner (private communication) to ask, "Does algebraic shifting preserve sequential Cohen-Macaulayness?" and "Does algebraic shifting preserve the h-triangle of sequentially Cohen-Macaulay simplicial complexes?"

Shifted complexes are nonpure shellable and hence sequentially Cohen-Macaulay, so $\Delta(K)$ is always sequentially Cohen-Macaulay. Thus, the "obvious" generalization, "K is sequentially Cohen-Macaulay if and only if $\Delta(K)$ is sequentially Cohen-Macaulay," is trivially false. Björner's first question may be restated as, "Can one use $\Delta(K)$ to tell if a simplicial complex K is sequentially Cohen-Macaulay?"

Our main result is to answer both of Björner's questions simultaneously, by showing that algebraic shifting preserves the h-triangle of a simplicial complex if and only if the complex is sequentially Cohen-Macaulay (Theorem 5.1).

In Section 2, we introduce basic definitions, including the f-triangle and the h-triangle. Cohen-Macaulayness and sequential Cohen-Macaulayness are discussed in Section 3, and algebraic shifting in Section 4. In Section 5, we prove our main result. Finally, Section 6 contains two corollaries concerning nonpure shellability and iterated Betti numbers (a nonpure generalization of homology Betti numbers), and a conjecture on partitions of sequentially Cohen-Macaulay complexes.

2 Degree and dimension

We start with some basic definitions that are used throughout. A **simplicial complex** K is a collection of finite sets (called faces) such that $F \in K$ and $G \subseteq F$ together imply that $G \in K$. We allow K to be the empty simplicial complex $\emptyset$ consisting of no faces, or the simplicial complex $\{\emptyset\}$ consisting of just the empty face, but we do distinguish between these two cases. A **subcomplex** of K is a subset of faces $L \subseteq K$ such that $F \in L$ and $G \subseteq F$ imply $G \in L$. A subcomplex is a simplicial complex in its own right. An **order filter** of K is a subset of faces $J \subseteq K$ such that $F \in J$ and $F \subseteq G \in K$ imply $G \in J$.

The **dimension** of a face $F \in K$ is $\dim F = |F| - 1$, and the **dimension** of K is $\dim K = \max\{\dim F\colon F \in K\}$. The maximal faces of K (under the set inclusion partial order) are called **facets**, and K is **pure** if all of its facets have the same dimension.

Following [BW2], we define the **degree** of a face $F \in K$ to be $\deg_K F = \max\{|G|\colon F \subseteq G \in K\}$. We further define the **degree** of K to be $\deg K = \min\{\deg_K F\colon F \in K\}$. Note that K is pure if and only if all of its faces have the same degree.

Definition (Björner-Wachs). Let K be a simplicial complex, and let $-1 \leq r, s \leq \dim K$. Then [BW2, Definition 2.8]

$$K^{(r,s)} = \{F \in K\colon \dim F \leq s,\ \deg_K F \geq r+1\}.$$

We may extend this by defining $K^{(r,s)}$ to be the empty simplicial complex when $r > \dim K$.

Clearly, $K^{(r,s)}$ is a subcomplex of K. We will frequently make use of the following special cases, the latter two first considered (though not named) in [BW2]: $K^{(s)} = K^{(-1,s)}$, the **s-skeleton** of K; $K^{<r>} = K^{(r,\dim K)}$, the **rth sequential layer**, the subcomplex of all faces of K whose degree is at least $r+1$ (equivalently, the subcomplex generated by all facets whose dimension is at least r); and $K^{[i]} = K^{(i,i)}$, the **pure i-skeleton**, the pure subcomplex generated by all i-dimensional faces. The notation $K^{[i]}$ is due to Wachs [Wa]. Other interpretations of $K^{(r,s)}$, then, are that $K^{(r,s)} = (K^{<r>})^{(s)}$ and, if $r \geq s$, that $K^{(r,s)} = (K^{[r]})^{(s)}$.

Lemma 2.1. *Let $L \subseteq K$ be a pair of simplicial complexes.*

(a) If $\deg L \geq i+1$, then $L \subseteq K^{<i>}$.

(b) $L^{<i>} \subseteq K^{<i>}$.

Proof. (a): Let $F \in L$. Because $\deg_L F \geq i+1$, there is a face $G \in L$ of dimension at least i containing F. But $G \in K$, too, so $\deg_K F \geq i+1$. Therefore, every face $F \in L$ has degree at least $i+1$ in K as well.

(b): Clearly, $L^{<i>} \subseteq L \subseteq K$ and $\deg L^{<i>} \geq i+1$, so by (a), $L^{<i>} \subseteq K^{<i>}$. $\square$

Let K_j denote the set of j-dimensional faces of K. The **f-vector** of K is the sequence $f(K) = (f_{-1}, \ldots, f_{d-1})$, where $f_j = f_j(K) = \#K_j$ and $d-1 = \dim K$. The **h-vector** of K is the sequence $h(K) = (h_0, \ldots, h_d)$ where

$$h_j = h_j(K) = \sum_{s=0}^{j} (-1)^{j-s} \binom{d-s}{j-s} f_{s-1}(K). \tag{1}$$

Inverting equation (1) gives

$$f_j(K) = \sum_{s=0}^{d} \binom{d-s}{j+1-s} h_s(K),$$

so knowing the h-vector of a simplicial complex is equivalent to knowing its f-vector.

Definition (Björner-Wachs). Let K be a $(d-1)$-dimensional simplicial complex. Define

$$f_{i,j} = f_{i,j}(K) = \#\{F \in K \colon \deg_K F = i,\ \dim F = j-1\}.$$

The triangular integer array $\boldsymbol{f}(K) = (f_{i,j})_{0\le j\le i\le d}$ is the f**-triangle** of K. Further define

$$h_{i,j} = h_{i,j}(K) = \sum_{s=0}^{j} (-1)^{j-s} \binom{i-s}{j-s} f_{i,s}(K). \tag{2}$$

The triangular array $\boldsymbol{h}(K) = (h_{i,j})_{0\le j\le i\le d}$ is the h**-triangle** of K [BW2, Definition 3.1].

Inverting equation (2) gives

$$f_{i,j}(K) = \sum_{s=0}^{i} \binom{i-s}{j+1-s} h_{i,s}(K), \tag{3}$$

so knowing the h-triangle of a simplicial complex is equivalent to knowing its f-triangle. If K is a pure $(d-1)$-dimensional simplicial complex, then every face has degree d, so

$$f_{i,j}(K) = \begin{cases} f_{j-1}(K), & \text{if } i = d, \\ 0, & \text{if } i \ne d \end{cases},$$

and similarly for the h's. Thus, when K is pure, the f-triangle and h-triangle essentially reduce to the f-vector and h-vector, respectively.

Clearly,

$$f_{j-1}(K^{<i-1>}) = \sum_{p=i}^{d} f_{p,j}(K) \tag{4}$$

for all $0 \le j, i \le d$. Inverting equation (4), we get

$$f_{i,j}(K) = f_{j-1}(K^{<i-1>}) - f_{j-1}(K^{<i>}) \tag{5}$$

for all $0 \le j \le i \le d$; this is essentially the same idea as [BW2, equation (3.2)]. In the case $i = d$, equation (5) relies upon the tail condition $f_{j-1}(K^{<d>}) = f_{j-1}(\emptyset) = 0$.

3 Cohen-Macaulayness

Cohen-Macaulayness is an important algebraic concept, but we will use the equivalent algebraic topological characterizations as our definitions. For all undefined topological terms, see [Mu]; for further details on Cohen-Macaulayness, see [St6].

The **pair** (K, L) will denote a pair of simplicial complexes $L \subseteq K$. Let $\boldsymbol{k}$ denote a field, fixed throughout the rest of the paper. Recall that $\widetilde{H}_p(K)$ refers to **reduced homology** of K (over $\boldsymbol{k}$), and $\widetilde{H}_p(K, L)$ denotes **reduced relative homology** of the pair (K, L) (over $\boldsymbol{k}$). For K a simplicial complex, $\widetilde{H}_p(K, \emptyset) = \widetilde{H}_p(K)$; for a pair (K, L) with L non-empty, $\widetilde{H}_p(K, L) = H_p(K, L)$.

The **link** of a face F in a simplicial complex K is defined to be the subcomplex

$$\operatorname{lk}_K F = \{G \in K\colon F \cup G \in K,\ F \cap G = \emptyset\}.$$

For $L \subseteq K$ a pair of subcomplexes and $F \in K$, define the **relative link** of F in L to be

$$\operatorname{lk}_L F = \{G \in L\colon F \cup G \in L,\ F \cap G = \emptyset\}$$

(see Stanley [St4, Section 5]). If $F \in L$, this matches the usual definition of $\operatorname{lk}_L F$, but we now allow the possibility that $F \notin L$, in which case $\operatorname{lk}_L F = \emptyset$.

Reisner [Re] showed that a simplicial complex K is **Cohen-Macaulay** (over $\boldsymbol{k}$) if, for every $F \in K$ (including $F = \emptyset$), $\widetilde{H}_p(\operatorname{lk}_K F) = 0$ for all $p < \dim \operatorname{lk}_K F$; it follows that K is pure. Stanley [St4, Theorem 5.3] showed that a pair of simplicial complexes (K, L) is **relative Cohen-Macaulay** (over $\boldsymbol{k}$) if and only if, for every $F \in K$ (including $F = \emptyset$), $\widetilde{H}_p(\operatorname{lk}_K F,\ \operatorname{lk}_L F) = 0$ for all $p < \dim \operatorname{lk}_K F$. We will take these conditions as our definitions of Cohen-Macaulayness and relative Cohen-Macaulayness, respectively.

It is a well-known consequence of Reisner's condition that every skeleton of a Cohen-Macaulay simplicial complex is again Cohen-Macaulay.

Lemma 3.1. *Let F be a face of a simplicial complex K, and let L be either the empty simplicial complex or a Cohen-Macaulay subcomplex of the same dimension as K. Then*

$$\widetilde{H}_p(\operatorname{lk}_K F) \cong \widetilde{H}_p(\operatorname{lk}_K F,\ \operatorname{lk}_L F)$$

for $p < \dim \operatorname{lk}_K F$.

Proof. If $\operatorname{lk}_L F = \emptyset$ (which is always the case if $L = \emptyset$), then $\widetilde{H}_p(\operatorname{lk}_K F) = \widetilde{H}_p(\operatorname{lk}_K F,\ \emptyset) = \widetilde{H}_p(\operatorname{lk}_K F,\ \operatorname{lk}_L F)$ for all p.

We may as well assume, then, that $\operatorname{lk}_L F \neq \emptyset$; let $G \in \operatorname{lk}_L F$, so $F \mathbin{\dot\cup} G \in L$ (where $\dot\cup$ denotes disjoint union). Because L has the same dimension as K and is pure, $F \mathbin{\dot\cup} G$ is contained in some facet of L of dimension $\dim K$, say $F \mathbin{\dot\cup} H$. Then $H \in \operatorname{lk}_L F$ and $\dim H = \dim \operatorname{lk}_K F$, so $\dim \operatorname{lk}_L F \geq \dim \operatorname{lk}_K F$. But $\operatorname{lk}_L F \subseteq \operatorname{lk}_K F$, and thus $\dim \operatorname{lk}_L F = \dim \operatorname{lk}_K F$.

Now let $p < \dim \operatorname{lk}_K F = \dim \operatorname{lk}_L F$. Because L is Cohen-Macaulay, $\widetilde{H}_p(\operatorname{lk}_L F)$ and $\widetilde{H}_{p-1}(\operatorname{lk}_L F)$ are trivial, so the relative homology long exact sequence of $(\operatorname{lk}_K F,\ \operatorname{lk}_L F)$,

$$\cdots \to \widetilde{H}_p(\operatorname{lk}_L F) \to \widetilde{H}_p(\operatorname{lk}_K F) \to \widetilde{H}_p(\operatorname{lk}_K F,\ \operatorname{lk}_L F) \to \widetilde{H}_{p-1}(\operatorname{lk}_L F) \to \cdots$$

(as in [Mu, Theorem 23.3], for example), becomes

$$\cdots \to 0 \to \widetilde{H}_p(\mathrm{lk}_K F) \to \widetilde{H}_p(\mathrm{lk}_K F,\ \mathrm{lk}_L F) \to 0 \to \cdots .$$

Therefore $\widetilde{H}_p(\mathrm{lk}_K F) \cong \widetilde{H}_p(\mathrm{lk}_K F,\ \mathrm{lk}_L F)$. □

Corollary 3.2. *Let K be a simplicial complex, and let L be either the empty simplicial complex or a Cohen-Macaulay subcomplex of the same dimension as K. Then K is Cohen-Macaulay if and only if (K, L) is relative Cohen-Macaulay.*

Proof. Let $F \in K$. By Lemma 3.1, all lower-dimensional ($p < \dim \mathrm{lk}_K F$) homology vanishes from all the links of K if and only if all lower-dimensional relative homology vanishes from all the relative links of (K, L), so K is Cohen-Macaulay if and only if (K, L) is relative Cohen-Macaulay. □

Definition (Stanley). Let K be a $(d-1)$-dimensional simplicial complex. Then K is **sequentially Cohen-Macaulay** if the pairs

$$\Omega_i(K) = (K^{[i]},\ (K^{[i+1]})^{(i)})$$

are relative Cohen-Macaulay for $-1 \le i \le d-1$ [St6, III.2.9]. In particular, when $i = d-1$, we require $\Omega_{d-1}(K) = (K^{[d-1]},\ \emptyset)$ to be relative Cohen-Macaulay, which is equivalent to $K^{<d-1>} = K^{[d-1]}$ being Cohen-Macaulay, by Corollary 3.2.

Remark. This definition is stated slightly differently from the one given by Stanley [St6], but it is entirely equivalent. In [St6], $\Omega_i^*(K) = (K_i^*,\ K_i^* \cap K^{<i+1>})$ is the pair that is required to be relative Cohen-Macaulay, where K_i^* denotes the subcomplex generated by the i-dimensional facets of K. But by remarks following [St4, Theorem 5.3], relative Cohen-Macaulayness of the pair (K, L) depends only on the difference $K \backslash L$. Both $K^{[i]} \backslash (K^{[i+1]})^{(i)}$ and $K_i^* \backslash K_i^* \cap K^{<i+1>}$ describe the set of faces in K whose degree in K is exactly $i+1$, so $\Omega_i(K)$ is relative Cohen-Macaulay precisely when $\Omega_i^*(K)$ is relative Cohen-Macaulay.

Theorem 3.3. *Let K be a $(d-1)$-dimensional simplicial complex. Then K is sequentially Cohen-Macaulay if and only if its pure i-skeleton $K^{[i]}$ is Cohen-Macaulay for all $-1 \le i \le d-1$.*

Proof. ($\Longrightarrow$): By induction on $(d-1) - i$.

$i = d-1$. By definition of sequential Cohen-Macaulayness, $\Omega_{d-1}(K) = (K^{[d-1]}, \emptyset)$ is relative Cohen-Macaulay. By Corollary 3.2, then, $K^{[d-1]}$ is Cohen-Macaulay.

induction step. Now assume, by way of induction, that $K^{[i+1]}$ is Cohen-Macaulay. Then $(K^{[i+1]})^{(i)}$ is the skeleton of a Cohen-Macaulay complex, and hence Cohen-Macaulay. Since K is sequentially Cohen-Macaulay, $\Omega_i(K) = (K^{[i]}, (K^{[i+1]})^{(i)})$ is relative Cohen-Macaulay, so by Corollary 3.2, $K^{[i]}$ is Cohen-Macaulay.

($\Longleftarrow$): To prove that K is sequentially Cohen-Macaulay, we need to show that every $\Omega_i(K)$ is relative Cohen-Macaulay. There are two cases. If $i = d-1$, then $\Omega_i(K) = (K^{[d-1]}, \emptyset)$ is relative Cohen-Macaulay by Corollary 3.2, since $K^{[d-1]}$ is Cohen-Macaulay.

If $i < d-1$, then $K^{[i+1]}$ and $K^{[i]}$ are Cohen-Macaulay. In that case, $(K^{[i+1]})^{(i)}$ is the skeleton of a Cohen-Macaulay complex, and hence Cohen-Macaulay. Then, by Corollary 3.2, $\Omega_i(K) = (K^{[i]}, (K^{[i+1]})^{(i)})$ is relative Cohen-Macaulay. □

See [Wa] for another characterization of sequential Cohen-Macaulayness, which relies upon Theorem 3.3.

4 Algebraic shifting

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with the same f-vector, and also preserves many algebraic properties of the original complex. Algebraic shifting was introduced by Kalai [Ka1]; our exposition is summarized from [BK1] (see also [BK2, Ka2]).

If $S = \{s_1 < \cdots < s_j\}$ and $T = \{t_1 < \cdots < t_j\}$ are j-subsets of integers, then:

- $S \leq_P T$ under the standard **partial order** if $s_p \leq t_p$ for all p; and
- $S <_L T$ under the **lexicographic order** if there is a q such that $s_q < t_q$ and $s_p = t_p$ for $p < q$.

A collection $\mathcal{C}$ of k-subsets is **shifted** if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex K is **shifted** if the set of j-dimensional faces of K is shifted for every j.

Definition (Kalai). Let K be a simplicial complex with vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. Let $\Lambda(\boldsymbol{k}V)$ denote the exterior algebra of the vector space $\boldsymbol{k}V$; it has a $\boldsymbol{k}$-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $S = \{e_{i_1} < \cdots < e_{i_j}\} \subseteq V$ (and $e_\emptyset = 1$). Let I_K be the ideal of $\Lambda(\boldsymbol{k}V)$ generated by $\{e_S \colon S \notin K\}$, and let $\tilde{x}$ denote the image modulo I_K of $x \in \boldsymbol{k}V$.

Let $\{f_1, \ldots, f_n\}$ be a "generic" basis of $\boldsymbol{k}V$, *i.e.*, $f_i = \sum_{j=1}^n \alpha_{ij} e_j$, where the α_{ij}'s are n^2 transcendentals, algebraically independent over $\boldsymbol{k}$. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $S = \{i_1 < \cdots < i_k\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(K) := \{S \subseteq [n] \colon \tilde{f}_S \notin \operatorname{span}\{\tilde{f}_R \colon R <_L S\}\}$$

be the **algebraically shifted complex** obtained from K. As the name implies, $\Delta(K)$ is a shifted simplicial complex, and it is independent of the numbering of the vertices of K or the choices of α_{ij}.

As is often the case with algebraic shifting, we do not use the definition directly, but rather some theorems that characterize the results of algebraic shifting.

Proposition 4.1 (Kalai). *Let K be a simplicial complex. Then $f_{j-1}(\Delta(K)) = f_{j-1}(K)$ for $j \geq 0$.*

Proof. This is [BK1, Theorem 3.1]. □

Proposition 4.2 (Kalai). *If $L \subseteq K$ are a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$.*

Proof. This is [Ka2, Theorem 2.2]. □

Corollary 4.3. *If $L \subseteq K$ are a pair of simplicial complexes, and L contains all the j-dimensional faces of K, then $\Delta(L)$ is a subcomplex of $\Delta(K)$ containing all the j-dimensional faces of $\Delta(K)$.*

Proof. This follows immediately from Propositions 4.1 and 4.2. □

The following result is the central property of algebraic shifting for our purposes.

Proposition 4.4 (Kalai). *Let K be a simplicial complex. Then K is Cohen-Macaulay if and only if $\Delta(K)$ is pure.*

Proof. This is [Ka2, Theorem 5.3]. □

Corollary 4.5. *Let L be a simplicial complex of dimension at least i ($i \geq -1$). Then $L^{(i)}$ is Cohen-Macaulay if and only if $\deg \Delta(L) \geq i+1$.*

Proof. By Proposition 4.4, $L^{(i)}$ is Cohen-Macaulay if and only if $\Delta(L^{(i)})$ is pure i-dimensional. But Corollary 4.3 implies that $\Delta(L^{(i)}) = \Delta(L)^{(i)}$. And $\Delta(L)^{(i)}$ is pure i-dimensional if and only if $\Delta(L)$ has no facets of dimension less than i, which is equivalent to $\deg \Delta(L) \geq i+1$. □

Theorem 4.6. *Let K be a simplicial complex of dimension at least i ($i \geq -1$). Then*

(a) $\Delta(K)^{<i>} \subseteq \Delta(K^{<i>})$, and

(b) equality holds in part (a) if and only if $\deg \Delta(K^{<i>}) \geq i+1$.

Proof. Because $K^{<i>}$ is a subcomplex of K, it follows that $\Delta(K^{<i>})$ is a subcomplex of $\Delta(K)$, making the complement $\Delta(K)\backslash\Delta(K^{<i>})$ an order filter of $\Delta(K)$. Furthermore, $K^{<i>}$ contains all the faces of K whose dimension is at least i, so by Corollary 4.3, $\Delta(K^{<i>})$ contains all the faces of $\Delta(K)$ whose dimension is at least i. Thus $\Delta(K)\backslash\Delta(K^{<i>})$ is an order filter of $\Delta(K)$, all of whose faces have dimension less than i. Every face in $\Delta(K)\backslash\Delta(K^{<i>})$ has degree in $\Delta(K)$ less than $i+1$, then, so

$$\Delta(K)\backslash\Delta(K^{<i>}) \subseteq \Delta(K)\backslash\Delta(K)^{<i>}.$$

Taking complements establishes part (a).

Next, $\deg \Delta(K)^{<i>} \geq i+1$, so Lemma 2.1(a) applied to the set inclusion in part (a) implies

$$\Delta(K)^{<i>} \subseteq \Delta(K^{<i>})^{<i>}; \tag{6}$$

on the other hand, $\Delta(K^{<i>}) \subseteq \Delta(K)$, so Lemma 2.1(b) implies

$$\Delta(K^{<i>})^{<i>} \subseteq \Delta(K)^{<i>}. \tag{7}$$

Combining inclusions (6) and (7), we get

$$\Delta(K^{<i>})^{<i>} = \Delta(K)^{<i>}. \tag{8}$$

It is easy to see that $\Delta(K^{<i>}) = \Delta(K^{<i>})^{<i>}$ holds precisely when $\deg \Delta(K^{<i>}) \geq i+1$; with equation (8), this establishes part (b). □

5 Main theorem

We now prove our main result.

Theorem 5.1. *Let K be a $(d-1)$-dimensional simplicial complex. Then K is sequentially Cohen-Macaulay if and only if*

$$h_{i,j}(\Delta(K)) = h_{i,j}(K)$$

for all $0 \leq j \leq i \leq d$.

Proof. We show that the following statements are all equivalent:

(a) K is sequentially Cohen-Macaulay;

(b) $K^{[i]} = (K^{<i>})^{(i)}$ is Cohen-Macaulay for all $-1 \leq i \leq d-1$;

(c) $\deg \Delta(K^{<i>}) \geq i+1$ for all $-1 \leq i \leq d-1$;

(d) $\Delta(K)^{<i>} = \Delta(K^{<i>})$ for all $-1 \leq i \leq d-1$;

(e) $f_j(\Delta(K)^{<i>}) = f_j(K^{<i>})$ for all $-1 \leq j, i \leq d-1$;

(f) $f_{i,j}(\Delta(K)) = f_{i,j}(K)$ for all $0 \leq j \leq i \leq d$; and

(g) $h_{i,j}(\Delta(K)) = h_{i,j}(K)$ for all $0 \leq j \leq i \leq d$.

(a) $\iff$ (b) $\iff$ (c) $\iff$ (d): These equivalences are Theorem 3.3, Corollary 4.5, and Theorem 4.6(b), respectively.

(d) $\iff$ (e): By Theorem 4.6(a), $\Delta(K)^{<i>} \subseteq \Delta(K^{<i>})$, so $\Delta(K)^{<i>} = \Delta(K^{<i>})$ if and only if $f_{j-1}(\Delta(K)^{<i>}) = f_{j-1}(\Delta(K^{<i>}))$ for all j. But, by Proposition 4.1, $f_{j-1}(\Delta(K^{<i>})) = f_{j-1}(K^{<i>})$.

(e) $\Longrightarrow$ (f): This follows from equation (5) applied to $\Delta(K)$ and K, respectively. (For the $i = d$ case, we also need that $\Delta(K)^{<d>} = \emptyset = K^{<d>}$ so $f_{j-1}(\Delta(K)^{<d>}) = 0 = f_{j-1}(K^{<d>})$ for all j.)

(f) $\Longrightarrow$ (e): This follows from equation (4) applied to $\Delta(K)$ and K, respectively.

(f) $\iff$ (g): This follows from equations (2) and (3). $\square$

6 Further results

We now discuss two corollaries that follow immediately from Theorem 5.1, and a conjecture suggested by Theorem 5.1. The first corollary is that the characterizations of the h-triangle of nonpure shellable, sequentially Cohen-Macaulay, and shifted complexes coincide. The second corollary extends a result about iterated Betti numbers (a nonpure generalization of reduced homology Betti numbers) from nonpure shellable to sequentially Cohen-Macaulay complexes. The conjecture is that sequentially Cohen-Macaulay complexes can be partitioned into Boolean intervals indexed by the h-triangle.

Shelling.

Many well-known combinatorially defined families of pure simplicial complexes are shellable, and this often provides the easiest way to verify that these complexes have certain nice properties, such as Cohen-Macaulayness (see, *e.g.*, [Bj1, BW1]). Björner and Wachs generalized shellability, simply by dropping the assumption of purity, and showed that many combinatorially interesting nonpure simplicial complexes are nonpure shellable [BW2, BW3]. It was this generalization of shellability that prompted Stanley to define sequentially Cohen-Macaulay complexes, and to design the definition so that nonpure shellable complexes are sequentially Cohen-Macaulay, generalizing the well-known pure result.

Definition (Björner-Wachs). A simplicial complex is **nonpure shellable** if it can be constructed by adding one facet at a time, so that as each facet is added, it intersects the existing complex (previous facets) in a union of codimension 1 faces [BW2, Definition 2.1]. Equivalently, as each facet F is added, a *unique* new minimal face, called the **restriction face** $R(F)$, is added. (Note that the dimension of $R(F)$ is one less than the number of codimension one faces in which F intersects the existing complex when it is added.)

This is the same as the earlier definition of shellability except only that we no longer *require* the complex to be pure, although we do *allow* it to be pure.

The restriction faces are counted by the h-triangle [BW2, Theorem 3.4]: If K is a nonpure shellable $(d-1)$-dimensional complex, then

$$h_{i,j}(K) = \#\{\text{facets } F \in K\colon\ \dim F = i-1,\ \dim R(F) = j-1\},$$

for $0 \le j \le i \le d$. This generalizes the well-known result that the restriction faces of a shellable complex are counted by the h-vector.

Our first application of Theorem 5.1 now follows easily.

Corollary 6.1. *Let $\boldsymbol{h} = (h_{i,j})_{0\le j\le i\le d}$ be an array of integers. Then the following are equivalent:*

(a) $\boldsymbol{h}$ is the h-triangle of a sequentially Cohen-Macaulay simplicial complex;

(b) $\boldsymbol{h}$ is the h-triangle of a nonpure shellable simplicial complex; and

(c) $\boldsymbol{h}$ is the h-triangle of a shifted simplicial complex.

Proof. (c) $\implies$ (b): A shifted complex is nonpure shellable [BW3, Theorem 11.3].

(b) $\implies$ (a): A nonpure shellable complex is sequentially Cohen-Macaulay [St6, Section III.2].

(a) $\implies$ (c): Let K be a sequentially Cohen-Macaulay simplicial complex. Theorem 5.1 implies that $h_{i,j}(K) = h_{i,j}(\Delta(K))$ for all $0 \le i \le j \le d$. Thus $\Delta(K)$ is a shifted complex with the same h-triangle as K. ∎

The pure case of Corollary 6.1 is due to Stanley [St1, Theorem 6]. The proof of Corollary 6.1 is a generalization of Kalai's proof of Stanley's result [Ka2, Corollary 5.2]. It follows from Corollary 6.1 that characterizing the h-triangle (equivalently, characterizing the f-triangle) of sequentially Cohen-Macaulay simplicial complexes is equivalent to characterizing the h-triangle of nonpure shellable complexes or even characterizing the h-triangle of shifted complexes. (See [BW2, Theorem 3.6] and the remarks that follow it, and also [Bj2].)

Iterated Betti numbers.

Iterated Betti numbers are a nonpure generalization of reduced homology Betti numbers ($\widetilde{\beta}_{i-1}(K) = \dim_{\boldsymbol{k}} \widetilde{H}_{i-1}(K)$) introduced in joint work with L. Rose. Although they can be defined as the Betti numbers of a certain chain complex [DR, Section 4], we will take the following equivalent formulation as our definition.

Definition. Let K be a simplicial complex. For a set F of positive integers, let $\mathrm{init}(F) = \max\{r\colon \{1, \ldots, r\} \subseteq F\}$ (so $\mathrm{init}(F)$ measures the largest "initial segment" in F, and is 0 if there is no initial segment, *i.e.*, if $1 \notin F$). Then by [DR, Theorem 4.1], the r**th iterated Betti numbers** of K are

$$\beta_{i-1}[r](K) = \#\{\text{facets } F \in \Delta(K)\colon \dim F = i-1,\ \mathrm{init}(F) = r\}.$$

A special case is $r = 0$; then $\beta_i[0](K) = \widetilde{\beta}_i(K)$, the (ordinary) Betti numbers of reduced homology.

Björner and Wachs [BW2, Theorem 4.1] showed that if K is nonpure shellable, then

$$\widetilde{\beta}_{i-1}(K) = h_{i,i}(K), \tag{9}$$

for $0 \le i \le d$. Equation (9) is generalized in [DR, Theorem 1.2] to

$$\beta_{i-1}[r](K) = h_{i,i-r}(K) \tag{10}$$

for nonpure shellable K. This algebraic interpretation of the h-triangle of nonpure shellable complexes was part of the motivation for iterated Betti numbers. Theorem 5.1 allows us to generalize even further, by weakening the assumption on K in equation (10) from being nonpure shellable to being merely sequentially Cohen-Macaulay.

Corollary 6.2. *If K is sequentially Cohen-Macaulay, then $\beta_{i-1}[r](K) = h_{i,i-r}(K)$.*

Proof. By [DR, Theorem 5.4], $\beta_{i-1}[r](K) = h_{i,i-r}(\Delta(K))$, for all simplicial complexes K. Then apply Theorem 5.1. In fact, Theorem 5.1 shows that the class of sequentially Cohen-Macaulay complexes is the largest class of complexes for which equation (10) holds for all i and r. $\square$

Collapsing.

Finally, we present a conjecture inspired by Theorem 5.1 and by collapsing, which is related to nonpure shelling.

Definition (Kalai). A face R of a simplicial complex K is **free** if it is included in a unique facet F. The empty set is a free face of K if K is a simplex. (This definition is slightly nonstandard in that facets are themselves free.) If $|R| = p$ and $|F| = q$, then we say R is of **type** (p, q). A (p, q)-**collapse step** is the deletion from K of a free face of type (p, q) and all faces containing it (*i.e.*, the deletion of the interval $[R, F]$). A **collapsing sequence** is a sequence of collapse steps that reduce K to the empty simplicial complex [Ka2, Section 4].

A nonpure shelling of K gives rise to a canonical collapsing (though not conversely): If $F_1, \ldots, F_t$ is a nonpure shelling order on the facets of K, then

$$[R(F_t), F_t], \ldots, [R(F_1), F_1]$$

is a collapsing sequence of K [DR, Lemma 5.5], [Ka2, Section 4]. Since $\Delta(K)$ is shifted and hence nonpure shellable, $\Delta(K)$ has a collapsing sequence whose types are given by $\boldsymbol{h}(\Delta(K))$. Kalai has conjectured that K must have a partition into Boolean intervals of the same type as a collapse sequence of $\Delta(K)$ [Ka2, Section 9.3]. Kalai's conjecture and Theorem 5.1 would then imply the following conjecture.

Conjecture 6.3. *A sequentially Cohen-Macaulay complex K can be partitioned into a collection of Boolean intervals (indexed by the set A)*

$$K = \dot{\cup}_{a \in A} [R_a, F_a], \tag{11}$$

such that

$$h_{i,j}(K) = \#\{a \in A \colon |F_a| = j,\ |R_a| = i\} \tag{12}$$

and every F_a is a facet in K.

It is not hard to see that if K is sequentially Cohen-Macaulay and has the partition (11), then the partition satisfies equation (12) if and only if every F_a is a facet.

This is the nonpure generalization of a conjecture made (separately) by Garsia [Ga, Remark 5.2] and Stanley [St2, p. 149], that a Cohen-Macaulay complex can be partitioned into Boolean intervals whose tops are facets (see also [St5, Du]). Conjecture 6.3 is equivalent to being able to partition a relative Cohen-Macaulay complex into Boolean intervals whose tops are facets.

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References

[Bj1] A. Björner, "Shellable and Cohen-Macaulay partially ordered sets," *Trans. Amer. Math. Soc.* **260** (1980), 159–183.

[Bj2] A. Björner, "Face numbers, Betti numbers and depth," in preparation.

[BK1] A. Björner and G. Kalai, "An extended Euler-Poincaré Theorem," *Acta Math.* **161** (1988), 279–303.

[BK2] A. Björner and G. Kalai, "On f-vectors and homology," in *Combinatorial Mathematics: Proceedings of the Third International Conference* (G. Bloom, R. Graham, J. Malkevitch, eds.); *Ann. N. Y. Acad. Sci.* **555** (1989), 63–80.

[BS] A. Björner and B. Sagan, "Subspace arrangements of type B_n and D_n," *J. Alg. Comb.*, to appear.

[BW1] A. Björner and M. Wachs, "On lexicographically shellable posets," *Trans. Amer. Math. Soc.* **277** (1983), 323–341.

[BW2] A. Björner and M. Wachs, "Shellable nonpure complexes and posets, I" *Trans. Amer. Math. Soc.* **348** (1996), 1299–1327.

[BW3] A. Björner and M. Wachs, "Shellable nonpure complexes and posets, II" *Trans. Amer. Math. Soc.*, to appear.

[Du] A. Duval, "A combinatorial decomposition of simplicial complexes," *Israel J. Math.* **87** (1994), 77–87.

[DR] A. Duval and L. Rose, "Iterated homology of simplicial complexes," preprint, 1996.

[Ga] A. Garsia, "Combinatorial methods in the theory of Cohen-Macaulay rings," *Adv. in Math.* **38** (1980), 229–266.

[Ka1] G. Kalai, "Characterization of f-vectors of families of convex sets in $\mathbf{R}^d$, Part I: Necessity of Eckhoff's conditions," *Israel J. Math.* **48** (1984), 175–195.

[Ka2] G. Kalai, Algebraic Shifting, unpublished manuscript (July 1993 version).

[Mu] J. Munkres, *Elements of Algebraic Topology,* Benjamin/Cummings, Menlo Park, CA, 1984.

[Re] G. Reisner, *Cohen-Macaulay quotients of polynomial rings,* thesis, University of Minnesota, 1974; *Adv. Math.* **21** (1976), 30–49.

[St1] R. Stanley, "Cohen-Macaulay complexes," in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Dordrecht and Boston, 1977, 51–62.

[St2] R. Stanley, "Balanced Cohen-Macaulay complexes," *Trans. Amer. Math. Soc.* **249** (1979), 139–157.

[St3] R. Stanley, "The number of faces of simplicial polytopes and spheres," in *Discrete Geometry and Convexity* (J. Goodman, *et. al.*, eds.); *Ann. N. Y. Acad. Sci.* **440** (1985), 212–223.

[St4] R. Stanley, "Generalized h-vectors, intersection cohomology of toric varieties, and related results," in *Commutative Algebra and Combinatorics* (M. Nagata and H. Matsumura, eds.), Advanced Studies in Pure Mathematics **11**, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, 187–213.

[St5] R. Stanley, "A combinatorial decomposition of acyclic simplicial complexes," *Disc. Math.* **120** (1993), 175–182.

[St6] R. Stanley, *Combinatorics and Commutative Algebra, 2nd ed.*, Birkhäuser, Boston, 1995.

[SWa] S. Sundaram and M. Wachs, "The homology representations of the k-equal partition lattice," *Trans. Amer. Math. Soc.*, to appear.

[SWe] S. Sundaram and V. Welker, "Group actions on arrangements of linear subspaces and applications to configuration spaces," *Trans. Amer. Math. Soc.*, to appear.

[Wa] M. Wachs, "Whitney homology of semipure shellable posets," preprint, 1996.